Exercise 5.2

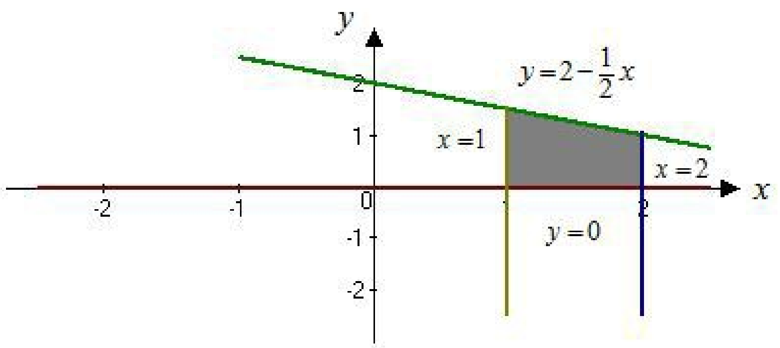
Answer 1E.

Consider the following equations of curves:

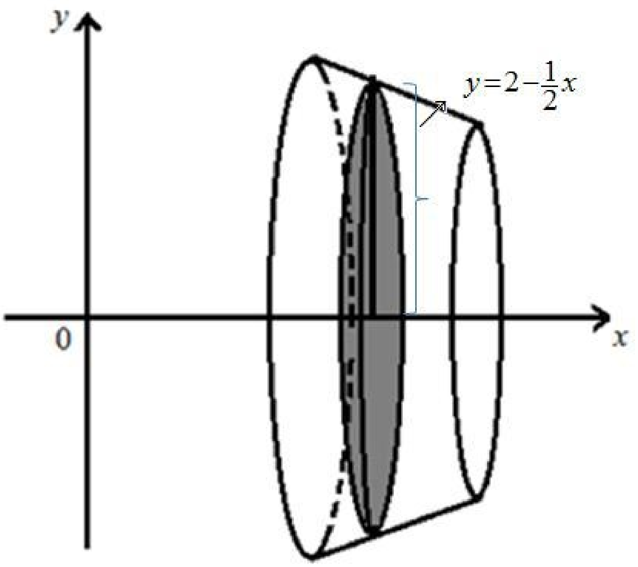
$y = 2 - \frac{1}{2}x$, $y = 0$, $x = 1$, $x = 2$

The objective is to find the volume of the solid obtained by rotating the region bounded by the given curves about *x*-axis.

The region bounded by the curves $y = 2 - \frac{1}{2}x$, $y = 0$, $x = 1$ and $x = 2$ is shown below:



The solid obtained by rotating the shaded region about the *x*-axis is shown below:



Radius of the typical disk is,

$$R = 2 - \frac{1}{2}x - 0$$
$$= 2 - \frac{1}{2}x$$

So, the area of a typical disk is calculated as follows:

$$A(x) = \pi R^2$$
$$= \pi\left(2 - \frac{1}{2}x\right)^2$$
$$= \pi\left[4 + \frac{x^2}{4} - 2x\right]$$

The solid region rotates between the line *x*=1 and *x*=2.

The volume of the solid obtained by rotating the shaded region about the *x*-axis is calculated as follows:

$$V = \int_1^2 A(x)dx$$
$$= \int_1^2 \left(4 + \frac{x^2}{4} - 2x\right)dx$$
$$= \pi\left[4x + \frac{1}{4}\left(\frac{x^3}{3}\right) - 2\left(\frac{x^2}{2}\right)\right]_1^2$$
$$= \pi\left[4x + \frac{x^3}{12} - x^2\right]_1^2$$
$$= \pi\left[\left(8 + \frac{8}{12} - 4\right) - \left(4 + \frac{1}{12} - 1\right)\right]$$
$$= \pi\left[\frac{56}{12} - \frac{37}{12}\right]$$
$$= \frac{19\pi}{12}$$

Therefore, the volume of the required solid is $\boxed{\frac{19\pi}{12}}$.

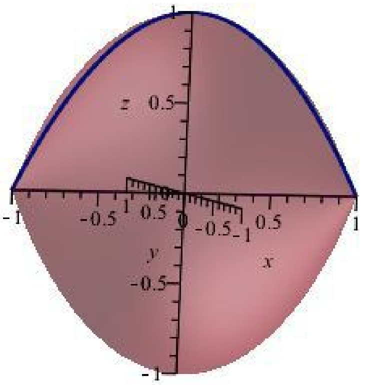
Answer 2E.

From the above Graph the point of intersection is $(-1,1)$.

The area of the cross section about x-axis is,

$$A(x)=\pi\left(1-x^2\right)^2$$

Graph of the solid is as shown below.



Calculate the volume of the solid as follows:

The volume of the solid is,

$$V=\int_{x=a}^{b} A(x)dx$$

The region lies between from $x=-1$ to $x=1$.

Substitute these values in $V=\int_{x=a}^{b} A(x)dx.$

That implies,

$$\begin{aligned}V&=\int_{x=-1}^{1} \pi\left(1-x^2\right)^2 dx\\&=\pi\int_{x=-1}^{1}\left(1-x^2\right)^2 dx\\&=\pi\int_{x=-1}^{1}\left(1-2x^2+x^4\right)dx\\&=2\pi\int_{x=0}^{1}\left(1-2x^2+x^4\right)dx\end{aligned}$$

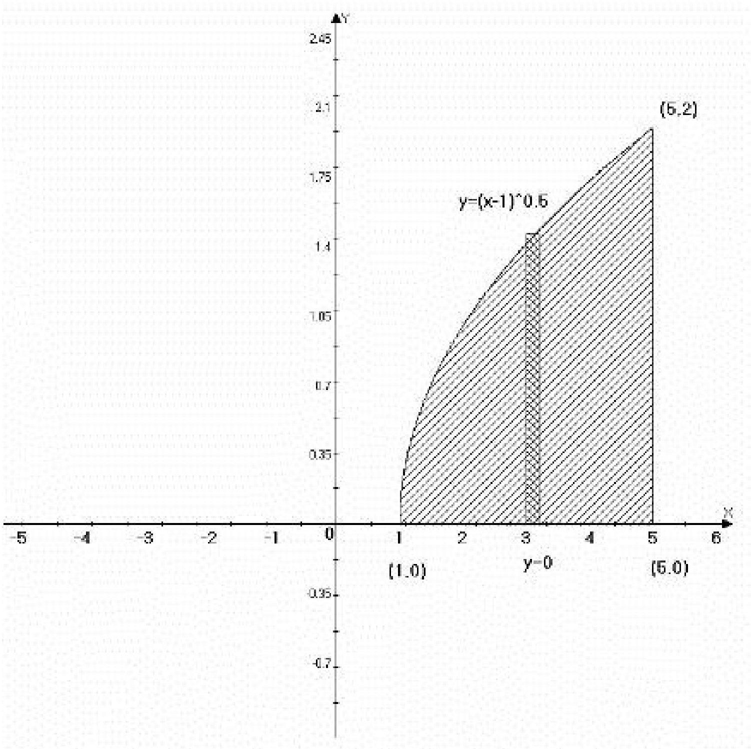
$$\begin{aligned}&=2\pi\left(x-2\left(\frac{x^3}{3}\right)+\frac{x^5}{5}\right)_0^1\\&=2\pi\left(1-\frac{2}{3}+\frac{1}{5}\right)\\&=2\pi\left(\frac{8}{15}\right)\\&=\frac{16\pi}{15}\end{aligned}$$

Therefore, the volume of the solid is $\boxed{V=\frac{16\pi}{15}}$.

Answer 3E.

Here we want to find the volume of solid obtained by rotating the given curve $y=\sqrt{x-1}, y=0, x=5$ about about x-axis.

The region is as follows:



In the graph y=(x-1) ^0.5 denotes $y=\sqrt{x-1}$.

Area of cross section through x is

$$A(x)=\pi y^2$$
$$=\pi(x-1)$$

And the volume of approximating cylinder is

$$A(x)\Delta x=\pi(x-1)\Delta x$$

Since the solid lies between $x=1$ and $x=5$, it volume

$$v=\int_{x=1}^{5} A(x)dx$$
$$=\int_{x=1}^{5} \pi(x-1)dx$$
$$=\pi\left(\frac{x^2}{2}-x\right)_1^5$$
$$=\pi\left(\frac{25}{2}-5-\frac{1}{2}+1\right)$$
$$=8\pi$$

$\boxed{\text{Volume} = 8\pi \text{ cubic units}}$

Answer 4E.

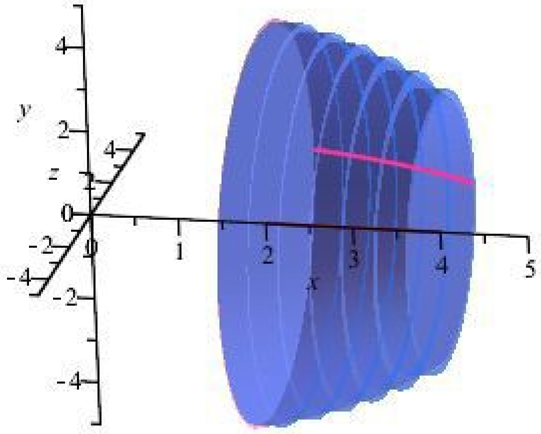
Consider the curves $y=\sqrt{25-x^2}, y=0, x=2$ and $x=4$.

The objective is to find the volume of the solid by rotation the region bounded by the curves about the $x-$ axis.

To find the volume generated by rotating the region about $x-$ axis, use the disk method formula $V=\int_a^b A(x)dx=\int_a^b \pi r^2 dx$.

Draw the following figure:

The 3 dimentional diagram is as shown below.



The area of the cross section is,

$$A(x)=\pi\left(\sqrt{25-x^2}\right)^2$$
$$=\pi\left(25-x^2\right)$$

The volume of the approximating a disk with thickness Δx is $A(x)\Delta x=\pi\left(25-x^2\right)\Delta x$.

The solid lies between $x=2$ and $x=4$.

The volume of the solid by rotating the region about $x-$ axis is,

$$V=\int_2^4 \pi\left(25-x^2\right)dx$$
$$=\pi\left(25x-\frac{x^3}{3}\right)_2^4$$
$$=\pi\left(\left(25(4)-\frac{(4)^3}{3}\right)-\left(25(2)-\frac{(2)^3}{3}\right)\right)$$
$$=\pi\left(\left(100-\frac{64}{3}\right)-\left(50-\frac{8}{3}\right)\right)$$
$$=\pi\left(\frac{236}{3}-\frac{142}{3}\right)$$
$$=\frac{94\pi}{3}$$

Hence, the volume of the solid is $\boxed{\frac{94}{3}\pi}$.

Answer 5E.

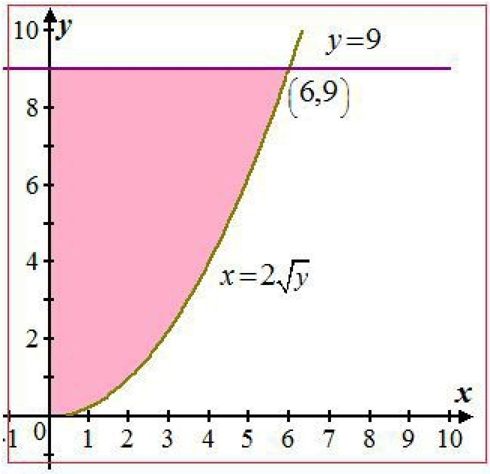
Consider the following curve:

$x = 2\sqrt{y}$, $x = 0, y = 9$

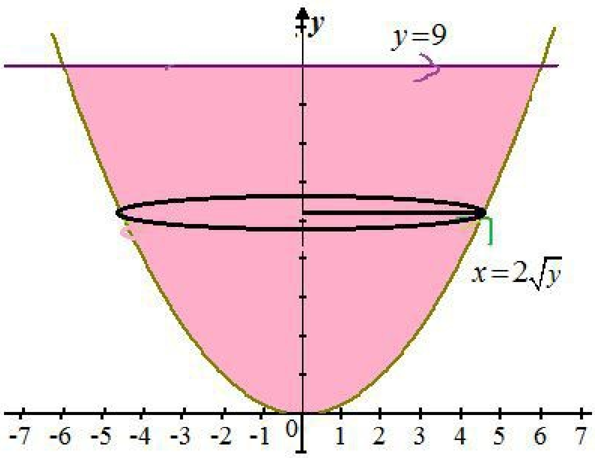
So, the region rotated about the *y*-axis.

The objective is to find the volume of the solid obtained by the region.

Graph of the area of the cross section.



After rotation of graph looks like this,



The area of the cross section about *y*-axis is,

$$A(y) = \pi\left(\left(2\sqrt{y}\right)^2\right)$$
$$= \pi(4y)$$

Calculate the volume of the solid is,

The region lies between $y = 0$ to 9

$$V = \int_{y=0}^{9} A(y)\,dy$$
$$= \int_{y=0}^{9} 4\pi y\,dy$$
$$= 4\pi \int_{y=0}^{9} y\,dy$$
$$= 4\pi\left[\frac{y^2}{2}\right]_0^9$$
$$= 4\pi\left[\frac{81}{2}\right]$$
$$= 162\pi$$

Therefore, the volume of the solid is, $\boxed{V = 162\pi}$

Answer 6E.

First we sketch the curves $x = y - y^2$ and $x = 0$

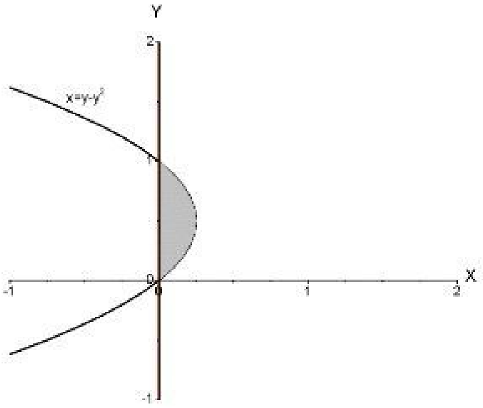


Fig.1

In this figure shaded region is enclosed by $x = y - y^2$ and $x = 0$
Y – Intercepts of the curve are $y = 0$ and $y = 1$

Now we sketch the solid obtained by rotating the shaded region about y – axis

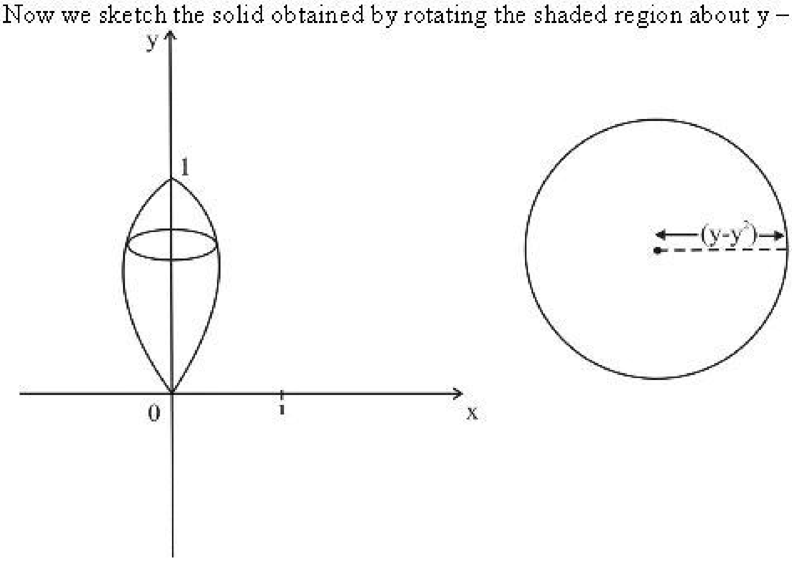


Fig. 2

This is figure shows the slope of solid obtained by rotating the region Bounded by given curves about y – axis and a typical disk

This cross sectional area of a typical disk is

$$A(y) = \pi \times (\text{radius})^2$$

Or $A(y) = \pi \times (y - y^2)^2$

Or $A(y) = \pi \times (y^2 + y^4 - 2y^3)$

Then volume of the solid is $v = \int_0^1 A(y)\,dy$

Or $v = \int_0^1 \pi (y^2 + y^4 - 2y^3)\,dy$

Or $v = \pi \int_0^1 (y^2 + y^4 - 2y^3)\,dy$

Or $v = \pi \left[\frac{y^3}{3} + \frac{y^5}{5} - \frac{2}{4} y^4 \right]_0^1$ [By Fundamental theorem of Calculus part 2]

Or $v = \pi \left[\left(\frac{1}{3} + \frac{1}{5} - \frac{1}{2} \right) - 0 \right]$

Or $\boxed{v = \frac{\pi}{30}}$

Answer 7E.

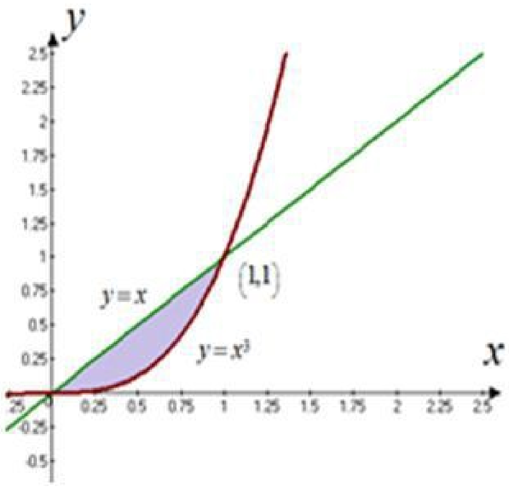
Consider the curves,

$y = x^3, y = x, x \geq 0$

The region rotated about the $x-$ axis.

The objective is to find the volume of the solid obtained by rotating the given curves about the $x-$axis.

The graph of the curve is as shown below.



Find the point of intersection to use the curves $y = x^3, y = x$ as follows:

$$x^3 = x$$
$$x(x^2 - 1) = 0$$
$$x = 0, x^2 - 1 = 0$$
$$x = 0, x = \pm 1$$

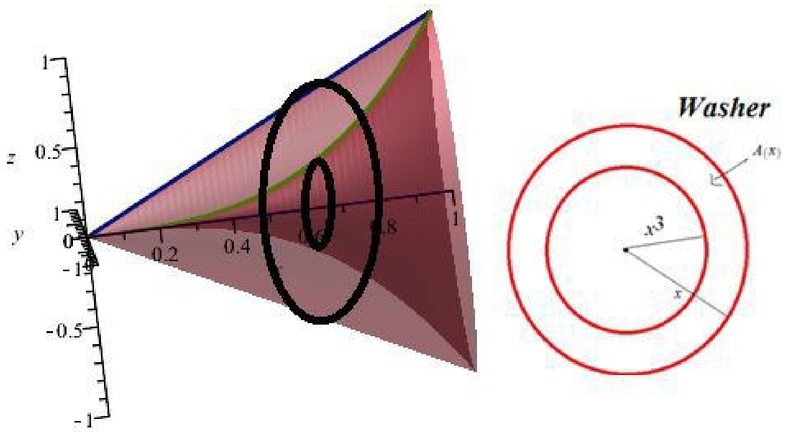
The point of intersection of these points is $(0,0),(1,1)$.

It makes sense to slice the solid perpendicular to the *x*-axis and therefore to integrate with respect to *x*.

So the area of a cross-section is,

$$\begin{aligned} A &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\ &= \pi(x)^2 - \pi(x^3)^2 \\ &= \pi x^2 - \pi x^6 \\ &= \pi(x^2 - x^6) \end{aligned}$$

Graph of the solid is as shown below.



Calculate the volume of the solid as follows:

The volume of the solid is,

$$V = \int_{x=a}^{b} A(x)\,dx$$

The region lies between from $x=0$ to $x=1$.

Substitute these values in $V = \int_{x=a}^{b} A(x)\,dx.$

That implies,

$$V = \int_0^1 \pi\left((x)^2 - (x^3)^2\right)dx$$

$$= \pi\int_0^1 \left(x^2 - x^6\right)dx$$

$$= \pi\left[\frac{x^3}{3} - \frac{x^7}{7}\right]_0^1$$

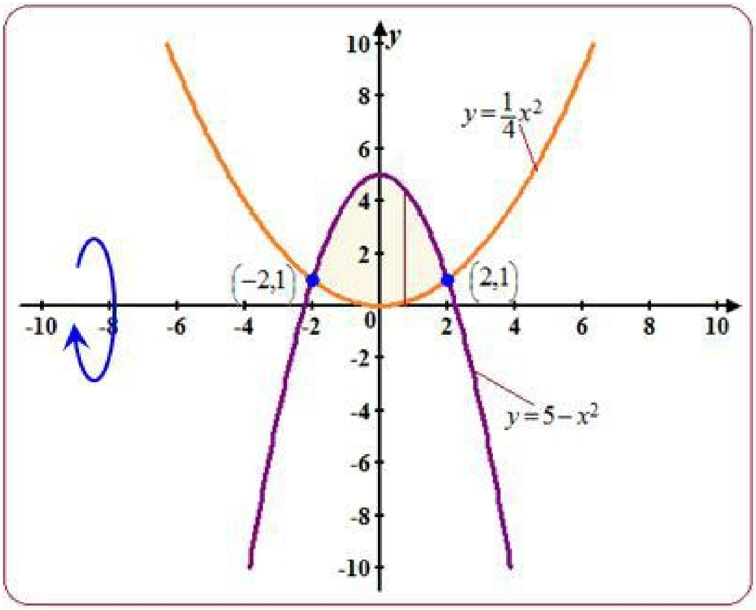
$$= \pi\left[\frac{1}{3} - \frac{1}{7}\right]$$

$$= \frac{4\pi}{21}$$

Therefore, the volume of the solid is, $\boxed{V = \frac{4\pi}{21}}$

Answer 8E.

Sketch the region bounded by the curves $y = \frac{1}{4}x^2$ and $y = 5 - x^2$ as shown below:

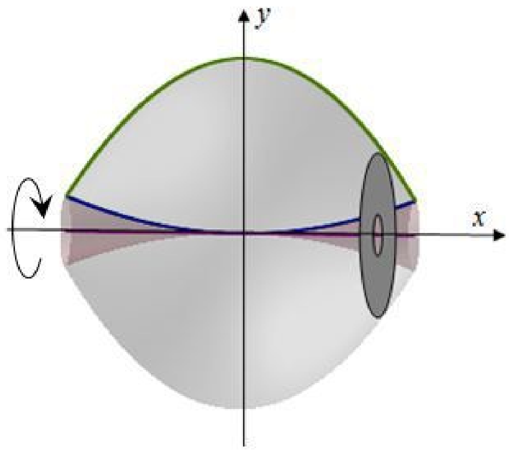


The curves $y = \frac{1}{4}x^2$ and $y = 5 - x^2$ intersect at the points $(2,1)$ and $(-2,1)$.

Find the volume of the solid of revolution about the line $y = 0$.

Rotate around a horizontal line, $y=0$ to obtain a vertical rectangle (perpendicular to the axis of rotation), which will create a disk or washer.

The solid of revolution is as shown below:



A cross-section in the plane P_x has the shape of a washer with inner radius $\frac{1}{4}x^2$ and outer radius $5-x^2$.

Find the cross-sectional area by subtracting the area of the inner circle from the area of the outer circle as,

$$A(x)=\pi(5-x^2)^2-\pi\left(\frac{1}{4}x^2\right)^2.$$

The volume of the solid is calculated as follows:

$$\begin{aligned}V&=\int_{-2}^{2}A(x)dx\\&=\int_{-2}^{2}\pi\left[(5-x^2)^2-\left(\frac{1}{4}x^2\right)^2\right]dx\\&=\int_{-2}^{2}\pi\left[25-10x^2+x^4-\frac{1}{16}x^4\right]dx\\&=\pi\int_{-2}^{2}\left[25-10x^2+\frac{15}{16}x^4\right]dx\end{aligned}$$

Use the identity $\int x^n dx=\frac{x^{n+1}}{n+1}+C.$

$$\begin{aligned}V&=\pi\left[25x-10\left(\frac{x^3}{3}\right)+\frac{15}{16}\left(\frac{x^5}{5}\right)\right]_{-2}^{2}\\&=\pi\left[25x-\frac{10}{3}x^3+\frac{3}{16}x^5\right]_{-2}^{2}\\&=\pi\left[25(2+2)-\frac{10}{3}(8+8)+\frac{3}{16}(32+32)\right]\\&=\pi\left[100-\frac{160}{3}+\frac{192}{16}\right]\\&=\pi\left[100-\frac{160}{3}+12\right]\\&=\frac{176}{3}\pi\end{aligned}$$

Therefore, volume of the solid obtained by rotating $x-$ axis is $\boxed{\frac{176}{3}\pi}$.

Answer 9E.

First we sketch the curve $x = y^2$ & $x = 2y$

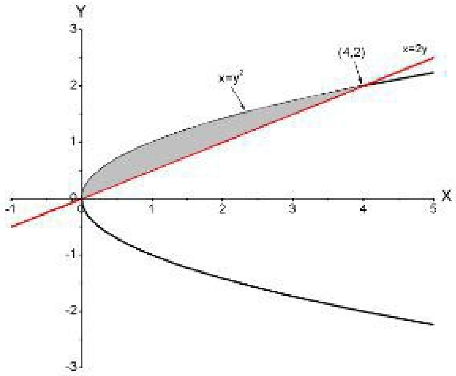


Fig.1

In this figure the shaded region is bounded by the given curves
Now we find the points of intersection of the curves
These curves will intersect, when

$$y^2 = 2y$$

Or $y^2 - 2y = 0$

Or $y(y-2) = 0$

Or $y = 0 \text{ or } y = 2$

So the shaded region lies in the interval $0 \le y \le 2$

Now we sketch the solid obtained by rotating the shaded region about y – axis

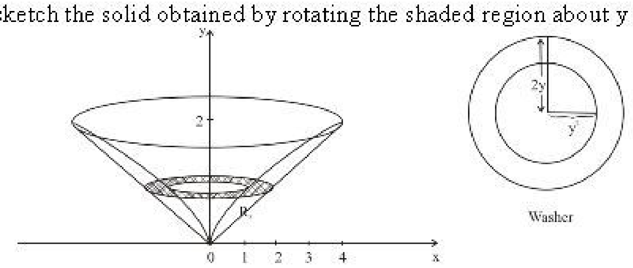


Fig. 2

In this figure the shape of solid obtained by rotating the shaded region about y – axis and a washer are shown

The cross-sectional area of washer is

$$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

$$A(y) = \pi(2y)^2 - \pi(y^2)^2$$

$$A(y) = 4\pi y^2 - \pi y^4$$

Or $A(y) = \pi(4y^2 - y^4)$

Then the volume of solid is

$$v = \int_0^2 A(y)\,dy$$

Or $v = \int_0^2 \pi(4y^2 - y^4)\,dy$

Or $v = \pi\int_0^2 (4y^2 - y^4)\,dy$

Or $v = \pi\left[\frac{4}{3}y^3 - \frac{y^5}{5}\right]_0^2$ [By FTC – 2]

Or $v = \pi\left[\frac{4}{3}(2)^3 - \frac{2^5}{5}\right]$

Or $\boxed{v = \frac{64}{15}\pi}$ or $\boxed{v \approx (4.27)\pi}$

Answer 10E.

Consider the bounded curves are $y=\frac{1}{4}x^2$, $y=0$ and $x=2$, and the rotation is about $y-$axis.

Need to find the volume of the solid obtained by rotating the region bounded by the curves about the given line:

Find the points of intersection of the given curves:

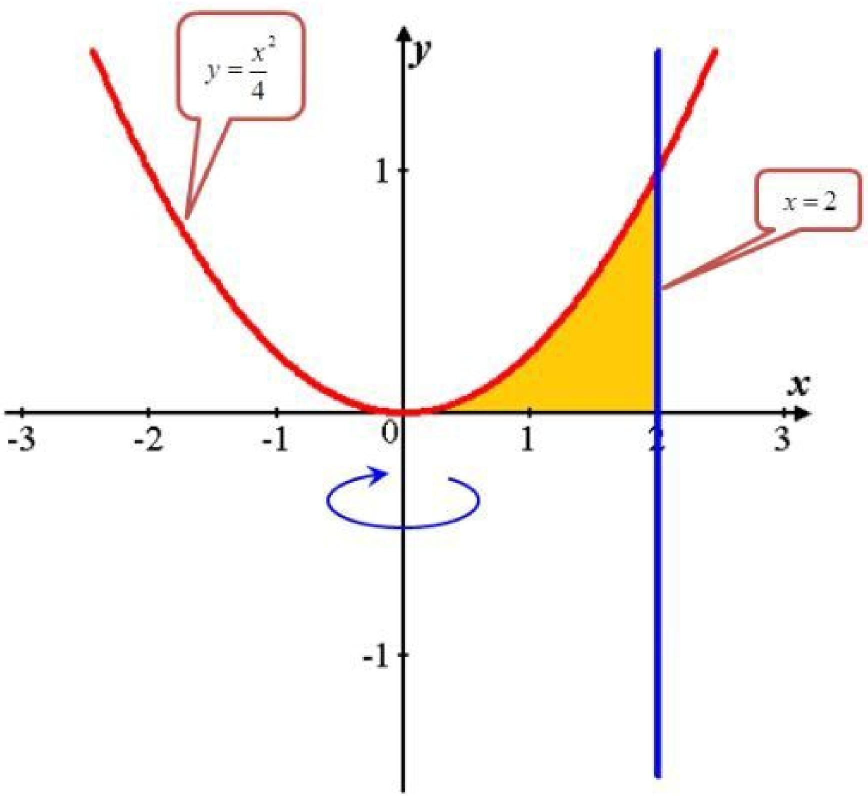
$$y=\frac{1}{4}(2)^2 \quad [\text{Substitute } x=2]$$
$$y=1$$

The curves $y=\frac{1}{4}x^2$ and $x=2$, intersect at the points $(0,0)$ and $(2,1)$. the region between them, the solid of rotation, and a cross-section perpendicular to the y-axis are shown in figure-1. Across-section in the plane has the shape of a **washer** (an annular ring) with inner radius $2\sqrt{y}$ and outer radius 2, so find the cross-sectional area by subtracting the area of the inner circle from the area of the outer circle.

$$A(y)=\pi(2)^2-\pi\left(2\sqrt{y}\right)^2$$

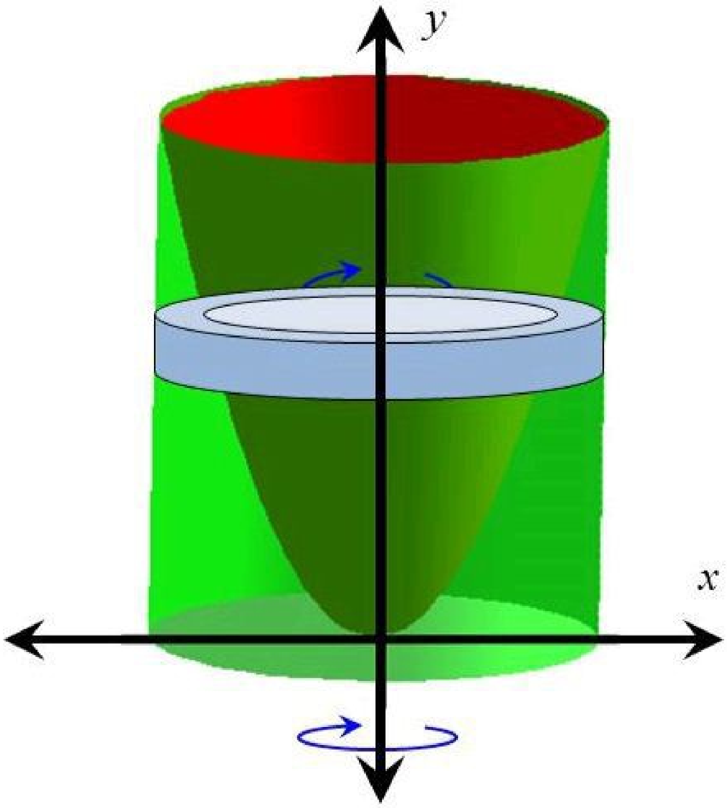
The bounded region is as follows:

Figure-1



The solid formed by rotating the above region is as shown below:

Figure-2



Since the solid lies between $y=0$ and $y=1$, its volume is calculated using the washer Formula which is

$$V=\int_{y=a}^{y=b} A(y)\,dy$$

$$V=\int_{y=a}^{y=b} \left(\pi r^2_{outer} - \pi r^2_{inner}\right) dy$$

$$=\int_0^1 \left(\pi(2)^2 - \pi\left(2\sqrt{y}\right)^2\right) dy$$

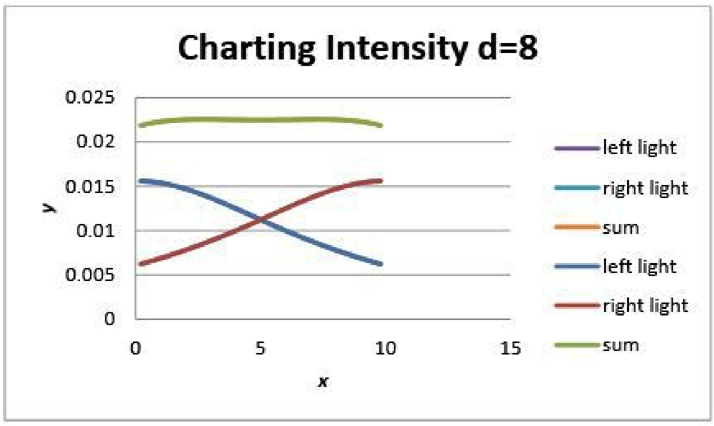
$$=\pi\int_0^1 \left(2^2 - \left(2\sqrt{y}\right)^2\right) dy$$

$$=\pi\left[4y-2y^2\right]_0^1$$

$$=\pi\left[4(1)-2(1)^2-4(0)+2(0)^2\right]$$

$$=\boxed{2\pi \text{ unit}^3}$$

For $d=8$, graph is as follows.



Now find the exact value of d which is a minimum. Take the derivative of the intensity function with respect to x. Then find the second derivative, to see where d has a sudden change, and a point of inflection.

$$I = \frac{F_1}{x_1^2 + d^2} + \frac{F_1}{100 + x_1^2 - 20x_1 + d^2}$$

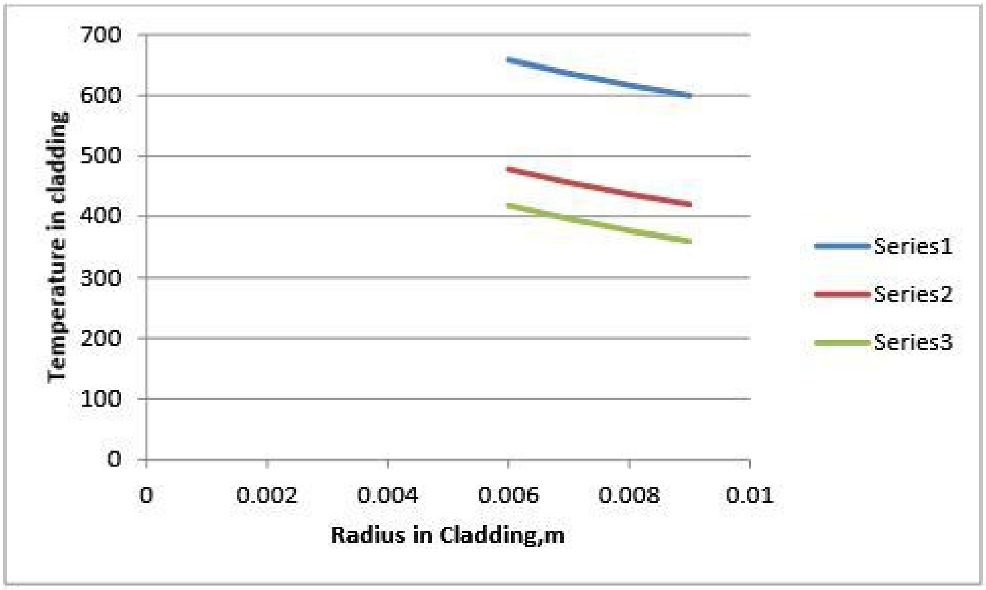
$$I'(x,d) = \frac{-2x_1F_1}{\left(x_1^2 + d^2\right)^2} - \frac{2F_1(x_1 - 10)}{\left((x_1 - 10)^2 + d^2\right)^2}$$

$$I''(x,d) = \frac{+4x_1^2F_1}{\left(x_1^2 + d^2\right)^3} + \frac{4F_1(x_1 - 10)^2}{\left((x_1 - 10)^2 + d^2\right)^3}$$

Continue to solve for d.

$$\boxed{0 = \frac{x_1^2}{\left(x_1^2 + d^2\right)^3} + \frac{(x_1 - 10)^2}{\left((x_1 - 10)^2 + d^2\right)^3}}$$

Plot the temperature distribution in the cladding for different values of heat transfer coefficient,

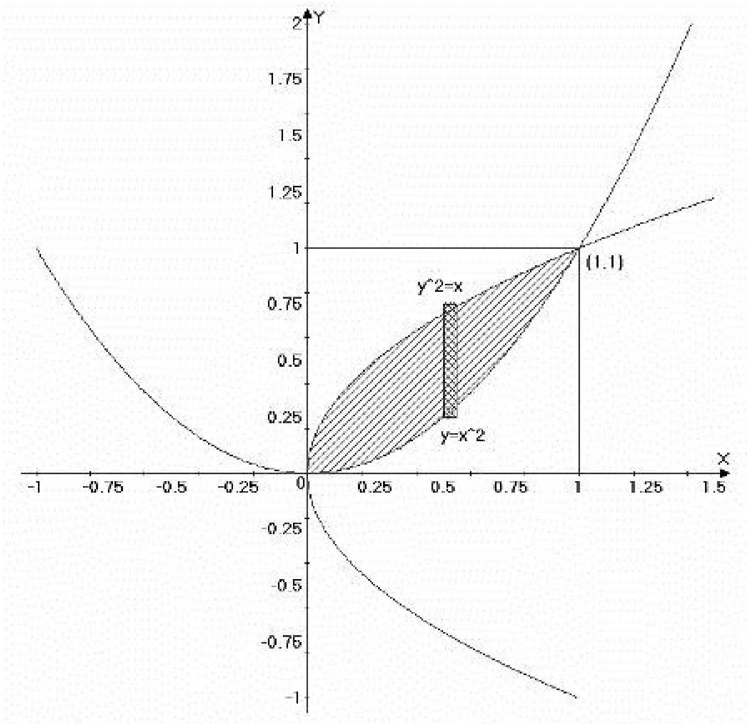


Answer 11E.

Here we want to find the volume of the solid obtained by rotating the region bounded by the given curves $y = x^2, x = y^2$ about the line $y = 1$.

The points of intersection of the curves are $(0,0),(1,1)$.

The region is as follows:



In the graph y^2 =x denotes $x=y^2$ and y=x^2 denotes $y=x^2$.
The figure shows outer cross-section. It is a washer with inner radius $y=1-\sqrt{x}$ and outer radius $y=1-x^2$, so the cross section area is

$$\begin{aligned} A(x) &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\ &= \pi\left(1-x^2\right)^2 - \pi\left(1-\sqrt{x}\right)^2 \\ &= \pi\left[1+x^4-2x^2-1+2\sqrt{x}-x\right] \\ &= \pi\left[x^4-2x^2-x+2\sqrt{x}\right] \end{aligned}$$

Hence volume $V=\int_0^1 A(x)\,dx$

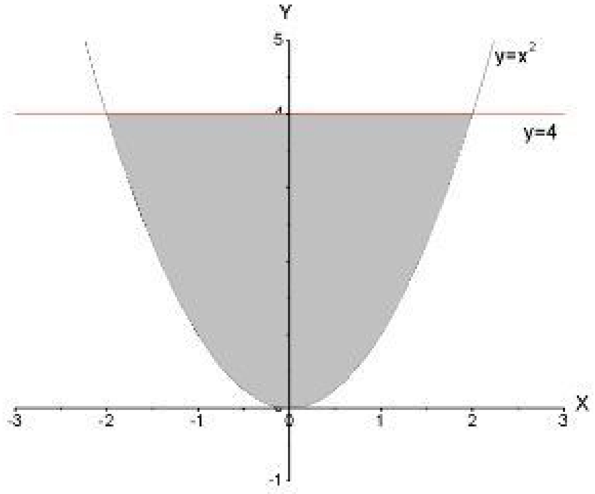
$$\begin{aligned} &= \int_0^1 \pi\left[x^4-2x^2-x+2\sqrt{x}\right]dx \\ &= \pi\left[\frac{x^5}{5}-2\cdot\frac{x^3}{3}-\frac{x^2}{2}+2\cdot\frac{x^{3/2}}{3/2}\right]_0^1 \\ &= \pi\left[\frac{x^5}{5}-\frac{2}{3}x^3-\frac{1}{2}x^2+\frac{4}{3}x^{3/2}\right]_0^1 \end{aligned}$$

$$\begin{aligned} &= \pi\left[\frac{1}{5}-\frac{2}{3}-\frac{1}{2}+\frac{4}{3}\right] \\ &= \frac{\pi}{30}[-20-15+40 \\ &= \frac{11\pi}{30} \end{aligned}$$

$$\boxed{\text{Volume} = \frac{11\pi}{30}\,\text{cubic units}}$$

Answer 12E.

First we sketch the curves $y = x^2$ and $y = 4$



In this figure the shaded region is bounded by the given curves. Clearly the points of intersection are (-2, 4) and (2, 4)
So this region lies in the interval [-2, 2]

Now we sketch the solid obtained by rotating this shaded region about y = 4

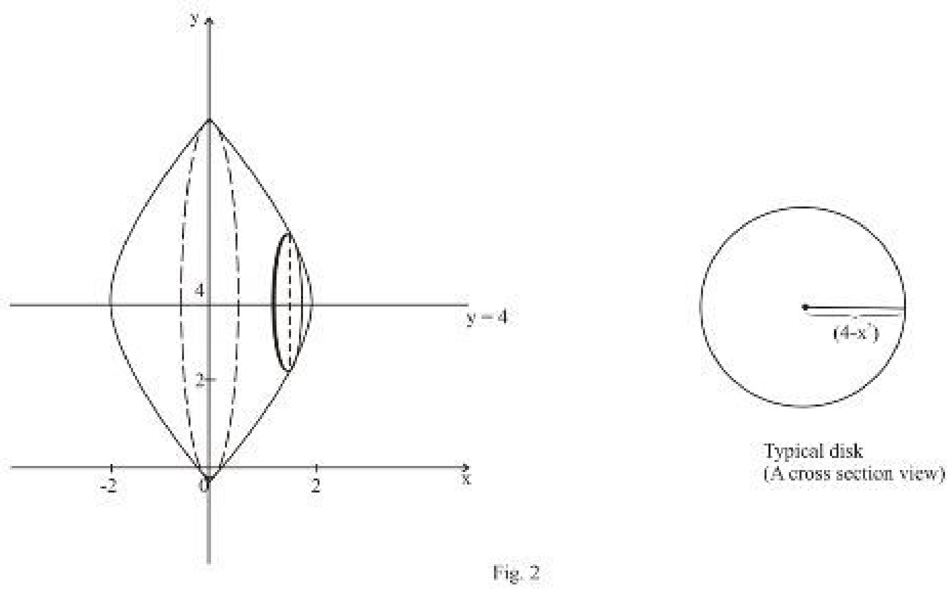


Fig. 2

This figure shows the shape of solid obtained by rotating the region bounded by $y = x^2$ and $y = 4$ about $y = 4$ and a typical disk

The radius of typical disk is $= 4 - x^2$
The cross sectional area of typical disk is

$$A(x) = \pi(\text{radius})^2$$

$$= \pi(4 - x^2)^2$$

Or $$A(x) = \pi(16 + x^4 - 8x^2)$$

Then the volume of solid is

$$v = \int_{-2}^{2} A(x)\,dx$$

$$= \int_{-2}^{2} \pi\left(16 + x^4 - 8x^2\right)dx$$

$$= \pi\int_{-2}^{2} \left(16 + x^4 - 8x^2\right)dx$$

$$= \pi\left[16x + \frac{x^5}{5} - \frac{8x^3}{3}\right]_{-2}^{2} \qquad \text{[By FTC - 2]}$$

$$= \pi\left[\left(16.(2) + \frac{2^5}{5} - \frac{8.(2^3)}{3}\right) - \left(16(-2) + \frac{(-2)^5}{5} - \frac{8(-2)^3}{3}\right)\right]$$

$$= \pi\left[32 + \frac{32}{5} - \frac{64}{3} + 32 + \frac{32}{5} - \frac{64}{3}\right]$$

$$= \pi\left[64 + \frac{64}{5} - \frac{128}{3}\right]$$

$$= \pi\left[\frac{512}{15}\right]$$

Or $\boxed{v = \frac{512\pi}{15}}$

Answer 13E.

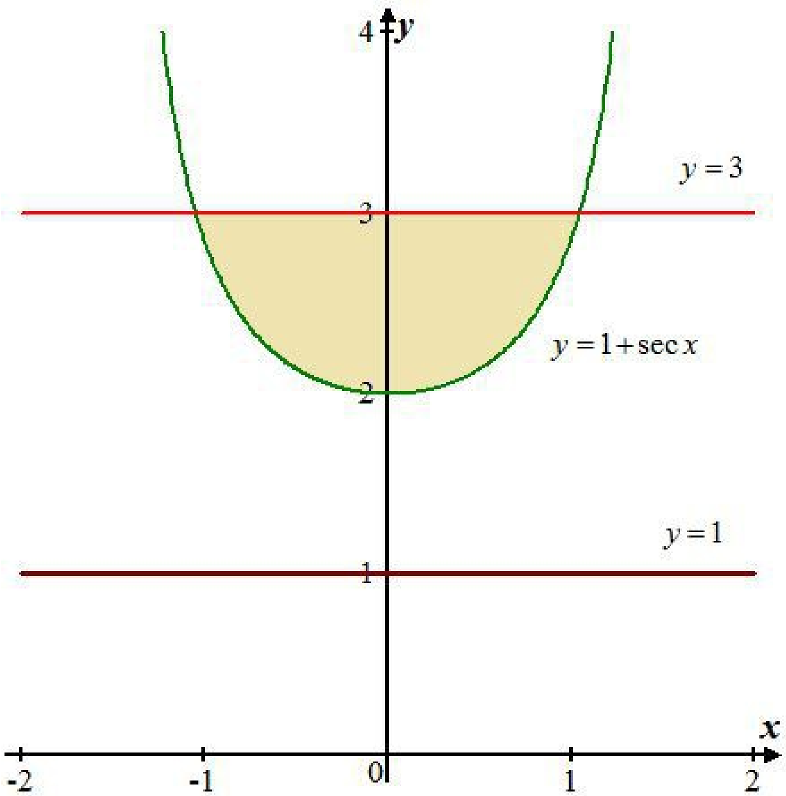
Consider the following curve:

$y = 1 + \sec x,\ y = 3$

So, the region rotated about the x-axis.

The objective is to find the volume of the solid obtained by the region.

Graph of the area of the cross section.

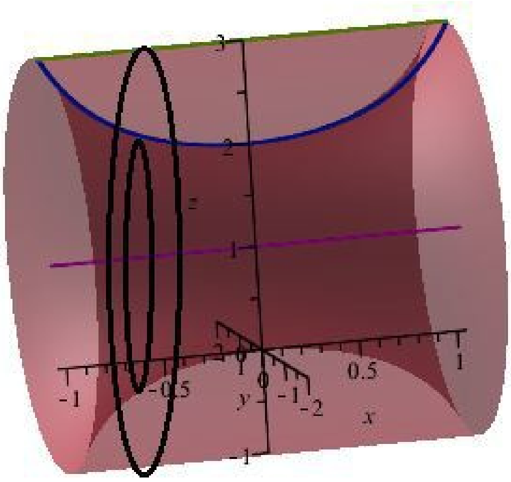


From the above graph the point intersection of the point is, $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$

The area of the cross section about *x*-axis is obtained by subtracting the area of the inner disc from the area of the outer disc.

$$\begin{aligned} A(x) &= \pi\left((3-1)^2 - (1+\sec x - 1)\right) \\ &= \pi\left((2)^2 - (\sec x)^2\right) \\ &= \pi\left(4 - \sec^2 x\right) \end{aligned}$$

Graph of the solid:



Calculate the volume of the solid is,

$$V = \int_{x=a}^{b} A(x)\,dx$$

The region lies between $x = \frac{-\pi}{3}$ to $x = \frac{\pi}{3}$

$$\begin{aligned} V &= \int_{x=\frac{-\pi}{3}}^{\frac{\pi}{3}} \pi\left(4 - \sec^2 x\right)dx \\ &= \pi \int_{x=\frac{-\pi}{3}}^{\frac{\pi}{3}} \left(4 - \sec^2 x\right)dx \\ &= \pi\left[\int_{x=-\frac{\pi}{3}}^{\frac{\pi}{3}} 4dx - \int_{x=-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec^2 x dx\right] \end{aligned}$$

$$= \frac{8}{3}\pi^2 - 2\pi\sqrt{3}$$
$$= 15.436$$

Therefore, the volume of the solid is, $\boxed{15.346}$

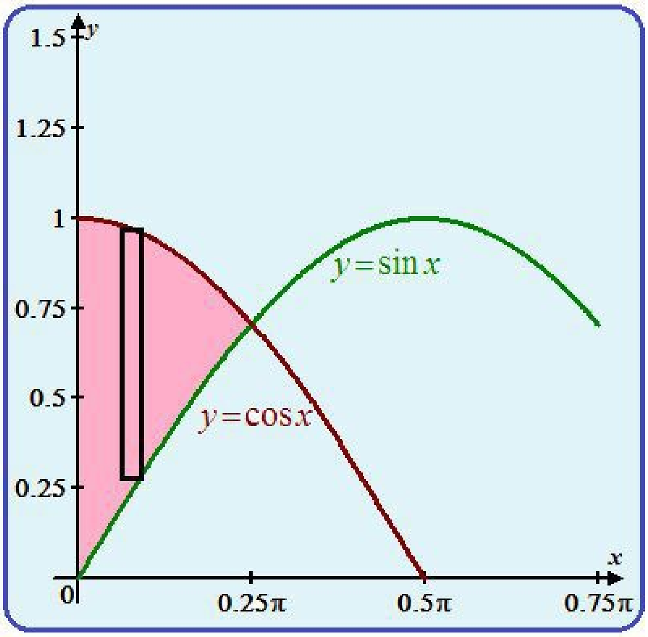
Answer 14E.

Consider the curves,

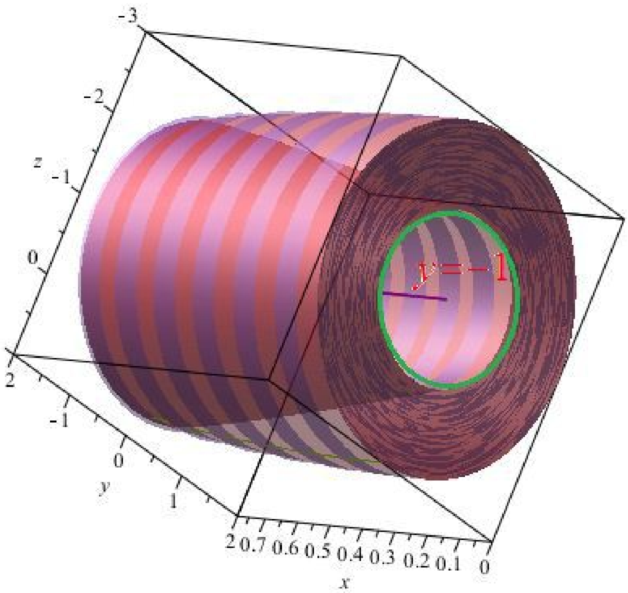
$y=\sin x, y=\cos x, 0\le x\le \pi/4$

Find the volume of the solid obtained by rotating the region bounded by the given curves about $y=-1$.

First sketch the region of the curves as follows:



The solid formed by rotating the above region is as shown below:



Now we want to find the volume generated by R about the line $y=-1$.

The figure shows outer-cross section. It is a washer with inner radius $y=1+\sin x$ and outer radius $y=1+\cos x$

So that the cross section is

$$A(x)=\pi(\text{outer radius})^2-\pi(\text{inner radius})^2$$

$$=\pi(1+\cos x)^2-\pi(1+\sin x)^2$$

$$=\pi\left[(1+\cos x)^2-(1+\sin x)^2\right]$$

$$=\pi\left[\cos^2 x+2\cos x-\sin^2 x-2\sin x\right]$$

$$=\pi\left[\cos 2x+2\cos x-2\sin x\right]$$

Hence the volume generated by R about $x=-1$ is

$$V=\int_0^{\pi/4} A(x)\,dx$$

$$=\int_0^{\pi/4} \pi\left[\cos 2x+2\cos x-2\sin x\right]dx$$

$$=\pi\left[\frac{\sin 2x}{2}+2\sin x+2\cos x\right]_0^{\pi/4}$$

$$=\pi\left[\frac{1}{2}+2\times\frac{1}{\sqrt{2}}+2\times\frac{1}{\sqrt{2}}-2\right]$$

$$=\pi\left[\frac{1}{2}+2\sqrt{2}-2\right]$$

$$=\left(\frac{1+4\sqrt{2}-4}{2}\right)\pi$$

$$=\left(\frac{4\sqrt{2}-3}{2}\right)\pi$$

Therefore,

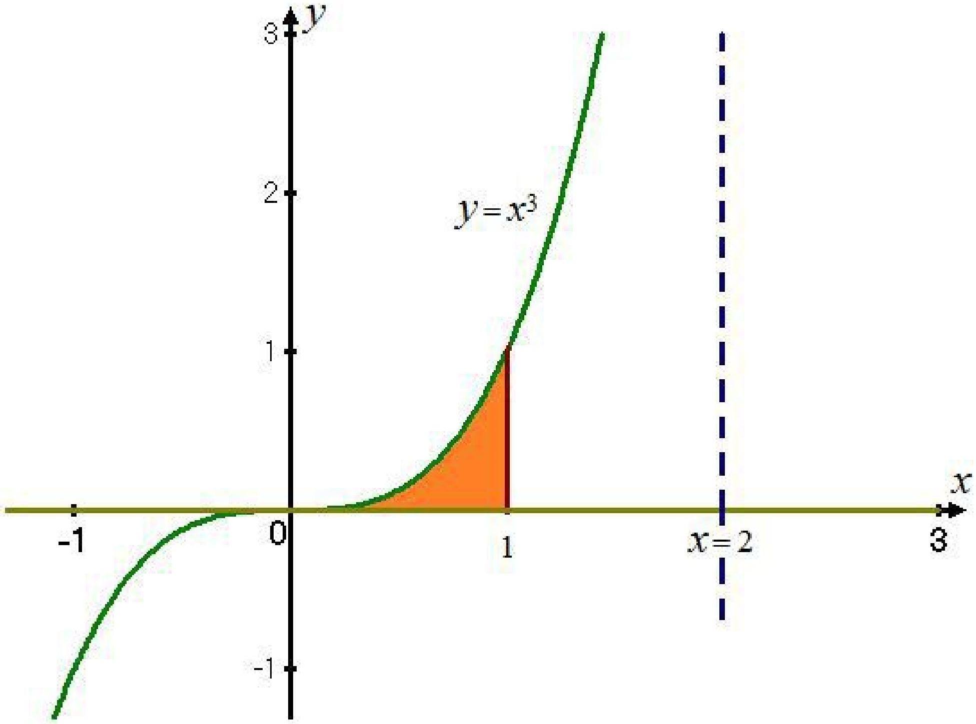
The volume of the solid is $\boxed{v=\left(\frac{4\sqrt{2}-3}{2}\right)\pi \text{ cubic units}}$.

Answer 15E.

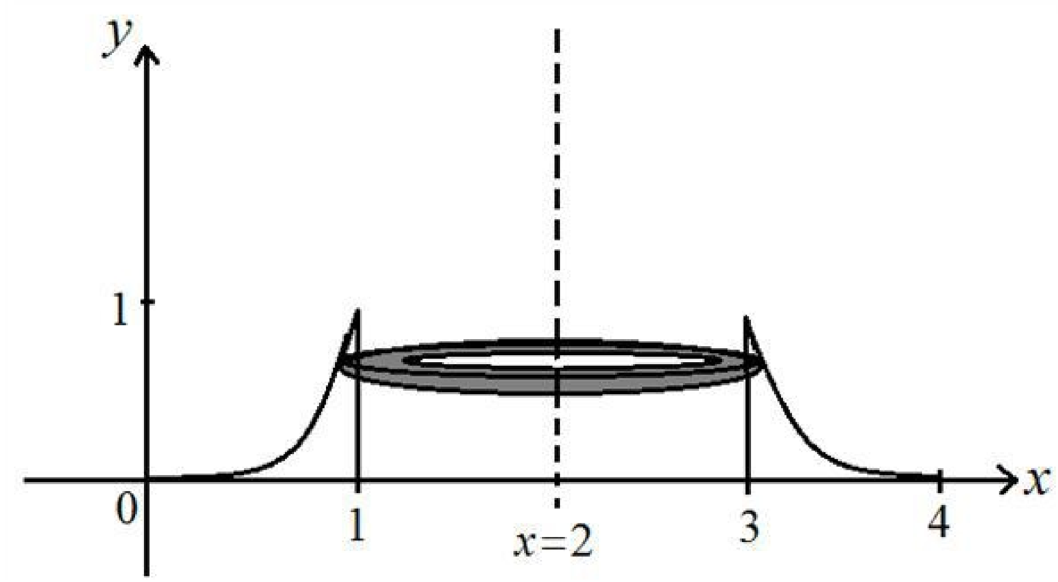
Consider the curves $y=x^3, y=0, x=1;$ about $x=2.$

Find the volume of the solid obtained by rotating the region bounded by the curves about the specified line.

Sketch the region is as follows:



Sketch the following figure:



The point of intersection of the curves $y = x^3$ and $x = 1$ is $(1,1)$ and the point of intersection of $y = x^3$ and $y = 0$ is $(0,0)$.

The curve $y = x^3$ can also be written as $x = \sqrt[3]{y}$.

The figure shows outer cross-section. It is a washer with inner radius $x = 1$ and outer radius $x = 2 - \sqrt[3]{y}$.

Hence the cross section area is as follows:

$$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$
$$= \pi\left[2 - \sqrt[3]{y}\right]^2 - \pi[1]$$
$$= \pi\left[4 - 4y^{1/3} + y^{2/3} - 1\right]$$
$$= \pi\left[3 - 4y^{1/3} + y^{2/3}\right]$$

Then volume of solid obtained by rotating the region about $x = 2$ is as follows:

$$V = \int_{y=0}^{1} A(y)\,dy$$
$$= \int_{y=0}^{1} \pi\left[3 - 4y^{1/3} + y^{2/3}\right]dy$$
$$= \pi\left[3y - 4\cdot\frac{3}{4}\cdot y^{4/3} + \frac{3}{5}y^{5/3}\right]_0^1$$
$$= \pi\left[3 - 3 + \frac{3}{5}\right]$$
$$= \frac{3\pi}{5}$$

Therefore, the volume of the solid obtained by rotating the region bounded by the curves is

$$\boxed{\frac{3\pi}{5}\text{ cubic units}}.$$

Answer 16E.

First we sketch the curves $y = x^2$ *and* $x = y^2$

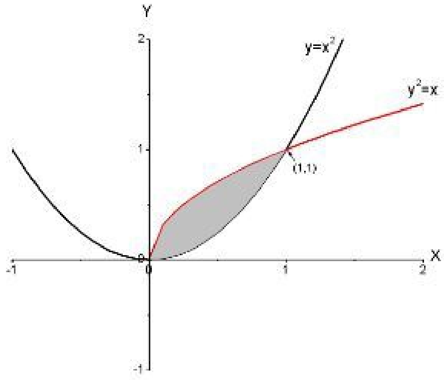


Fig.1

The shaded region is bounded by given curves, the points of intersection are (0, 0) and (1, 1)

Now we sketch the solid obtained by rotating shaded region about x = -1

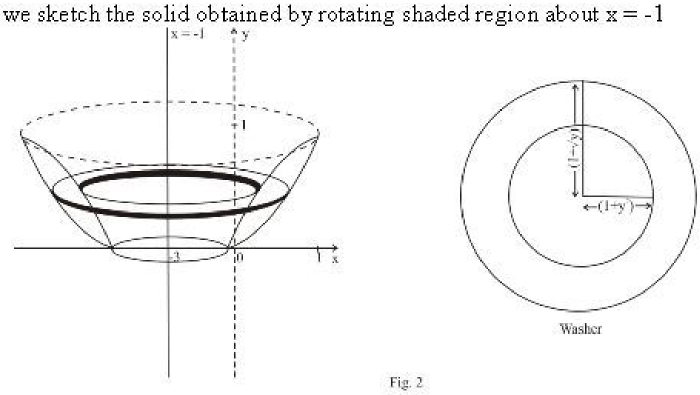


Fig. 2

Outer radius of washer $= 1+\sqrt{y}$ $\quad\left(\text{since } x^2 = y \text{ or } x = \sqrt{y}\right)$

Inner radius of washer $= 1+y^2$

Cross sectional area of washer is

$$A(y) = \pi\left[(outer\ radius)^2 - (inner\ radius)^2\right]$$
$$= \pi\left[\left(1+\sqrt{y}\right)^2 - \left(1+y^2\right)^2\right]$$
$$= \pi\left[1+y+2\sqrt{y}-1-y^4-2y^2\right]$$

Or $\quad A(y) = \pi\left[2y^{1/2}+y-2y^2-y^4\right]$

Then volume of solid is $v = \int_0^1 A(y)dy$

Or $\quad v = \pi\int_0^1\left(2y^{1/2}+y-2y^2-y^4\right)dy$

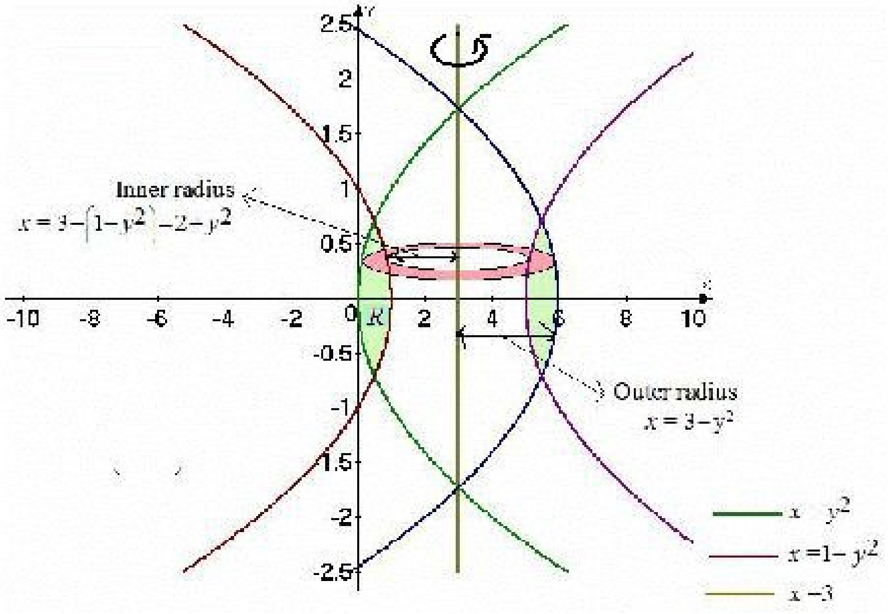
$$= \pi\left[\frac{4}{3}y^{3/2}+\frac{y^2}{2}-\frac{2}{3}y^3-\frac{y^5}{5}\right]_0^1 \qquad \text{[By FTC - 2]}$$
$$= \pi\left[\frac{4}{3}+\frac{1}{2}-\frac{2}{3}-\frac{1}{5}\right]$$
$$= \pi\left[\frac{2}{3}+\frac{1}{2}-\frac{1}{5}\right]$$

Or $\quad \boxed{v = \frac{29}{30}\pi}$

Answer 17E.

Here we want to find the volume of the solid obtained by rotating the region bounded by the given curves $x = y^2, x = 1-y^2$; about the line $x = 3$.

Bounded region R is shown in the below figure.



First we will find the intersection of the two curves $x = y^2, x = 1 - y^2$.

$$y^2 = 1 - y^2$$
$$\Rightarrow 2y^2 = 1$$
$$\Rightarrow y^2 = \frac{1}{2}$$
$$\Rightarrow y = \pm\frac{1}{\sqrt{2}}$$

Here we want to find the volume generated by R about the line $x = 3$.
The bounded region, the solid of rotation and a cross-section perpendicular to the y-axis are shown in the above figure. A cross section in the plane has the shape of a washer with inner radius

$$x = 3 - \left(1 - y^2\right)$$
$$= 2 + y^2$$

and outer radius

$$x = 3 - y^2$$

So that the cross sectional area is

$$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$
$$= \pi\left(3 - y^2\right)^2 - \pi\left(2 + y^2\right)^2$$
$$= \pi\left(9 + y^4 - 6y^2 - 4 - y^4 - 4y^2\right)$$
$$= \pi\left(5 - 10y^2\right)$$

And y varies from $-\frac{1}{\sqrt{2}}$ to $\frac{1}{\sqrt{2}}$.

And the volume generated by R about $x = 3$ is

$$V = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} A(y)\,dy$$

$$= \pi \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left(5 - 10y^2\right) dy$$

$$= \pi \left(5y - 10 \cdot \frac{y^3}{3}\right)_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}$$

$$= \pi \left[\frac{5}{\sqrt{2}} - \frac{10}{3} \cdot \frac{1}{2\sqrt{2}} + \frac{5}{\sqrt{2}} - \frac{10}{3} \cdot \frac{1}{2\sqrt{2}}\right]$$

$$= \frac{10\pi\sqrt{2}}{3}$$

Hence volume is $\boxed{V = \frac{10\pi\sqrt{2}}{3}}$.

Answer 18E.

We sketch the curves $y = x,\ y = 0,\ x = 2$ *and* $x = 4$

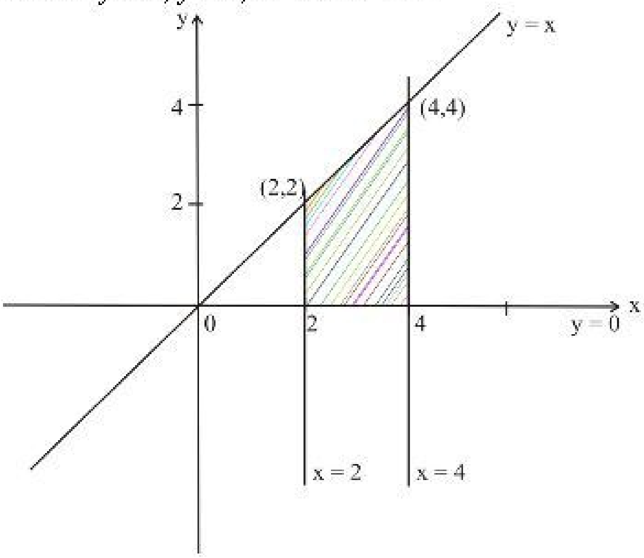


Fig. 1

Shaded region is bounded by given curves

Now we sketch the slid obtained by rotating shaded region about x = 1

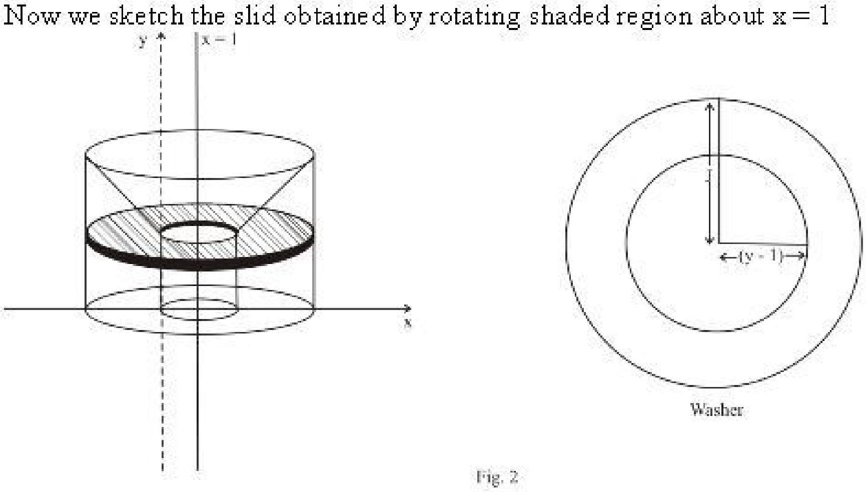


Fig. 2

Here we have to find the volume of solid in to two parts

First we find the volume in the interval $0 \le y \le 2$ in this interval

Outer radius of washer is = 3

And inner radius is = 1

Then cross section area of washer is $A(y) = \pi.3^2 - \pi.1^2$ or $A(y) = 8\pi$

So volume of solid in the interval [0, 2]

$$v_1 = \int_0^2 8\pi dy$$

Or $v_1 = 8\pi[y]_0^2$ [By FTC - 2]

Or $v_1 = 16\pi$

Now we find the volume of solid in the interval [2, 4] in this interval

Outer radius of washer is = 3

And inner radius of washer is = y – 1

Then cross sectional area of washer is

$$A(y) = \pi.(3^2) - \pi(y-1)^2$$

$$= 9\pi - \pi(y^2 + 1 - 2y)$$

Or $A(y) = \pi[8 - y^2 + 2y]$

Then volume of solid in the interval [2, 4] is

$$v_2 = \int_2^4 A(y)dy$$

Or $v_2 = \pi\int_2^4 (8 - y^2 + 2y)dy$

$$= \pi\left[8y - \frac{y^3}{3} + y^2\right]_2^4$$

$$= \pi\left[32 - \frac{64}{3} + 16 - 16 + \frac{8}{3} - 4\right]$$

$$= \pi\left[8y - \frac{y^3}{3} + y^2\right]_2^4$$

$$= \pi\left[32 - \frac{64}{3} + 16 - 16 + \frac{8}{3} - 4\right]$$

Or $v_2 = \dfrac{28\pi}{3}$

Then total volume $v = v_1 + v_2 = 16\pi + \dfrac{28\pi}{3}$ or $\boxed{v = \dfrac{76\pi}{3}}$

We sketch the curves $y = x$, $y = 0$, $x = 2$ and $x = 4$

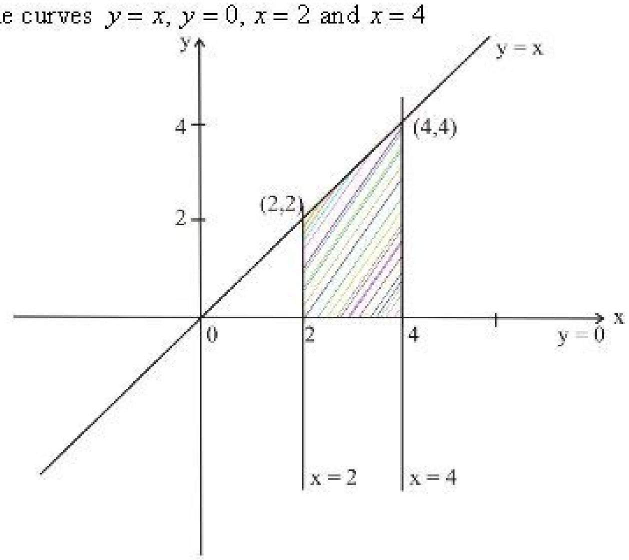


Fig. 1

Shaded region is bounded by given curves

Now we sketch the slid obtained by rotating shaded region about $x = 1$

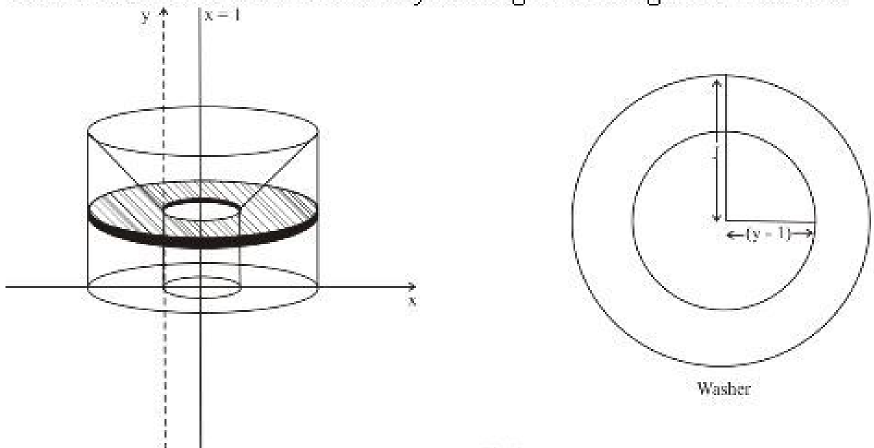


Fig. 2

Here we have to find the volume of solid in to two parts
First we find the volume in the interval $0 \le y \le 2$ in this interval

Outer radius of washer is = 3

And inner radius is = 1

Then cross section area of washer is $A(y) = \pi.3^2 - \pi.1^2$ or $A(y) = 8\pi$

So volume of solid in the interval [0, 2]

$$v_1 = \int_0^2 8\pi dy$$

Or $v_1 = 8\pi[y]_0^2$ [By FTC - 2]

Or $v_1 = 16\pi$

Now we find the volume of solid in the interval [2, 4] in this interval
Outer radius of washer is = 3
And inner radius of washer is = y – 1
Then cross sectional area of washer is

$$A(y) = \pi.(3^2) - \pi(y-1)^2$$

$$= 9\pi - \pi(y^2 + 1 - 2y)$$

Or $A(y) = \pi[8 - y^2 + 2y]$

Then volume of solid in the interval [2, 4] is

$$v_2 = \int_2^4 A(y)dy$$

Or $v_2 = \pi\int_2^4 (8 - y^2 + 2y)dy$

$$= \pi\left[8y - \frac{y^3}{3} + y^2\right]_2^4$$

$$= \pi\left[32 - \frac{64}{3} + 16 - 16 + \frac{8}{3} - 4\right]$$

$$= \pi\left[8y - \frac{y^3}{3} + y^2\right]_2^4$$

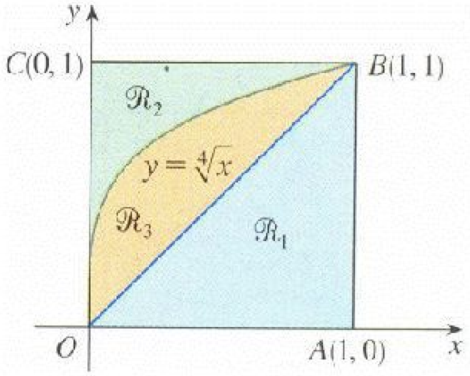
$$= \pi\left[32 - \frac{64}{3} + 16 - 16 + \frac{8}{3} - 4\right]$$

Or $v_2 = \frac{28\pi}{3}$

Then total volume $v = v_1 + v_2 = 16\pi + \frac{28\pi}{3}$ *or* $\boxed{v = \frac{76\pi}{3}}$

Answer 19E.

Given graph:



The graph of $\Re_1$ is shown below:

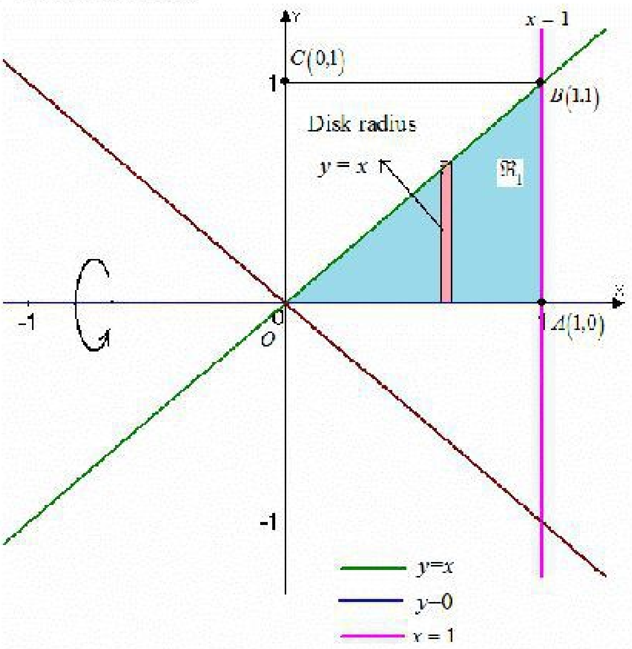


Figure (1)

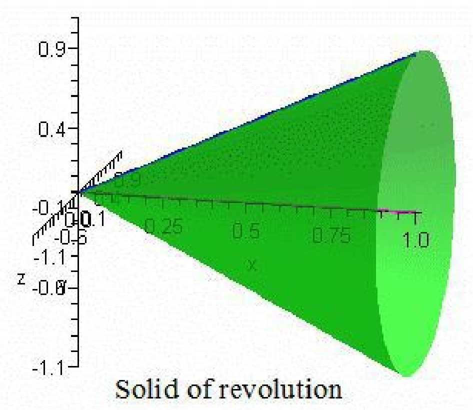


Figure (2)

Now we have to find the volume generated by rotating the region $\Re_1$ about the line joining points O and A.
Equation of the line joining points O and A is $y=0$.
The region is shown in the figure (1) and the resulting solid is shown in the figure (2).
When we slice through the point x, we get a disk with radius $y = x$.
And by observing graph, x varies from 0 to 1
The area of this cross-section is

$$A(x) = \pi y^2$$
$$= \pi(x^2)$$

And hence the volume generated by $\Re_1$ about OA is

$$V = \int_{x=0}^{1} A(x)\,dx$$

$$= \int_0^1 \pi\left(x^2\right)dx$$

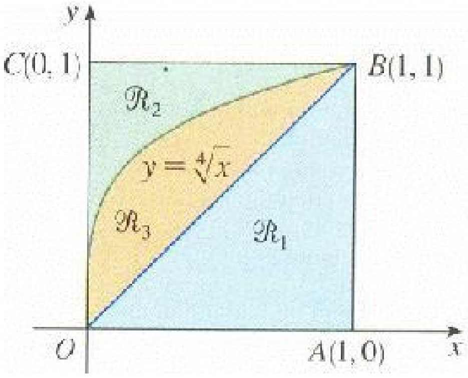
$$= \pi\left(\frac{x^3}{3}\right)_0^1$$

$$= \frac{\pi}{3}$$

Hence the volume is $\boxed{V = \frac{\pi}{3}}$.

Answer 20E.

Given graph:



The graph of $\Re_1$ is shown below:

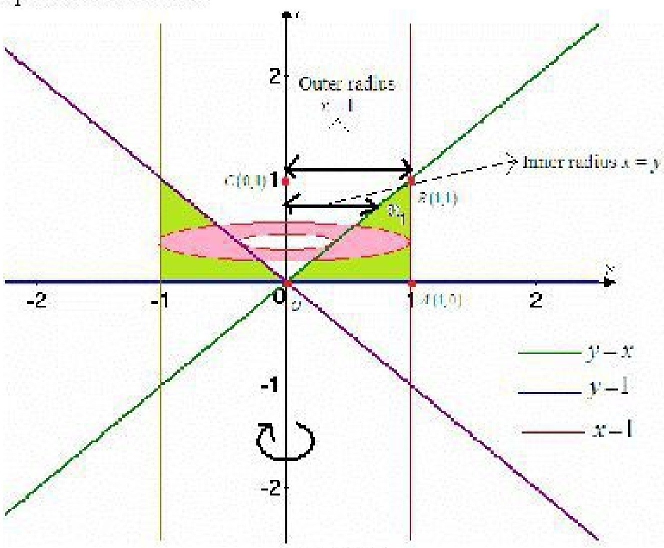


Figure (1)

Now we have to find the volume generated by rotating the region $\Re_1$ about the line joining points O and C.

Equation of the line joining points O and C is $x = 0$.

The region and a cross section perpendicular to the y-axis are shown in the figure (1).

A cross section in the plane has a shape of a washer with inner radius $x = y$ and outer radius $x = 1$.

Hence the cross sectional area is

$$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$

$$= \pi(1)^2 - \pi(y)^2$$

$$= \pi\left(1 - y^2\right)$$

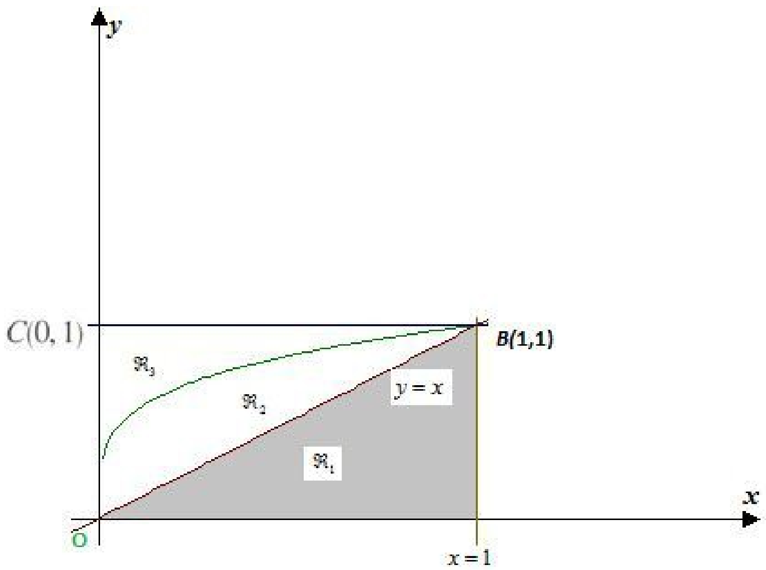
And the volume generated by $\Re_1$ about OC is

$$V = \int_0^1 A(y)\,dy$$
$$= \pi\int_0^1 \left(1-y^2\right)dy$$
$$= \pi\left(y-\frac{y^3}{3}\right)_0^1$$
$$= \pi\left(1-\frac{1}{3}\right)$$
$$= \frac{2\pi}{3}$$

Hence volume is $\boxed{V = \frac{2\pi}{3}}$.

Answer 21E.

Consider the following graph:



The graph of $\Re_1$ is shown below:

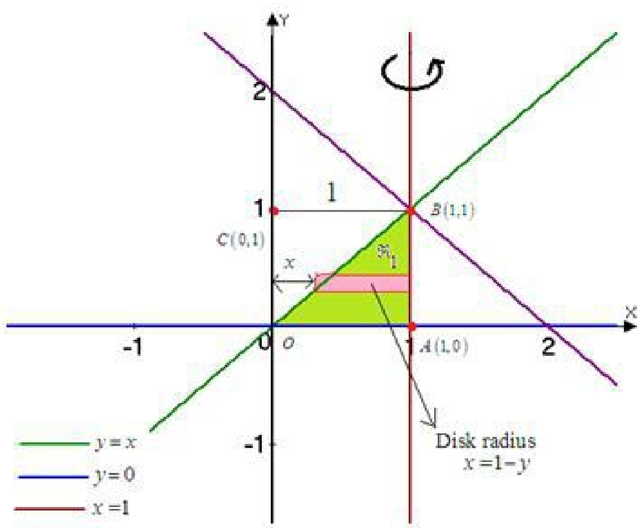
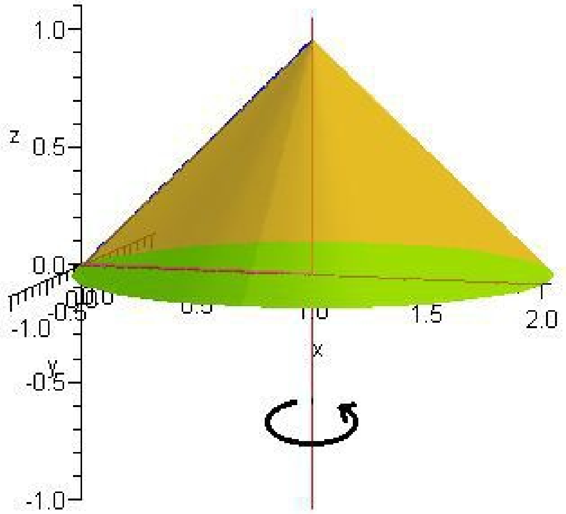


Figure (1)



Solid of revolution

Figure (2)

The objective is to find the volume generated by rotating the region $\Re_1$ about the line joining points *A* and *B*.

Equation of the line joining points *A* and *B* is $x = 1$.

The region is shown in the figure (1) and the resulting solid is shown in the figure (2).

Because the region is rotated about the line $x = 1$, it makes sense to slice the solid perpendicular to the *y*-axis and therefore to integrate with respect to *y*. If we slice at height *y*, we get a circular disk with radius *x*, where $x = 1 - y$.

So, the area of a cross-section through *y* is

$$A(y) = \pi(x)^2$$
$$= \pi(1-y)^2$$

And by observing the graph, *y* varies from 0 to 1.

The volume generated by $\Re_1$ about AB is

$$V = \int_0^1 A(y)\,dy$$

$$= \int_0^1 \pi(1-y)^2\,dy$$

$$= \pi\int_0^1 \left(1+y^2-2y\right)dy$$

$$= \pi\left(y+\frac{y^3}{3}-y^2\right)_0^1$$

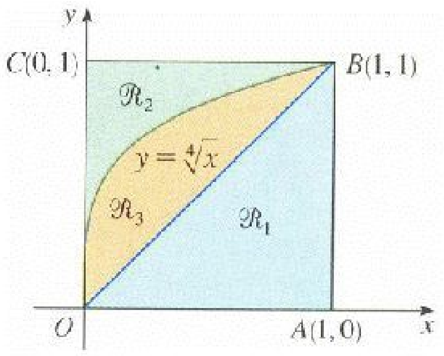
$$= \pi\left(1+\frac{1}{3}-1\right)$$

$$= \frac{\pi}{3}$$

Hence the volume is $\boxed{V = \frac{\pi}{3}}$.

Answer 22E.

Given graph:



The graph of $\mathfrak{R}_1$ is shown below:

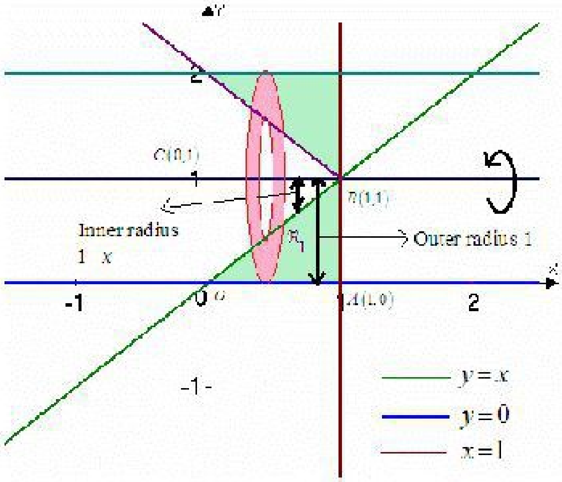


Figure (1)

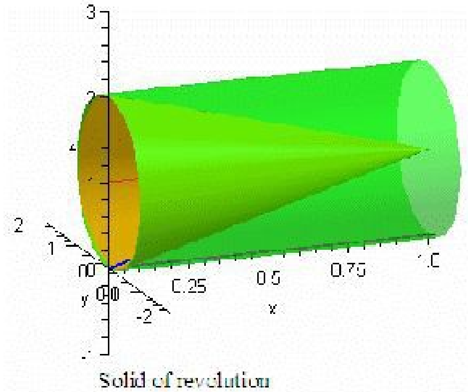


Figure (2)

Now we have to find the volume generated by rotating the region $\mathfrak{R}_1$ about the line joining points B and C.
Equation of the line joining points B and C is $y=1$.
The region and a cross section perpendicular to the x-axis are shown in the figure (1) and the resulting solid is shown in the figure (2).
A cross section in the plane has a shape of a washer with inner radius $y=1-x$ and outer radius $y=1$.

So that the cross sectional area is

$$A(x)=\pi(\text{outer radius})^2-\pi(\text{inner radius})^2$$
$$=\pi(1)^2-\pi(1-x)^2$$
$$=\pi\left(1-1-x^2+2x\right)$$
$$=\pi\left(-x^2+2x\right)$$

By observing the graph x varies from 0 to 1.

And hence the volume generated by $\Re_1$ about BC is

$$V = \int_{x=0}^{1} A(x) \cdot dx$$

$$= \int_0^1 \pi\left(-x^2 + 2x\right) dx$$

$$= \pi\left(-\frac{x^3}{3} + x^2\right)_0^1$$

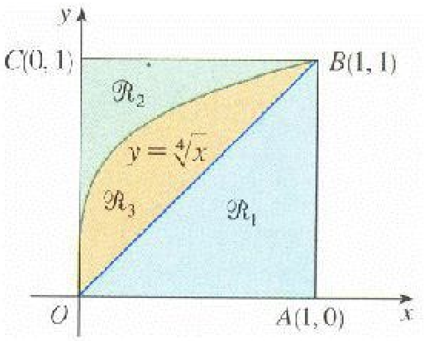
$$= \pi\left(-\frac{1}{3} + 1\right)$$

$$= \frac{2\pi}{3}$$

Hence the volume is $\boxed{V = \frac{2\pi}{3}}$.

Answer 23E.

Given graph:



The graph of $\Re_2$ is shown below:

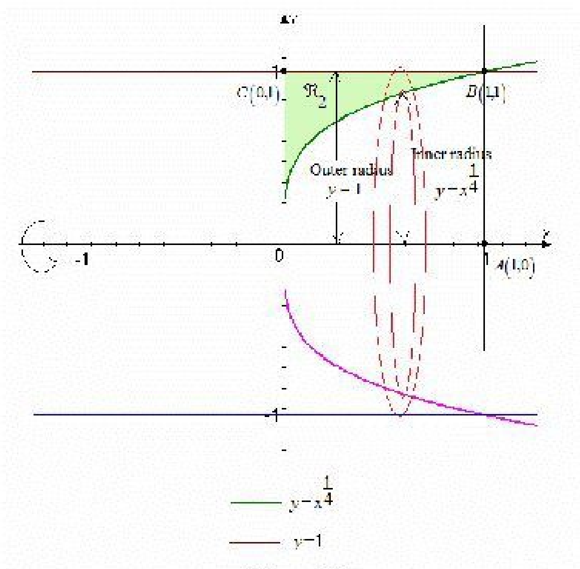


Figure (1)

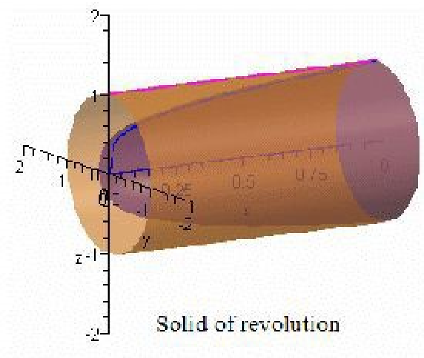


Figure (2)

Now we have to find the volume generated by rotating the region $\Re_2$ about the line joining points O and A.
Equation of the line joining points O and A is $y=0$.
The region and a cross section perpendicular to the x-axis are shown in the figure (1) and the resulting solid is shown in the figure (2).

A cross section in the plane has a shape of a washer with inner radius $y=x^{\frac{1}{4}}$ and outer radius $y=1$.
So that the cross sectional area is

$$A(x)=\pi(\text{outer radius})^2-\pi(\text{inner radius})^2$$
$$=\pi(1)^2-\pi\left(x^{\frac{1}{4}}\right)^2$$
$$=\pi\left(1-x^{\frac{1}{2}}\right)$$

By observing the graph x varies from 0 to 1.

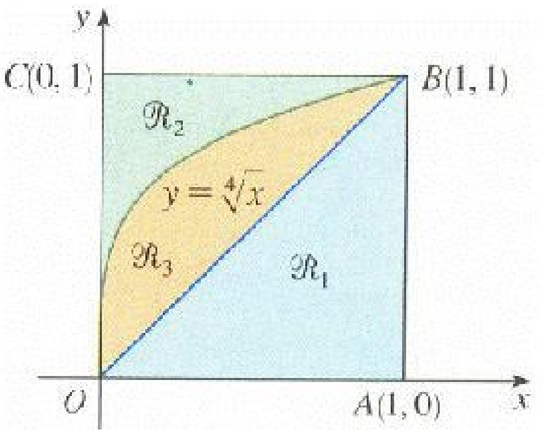
And hence the volume generated by $\Re_2$ about OA is

$$V=\int_{x=0}^{1} A(x)\cdot dx$$
$$=\int_0^1 \pi\left(1-x^{\frac{1}{2}}\right)dx$$
$$=\pi\left(x-\frac{2}{3}x^{\frac{3}{2}}\right)_0^1$$
$$=\pi\left(1-\frac{2}{3}\right)$$
$$=\frac{\pi}{3}$$

Hence the volume is $\boxed{V=\frac{\pi}{3}}$.

Answer 24E.

Given graph:



The graph of $\Re_2$ is shown below:

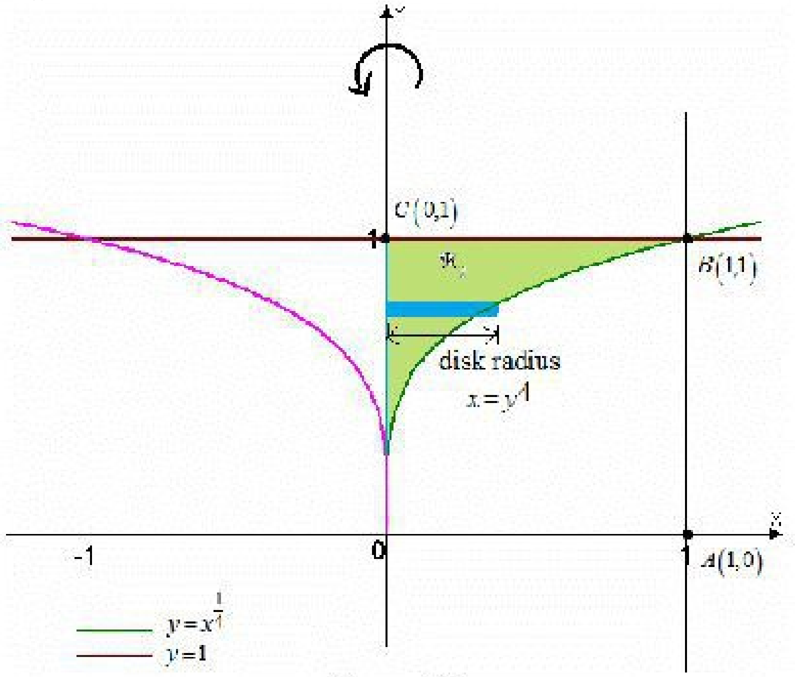


Figure (1)

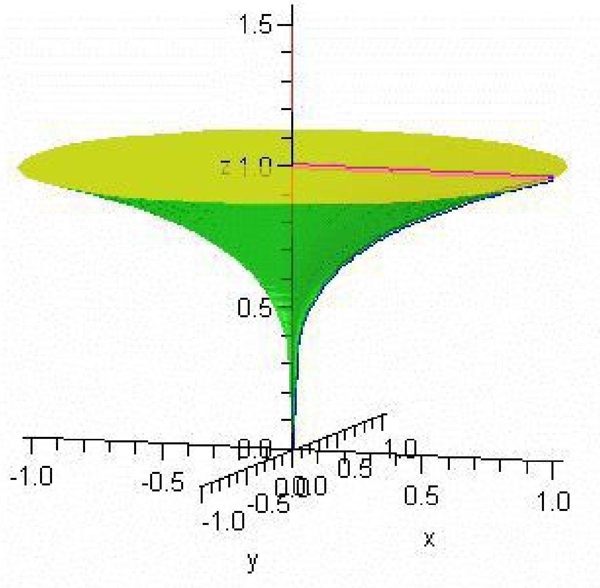


Figure (2)

Now we have to find the volume generated by rotating the region $\Re_2$ about the line joining points O and C.

Equation of the line joining points O and C is $x = 0$.

The region is shown in the figure (1) and the resulting solid is shown in the figure (2).

When we slice through the point y, we get a disk with radius $x = y^4$.

And by observing graph, y varies from 0 to 1

The area of this cross-section is

$$A(y) = \pi x^2$$

$$= \pi (y^4)^2$$

$$= \pi y^8$$

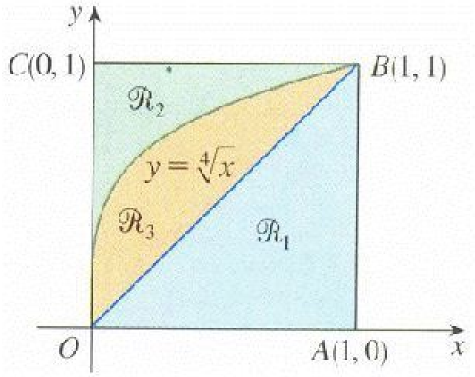
And hence the volume generated by $\Re_2$ about OC is

$$V = \int_{y=0}^{1} A(y)\,dy$$
$$= \int_0^1 \pi(y^8)\,dy$$
$$= \pi\left(\frac{y^9}{9}\right)_0^1$$
$$= \pi\left(\frac{1}{9}\right)$$
$$= \frac{\pi}{9}$$

Hence the volume is $\boxed{V = \frac{\pi}{9}}$.

Answer 25E.

Given graph:



The graph of $\Re_2$ is shown below:

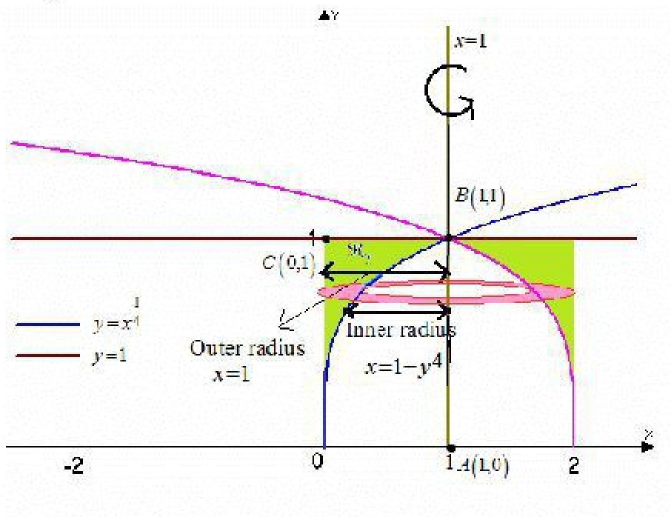


Figure (1)

Now we have to find the volume generated by rotating the region $\Re_2$ about the line joining points A and B.
Equation of the line joining points A and B is $x = 1$.
The region and a cross-section perpendicular to y-axis are shown in the figure (1) and the resulting solid is shown in the figure (2).
A cross section in the plane has a shape of a washer with inner radius $x = 1 - y^4$ and outer radius $x = 1$.
So that the cross section area is

$$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$
$$= \pi(1)^2 - \pi(1 - y^4)^2$$
$$= \pi(2y^4 - y^8) \text{ and } y = 0 \text{ to } 1$$

And hence the volume generated by $\Re_2$ about AB is

$$V = \int_0^1 A(y)\,dy$$

$$= \int_0^1 \pi\left(2y^4 - y^8\right)dy$$

$$= \pi\left[2\cdot\frac{y^5}{5} - \frac{y^9}{9}\right]_0^1$$

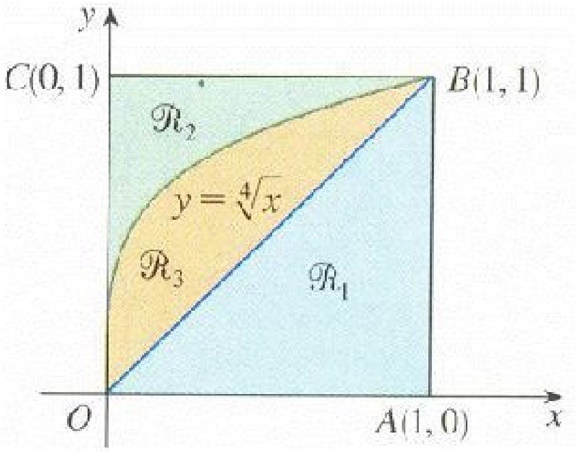
$$= \pi\left[\frac{2}{5} - \frac{1}{9}\right]$$

$$= \frac{13\pi}{45}$$

Hence the volume is $\boxed{V = \frac{13\pi}{45}}$

Answer 26E.

Given graph:



The graph of $\Re_2$ is shown below:

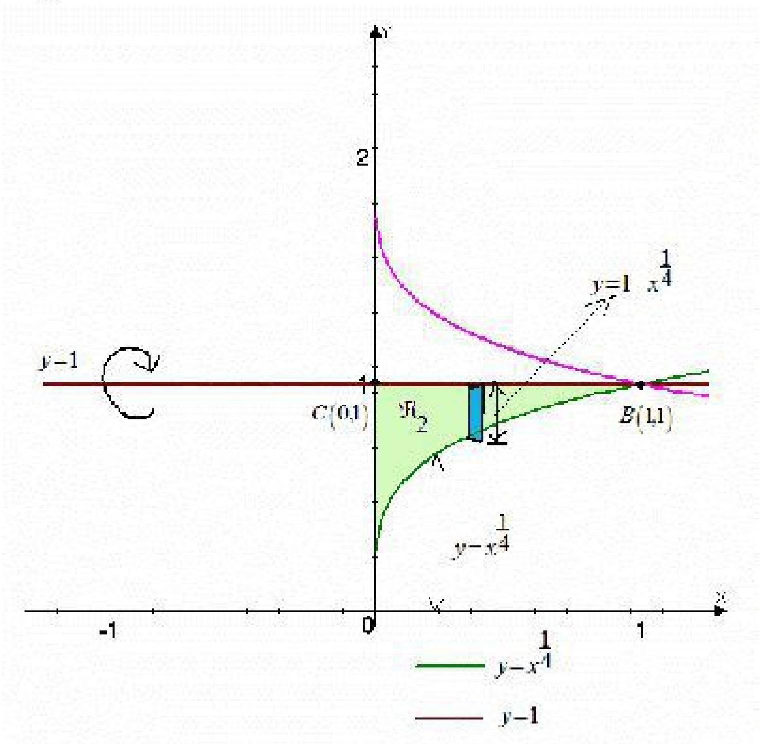


Figure (1)

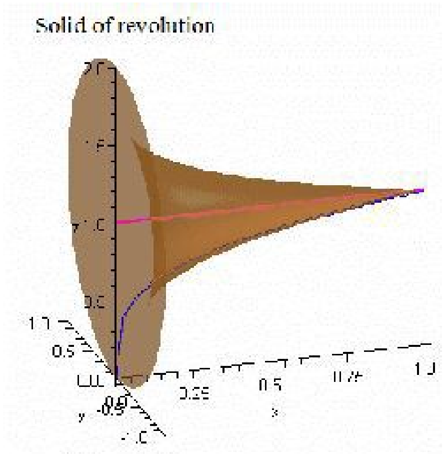


Figure (2)

Now we have to find the volume generated by rotating the region $\Re_2$ about the line joining points B and C.
Equation of the line joining points B and C is $y=1$.
The region is shown in the figure (1) and the resulting solid is shown in the figure (2).

When we slice through the point x, we get a disk with radius $y=1-x^{\frac{1}{4}}$.
And by observing graph, x varies from 0 to 1
The area of this cross-section is

$$A(x)=\pi\left(1-x^{\frac{1}{4}}\right)^2$$
$$=\pi\left(1+x^{\frac{1}{2}}-2x^{\frac{1}{4}}\right)$$

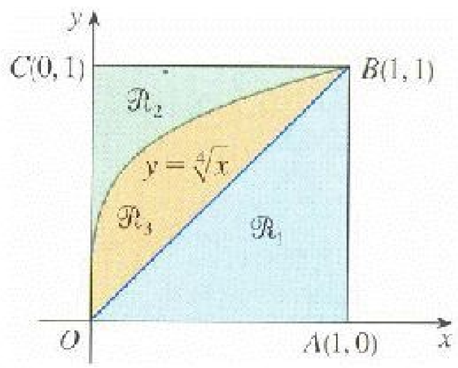
And hence the volume generated by $\Re_2$ about BC is

$$V=\int_{x=0}^{1} A(x)dx$$
$$=\int_0^1 \pi\left(1+x^{\frac{1}{2}}-2x^{\frac{1}{4}}\right)dx$$
$$=\pi\left(x+\frac{2}{3}x^{\frac{3}{2}}-\frac{8}{5}x^{\frac{5}{4}}\right)_0^1$$
$$=\pi\left(1+\frac{2}{3}(1)^{\frac{3}{2}}-\frac{8}{5}(1)^{\frac{5}{4}}\right)$$
$$=\frac{\pi}{15}$$

Hence the volume is $\boxed{V=\frac{\pi}{15}}$.

Answer 27E.

Given graph:



The graph of $\Re_3$ is shown below:

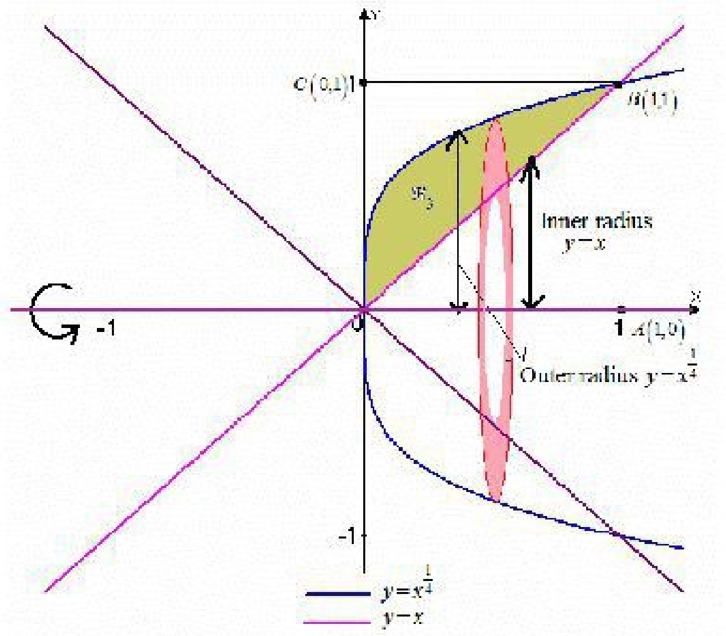


Figure (1)

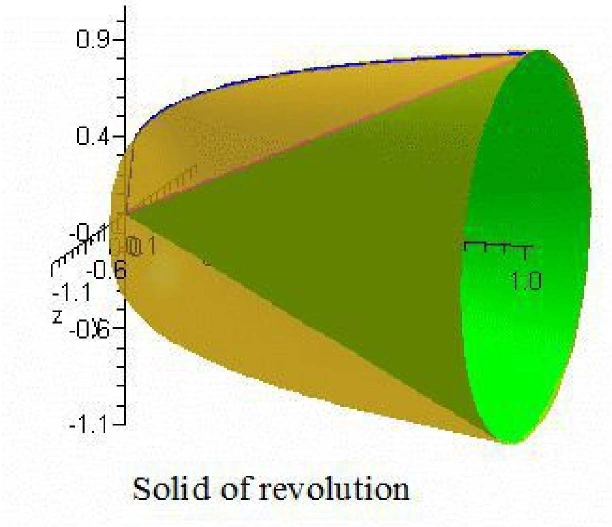


Figure (2)

Now we have to find the volume generated by rotating the region $\Re_3$ about the line joining points O and A.
Equation of the line joining points O and A is $y=0$.
The region and a cross-section perpendicular to the x-axis are shown in the figure (1) and the resulting solid is shown in the figure (2).
A cross section in the plane has a shape of a washer with inner radius $y = x$ and outer radius $y = x^{\frac{1}{4}}$.
So that the cross sectional area is

$$\begin{aligned} A(x) &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\ &= \pi\left(x^{\frac{1}{4}}\right)^2 - \pi(x)^2 \\ &= \pi\left(x^{\frac{1}{2}} - x^2\right) \end{aligned}$$

Then the volume generated by $\Re_3$ about OA is

$$V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \pi\left(x^{\frac{1}{2}} - x^2\right)dx$$

$$= \pi\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

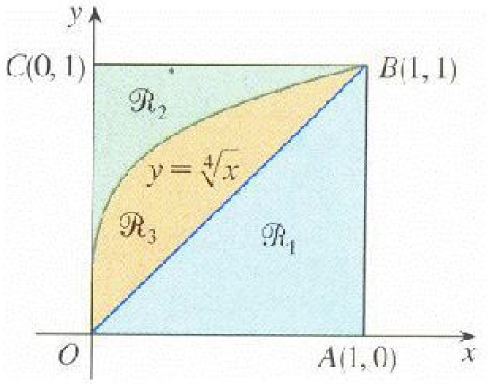
$$= \pi\left[\frac{2}{3} - \frac{1}{3}\right]$$

$$= \frac{\pi}{3}$$

Hence the volume is $\boxed{V = \frac{\pi}{3}}$.

Answer 28E.

Given graph:



The graph of $\Re_3$ is shown below:

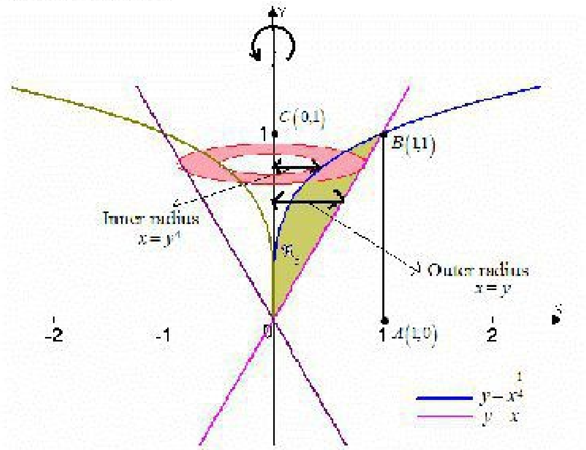


Figure (1)

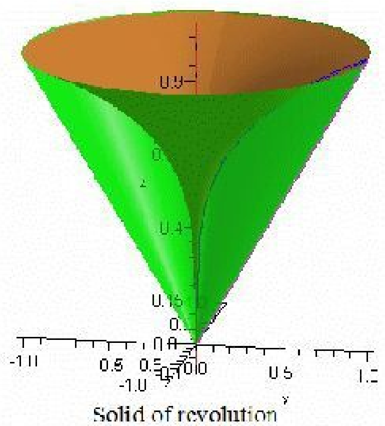


Figure (2)

Now we have to find the volume generated by rotating the region $\Re_3$ about the line joining points O and C.
Equation of the line joining points O and C is $x = 0$.
The region and a cross-section perpendicular to y-axis are shown in the figure (1) and the resulting solid is shown in the figure (2).
A cross section in the plane has a shape of a washer with inner radius $x = y^4$ and outer radius $x = y$.
And by observing graph, y varies from 0 to 1
The area of this cross-section is

$$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$
$$= \pi(y)^2 - \pi(y^4)^2$$
$$= \pi(y^2 - y^8)$$

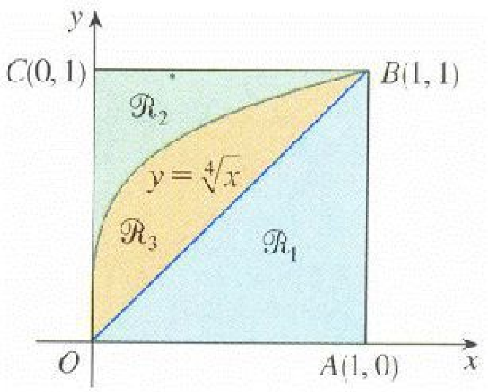
Then the volume generated by $\Re_3$ about OC is

$$V = \int_{y=0}^{1} A(y)\,dy$$
$$= \int_0^1 \pi(y^2 - y^8)\,dy$$
$$= \pi\left[\frac{y^3}{3} - \frac{y^9}{9}\right]_0^1$$
$$= \pi\left[\frac{1}{3} - \frac{1}{9}\right]$$
$$= \frac{2\pi}{9}$$

Hence the volume is $\boxed{V = \frac{2\pi}{9}}$.

Answer 29E.

Given graph:



The graph of $\Re_3$ is shown below:

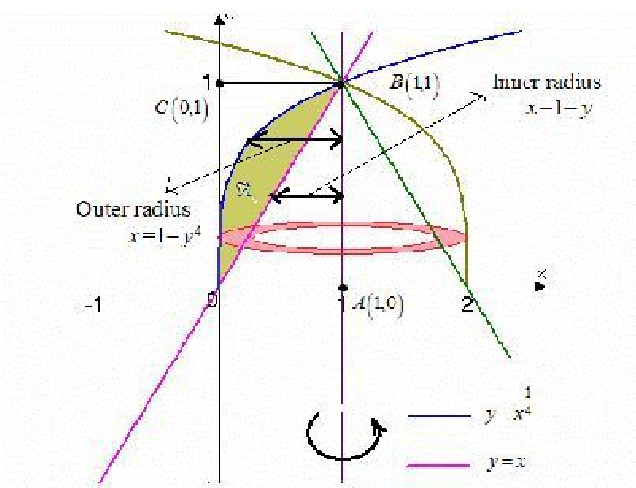


Figure (1)

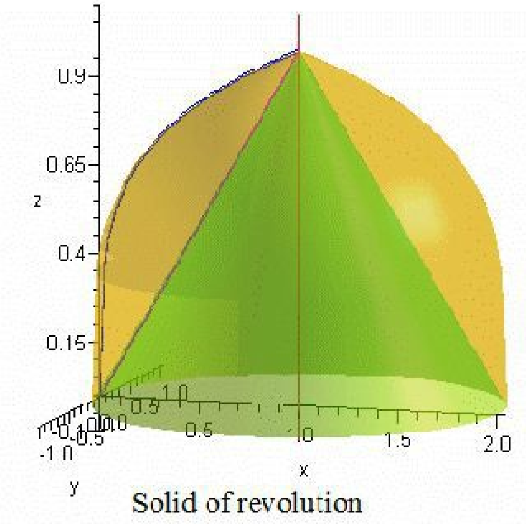


Figure (2)

Now we have to find the volume generated by rotating the region $\Re_3$ about the line joining points A and B.
Equation of the line joining points A and B is $x=1$.
The region is shown in the figure (1) and the resulting solid is shown in the figure (2).
A cross section in the plane has a shape of a washer with inner radius $x=1-y$ and outer radius $x=1-y^4$.
The cross sectional area is

$$A(y)=\pi(\text{outer radius})^2-\pi(\text{inner radius})^2$$
$$=\pi\left(1-y^4\right)^2-\pi(1-y)^2$$
$$=\pi\left[y^8-2y^4-y^2+2y\right]$$

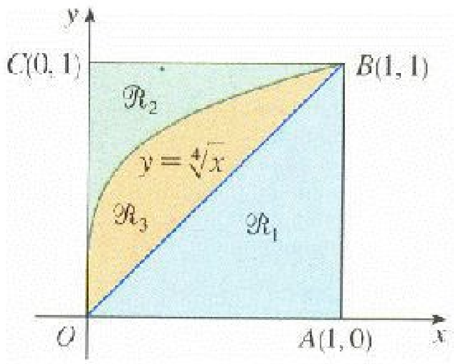
Then the volume generated by $\Re_3$ about AB is

$$V=\int_0^1 A(y)\,dy$$
$$=\int_0^1 \pi\left[y^8-2y^4-y^2+2y\right]dy$$
$$=\pi\left[\frac{y^9}{9}-2\cdot\frac{y^5}{5}-\frac{y^3}{3}+2\cdot\frac{y^2}{2}\right]_0^1$$
$$=\pi\left[\frac{1}{9}-\frac{2}{5}-\frac{1}{3}+1\right]$$
$$=\frac{17\pi}{45}$$

Hence the volume is $\boxed{V=\frac{17\pi}{45}}$.

Answer 30E.

Given graph:



The graph of $\Re_3$ is shown below:

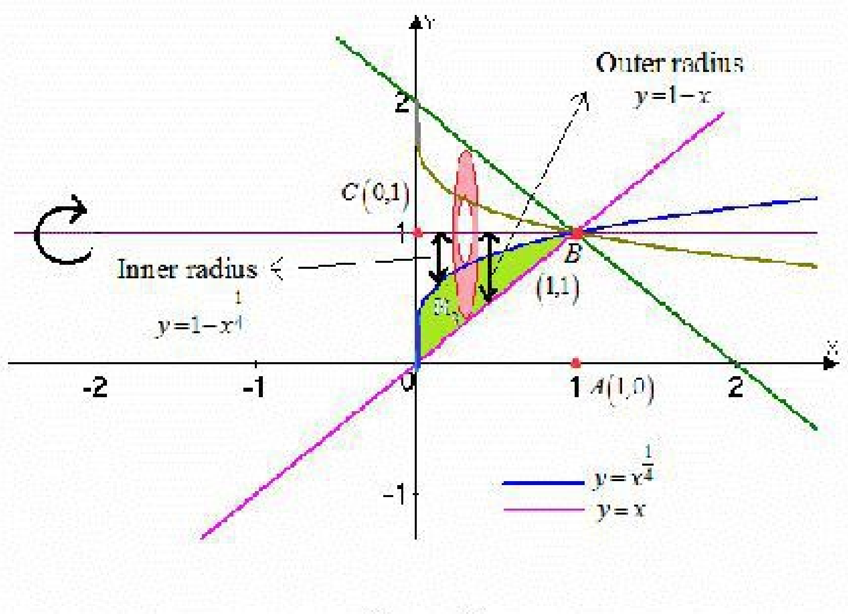


Figure (1)

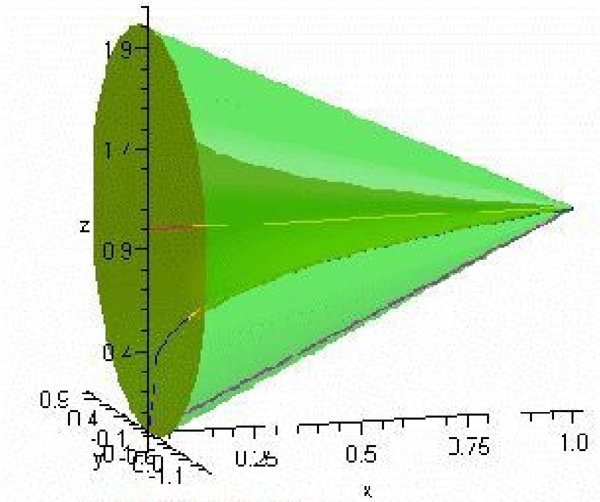


Figure (2)

Now we have to find the volume generated by rotating the region $\Re_3$ about the line joining points B and C.

Equation of the line joining points B and C is $y=1$.

The region and a cross-section perpendicular to the x-axis are shown in the figure (1) and the resulting solid is shown in the figure (2).

A cross section in the plane has a shape of a washer with inner radius $y=1-x^{\frac{1}{4}}$ and outer radius $y=1-x$.

By observing the graph x varies from 0 to 1

The cross sectional area is

$$A(x)=\pi(\text{outer radius})^2-\pi(\text{inner radius})^2$$

$$=\pi(1-x)^2-\pi\left(1-x^{\frac{1}{4}}\right)^2$$

$$=\pi\left(x^2-2x-x^{\frac{1}{2}}+2x^{\frac{1}{4}}\right)$$

Then the volume generated by $\Re_3$ about BC is

$$V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \pi\left(x^2 - 2x - x^{\frac{1}{2}} + 2x^{\frac{1}{4}}\right)dx$$

$$= \pi\left(\frac{x^3}{3} - \frac{2x^2}{2} - \frac{2}{3}x^{\frac{3}{2}} + 2\left(\frac{4}{5}x^{\frac{5}{4}}\right)\right)\Bigg|_0^1$$

$$= \pi\left(\frac{1}{3} - 1 - \frac{2}{3} + \frac{8}{5}\right)$$

$$= \frac{4\pi}{15}$$

Hence the volume is $\boxed{V = \frac{4\pi}{15}}$.

Answer 31E.

(a) Given curves are $y = \tan x, y = 0, x = \frac{\pi}{4}$.

Now we have to find the volume of the region formed by the given curves about the x-axis.

The graph of the bounded region R is shown below:

Graph of the bounded region R:

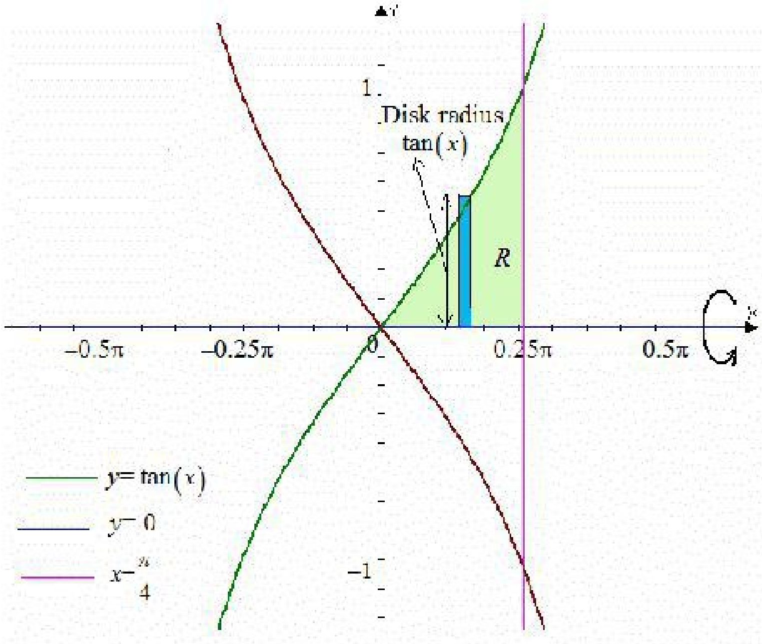
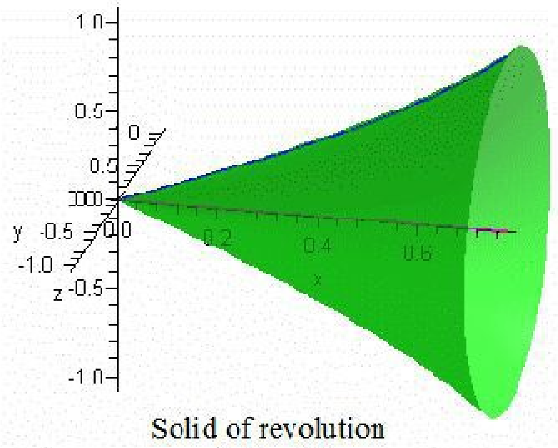


Figure (1)



Solid of revolution

Figure (2)

The region is shown in the figure (1) and the resulting solid is shown in the figure (2).
When we slice through the point x, we get a disk with radius $y = \tan(x)$.

And by observing graph, x varies from 0 to $\frac{\pi}{4}$.

The area of this cross-section is

$$\begin{aligned} A(x) &= \pi(y)^2 \\ &= \pi(\tan(x))^2 \\ &= \pi\tan^2(x) \end{aligned}$$

And hence the volume generated by R about the x-axis is

$$\begin{aligned} V &= \int_{x=0}^{\frac{\pi}{4}} A(x)\,dx \\ &= \pi\int_{x=0}^{\frac{\pi}{4}} \tan^2(x)\,dx \\ &= \pi\int_{x=0}^{\frac{\pi}{4}} \left(\sec^2(x)-1\right)dx \\ &= \pi\left(\int_{x=0}^{\frac{\pi}{4}} \sec^2(x)\,dx - \int_{x=0}^{\frac{\pi}{4}} dx\right) \end{aligned}$$

$$\begin{aligned} &= \pi\left(\tan(x)-x\right)_0^{\frac{\pi}{4}} \\ &= \pi\left(\tan\left(\frac{\pi}{4}\right)-\frac{\pi}{4}-\tan(0)+0\right) \\ &= \pi\left(1-\frac{\pi}{4}\right) \\ &\approx 0.67419 \qquad \text{(Since using calculator)} \end{aligned}$$

Hence the volume is $\boxed{V \approx 0.67419}$.

(b) Here we want to find the volume generated by R about the line $y = -1$.
The graph of the bounded region R is shown below:

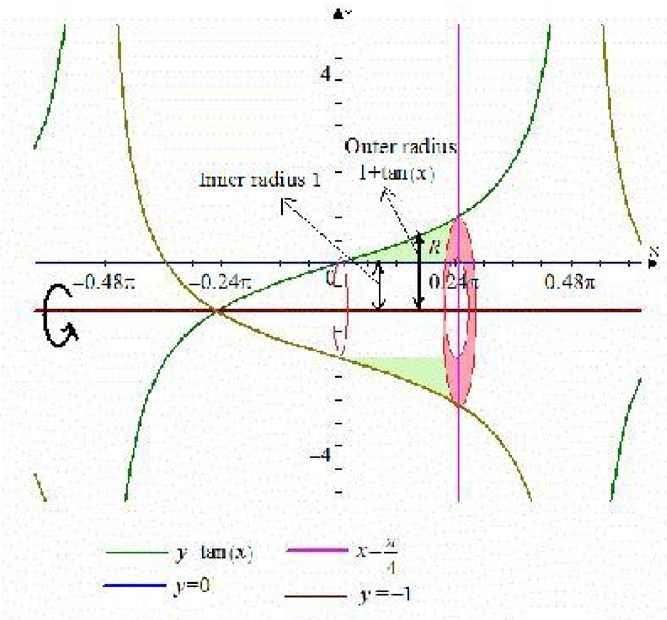


Figure (1)

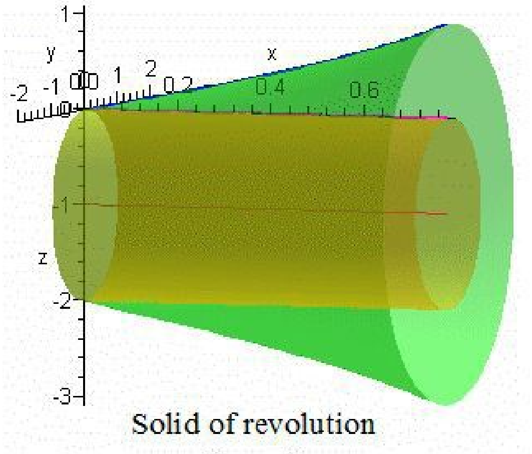


Figure (2)

The region and a cross section perpendicular to the x-axis are shown in the figure (1) and the resulting solid is shown in the figure (2).
A cross section in the plane has a shape of a washer with inner radius $y=1$ and outer radius $y=1+\tan x$.
So that the cross sectional area is

$$\begin{aligned} A(x) &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\ &= \pi(1+\tan x)^2 - \pi(1)^2 \\ &= \pi\left(2\tan x + \tan^2 x\right) \\ &= \pi\left(2\tan x + \sec^2 x - 1\right) \end{aligned}$$

By observing the graph x varies from 0 to $\frac{\pi}{4}$.

And the volume generated by R about $y=-1$ is

$$\begin{aligned} V &= \int_0^{\frac{\pi}{4}} A(x)\,dx \\ &= \int_0^{\frac{\pi}{4}} \pi\left(2\tan x + \sec^2 x - 1\right)dx \\ &= \pi\left(2\ln(\sec x) + \tan x - x\right)_0^{\frac{\pi}{4}} \\ &= \pi\left(2\ln\left(\sec\frac{\pi}{4}\right) + \tan\frac{\pi}{4} - \frac{\pi}{4} - 2\ln(\sec 0) - \tan 0 + 0\right) \\ &= \pi\left(2\ln\left(\sqrt{2}\right) + 1 - \frac{\pi}{4} - 2\ln(1)\right) \\ &= \pi\left(2\ln\left(\sqrt{2}\right) + 1 - \frac{\pi}{4}\right) \qquad (\text{Since } \ln(1) = 0) \\ &= \pi\left[\ln 2 + 1 - \frac{\pi}{4}\right] \\ &\approx \pi(0.90780) \\ &\approx 2.8519 \end{aligned}$$

Therefore volume of the bounded region R rotating about the line $y=-1$ is

$$\boxed{V \approx 2.8519}$$

Answer 32E.

(a) Given curves are $y=\cos^2 x, y=0, x=-\frac{\pi}{2}, x=\frac{\pi}{2}$.

Now we have to find the volume of the region formed by the given curves about the x-axis.
The graph of the bounded region R is shown below:

Graph of the bounded region R:

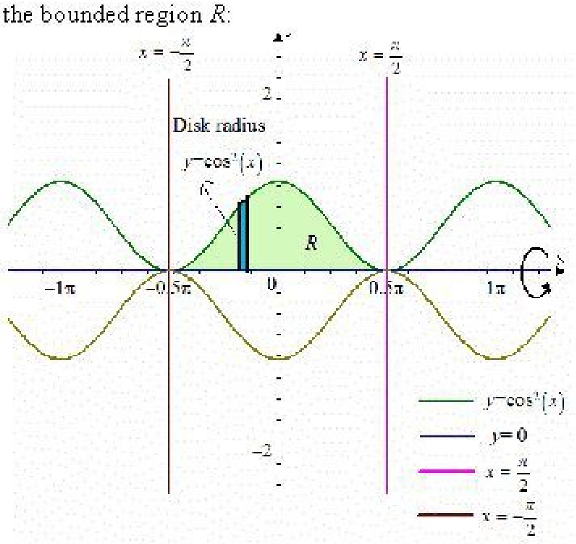


Figure (1)

The region is shown in the figure (1) and the resulting solid is shown in the figure (2).

When we slice through the point x, we get a disk with radius $y = \cos^2(x)$.

And by observing graph, x varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

The area of this cross-section is

$$
\begin{aligned}
A(x) &= \pi y^2 \\
&= \pi\left(\cos^2(x)\right)^2 \\
&= \pi\left(\frac{1+\cos 2x}{2}\right)^2 \\
&= \frac{\pi}{4}\left(1+\cos^2(2x)+2\cos(2x)\right) \\
&= \frac{\pi}{4}\left(1+\frac{1+\cos(4x)}{2}+2\cos(2x)\right) \\
&= \frac{\pi}{8}\left(3+\cos(4x)+4\cos(2x)\right)
\end{aligned}
$$

And hence the volume generated by R about the x-axis is

$$
\begin{aligned}
V &= \int_{x=0}^{\frac{\pi}{4}} A(x)\,dx \\
&= \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{8}\left(3+\cos(4x)+4\cos(2x)\right)dx \\
&= \frac{\pi}{8}(2)\int_{x=0}^{\frac{\pi}{2}} \left(3+\cos(4x)+4\cos(2x)\right)dx \\
&= \frac{\pi}{4}\left(3x+\frac{\sin(4x)}{4}+\frac{4\sin(2x)}{2}\right)_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{4}\left(3\left(\frac{\pi}{2}\right)+\frac{\sin(2\pi)}{4}+\frac{4\sin(\pi)}{2}\right) \\
&= \frac{\pi}{4}\left(3\left(\frac{\pi}{2}\right)\right) \\
&= \frac{3\pi^2}{8} \\
&\approx 3.7011
\end{aligned}
$$

Hence the volume is $\boxed{V \approx 3.7011}$.

(b) Here we want to find the volume generated by R about the line $y=1$.

The graph of the bounded region R is shown below:

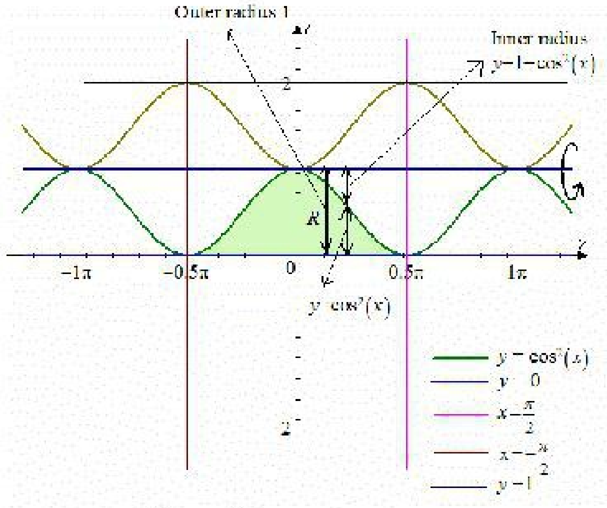


Figure (1)

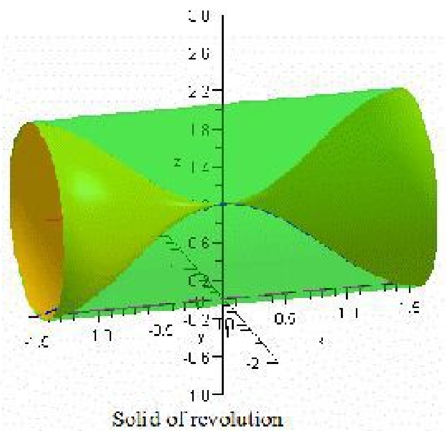


Figure (2)

The region is shown in the figure (1) and the resulting solid is shown in the figure (2).

A cross section in the plane has a shape of a washer with inner radius $y=1-\cos^2(x)$ and outer radius $y=1$.

So that the cross sectional area is

$$\begin{aligned}
A(x) &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\
&= \pi(1)^2 - \pi\left(1-\cos^2(x)\right)^2 \\
&= \pi\left(-\cos^4(x) + 2\cos^2(x)\right) \\
&= \pi\cos^2(x)\left(-\cos^2 x + 2\right) \\
&= \pi\left(\frac{1+\cos(2x)}{2}\right)\left(\frac{-1-\cos(2x)}{2}+2\right) \\
&= \pi\left(\frac{1+\cos(2x)}{2}\right)\left(\frac{3-\cos(2x)}{2}\right) \\
&= \frac{\pi}{4}\left(-\cos(2x) - \cos^2(2x) + 3 + 3\cos(2x)\right) \\
&= \frac{\pi}{4}\left(-\cos(2x) - \frac{1+\cos(2x)}{2} + 3 + 3\cos(2x)\right) \\
&= \frac{\pi}{8}\left(5+3\cos(2x)\right)
\end{aligned}$$

By observing the graph x varies from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

And the volume generated by R about $y=1$ is

$$V=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A(x)\,dx$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{8}\left(5+3\cos(2x)\right)dx$$

$$=\frac{\pi}{8}(2)\int_{0}^{\frac{\pi}{2}}\left(5+3\cos(2x)\right)dx$$

$$=\frac{\pi}{4}\left(5x+\frac{3\sin(2x)}{2}\right)_{0}^{\frac{\pi}{2}}$$

$$=\frac{\pi}{4}\left(5\left(\frac{\pi}{2}\right)+\frac{3\sin(\pi)}{2}\right)$$

$$=\frac{5\pi^2}{8}$$

$$\approx 6.1686$$

Therefore volume of the bounded region R rotating about the line $y=1$ is $\boxed{V\approx 6.1686}$.

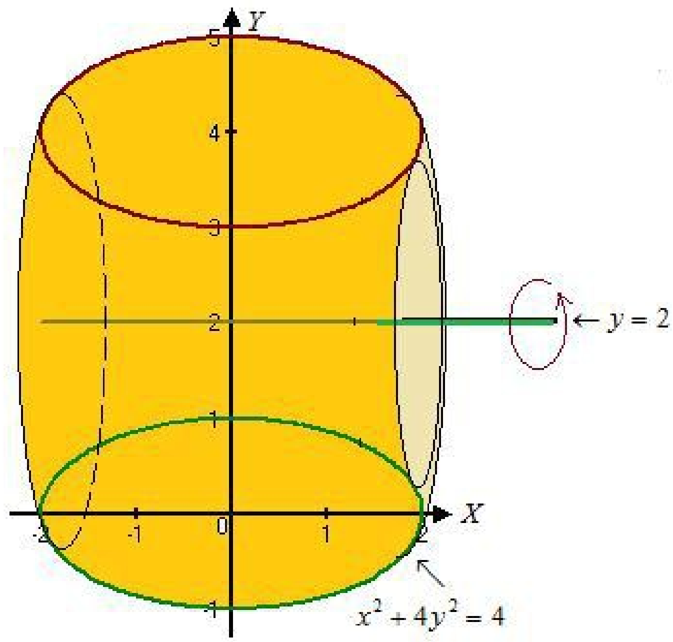
Answer 33E.

(a)

Consider the curve $x^2+4y^2=4$

$$\frac{x^2}{4}+\frac{y^2}{1}=1$$

Which represents ellipse and when this curve is rotated about $y=2$, the formed region looks like



Here need to find the volume generated by R about $y=2$

Since $\frac{x^2}{4}+\frac{y^2}{1}=1$, then $y=\pm\sqrt{1-\frac{x^2}{4}}$ and $-2\le x\le 2$.

The figure shows outer cross section.

It is a washer with inner radius $2-\sqrt{1-\frac{x^2}{4}}$ and outer radius $2+\sqrt{1-\frac{x^2}{4}}$

The cross section area is

$$A(x)=\pi(\text{outer radius})^2-\pi(\text{inner radius})^2$$
$$=\pi\left(2+\sqrt{1-\frac{x^2}{4}}\right)^2-\pi\left(2-\sqrt{1-\frac{x^2}{4}}\right)^2$$
$$=\pi\left[\left(2+\sqrt{1-\frac{x^2}{4}}\right)^2-\left(2-\sqrt{1-\frac{x^2}{4}}\right)^2\right]$$
$$=\pi\left(4\cdot 2\cdot\sqrt{1-\frac{x^2}{4}}\right)$$
$$=8\pi\sqrt{1-\frac{x^2}{4}}$$

Hence, the volume generated by R about $y=2$ is

$$V=\int_{x=-2}^{2}A(x)dx$$
$$=\int_{x=-2}^{2}8\pi\sqrt{1-\frac{x^2}{4}}dx$$
$$=2\int_{x=0}^{2}8\pi\sqrt{1-\frac{x^2}{4}}dx \qquad \text{Since } \left(1-\frac{x^2}{4}\right) \text{ is an even function.}$$
$$=\boxed{2\pi\int_{x=0}^{2}8\sqrt{1-\frac{x^2}{4}}dx}$$

Now evaluate this integral using a calculator.

Maple software can be used to find the definite integral.

Key board strokes of the command are as follows:

Maple command:

```
int (2*Pi*8((1-x^2/4)^1/2),x=0..2,numeric);
```

Maple command and output:

$> int\left(2\cdot \text{Pi}\cdot 8\cdot \text{sqrt}\left(1-\frac{x^2}{4}\right), x=0..2, numeric\right);$

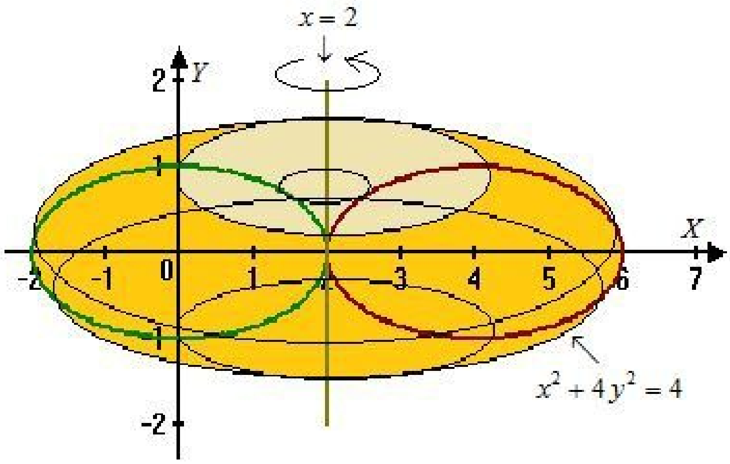
78.95683521

Thus, the volume

$$V=2\pi\int_{x=0}^{2}8\sqrt{1-\frac{x^2}{4}}dx$$
$$\approx\boxed{78.95683521}$$

(b)

Ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$ and when this curve is rotated about $x=2$, the formed region looks like



Here need to find the volume generated by R about $x=2$

Since $\frac{x^2}{4}+\frac{y^2}{1}=1$, then $x=\pm\sqrt{4-4y^2}$ and $-1\le y\le 1$.

The figure shows outer cross section.

It is washer with inner radius $x=2-\sqrt{4-4y^2}$ and outer radius $x=2+\sqrt{4-4y^2}$

The cross section area is

$$A(y)=\pi(\text{outer radius})^2-\pi(\text{inner radius})^2$$

$$=\pi\left[2+\sqrt{4-4y^2}\right]^2-\pi\left[2-\sqrt{4-4y^2}\right]^2$$

$$=\pi\left[\left(2+\sqrt{4-4y^2}\right)^2-\left(2-\sqrt{4-4y^2}\right)^2\right]$$

$$=\pi\left[4\cdot 2\cdot\sqrt{4-4y^2}\right]$$

$$=8\pi\sqrt{4-4y^2}$$

Hence, the volume generated by R about $x=2$ is

$$V=\int_{y=-1}^{1}A(y)\,dy$$

$$=\int_{-1}^{1}8\pi\sqrt{4-4y^2}\,dy$$

$$=8\pi\cdot 2\int_{0}^{1}\sqrt{4-4y^2}\,dy \quad \left(\text{because } 1-y^2 \text{ is an even function}\right)$$

$$=\boxed{2\pi\int_{0}^{1}8\sqrt{4-4y^2}\,dy}$$

Now evaluate this integral using a calculator.

Maple software can be used to find the definite integral.

Key board strokes of the command are as follows:

Maple command:

```
int (2*Pi*8((1-x^2/4)^1/2),x=0..2,numeric);
```

Maple command and output:

> $int\left(2\cdot \text{Pi}\cdot 8\cdot \text{sqrt}\left(4-4y^2\right), y=0..1, numeric\right);$

78.95683521

Thus, the volume

$$V = 2\pi\int_0^1 8\sqrt{4-4y^2}\,dy$$

$$\approx \boxed{78.95683521}$$

Answer 34E.

a)

The objective is to setup an integral to find the volume formed by rotating the region surroun by the curves $x^2+y^2=1, y=x^2$, $y\geq 0$ about x-axis.
At first sketch these curves, the first equation $x^2+y^2=1$ is a circle centered at origin with ra 1, the second curve is a parabola with vertex at the origin and the inequality $y\geq 0$ represents the area is above the x-axis.

Next find the points of intersection of the curves $x^2+y^2=1, y=x^2$

Substitute $y=x^2$ in $x^2+y^2=1$ to obtain,

$$x^2+\left(x^2\right)^2=1$$
$$x^4+x^2-1=0$$

It is difficult to solve the equation $x^4+x^2-1=0$ so use CAS to solve this equation.
Type the following command in Maple, and then press ENTER to obtain the final result.

$solve\left(x^4+x^2-1=0, x\right);$

$$-\frac{1}{2}\sqrt{-2+2\sqrt{5}}, \frac{1}{2}\sqrt{-2+2\sqrt{5}},$$
$$-\frac{1}{2}\,\mathrm{I}\sqrt{2+2\sqrt{5}}, \frac{1}{2}\,\mathrm{I}\sqrt{2+2\sqrt{5}}$$

There are two real roots for the equation $x^4+x^2-1=0$

They are $x=\frac{1}{2}\sqrt{-2+2\sqrt{5}}\approx -0.78615, -\frac{1}{2}\sqrt{-2+2\sqrt{5}}\approx 0.78615$

Now substitute $x=0.78615$ and $x=-0.78615$ in $y=\sqrt{1-x^2}$

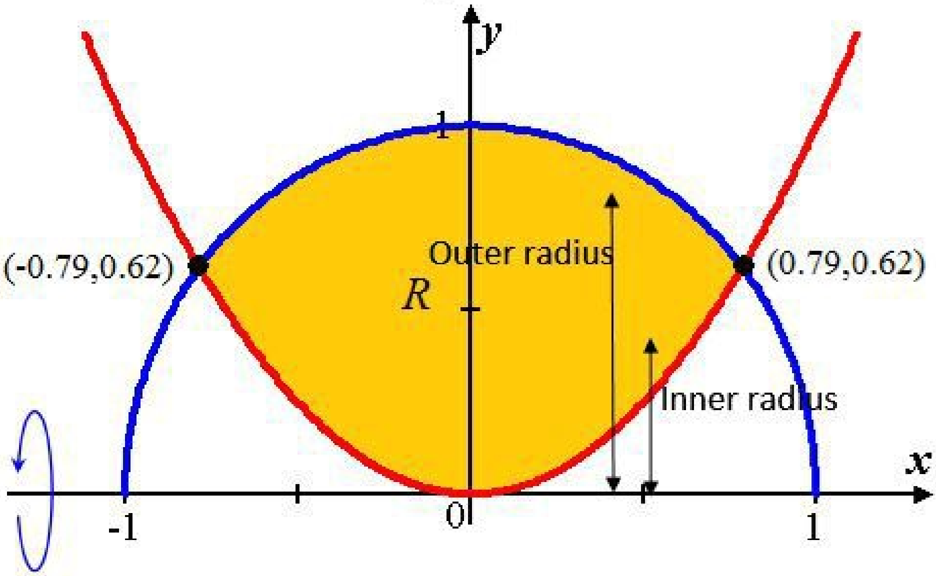
$$y=\sqrt{1-0.79^2}$$
$$=0.62$$

So, the points of intersection are $(-0.79, 0.62)$, and $(0.79, 0.62)$.

Hence, over the given region x varies from -0.79 to 0.79.

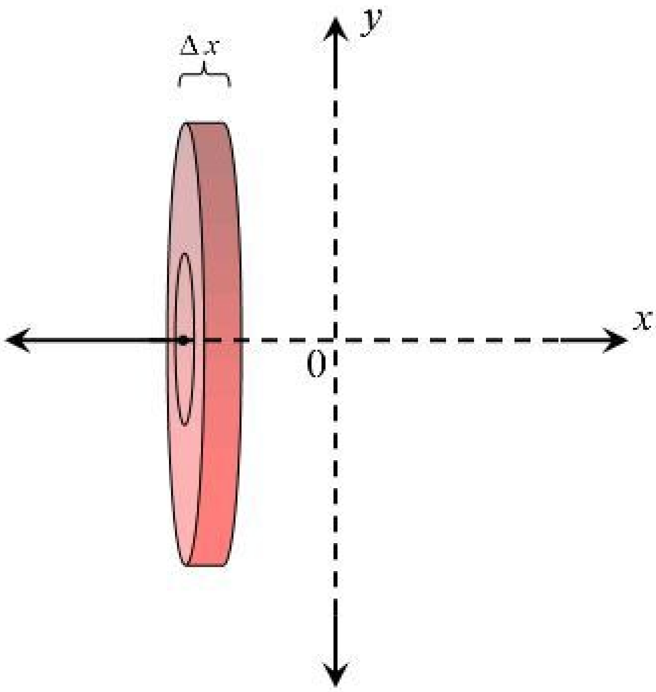
Use this information to sketch these curves, and then the region will be as shown Will be as shown below:

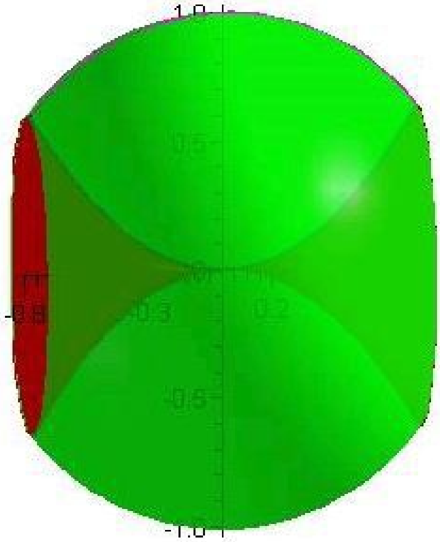
Figure-1



The solid formed by rotating the above region R is as shown below:

Figure-2





Inner radius of the approximating cylinder is given by

Inner radius = Lower curve − axis of rotation

$$r_{\text{Inner}} = x^2 - 0$$

$$r_{\text{Inner}} = x^2$$

Inner radius of the approximating cylinder is given by

Outer radius = Upper curve − axis of rotation

$$r_{\text{Outer}} = \sqrt{1-x^2} - 0$$

$$r_{\text{Outer}} = \sqrt{1-x^2}$$

Area of cross-section of the approximating cylinder is as follows:

$$A(x) = \pi r^2{}_{\text{outer}} - \pi r^2{}_{\text{inner}}$$

$$A(x) = \pi\left(\sqrt{1-x^2}\right)^2 - \pi\left(x^2\right)^2$$

Use the washer formula which is

$$V = \int A(x)dx$$

$$V = \int_{x=-0.79}^{x=0.79}\left[\pi\left(\sqrt{1-x^2}\right)^2 - \pi\left(x^2\right)^2\right]dx$$

It is difficult to solve the integral manually so use CAS to solve this integral.
Type the following command in Maple, and then press ENTER to obtain the final result.

$$int\left(\left(\pi\cdot\left(\sqrt{1-x^2}\right)^2 - \pi\cdot\left(x^2\right)^2\right), x=-0.79\,..0.79\right);$$

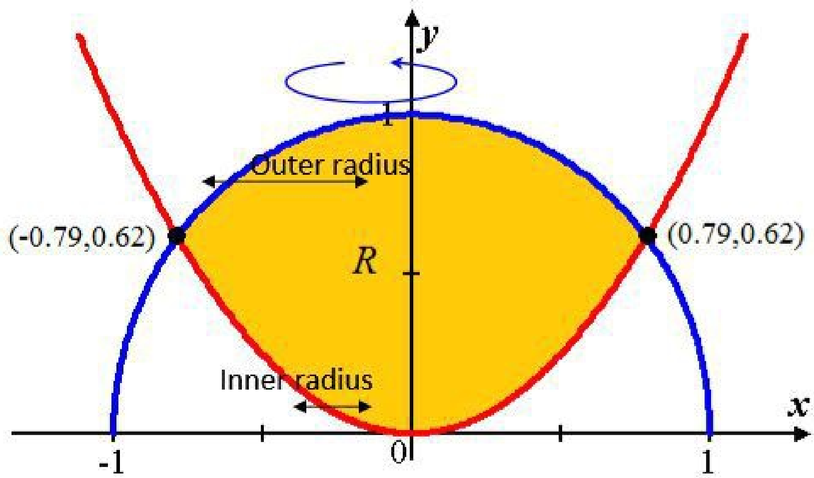
3.544423615

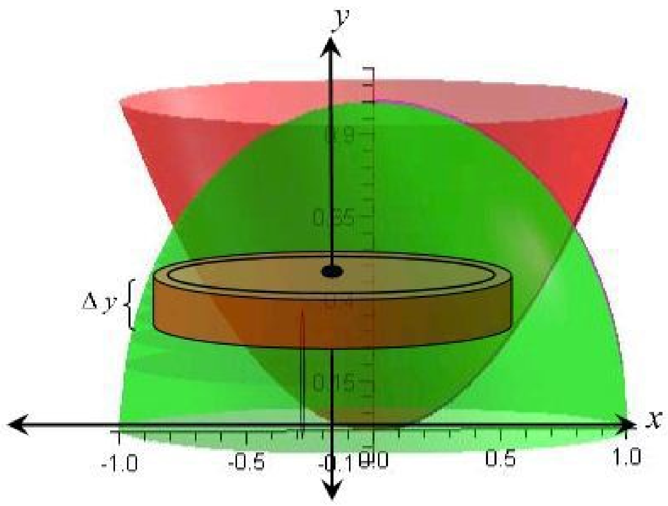
Therefore, the volume of the solid obtained is $\boxed{3.54 \text{ unit}^3}$

b)

If the region R is rotated about y-axis the obtained figure will be as shown in the figure-3:

Figure-3





Inner radius of the approximating cylinder is given by

$$\text{Inner radius} = \text{Lower curve} - \text{axis of rotation}$$
$$r_{\text{Inner}} = \sqrt{y} \quad - \quad 0$$
$$r_{\text{Inner}} = \sqrt{y}$$

Inner radius of the approximating cylinder is given by

$$\text{Outer radius} = \text{Upper curve} - \text{axis of rotation}$$
$$r_{\text{Outer}} = \sqrt{1-y^2} \quad - \quad 0$$
$$r_{\text{Outer}} = \sqrt{1-y^2}$$

Over the given region y varies from 0 to 1.

Area of cross-section of the approximating cylinder is as follows:

$$A(y) = \pi r^2_{\text{outer}} - \pi r^2_{\text{inner}}$$
$$A(y) = \pi\left(\sqrt{1-y^2}\right)^2 - \pi\left(\sqrt{y}\right)^2$$

Use the washer formula to find the volume.

$$V = \int A(y)\,dy$$

$$V = \int_{y=0}^{y=1} \left[\pi\left(\sqrt{1-y^2}\right)^2 - \pi\left(y^2\right)^2 \right] dy$$

Now use CAS to evaluate the integral,

Type the following command in Maple, and then press ENTER to obtain the final result.

$$int\left(\left(\pi\cdot\left(\sqrt{1-y^2}\right)^2 - \pi\cdot\left(\sqrt{y}\right)^2\right), y=0..1\right);$$

$$\frac{1}{6}\pi$$

Therefore, the volume of the solid obtained is $\boxed{\frac{\pi}{6} \approx 0.5231 \text{ unit}^3}$.

Answer 35E.

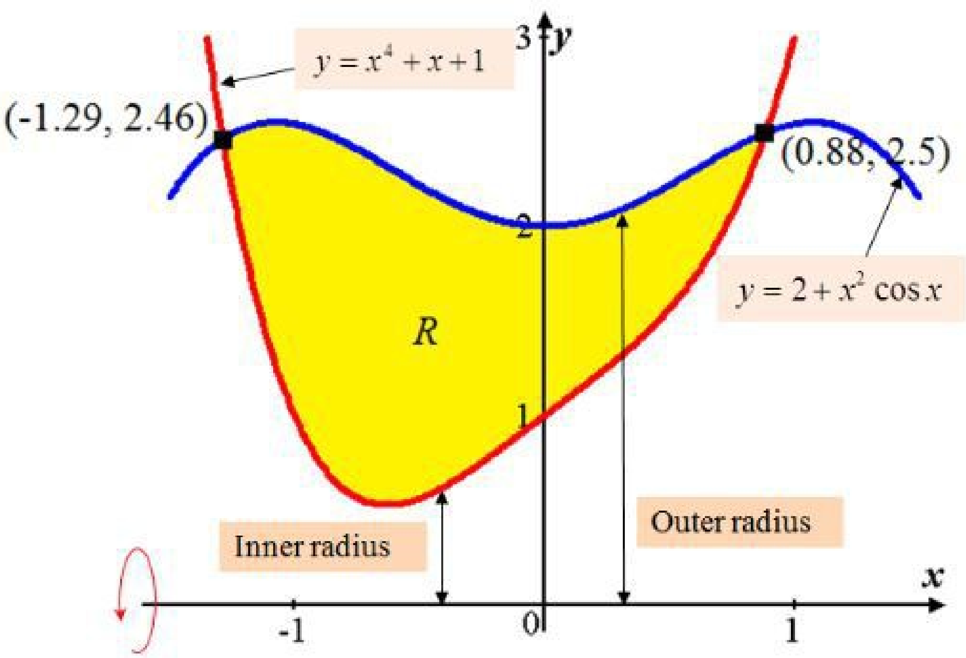
Consider the region bounded by the curves

$$y = 2 + x^2 \cos x$$
$$y = x^4 + x + 1$$

Need to find approximate -coordinates of the points of intersection of the given curves And then use calculator to find (approximately) the volume of the solid obtained by rotating about the x-axis the region bounded by the given curves.

$$y = 2 + x^2 \cos x$$
$$y = x^4 + x + 1$$

At first sketch these two functions to find the points of intersection.



From the figure, the points of intersection are $(-1.29, 2.46)$ and $(0.88, 2.5)$

The region R surrounded by the curves $y = 2 + x^2 \cos x$, $y = x^4 + x + 1$ is rotating about the x-axis.

The upper curve of the region R is $y = 2 + x^2 \cos x$ so the outer radius of the solid formed by rotating the region R is given by

$$\text{Outer radius } (r_{\text{outer}}) = 2 + x^2 \cos x$$

The lower curve of the region R is $y = x^4 + x + 1$ so the inner radius of the solid formed by rotating the region R is given by

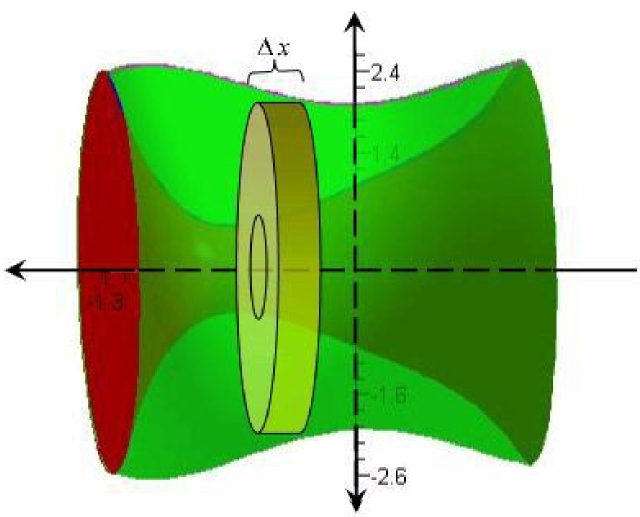
$$\text{Inner radius } (r_{\text{inner}}) = x^4 + x + 1$$

Area of the cross-section of the solid formed is given by

$$\begin{aligned} A(x) &= \pi (r_{\text{outer}})^2 - \pi (r_{\text{inner}})^2 \\ &= \pi (2 + x^2 \cos x)^2 - \pi (x^4 + x + 1)^2 \end{aligned}$$

Over the given region the variable x takes values from -1.29 to 0.88.
So the integration can evaluated with the limits -1.29 and 0.88.

The solid formed by rotating the region R about the axis $y = 0$ is as shown below:



Now the volume of the solid can be calculated using washer formula as show below:

$$\begin{aligned} V &= \int A(x)\,dx \\ &= \int_{x=a}^{x=b} \left[\pi (r_{\text{outer}})^2 - \pi (r_{\text{inner}})^2 \right] dx \\ &= \int_{x=-1.29}^{x=0.88} \left[\pi (2 + x^2 \cos x)^2 - \pi (x^4 + x + 1)^2 \right] dx \end{aligned}$$

It difficult to solve this integral manually, computers are well suited for this task.

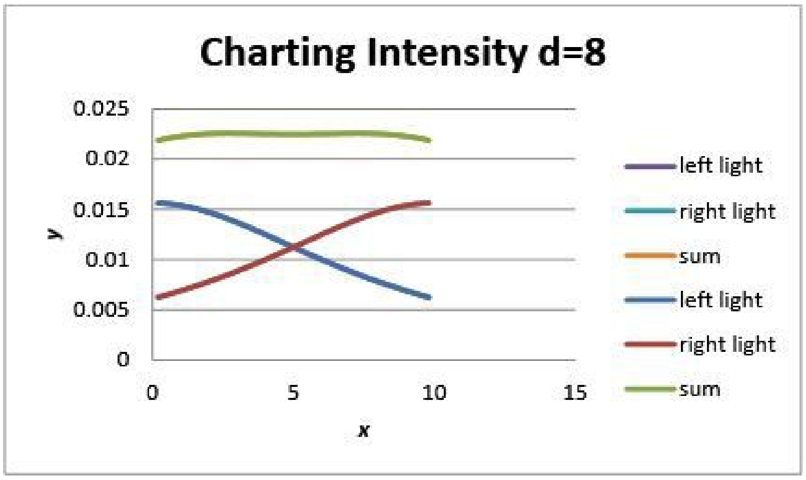
Type the following command in **MAPLE** and then press ENTER to obtain the final result.

$> int\left(\left(\text{pi}\cdot(2 + x^2\cdot\cos(x))^2 - \text{pi}\cdot(x^4 + x + 1)^2\right), x = -1.29 \,..\, 0.88\right);$

$$7.569221661\,\pi$$

Hence the volume of solid is approximately equal to $\boxed{7.57\pi.}$

For $d = 8$, graph is as follows.



Now find the exact value of d which is a minimum. Take the derivative of the intensity function with respect to x. Then find the second derivative, to see where d has a sudden change, and a point of inflection.

$$I = \frac{F_1}{x_1^2 + d^2} + \frac{F_1}{100 + x_1^2 - 20x_1 + d^2}$$

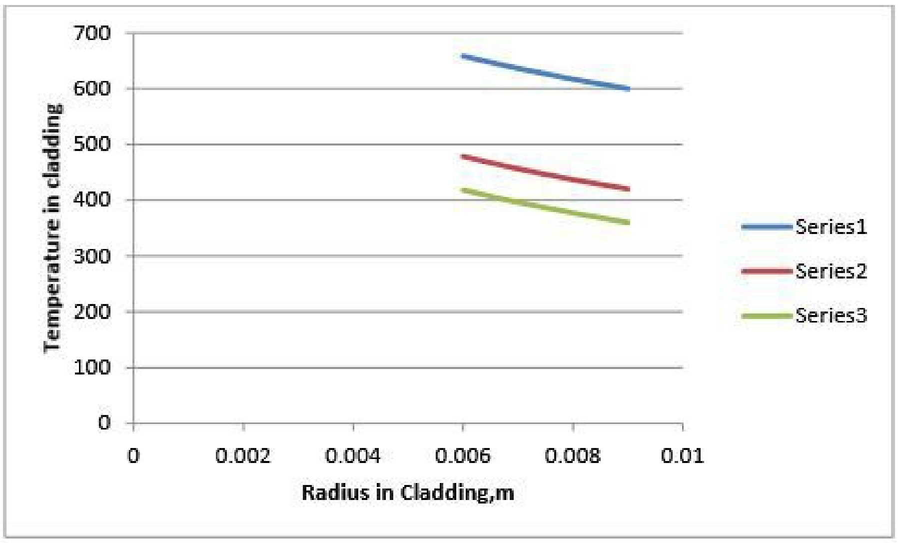
$$I'(x,d) = \frac{-2x_1F_1}{\left(x_1^2 + d^2\right)^2} - \frac{2F_1(x_1 - 10)}{\left((x_1 - 10)^2 + d^2\right)^2}$$

$$I''(x,d) = \frac{+4x_1^2F_1}{\left(x_1^2 + d^2\right)^3} + \frac{4F_1(x_1 - 10)^2}{\left((x_1 - 10)^2 + d^2\right)^3}$$

Continue to solve for d.

$$0 = \frac{x_1^2}{\left(x_1^2 + d^2\right)^3} + \frac{(x_1 - 10)^2}{\left((x_1 - 10)^2 + d^2\right)^3}$$

Plot the temperature distribution in the cladding for different values of heat transfer coefficient,



Answer 36E.

First we sketch the curves $y = x^4$ *and* $y = 2x - x^3$ with the help of computer

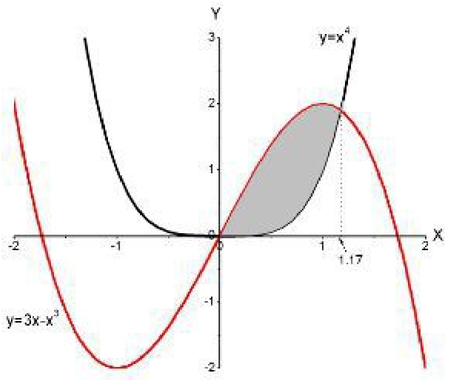


Fig.1

Now we move the cursor to the points of intersection we see that x –coordinates of the these points are x = 0 and $x \approx 1.17$

Now we sketch the solid obtained rotating the shaded region about x – axis

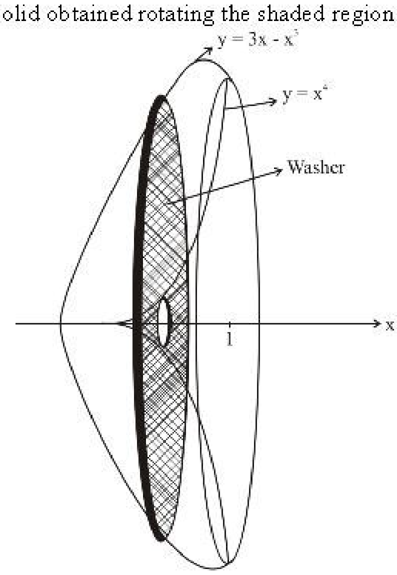


Fig. 2

Outer radius of washer is $= 3x - x^3$
Inner radius of washer is $= x^4$
Then cross sectional area of washer is

$$A(x) = \pi\left[(outer\ radius)^2 - (inner\ radius)^2\right]$$

Or $$A(x) = \pi\left[\left(3x - x^3\right)^2 - \left(x^4\right)^2\right]$$

$$A(x) = \pi\left[9x^2 + x^6 - 6x^4 - x^8\right]$$

Then the volume of solid is $v = \int_0^{1.17} A(x)dx$

Or $$v = \int_0^{1.17} \pi\left[9x^2 + x^6 - 6x^4 - x^8\right]dx$$

$$v = \pi\int_0^{1.17}\left[9x^2 + x^6 - 6x^4 - x^8\right]dx$$

$$v = \pi\left[3x^3 + \frac{1}{7}x^7 - \frac{6}{5}x^5 - \frac{x^9}{9}\right]_0^{1.17} \quad \text{[By FTC - 2]}$$

$$v = \pi\left[3(1.17)^3 + \frac{1}{7}(1.17)^7 - \frac{6}{5}(1.17)^5 - \frac{(1.17)^9}{9}\right]_0^{1.17}$$

$$\approx \pi\left[4.80 + 0.43 - 2.63 - 0.46\right]$$

$$\approx \pi\left[2.14\right]$$

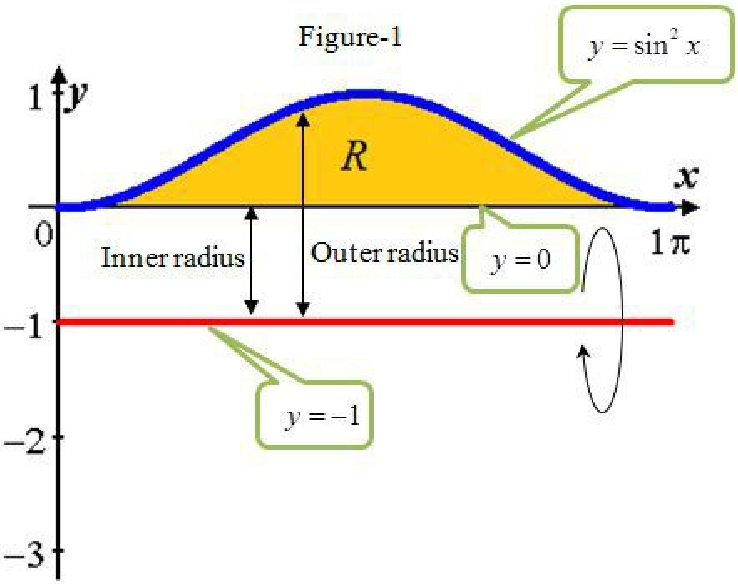
Or $\boxed{v \approx 6.72}$

Answer 37E.

Find the volume of the solid obtained by rotating the region surrounded by the curves

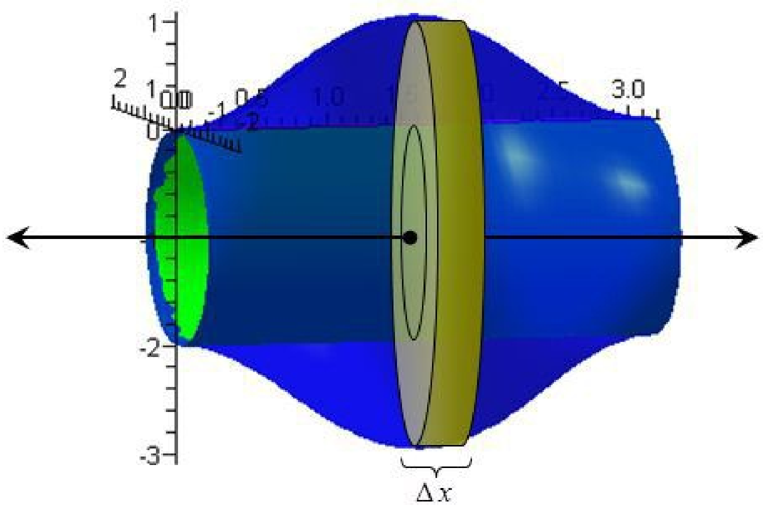
$y = \sin^2 x, y = 0; 0 \le x \le \pi$

The region is as shown below:



The solid formed by rotating the above region *R* is as shown below:

Figure-2



Inner radius of the approximating cylinder is the distance between the axis of rotation and the lower curve.

$$r_{\text{Inner}} = 0 - (-1)$$
$$= 1$$

Outer radius of the approximating cylinder is the distance between the axis of rotation and the upper curve.

$$r_{\text{Outer}} = \sin^2 x - (-1)$$
$$= \sin^2 x + 1$$

Area of the cross section of the approximating cylinder is given by

$$A(x) = \pi \cdot r^2_{\text{Outer}} - \pi \cdot r^2_{\text{Inner}}$$

By the washer method, the volume of the solid can be evaluated as follows:

$$V = \int_{x=a}^{x=b} A(x)\,dx$$
$$= \int_{x=a}^{x=b} \left[\pi \cdot r^2_{\text{Outer}} - \pi \cdot r^2_{\text{Inner}}\right] dx$$
$$= \int_0^{\pi} \left[\pi \cdot \left(\sin^2 x + 1\right)^2 - \pi \cdot (1)^2\right] dx$$
$$= \pi \int_0^{\pi} \left[\sin^4 x + 2\sin^2 x\right] dx$$

It difficult to solve this integral manually, computers are well suited for this task.

Type the following command in **MAPLE** and then press ENTER to obtain the final result.

$> int\left(\pi \cdot \left((\sin(x))^4 + 2 \cdot (\sin(x))^2 \right), x = 0 .. \pi\right);$

$$\frac{11}{8}\pi^2$$

Therefore, volume of the required solid is

$$\boxed{\frac{11}{8}\pi^2 \text{ Unit}^3}$$

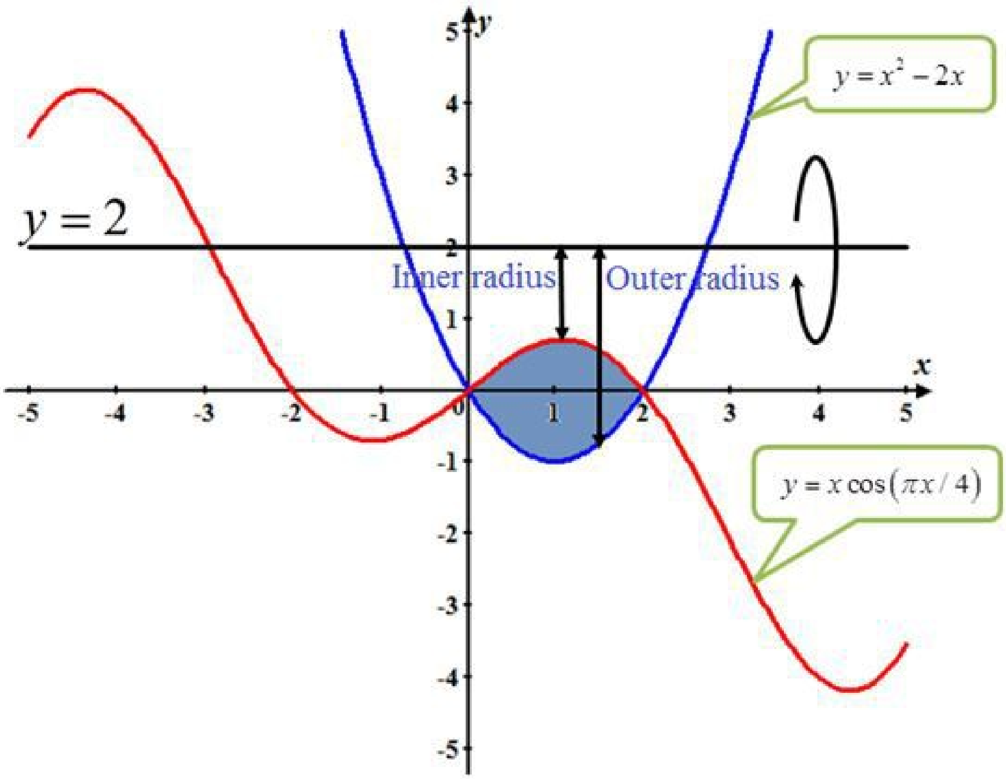
Answer 38E.

Find the volume of the solid obtained by rotating the region surrounded by the curves

$y = x^2 - 2x,\ y = x\cos(\pi x/4);$ about $y = 2$.

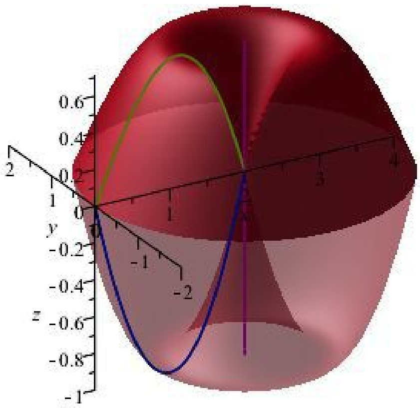
The region is as shown below:

Figure-1



The solid formed by rotating the above region R is as shown below:

Figure-2



Inner radius of the approximating region is the distance between the axis of rotation and the lower curve.

$$r_{\text{Inner}} = 2 - x\cos\left(\frac{\pi x}{4}\right)$$

Outer radius of the approximating cylinder is the distance between the axis of rotation and the upper curve.

$$\begin{aligned} r_{\text{Outer}} &= 2 - \left(x^2 - 2x\right) \\ &= 2 - x^2 + 2x \end{aligned}$$

Area of the cross section of the approximating region is given by

$$A(x) = \pi \cdot r^2{}_{\text{Outer}} - \pi \cdot r^2{}_{\text{Inner}}$$

By the washer method, the volume of the solid can be evaluated as follows:

$$\begin{aligned} V &= \int_{x=a}^{x=b} A(x)\,dx \\ &= \int_{x=a}^{x=b} \left[\pi \cdot r^2{}_{\text{Outer}} - \pi \cdot r^2{}_{\text{Inner}}\right] dx \\ &= \int_0^2 \left[\pi \cdot \left(2 - x^2 + 2x\right)^2 - \pi \cdot \left(2 - x\cos\left(\frac{\pi x}{4}\right)\right)^2\right] dx \end{aligned}$$

It difficult to solve this integral manually, computers are well suited for this task.

Type the following command in **MAPLE** and then press ENTER to obtain the final result.

$$> int\left(\pi \cdot (2 - x^2 + 2 \cdot x)^2 - \pi \cdot \left(2 - x \cdot \cos\left(\frac{\pi \cdot x}{4}\right)\right)^2, x = 0..2\right);$$

$$\frac{4}{15}\,\frac{19\pi^2 + 120\pi - 210}{\pi}$$

Therefore volume of the required solid is

$$\boxed{30.019 \text{ Unit}^3}$$

Answer 39E.

Given the integral $\pi\int_0^{\pi} \sin x\,dx$

We write the integral as

$$\pi\int_0^{\pi} \sin x\,dx = \pi\int_0^{\pi} \left(\sqrt{\sin x}\right)^2 dx$$

So the integral represents the volume of the solid obtained by rotating the region $0 \le x \le \pi, 0 \le y \le \sqrt{\sin x}$ about the $x-$axis

Answer 40E.

Given the integral $\pi\int_{-1}^{1}\left(1-y^2\right)^2\,dy$

This describes the volume of the solid obtained by rotating the region

$-1\le y\le 1,\ 0\le x\le 1-y^2$ about $y-$axis

Answer 41E.

$$\text{Since volume } v=\pi\int_0^1\left(y^4-y^8\right)dy$$
$$=\int_0^1\pi\left(y^4-y^8\right)dy$$
$$=\int_0^1\left(\pi y^4-\pi y^8\right)dy$$
$$=\int_0^1\left(\pi\left(y^2\right)^2-\pi\left(y^4\right)^2\right)dy$$

Since we have volume of solid is $v=\int_0^1 A(y)dy$

And cross sectional area of washer is

$$A(y)=\pi(\text{outer radius})^2-\pi(\text{inner radius})^2$$

By comparing, we have

Outer radius $=y^2$ and inner radius $=y^4$

So solid is obtained by rotating the region bounded by curves $x=y^2$ and $x=y^4$

from $y=0$ to $y=1$ about $y-$axis

$\boxed{y^4\le x\le y^2,\ 0\le y\le 1}$ About $y-$axis

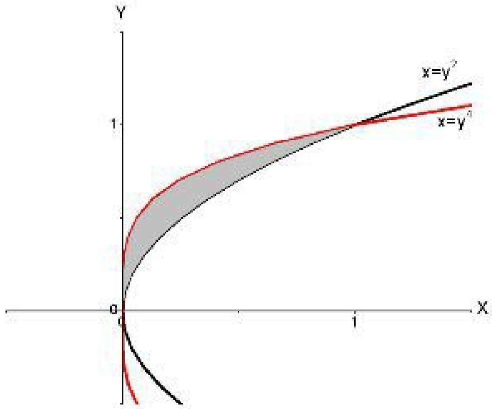


Fig.1

Answer 42E.

$$\text{Since volume } v=\pi\int_0^{\pi/2}\left[(1+\cos x)^2-1^2\right]dx$$

$$\text{Or}\quad v=\int_0^{\pi/2}\pi\left[(1+\cos x)^2-(1+0)^2\right]dx$$

Since $v=\int_0^{\pi/2}A(x)dx$

And area of washer is $A(y)=\pi\left[(\text{outer radius})^2-\pi(\text{inner radius})^2\right]$

So by comparing we have

Outer radius $=1+\cos x$

And inner radius $=1+0=1$

So the solid is obtained by rotating the region bounded by the curves $y=\cos x$

and $y=0$ about $y=-1$

Or in other words the solid is obtained by rotating the region

$\boxed{0\le y\le\cos x,\ 0\le x\le\frac{\pi}{2}}$ about $y=-1$

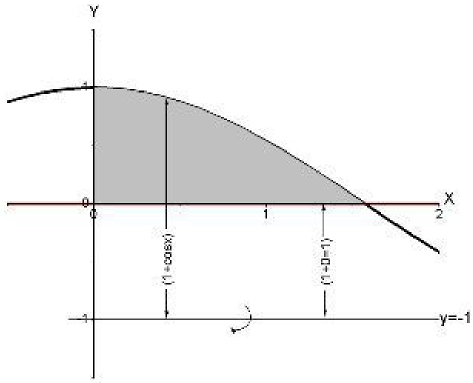


Fig.1

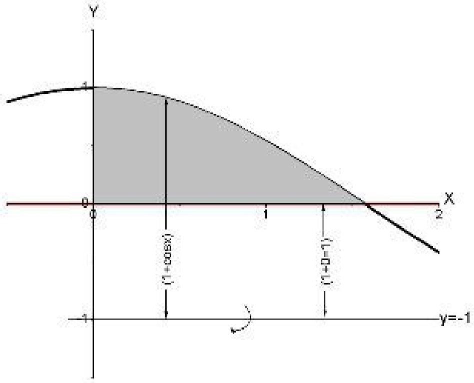


Fig.1

Answer 43E.

First we make a table for given areas of the cross section spaced 1.5cm a part of a human liver

cm	Space x	0	1.5	3	4.5	6	7.5	9	10.5	12	13.5	15
cm^2	Area $A(x)$	0	18	58	79	94	106	117	128	63	39	0

We have the interval $0 \le x \le 15$, we divide this interval 5 subintervals so n = 5 and width of the sub intervals is $\Delta x = \frac{15-0}{5} = 3$

Then sub intervals are [0, 3], [3, 6], [6, 9], [9, 12] and [12, 15] then mid points of the sub intervals are 1.5, 4.5, 7.5, 10.5, 13.5

We can approximate the values of liver as

$$V \approx \sum_{i=1}^{5} A(x_i^*)\Delta x \qquad \text{[By mid point rule]}$$

Where x_i^* is the mid point of the sub interval $[x_{i-1}, x_i]$ then we have

$$V \approx \Delta x[A(1.5)+A(4.5)+A(7.5)+A(10.5)+A(13.5)]$$

Or $V \approx 3[18+79+106+128+39]\ \ cm^3$

Or $\boxed{V \approx 1110\ cm^3}$

Answer 44E.

The cross sectional areas at a distance x from the end of the log as given in the table

$x(m)$	0	1	2	3	4	5	6	7	8	9	10
$A(m^2)$	0.68	0.65	0.64	0.61	0.58	0.59	0.53	0.55	0.52	0.50	0.48

Taking n = 5, width of sub intervals is = $\Delta x = \frac{10}{5} = 2$

So sub intervals are [0, 2], [2, 4], [4, 6], [6, 8] and [8, 10]

Then mid points are 1, 3, 5, 7 and 9

By mid points rule we can estimate the volume of log as

$$V \approx \sum_{i=1}^{5} A(x_i^*)\Delta x$$

Where x_i^* is the mid point of the sub interval $[x_{i-1}, x_i]$

Then we have

$$V \approx \Delta x[A(1)+A(3)+A(5)+A(7)+A(9)]\text{m}^3$$

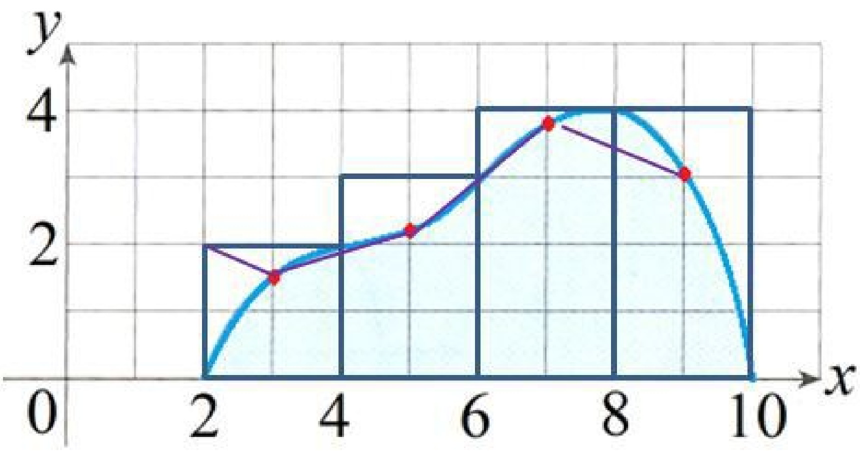
$$V \approx 2[0.65+0.61+0.59+0.55+0.50]\ \text{m}^3$$

$$V \approx 2\times 2.9\ \ \text{m}^3$$

Or $\boxed{V \approx 5.80\ \ \text{m}^3}$

Answer 45E.

(a) Consider the graph



From the graph the approximate estimated values are

$(3,1.5),(5,2.2),(7,3.9),(9,3)$

That is $f(3)=1.5, f(5)=2.2, f(7)=3.9,$ and $f(9)=3$

Here $a=2$ and $b=10$ and $n=4$

By definition

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{10-2}{4}$$
$$= 2$$

By Midpoint rule Area about $x-$axis is

$$A = \int_2^{10} f(x)dx$$
$$= \Delta x[f(3)+f(5)+f(7)+f(9)]$$
$$= 2[1.5+2.2+3.9+3]$$
$$= 2[10.6]$$
$$= 21.2$$

Since by algebraic simplification.

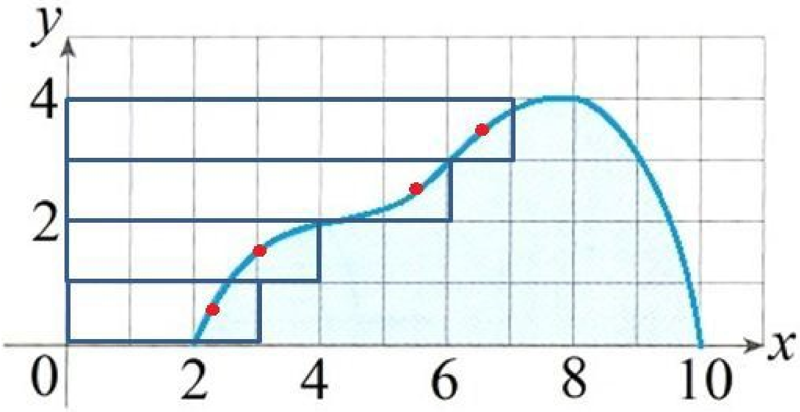
By definition of volume $V=\int_2^{10} A(x)\,dx$

$$V=\int_2^{10}(21.2)\,dx$$
$$=(21.2)[x]_2^{10}$$
$$=(21.2)[10-2]$$
$$=(21.2)(8)$$
$$=169.6$$

Therefore, Volume $V=\boxed{169.6}$

(b)

Consider the graph



From the graph the approximate estimated values are

$(2.3,0.5),(2.5,1.5),(5.5,2.5),(6.5,3.5)$

That is $f(2.3)=0.5, f(2.5)=1.5, f(5.5)=2.5,$ and $f(6.5)=3.5$

Here $a=0$ and $b=4$ and $n=4$

By definition

$$h=\frac{b-a}{n}$$
$$=\frac{4-0}{4}$$
$$=1$$

By Midpoint rule Area about $y-$axis is

$$A=\int_0^4 f(x)dx$$
$$=1[2.3+2.5+5.5+6.5]$$
$$=1[16.8]$$
$$=16.8$$

By Midpoint rule Area about $y-$axis is

By definition of volume $V=\int_0^4 A(x)\,dx$

$$V=\int_0^4(16.8)\,dx$$
$$=(16.8)[x]_0^4$$
$$=(16.8)[4-0]$$
$$=67.2$$

Therefore, Volume $V=\boxed{67.2}$

Answer 46E.

(a) Consider the function $f(x)=(ax^3+bx^2+cx+d)\sqrt{1-x^2}$

Use the formula for the volume of a solid. Let S be a solid that lies between $x=a$ and $x=b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is an integrable function, then the volume of S is

$$V=\lim_{\max \Delta x_i \to 0}\sum_{i=1}^{n} A(x_i^*)\Delta x_i=\int_a^b A(x)dx$$

Any plane P_x that passes through x and is perpendicular to the x-axis intersects the solid in a circular disk. Let r be the radius of the circle. Write an expression for r.

$$r=(ax^3+bx^2+cx+d)\sqrt{1-x^2}$$

Then find the area of the cross-sections using the formula for the area of a circle.

$$\begin{aligned}A(x)&=\pi r^2\\&=\pi(1-x^2)(ax^3+bx^2+cx+d)^2\end{aligned}$$

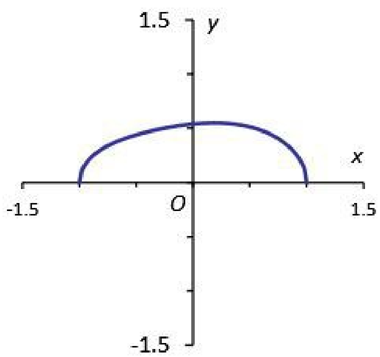
The graph of $f(x)=(ax^3+bx^2+cx+d)\sqrt{1-x^2}$ is above the x-axis between $x=-1$ and $x=1$. Use this area to write an integral for the volume of the solid.

$$V=\int_{-1}^{1}\left[\pi(1-x^2)(ax^3+bx^2+cx+d)^2\right]dx$$

Then use a CAS to simplify V.

$$V=\boxed{\frac{4}{315}\pi(5a^2+18ac+9b^2+42bd+21c^2+105d^2)}$$

(b) Graph f with $a=-0.06$, $b=0.04$, $c=0.1$, and $d=0.54$ from $x=-1$ to $x=1$.



Then find the volume of the egg using the formula you found above.

$$\begin{aligned}V&=\frac{4}{315}\pi\left[5(-0.06)^2+18(-0.06)(0.1)+9(0.04)^2\right.\\&\qquad\left.+42(0.04)(0.54)+21(0.1)^2+105(0.54)^2\right]\\&\approx\boxed{1.263}\end{aligned}$$

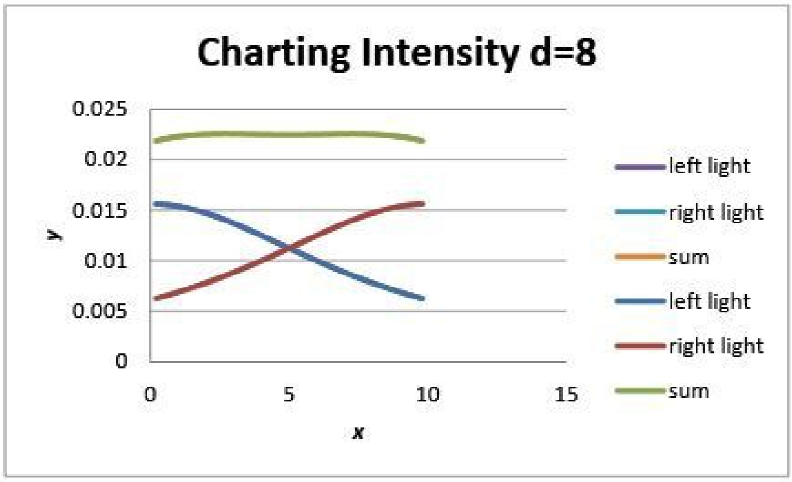
Type the following command in **MAPLE** and then press ENTER to obtain the final result.

```
> int((pi·(2 + x^2·cos(x))^2 − pi·(x^4 + x + 1)^2), x =
  -1.29 ..0.88);
```

7.569221661 π

Hence the volume of solid is approximately equal to $\boxed{7.57\pi}$.

For $d = 8$, graph is as follows.



Now find the exact value of d which is a minimum. Take the derivative of the intensity function with respect to x. Then find the second derivative, to see where d has a sudden change, and a point of inflection.

$$I = \frac{F_1}{x_1^2 + d^2} + \frac{F_1}{100 + x_1^2 - 20x_1 + d^2}$$

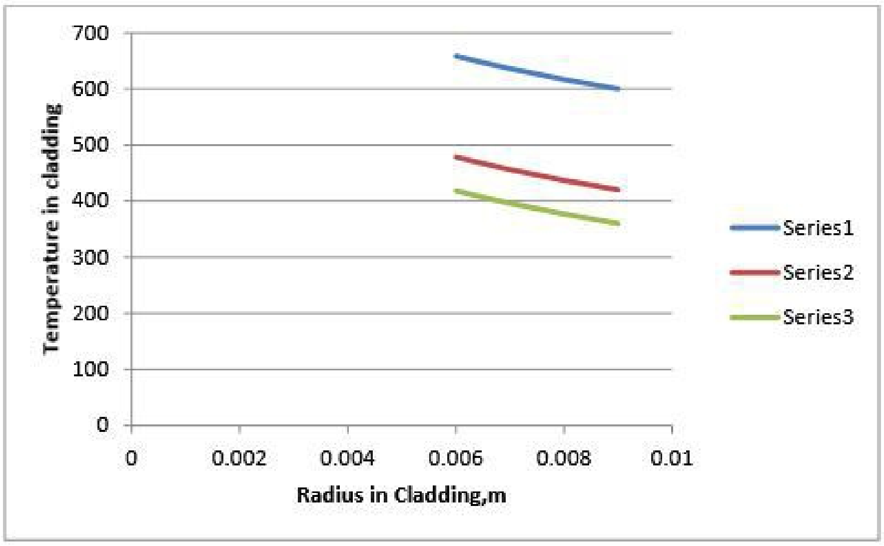
$$I'(x,d) = \frac{-2x_1F_1}{\left(x_1^2 + d^2\right)^2} - \frac{2F_1(x_1 - 10)}{\left((x_1 - 10)^2 + d^2\right)^2}$$

$$I''(x,d) = \frac{+4x_1^2F_1}{\left(x_1^2 + d^2\right)^3} + \frac{4F_1(x_1 - 10)^2}{\left((x_1 - 10)^2 + d^2\right)^3}$$

Continue to solve for d.

$$\boxed{0 = \frac{x_1^2}{\left(x_1^2 + d^2\right)^3} + \frac{(x_1 - 10)^2}{\left((x_1 - 10)^2 + d^2\right)^3}}$$

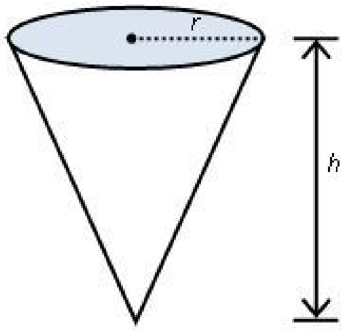
Plot the temperature distribution in the cladding for different values of heat transfer coefficient,



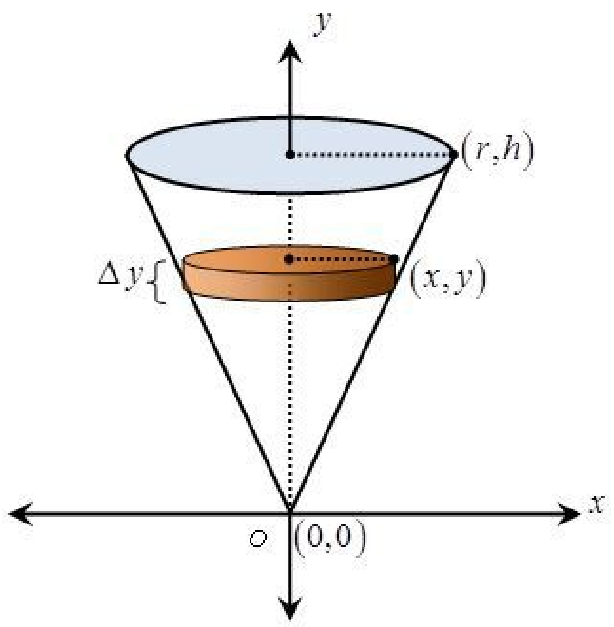
Answer 47E.

Find the volume of the described solid *S*.

A right circular cone with height *h* and base radius *r* is as shown below:



Let the vertex of the solid cone is at the origin and take *y*-axis as the axis of rotation then the cone will be as shown below:



The variable radius *x* of the approximating cylinder is given by the equation of the line passing through the points (r,h) and $(0,0)$ which is given by

$$y-0=\left(\frac{h-0}{r-0}\right)(x-0) \quad \left[\begin{array}{l}\text{Use the formula: Equation to the line} \\ \text{passing through the points} \\ (x_1,y_1) \text{ and } (x_2,y_2) \text{ is } y-y_1=\left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)\end{array}\right]$$

$$\Rightarrow x=\frac{r}{h}y$$

The area of the cross section of the approximating cylinder is given by

$$A(y)=\pi x^2$$

$$=\pi\left(\frac{r}{h}y\right)^2$$

Since the height of the right circular cone is h. it means, the variable y takes limits from 0 to h.

Now use the disk formula to obtain the volume of the given right circular cone.

$$V = \int A(y)\,dy$$
$$= \int_{y=a}^{y=b} \pi x^2\,dy$$
$$= \pi \int_0^h \left[\frac{r\,y}{h}\right]^2 dy$$
$$= \pi \frac{r^2}{h^2} \int_0^h y^2 dy$$
$$= \pi \frac{r^2}{h^2} \cdot \left[\frac{y^3}{3}\right]_0^h$$
$$= \pi \frac{r^2}{h^2} \cdot \frac{h^3}{3}$$
$$= \frac{1}{3}\pi r^2 h$$

Therefore, the volume of the right circular cone with height h and base radius r is,

$$\boxed{\frac{1}{3}\pi r^2 h}.$$

Answer 48E.

Let the centre of top of frustum be at the origin O and the centre of lower base of the frustum be at x – axis
Let the center of lower base be A
then OA = h (Given = height of frustum)
And suppose B and C are any points on the top and lower base of the frustum respectively. Then OC = r and AB = R are given

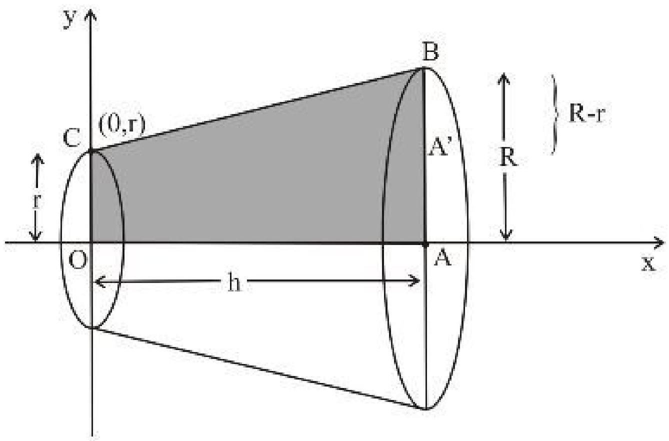


Fig. 1

If we draw a line parallel to x – axis from the point C then this line interest AB at A' (Let)
And $A'B = R - r$

Then slope of line $BC = \frac{A'B}{OA} = \frac{R-r}{h}$

Then equation of line BC, passing though the point (0, r), is

$$(y - r) = \frac{R-r}{h}.x$$

Or $$y = \left(\frac{R-r}{h}\right)x + r$$

Now if we rotate the region OABC about x-axis then we get the frustum as given
If we choose any typical disk so the cross sectional area of the disk is

$$A(x) = \pi\left[\left(\frac{R-r}{h}\right)x + r\right]^2 \qquad [y = \text{Radius}]$$

Or $$A(x) = \pi\left[\left(\frac{R-r}{h}\right)^2 x^2 + r^2 + 2r\frac{(R-r)}{h}x\right]$$

Then the volume of frustum is $V = \int_0^h A(x)dx$

Or $$V = \pi\int_0^h \left[\left(\frac{R-r}{h}\right)^2 x^2 + r^2 + 2r\frac{(R-r)}{h}x\right]dx$$

$$= \pi\left[\frac{1}{3}\frac{(R-r)^2}{h^2}x^3 + r^2x + \frac{r(R-r)}{h}x^2\right]_0^h \qquad [\text{By FTC - 2}]$$

$$= \pi\left[\frac{1}{3}\frac{(R-r)^2}{h^2}h^3 + r^2h + \frac{r(R-r)}{h}h^2\right]$$

$$= \pi\left[\frac{1}{3}h(R-r)^2 + r^2h + rh(R-r)\right]$$

$$= \frac{1}{3}\pi h\left[(R-r)^2 + 3r^2 + 3r(R-r)\right]$$

$$= \frac{1}{3}\pi h\left[R^2 + r^2 - 2rR + \cancel{3r^2} + 3rR - \cancel{3r^2}\right]$$

Or $$\boxed{V = \frac{1}{3}\pi h\left[R^2 + r^2 + Rr\right]}$$

Answer 49E.

First we consider a circle of radius r whose center at the origin

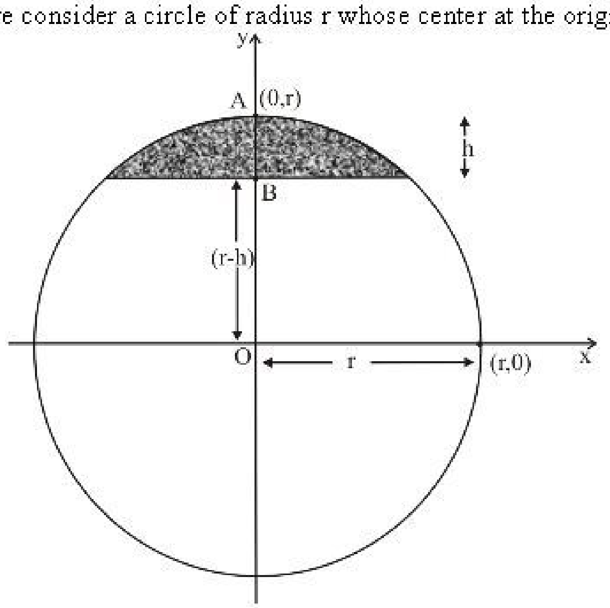


Fig. 1

Since OA = r (radius of circle) and AB = h (given)
Then OB = r-h
So y-coordinates of the points A and B are r and (r-h) respectively
Since equation of the circle is $x^2 + y^2 = r^2$

Then $x = \sqrt{r^2 - y^2}$

If we rotate the shaded region about y – axis then we get a cap of a sphere with height h.

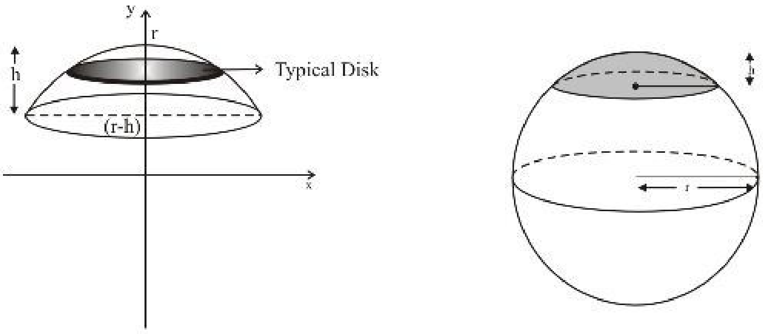


Fig. 2

Radius of typical disk is $= \sqrt{r^2 - y^2}$

Then cross sectional area of typical disk is

$$A(y) = \pi(radius)^2$$
$$= \pi\left(\sqrt{r^2 - y^2}\right)^2$$
$$= \pi\left(r^2 - y^2\right)$$

Then the volume of cap of a sphere of radius r

$$V = \int_{r-h}^{r} A(y)\,dy$$
$$= \int_{r-h}^{r} \pi\left(r^2 - y^2\right)dy$$
$$= \pi\int_{r-h}^{r} \left(r^2 - y^2\right)dy$$
$$= \pi\left[r^2 y - \frac{y^3}{3}\right]_{r-h}^{r}$$
$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2(r-h) - \frac{(r-h)^3}{3}\right)\right]$$
$$= \pi\left[r^3 - \frac{r^3}{3} - (r-h)\left(r^2 - \frac{(r-h)^2}{3}\right)\right]$$

Or $$V = \pi\left[\frac{2r^3}{3} - (r-h)\left(r^2 - \frac{\left(r^2 + h^2 - 2rh\right)}{3}\right)\right]$$
$$= \pi\left[\frac{2r^3}{3} - \frac{(r-h)}{3}\left(2r^2 + 2rh - h^2\right)\right]$$
$$= \frac{1}{3}\pi\left[2r^3 - (r-h)\left(2r^2 + 2rh - h^2\right)\right]$$
$$= \frac{\pi}{3}\left[2r^3 - \left(2r^3 - 3rh^2 + h^3\right)\right]$$
$$= \frac{\pi}{3}\left[2r^3 - 2r^3 + 3rh^2 - h^3\right]$$
$$= \frac{\pi}{3}\left[3rh^2 - h^3\right]$$

Or $\boxed{V = \frac{\pi}{3}h^2\left[3r - h\right]}$ or $\boxed{V = \pi h^2\left(r - \frac{h}{3}\right)}$

Answer 50E.

We place the origin O at the center of the top of the frustum and x – axis along tis central axis

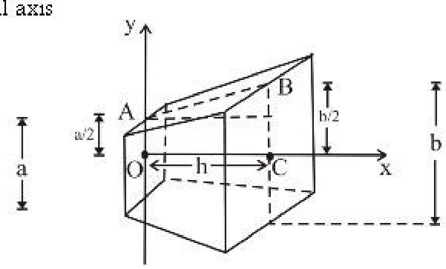


Fig. 1

From figure $BC = \frac{b}{2}$, $OA = \frac{a}{2}$ and $OC = h$

Slope of line $AB = \frac{1}{2}\frac{(b-a)}{h}$

Since the line AB passes though the point A whose co-ordinates are $\left(0, \frac{a}{2}\right)$ then the equation of line AB is

$$\left(y - \frac{a}{2}\right) = \frac{(b-a)}{2h}(x-0)$$

Or $$y = \frac{(b-a)}{2h}x + \frac{a}{2}$$

Now we consider a square any where in the frustum with side = 2y

Or $$\text{side} = 2.\left[\frac{(b-a)}{2h}x + \frac{a}{2}\right]$$

$$= \left[\frac{(b-a)}{h}x + a\right]$$

Then the area of square $= (\text{side})^2$

Or $$A(x) = \left[\frac{(b-a)}{h}x + a\right]^2$$

Or $$A(x) = \left[\frac{(b-a)^2}{h^2}x^2 + a^2 + \frac{2a(b-a)}{h}x\right]$$

$\left[\textit{we used } (A+B)^2 = A^2 + B^2 + 2AB\right]$

Then the volume of frustum is $V = \int_0^h A(x)dx$

Or $$V = \int_0^h \left[\frac{(b-a)^2}{h^2}x^2 + a^2 + \frac{2a(b-a)}{h}x\right]dx$$

Or $$V = \left[\frac{(b-a)^2}{h^2}\frac{x^3}{3} + a^2x + \frac{2a(b-a)}{h}\frac{x^2}{2}\right]_0^h \quad \text{[By FTC - 2]}$$

Or $$V = \left[\frac{(b-a)^2}{h^2}\frac{h^3}{3} + a^2h + \frac{2a(b-a)}{h}\frac{h^2}{2}\right]$$

Or $$V = \left[\frac{(b-a)^2 h}{3} + a^2h + a(b-a)h\right]$$

Or $$V = \left[\left(b^2 + a^2 - 2ab\right)\frac{h}{3} + a^2h + abh - a^2h\right]$$

Or $$V = \frac{1}{3}h\left[b^2 + a^2 - 2ab + 3a^2 + 3ab - 3a^2\right]$$

Or $$V = \frac{1}{3}h\left[b^2 + a^2 + ab\right]$$

Or volume of the frustum is $\boxed{V = \frac{1}{3}h\left[b^2 + a^2 + ab\right]}$

Answer 51E.

W place the origin O at the vertex of the pyramid and x – axis along its central axis and y – axis parallel to the side of dimension 2b of the base of pyramid

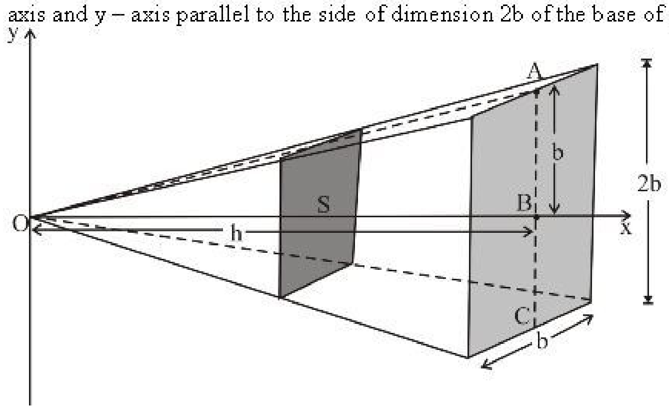


Fig. 1

From figure AC = 2b then AB = b and OB = h

Then slope of line $OA = \frac{b}{h}$

So the equation of line OB which is passing through the origin

$$y = \frac{b}{h}x$$

Now we consider a rectangle S

Then side of S parallel to $AC = 2.\frac{b}{h}x$ (length)

And another side of $S = \frac{b}{h}x$ (width)

Then the area of rectangle = length × width

Or $A(x) = \frac{2bx}{h} \times \frac{b}{h}x$

Or $A(x) = 2\frac{b^2}{h^2}x^2$

Then the volume of pyramid is $V = \int_0^h A(x)\,dx$

Or $V = \int_0^h 2\frac{b^2}{h^2}x^2\,dx$

Or $V = 2\frac{b^2}{h^2}\int_0^h x^2\,dx$

Or $V = 2\frac{b^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$ [By FTC - 2]

Or $V = 2\frac{b^2}{h^2}.\frac{h^3}{3}$

Or $V = \frac{2}{3}b^2h$

Or $\boxed{V = \frac{2}{3}b^2h}$

Answer 52E.

First we consider the properties of equilateral triangle ABC

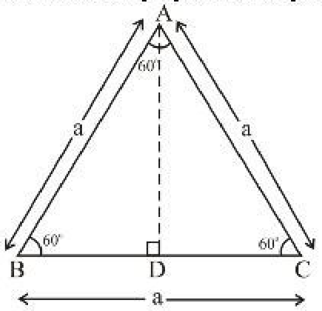


Fig. 1

In equilateral triangle each angle = 60° means $\angle A = \angle B = \angle C = 60°$ and perpendicular AD from A will bisect BC so $BD = \frac{a}{2}$

Then by the trigonometric identity we have $\tan 60° = \frac{AD}{BD}$ --- (1)

Or $\sqrt{3} = \frac{AD}{\frac{a}{2}}$

Or $\boxed{AD = \frac{\sqrt{3}}{2}.a}$ -- (2) which is height of the triangle ABC

Now we place the origin O at the vertex of the pyramid and x 0 axis along OE where E is the centroid of the triangle ABC

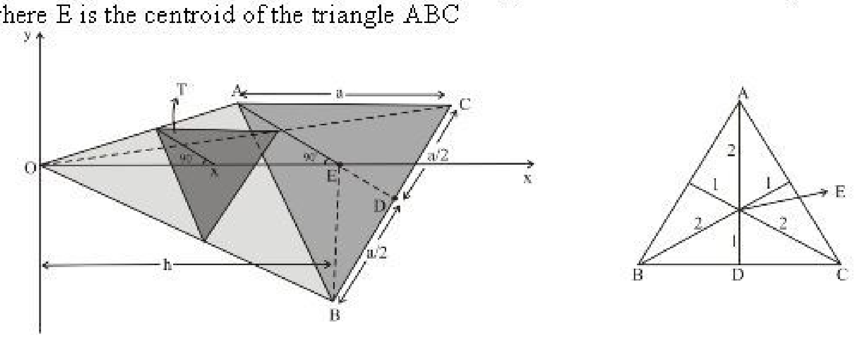


Fig. 2

Since a centroid of the triangle divides the median in to ration 2:1
So AE: ED = 2:1

From step 1 we have $AD = \frac{\sqrt{3}}{2}a$ then $AE = \frac{\sqrt{3}}{2}a.\frac{2}{3} = \frac{a}{\sqrt{3}}$

Then the slope of line $OA = \frac{AE}{OE} = \frac{a/\sqrt{3}}{h} = \frac{a}{\sqrt{3}.h}$

Since line OA passes though the origin so equation of line OA is

$$y = \frac{a}{\sqrt{3}.h}x$$

Now we take any point x on OE which will be a centroid of the triangle T (Let)

Then distance of the line OA form x is $= \frac{a}{\sqrt{3}.h}x$

So the median of the triangle T is $= \frac{a}{\sqrt{3}.h}x\frac{3}{2} = \frac{\sqrt{3}a}{2h}x$

In a equilateral triangle the median of the triangle is a perpendicular also

So height of the triangle $T = \frac{\sqrt{3}a}{2h}x$ -- (1)

And the base of equilateral triangle $= \frac{2}{\sqrt{3}} \times$ height of the triangle

So base of the triangle T is $= \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{2}\frac{ax}{h}$

Or base $= \frac{ax}{h}$ --- (2)

Now the area of the triangle $= \frac{1}{2} \times base \times height$

So the area of triangle T is $A(x) = \frac{1}{2} \times \frac{ax}{h} \times \frac{\sqrt{3}}{2}\frac{ax}{h}$ [From (1) & (2)]

Or $A(x) = \frac{\sqrt{3}}{4}\frac{a^2x^2}{h^2}$

Then the volume of the pyramid is

$$V = \int_0^h A(x)\,dx$$

$$= \int_0^h \frac{\sqrt{3}}{4}\frac{a^2x^2}{h^2}\,dx$$

$$= \frac{\sqrt{3}}{4}\frac{a^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{\sqrt{3}}{4}\frac{a^2}{h^2}\left[\frac{x^3}{3}\right]_0^h \qquad \text{[By FTC - 2]}$$

$$= \frac{\sqrt{3}}{4}\frac{a^2}{h^2}\frac{h^3}{3}$$

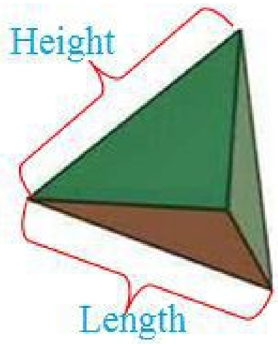
Or $\boxed{V = \frac{\sqrt{3}a^2h}{12}}$

Answer 53E.

Consider a tetrahedron with three mutually perpendicular faces and three mutually perpendicular edges with lengths 3 cm, 4 cm, and 5 cm.

Need to find the volume of the tetrahedron.

From the given data to construct the below figure:



To find volume, use the formula for the volume of a solid.

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where *A* is an integrable function, then the volume of *S* is

$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)\,dx$$

Place the tetrahedron on the coordinate axis so the 3 cm edge is on the *x*-axis, the 4 cm edge is on the *y*-axis, and the 5 cm edge is on the *z*-axis. Any plane P_x that passes through *x* and is perpendicular to the *x*-axis intersects the tetrahedron in a right triangle.

Let *h* be height of the triangular cross-section (in the direction of the *y*-axis) and *b* be its length (in the direction of the *z*-axis).

Write an expression for *h* and *b* as *x* goes from 0 to 3.

$h = \dfrac{4(3-x)}{3}$ and $b = \dfrac{5(3-x)}{3}$

Then find the area of the cross-sections using the formula for the area of triangle.

$$A(x)=\frac{1}{2}bh$$

Now substitute the values of b and h into the formula, obtain

$$\begin{aligned}A(x)&=\frac{1}{2}bh\\&=\frac{1}{\not{2}}\cdot\frac{5(3-x)}{3}\cdot\frac{\overset{2}{\not{4}}(3-x)}{3}\\&=\frac{10(3-x)^2}{9}\end{aligned}$$

The tetrahedron lies between $x=0$ and $x=3$.

Write an integral for the volume of the tetrahedron.

$$V=\int_0^3\frac{10(3-x)^2}{9}dx$$

$$V=-\frac{10(3-x)^3}{9\cdot3}\Bigg]_0^3$$ Use $\int x^n dx=\frac{x^{n+1}}{n+1}$

$$=-\left[\frac{10(3-x)^3}{27}\right]_0^3$$

$$=-\left[\frac{10(3-3)^3}{27}-\frac{10(3-0)^3}{27}\right]$$

$$=-\left[\frac{10(0)^3}{27}-\frac{10(3)^3}{27}\right]$$

$$=-\left[-\frac{10(3)^3}{27}\right]$$

$$=0-\left[-\frac{10(\not{27})}{\not{27}}\right]$$ Apply limits

$=10$ Simplify

Therefore,

The volume of the tetrahedron is $\boxed{V=10\text{ cm}^3}$

Answer 54E.

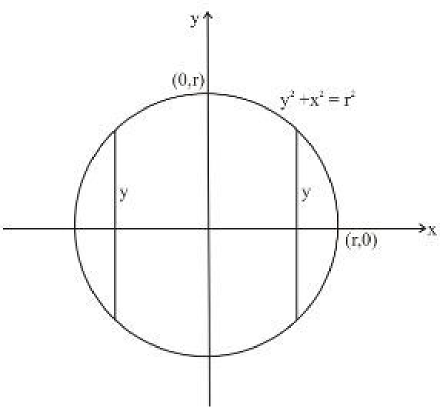


Fig. 1

$$A(x)=(2y)^2=4y^2$$
$$A(x)=4\left(r^2-x^2\right)$$

$$\begin{aligned} Volume &= \int_{-r}^{r} 4\left(r^2 - x^2\right)dx \\ &= 4\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} \\ &= 4\left[r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right] \\ &= 4\left[2r^3 - \frac{2r^3}{3}\right] \\ &= 4\left[\frac{4}{3}r^3\right] \\ &= \boxed{\frac{16}{3}r^3} \end{aligned}$$

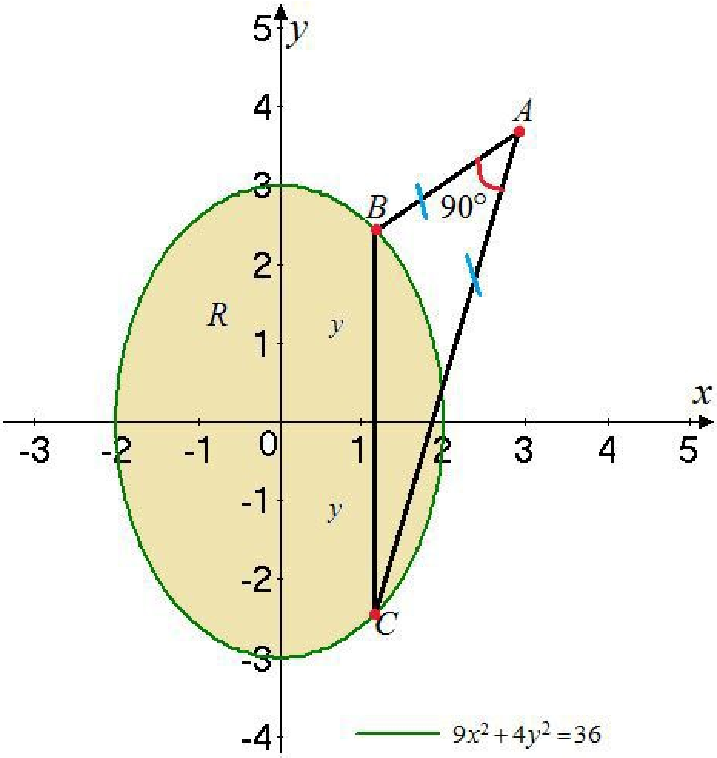
Answer 55E.

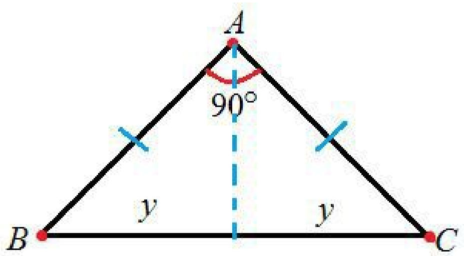
The base of solid is an elliptical region *R* with a boundary curve $9x^2 + 4y^2 = 36$ is shown below:

And the cross sections are isosceles right triangles, which are perpendicular to the *x*-axis.

And the hypotenuse of the triangle is in the base.

Consider the following figure:





Here, *BC* is the hypotenuse of the isosceles right triangles.

Consider the equation $9x^2 + 4y^2 = 36.$

$$9x^2+4y^2=36$$
$$4y^2=36-9x^2$$
$$y^2=\frac{36-9x^2}{4}$$
$$y=\frac{3}{2}\sqrt{4-x^2}$$

Calculate the length of the hypotenuse *BC*.

$$\begin{aligned}L&=2y\\&=2\left(\frac{3}{2}\sqrt{4-x^2}\right)\\&=3\sqrt{4-x^2}\end{aligned}$$

Calculate the area of the isosceles right triangle in terms of hypotenuse.

$$\begin{aligned}A&=\frac{L^2}{4}\\&=\frac{\left(3\sqrt{4-x^2}\right)^2}{4}\\&=\frac{9\left(4-x^2\right)}{4}\end{aligned}$$

Therefore, the cross sectional area is $A(x)=\frac{9\left(4-x^2\right)}{4}$.

And the x varies from -2 to 2.

Calculate the Volume of the solid as follows:

$$\begin{aligned}V&=\int_{-2}^{2}A(x)dx\\&=\int_{-2}^{2}\frac{9\left(4-x^2\right)}{4}dx\\&=\frac{9}{4}\int_{-2}^{2}\left(4-x^2\right)dx\\&=\frac{9}{4}\left[4x-\frac{x^3}{3}\right]_{-2}^{2}\end{aligned}$$

$$\begin{aligned}&=\frac{9}{4}\left[\left(4(2)-\frac{2^3}{3}\right)-\left(4(-2)-\frac{(-2)^3}{3}\right)\right]\\&=\frac{9}{4}\left[\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)\right]\\&=\frac{9}{4}\left[\frac{32}{3}\right]\\&=24\end{aligned}$$

Hence, the required volume of the solid is $\boxed{24}$.

Answer 56E.

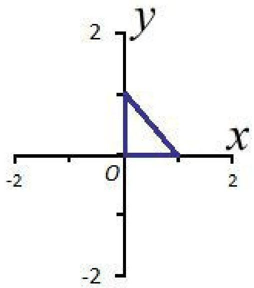
Consider a triangular region with vertices $(0,0),(1,0)$, and $(0,1)$ and base *b*.

Need to find the volume.

To find the volume use the definition for the volume of a solid. Let *S* be a solid that lies between $x=a$ and $x=b$. If the cross-sectional area of *S* in the plane *P*x, through *x* and perpendicular to the *x*-axis, is $A(x)$, where *A* is an integrable function, then the volume of *S* is

$$V=\lim_{\max\Delta x_i\to 0}\sum_{i=1}^{n}A\left(x_i^*\right)\Delta x_i=\int_a^b A(x)dx$$

Graph the triangular region as follows:



Any plane *Px* that passes through *x* and is perpendicular to the *x*-axis intersects *S* in an equilateral triangle. Let *s* be the length of the sides of the triangle. Write an expression for *s* as *x* goes from 0 to 1. Since the triangles are equilateral, *s* is equal the height of the triangular region for any given *x*.

$s = 1 - x$

Then find the area of the cross-sections using the formula for the area of an equilateral triangle.

$A(x) = \frac{\sqrt{3}}{4}s^2 \quad = \frac{\sqrt{3}}{4}(1-x)^2$ Since $s = 1 - x$

The solid lies between $x = 0$ and $x = 1$.

Write an integral for the volume of the solid.

$$V = \int_0^1 \left[\frac{\sqrt{3}}{4}(1-x)^2\right] dx$$

$$V = -\frac{\sqrt{3}}{4} \cdot \frac{(1-x)^3}{3}\Bigg]_0^1$$ Use $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$= -\frac{\sqrt{3}}{12}(1-x)^3\Bigg]_0^1$$

$$= \left[-\frac{\sqrt{3}}{12}(1-1)^3 + \frac{\sqrt{3}}{12}(1-0)^3\right]$$

$$= \left[-\frac{\sqrt{3}}{12}(0)^3 + \frac{\sqrt{3}}{12}\right]$$

$$= \frac{\sqrt{3}}{12}$$

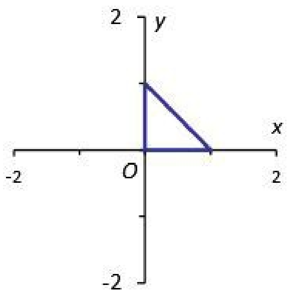
Therefore,

The volume of the triangular region is $\boxed{V = \frac{\sqrt{3}}{12}}$.

Answer 57E.

Consider a triangular region with vertices $(0,0), (1,0)$, and $(0,1)$ and base b.
To find the volume, use the formula for the volume of a solid.
Let S be a solid that lies between and
If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is where A is an integrable function then the volume of S is,

Graph of the triangular region is as shown below:



Any plane P_x that passes through x and is perpendicular to the x-axis intersects S in a square.
Let s be the length of the sides of the square. Write an expression for s as x goes from 0 to 1.
Since the cross-sections are square, s is equal the height of the triangular region for any given x.
Then,

Then find the area of the cross-sections using the formula for the area of a square.

The solid lies between and .
The volume of the solid computed as follows:

$$V = -\frac{1}{3}(1-x)^3\Big]_0^1 \qquad \text{Use } \int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= 0 - \left(-\frac{1}{3}\right) \qquad \text{Apply limits}$$

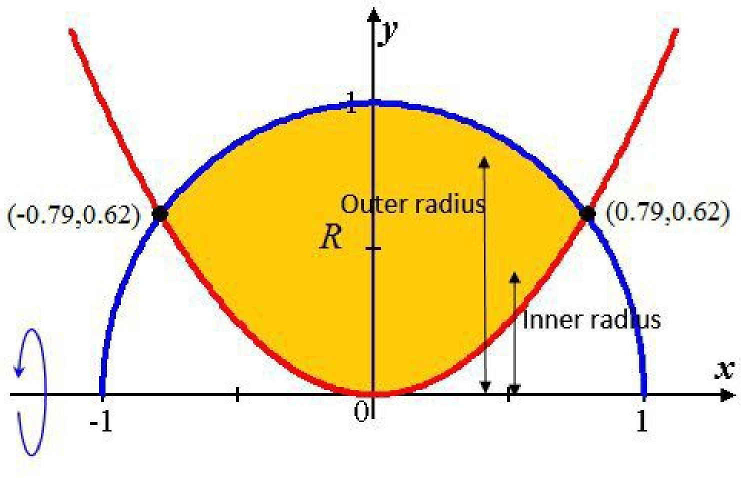
$$= \frac{1}{3}$$

Therefore, the volume of the triangular region is $\boxed{\frac{1}{3}}$.

Hence, over the given region x varies from -0.79 to 0.79.

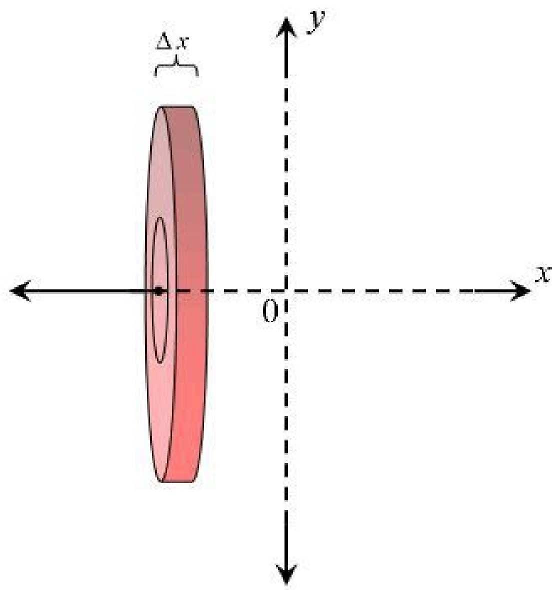
Use this information to sketch these curves, and then the region will be as shown
Will be as shown below:

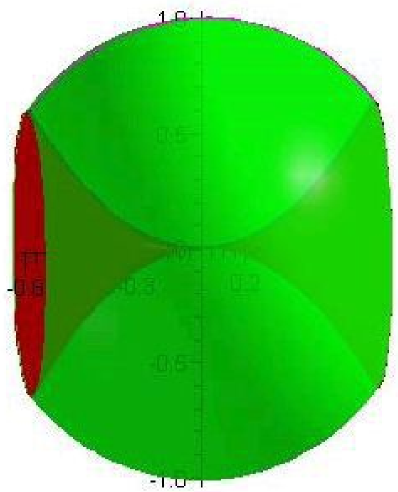
Figure-1



The solid formed by rotating the above region R is as shown below:

Figure-2





Inner radius of the approximating cylinder is given by

$$\text{Inner radius} = \text{Lower curve} - \text{axis of rotation}$$
$$r_{\text{Inner}} = \quad x^2 \quad - \quad 0$$
$$r_{\text{Inner}} = x^2$$

Inner radius of the approximating cylinder is given by

$$\text{Outer radius} = \text{Upper curve} - \text{axis of rotation}$$
$$r_{\text{Outer}} = \quad \sqrt{1-x^2} \quad - \quad 0$$
$$r_{\text{Outer}} = \sqrt{1-x^2}$$

Area of cross-section of the approximating cylinder is as follows:

$$A(x) = \pi r^2{}_{\text{outer}} - \pi r^2{}_{\text{inner}}$$
$$A(x) = \pi\left(\sqrt{1-x^2}\right)^2 - \pi\left(x^2\right)^2$$

Use the washer formula which is

$$V = \int A(x)dx$$

$$V = \int_{x=-0.79}^{x=0.79} \left[\pi\left(\sqrt{1-x^2}\right)^2 - \pi\left(x^2\right)^2 \right] dx$$

It is difficult to solve the integral manually so use CAS to solve this integral.
Type the following command in Maple, and then press ENTER to obtain the final result.

$$int\left(\left(\pi\cdot\left(\sqrt{1-x^2}\right)^2 - \pi\cdot\left(x^2\right)^2\right), x=-0.79..0.79\right);$$

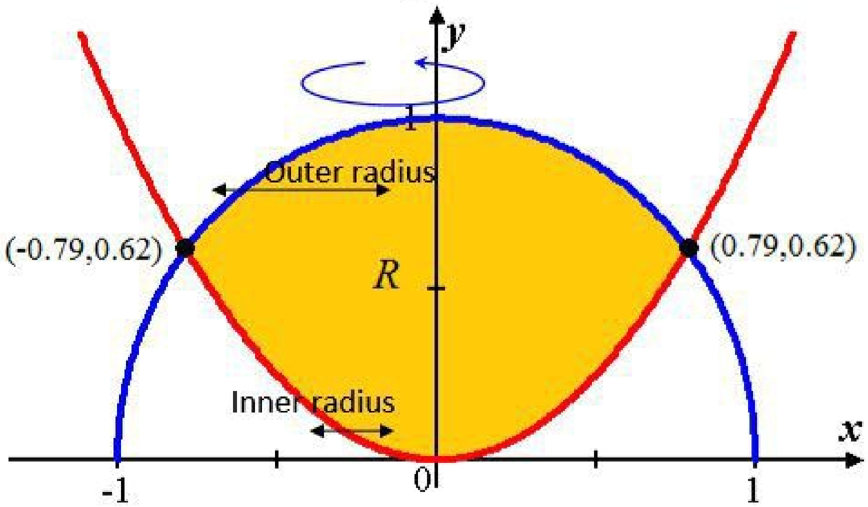
3.544423615

Therefore, the volume of the solid obtained is $\boxed{3.54 \text{ unit}^3}$

b)

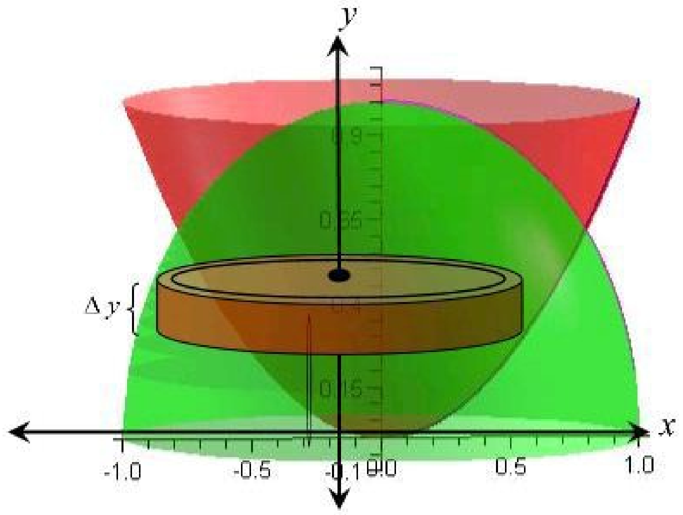
If the region R is rotated about y-axis the obtained figure will be as shown in the figure-3:

Figure-3



The solid formed by rotating the above region R is as shown below:

Figure-2



Inner radius of the approximating cylinder is given by

Inner radius = Lower curve − axis of rotation

$$r_{\text{Inner}} = \sqrt{y} \qquad - \qquad 0$$
$$r_{\text{Inner}} = \sqrt{y}$$

Inner radius of the approximating cylinder is given by

Outer radius = Upper curve − axis of rotation

$$r_{\text{Outer}} = \sqrt{1-y^2} \qquad - \qquad 0$$
$$r_{\text{Outer}} = \sqrt{1-y^2}$$

Over the given region y varies from 0 to 1.

Area of cross-section of the approximating cylinder is as follows:

$$A(y) = \pi r^2{}_{\text{outer}} - \pi r^2{}_{\text{inner}}$$
$$A(y) = \pi\left(\sqrt{1-y^2}\right)^2 - \pi\left(\sqrt{y}\right)^2$$

Use the washer formula to find the volume.

$$V = \int A(y)\,dy$$
$$V = \int_{y=0}^{y=1}\left[\pi\left(\sqrt{1-y^2}\right)^2 - \pi\left(y^2\right)^2\right]dy$$

Now use CAS to evaluate the integral,

Type the following command in Maple, and then press ENTER to obtain the final result.

$$int\left(\left(\pi\cdot\left(\sqrt{1-y^2}\right)^2 - \pi\cdot\left(\sqrt{y}\right)^2\right), y = 0\,..1\right);$$
$$\frac{1}{6}\pi$$

Therefore, the volume of the solid obtained is $\boxed{\frac{\pi}{6} \approx 0.5231 \text{ unit}^3}$.

Answer 59E.

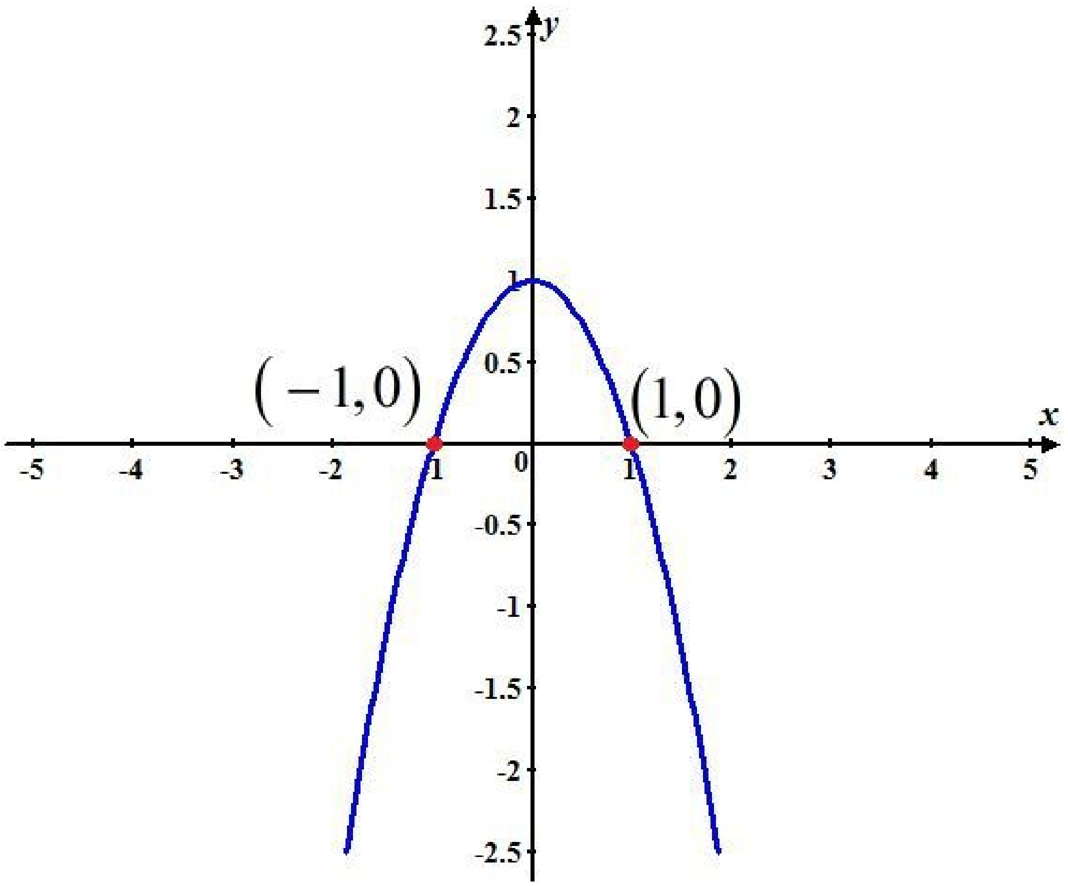
Consider the parabola $y = 1 - x^2$ with base *S*.

Need to find the volume of the parabola.

To find volume, use the definition for the volume of a solid. Let *S* be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of *S* in the plane *Px*, through *x* and perpendicular to the *x*-axis, is $A(x)$, where *A* is an integrable function, then the volume of *S* is

$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i^*) \Delta x_i = \int_a^b A(x)\,dx$$

Graph the region enclosed by the parabola $y = 1 - x^2$ and the *x*-axis.



Any plane *Px* that passes through *x* and is perpendicular to the *x*-axis intersects *S* in an isosceles triangle. Let *b* be the base of the triangle. Write an expression for *b* as *x* goes from 1 (the left boundary of the region) to 1 (the right boundary).

$$s = 1 - x^2$$

Then find the area of the cross-sections using the fact that the height of each triangle is equal to its base.

$$\begin{aligned} A(x) &= \frac{1}{2}bh \\ &= \frac{1}{2}(1 - x^2)^2 \\ &= \frac{1}{2}(x^4 - 2x^2 + 1) \end{aligned}$$

The solid lies between $x=-1$ and $x=1$.

Write an integral for the volume of the solid.

$$V=\int_{-1}^{1}\left[\frac{1}{2}\left(x^4-2x^2+1\right)\right]dx$$
$$=\frac{1}{2}\int_{-1}^{1}\left(x^4-2x^2+1\right)dx$$

$$V=\frac{1}{2}\left[\frac{x^5}{5}-\frac{2x^3}{3}+x\right]_{-1}^{1}$$ Use $\int x^n dx=\frac{x^{n+1}}{n+1}$

$$=\frac{1}{2}\left[\left(\frac{1}{5}-\frac{2}{3}+1\right)-\left(-\frac{1}{5}+\frac{2}{3}-1\right)\right]$$ Apply limits

$$=\frac{1}{2}\left[\frac{3-10+15}{15}-\left(\frac{-3+6-1}{15}\right)\right]$$ Simplify

$$=\frac{1}{2}\left[\frac{8}{15}+\frac{8}{15}\right]$$
$$=\frac{1}{2}\left(\frac{16}{15}\right)$$
$$=\frac{8}{15}$$

Therefore,

The volume of the parabola is $\boxed{V=\frac{8}{15}}$.

Answer 60E.

(a)

Consider a circular disk with radius *r*

The formula for the volume of a solid:

Let *S* be a solid that lies between $x=a$ and $x=b$. If the cross-sectional area of *S* in the plane P_x, through *x* and perpendicular to the *x*-axis, is $A(x)$, where *A* is an integrable function, then the volume of *S* is

$$V=\lim_{\max \Delta x_i\to 0}\sum_{i=1}^{n}A\left(x_i^*\right)\Delta x_i=\int_a^b A(x)dx$$

Place the origin *O* at the center of the circular disk.

Any plane *Px* that passes through *x* and is perpendicular to the *x*-axis intersects the solid in an isosceles triangle. Let *b* be the base of the triangle. Use the Pythagorean Theorem to write an expression for *b* as *x* goes from *r* to *r*.

$$b=2\sqrt{r^2-x^2}$$

Then find the cross-sectional area using the formula for the area of a triangle.

$$A(x)=\frac{1}{2}bh$$
$$=h\sqrt{r^2-x^2}$$

Use this area to write an integral for the volume of the solid.

$$V=\boxed{\int_{-r}^{r}\left(h\sqrt{r^2-x^2}\right)dx}$$

(b)

From the graph of the circle above, the area of a circle can be found using the following integral.

$$A = \int_{-r}^{r} 2\sqrt{r^2 - x^2}$$

Therefore, interpret the volume of the solid as the area of a circle *A* times $\frac{1}{2}h$, which is a constant.

$$\begin{aligned} V &= \int_{-r}^{r} \left(h\sqrt{r^2 - x^2}\right) dx \\ &= \frac{1}{2}h \times \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx \\ &= \frac{1}{2}hA \end{aligned}$$

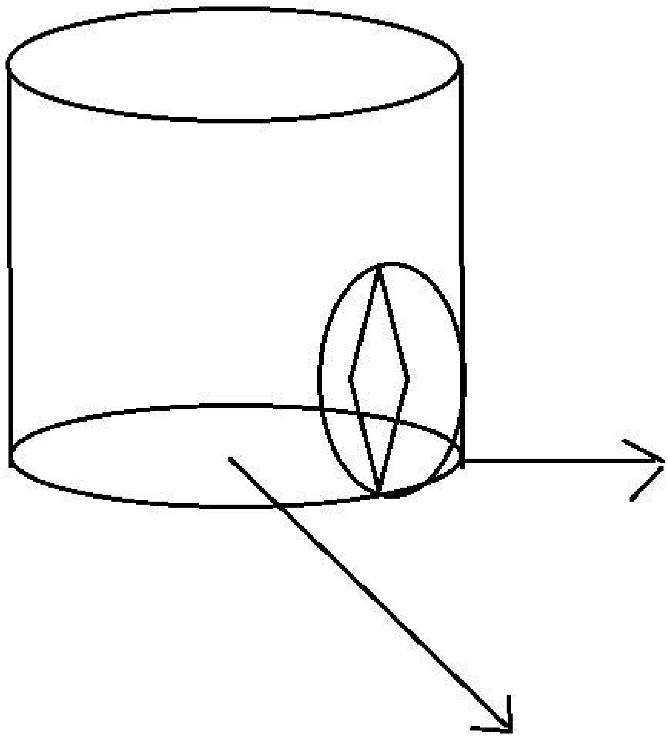
The area of a circle is also given by the formula $A = \pi r^2$.

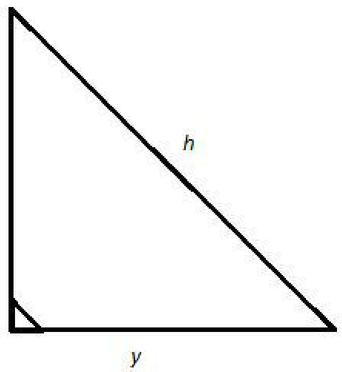
Therefore, volume of the solid is

$$\begin{aligned} V &= \frac{1}{2}h \times \pi r^2 \\ &= \boxed{\frac{1}{2}\pi r^2 h} \end{aligned}$$

Answer 62E.

Consider the following diagram:





If cross sections are parallel to the line of intersection of the planes, then the cross sections will be rectangular.

AB is the breadth of such a cross sectional rectangle

Length of rectangle = $2\sqrt{16-x^2}$

Also, $\frac{y}{x} = \tan 30^o$

$y = \frac{x}{\sqrt{3}}$

The area is calculated as follows:

$$A(x) = 2\sqrt{16-x^2}\,\frac{x}{\sqrt{3}}$$

Then the volume is calculated as follows:

$$\text{Volume} = \frac{1}{\sqrt{3}}\int_0^4 2x\sqrt{16-x^2}\,dx$$

$$= \frac{1}{\sqrt{3}}\left[-\frac{2}{3}\left(16-x^2\right)^{3/2}\right]_0^4$$

$$= \frac{1}{\sqrt{3}}\left[\frac{-2}{3}(0-64)\right]$$

$$= \frac{1}{\sqrt{3}}\left[\frac{128}{3}\right]$$

$$= \boxed{\frac{128}{3\sqrt{3}}}$$

Therefore, the volume is $\boxed{\frac{128}{3\sqrt{3}}}$.

Answer 63E.

(a)

If the cross-sectional area of S_1 is given by the function $A_1(x)$ and the cross-sectional area of S_2 is given by the function $A_2(x)$ as *x* goes from *a* to *b*, then the volumes of S_1 and S_2 are

$$V_1 = \int_a^b A_1(x)dx$$

$$V_2 = \int_a^b A_2(x)dx$$

The problem states that a family of parallel planes gives equal cross-sectional areas for the two solids. Say that the planes perpendicular to the *x*-axis give equal areas (you could always move or rotate the solids to make this true).

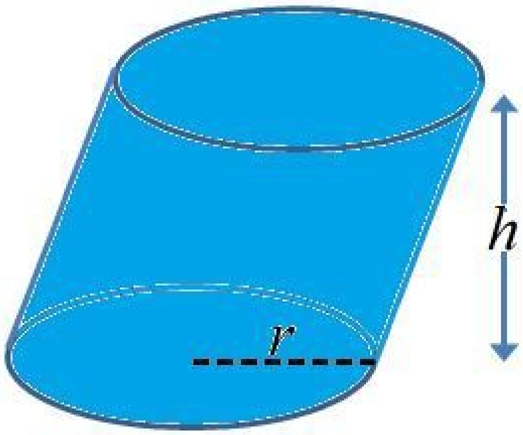
Therefore, $A_1(x) = A_2(x)$ for any value of *x*.

Thus, their volumes are equal as well.

$$\int_a^b A_1(x)dx = \int_a^b A_2(x)dx$$

(b)

Consider the oblique cylinder:



In the oblique cylinder shown in the figure, notice that as you go down from the top of the cylinder to the bottom, the cross-sections are circular disks.

These disks have the same area as the cross-sections of a right cylinder with height *h* and radius *r*.

By Cavalieri's Principle, this means that volume of the oblique cylinder is equal to the volume of the right cylinder.

The volume of a right cylinder is given by the formula $V = \pi r^2 h$.

Therefore, the volume of the given cylinder is $\boxed{\pi r^2 h}$ as well.

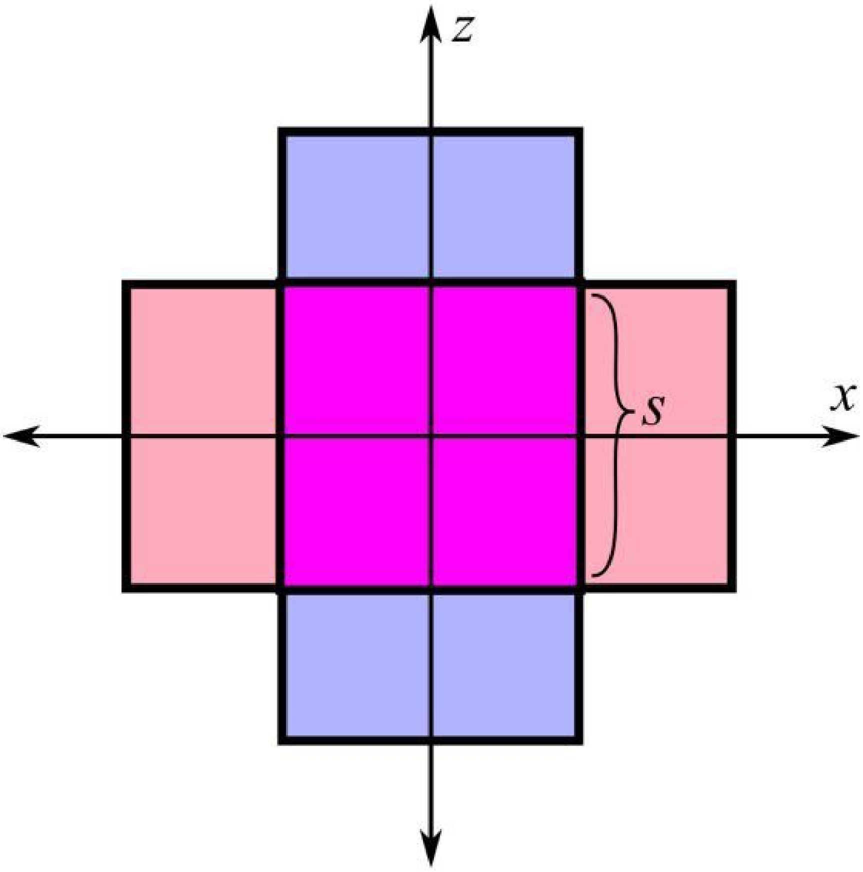
Answer 64E.

Find the volume of the solid formed by the intersection of two circular cylinders.

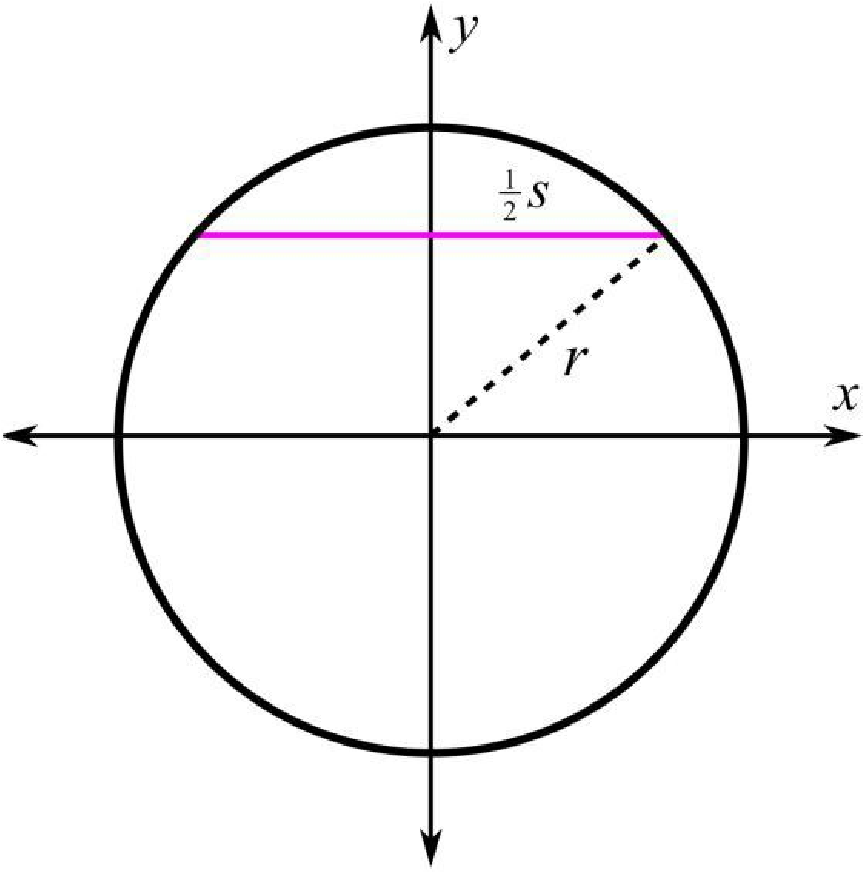
Let *S* be a solid that lies between $y = c$ and $y = d$. If the cross-sectional area of *S* in the plane *Py*, through *y* and perpendicular to the *y*-axis, is $A(y)$, where *A* is an integrable function, then the volume of *S* is

$$V = \lim_{\max \Delta y_i \to 0} \sum_{i=1}^{n} A(y_i^*)\Delta y_i = \int_c^d A(y)dy$$

As you go from the bottom of the solid to its top, the cross sections are squares as shown in the diagram below. In the diagram, the sides of the square are *s*, so its area is $A(s) = s^2$



Now, draw a side view of the solid. From the diagram, the value of *s* is $s = 2\sqrt{r^2 - y^2}$.



Next write an expression for the area of the squares in terms of *y* by substituting the function for *s* into the area formula.

$$A(y) = s^2$$
$$= \left(2\sqrt{r^2 - y^2}\right)^2$$
$$= 4r^2 - 4y^2$$

The solid lies between $y = -r$ (the bottom of the solid) and $y = r$ (the top).

Write an integral for the volume of the solid.

$$V = \int_{-r}^{r} \left(4r^2 - 4y^2\right) dy$$

Then simplify the result.

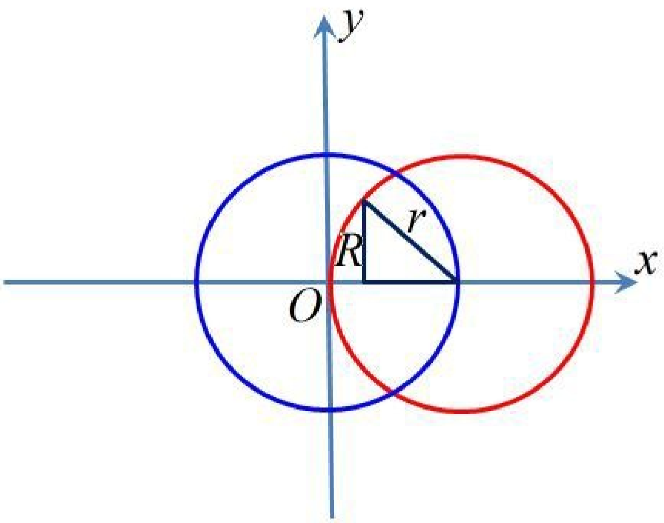
$$V = \left[4r^2 y - \tfrac{4}{3} y^3\right]_{-r}^{r}$$
$$= \left[4r^3 - \tfrac{4}{3} r^3\right] - \left[-4r^3 + \tfrac{4}{3} r^3\right]$$
$$= \boxed{\frac{16}{3} r^3}$$

Answer 65E.

The formula for the volume of a solid: Let *S* be a solid that lies between $x = a$ and $x = b$ If the cross-sectional area of *S* in the plane *P*x, through *x* and perpendicular to the *x*-axis, is $A(x)$, where *A* is an integrable function, then the volume of *S* is

$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i^*) \Delta x_i = \int_a^b A(x) dx$$

Place the center of one sphere at the origin *O*.



Any plane *P*x that passes through *x* and is perpendicular to the *x*-axis intersects the solid in a circular disk. Notice that the volume of the solid from $x = 0$ (the origin) to $x = r/2$ (the midpoint between the spheres) is equal to the volume from $x = r/2$ to $x = r$ (the center of the right sphere). Therefore, find just one of these volumes and double it. For this problem, find the volume of the solid from $x = 0$ to $x = r/2$.

Let *R* be the radius of the circular cross sections. Use the Pythagorean Theorem to write an expression for *R* as *x* goes from 0 to $r/2$.

$$R = \sqrt{r^2 - (r - x)^2}$$
$$= \sqrt{2rx - x^2}$$

Then find the cross-sectional area using the formula for the area of a circle.

$$\begin{aligned} A(x) &= \pi R^2 \\ &= \pi\left(\sqrt{2rx - x^2}\right)^2 \\ &= 2\pi rx - \pi x^2 \end{aligned}$$

The solid lies between $x = 0$ and $x = r/2$. Write an integral for the volume of the solid. Since this only accounts for half of the total volume of the solid, double the integral.

$$V = 2\int_0^{r/2} \left(2\pi rx - \pi x^2\right) dx$$

Then simplify the result.

$$\begin{aligned} V &= 2\left[\pi rx^2 - \tfrac{1}{3}\pi x^3\right]_0^{r/2} \\ &= 2\left[\pi r\left(\frac{r}{2}\right)^2 - \tfrac{1}{3}\pi\left(\frac{r}{2}\right)^3 - \pi r(0)^2 + \tfrac{1}{3}\pi(0)^3\right] \\ &= 2\left(\frac{1}{4}\pi r^3 - \frac{1}{24}\pi r^3\right) \\ &= 2\left(\frac{6\pi r^3 - \pi r^3}{24}\right) \\ &= \frac{5}{12}\pi r^3 \end{aligned}$$

Thus, the required volume is $\boxed{\frac{5}{12}\pi r^3}$

Answer 66E.

The formula to find the volume of a solid is stated as follows:

Let *S* be a solid that lies between $x = a$ and $x = b$.

Let $A(x)$ be the cross-sectional area of the solid *S* in the plane P_x and perpendicular to *x*-axis.Here, *A* is an integral function.

Then, the volume of the solid *S* is calculated by the following formula:

$$\begin{aligned} V &= \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A\left(x_i^*\right)\Delta x_i \\ &= \int_a^b A(x)\,dx \end{aligned}$$

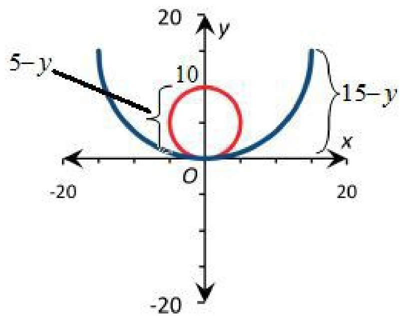
Consider the diameter of the bowl is 30 cm.

So, the radius of the bowl is $r = \frac{30}{2} = 15 \text{ cm}$.

Consider the diameter of the ball is 10 cm.

So, the radius of the ball is $r = \frac{10}{2} = 5 \text{ cm}$.

As shown in the following figure, place the bottom of the bowl on the origin *O*.



Any plane P_y, that passes through *y* and is perpendicular to the *y*-axis, intersects the solid in a washer.

Let *R* be the radius of the circular cross sections.

Find the inner radius and outer radius of the washer, as *y* goes from 0 (the bottom of the bowl) to 10 (the top of the ball).

$$\text{inner radius} = \sqrt{5^2-(5-y)^2}$$
$$= \sqrt{25-(5-y)^2}$$

$$\text{outer radius} = \sqrt{(15)^2-(15-y)^2}$$
$$= \sqrt{225-(15-y)^2}$$

Then, find the cross-sectional area using the area formula of a washer.

$$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$
$$= \pi\left(225-(15-y)^2\right) - \pi\left(25-(5-y)^2\right)$$
$$= \pi\left(225-225+30y-y^2\right) - \pi\left(25-25+10y-y^2\right)$$
$$= \pi\left(30y-y^2\right) - \pi\left(10y-y^2\right)$$
$$= 30\pi y - \pi y^2 - 10\pi y + \pi y^2$$
$$= 20\pi y$$

The water in the bowl lies between $y=0$ and $y=h$.

The volume of the water can be calculated as follows:

$$V = \int_0^h A(y)\, dy$$
$$= \int_0^h 20\pi y\, dy \qquad \text{Substitute } A(y) = 20\pi y$$
$$= 20\pi \int_0^h y\, dy$$
$$= 20\pi\left[\frac{y^2}{2}\right]_0^h \qquad \text{Use } \int x^n dx = \frac{x^{n+1}}{n+1} + C$$
$$= 20\pi\left[\frac{h^2}{2} - 0\right]$$
$$= 20\pi\left[\frac{h^2}{2}\right]$$
$$= 10\pi h^2$$

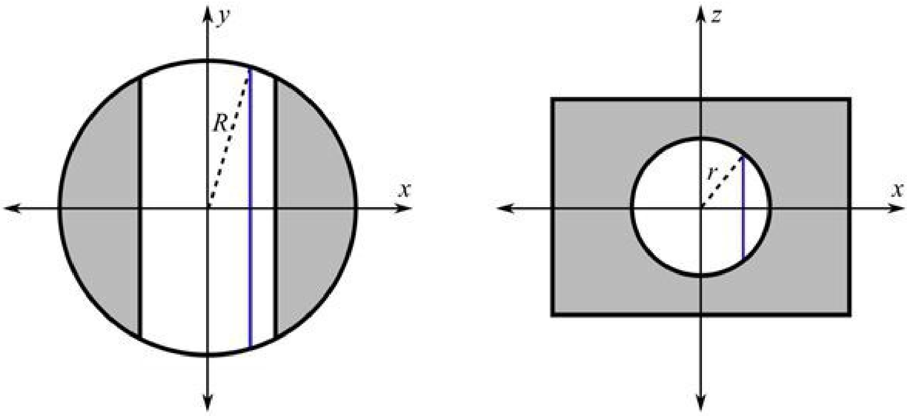
Therefore, the volume of the water in the bowl is $\boxed{10\pi h^2 \text{ cm}^3}$.

Answer 67E.

The formula for the volume of a solid. Let *S* be a solid that lies between $x=a$ and $x=b$. If the cross-sectional area of *S* in the plane *Px*, through *x* and perpendicular to the *x*-axis, is $A(x)$, where *A* is an integrable function, then the volume of *S* is

$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)dx$$

Draw a diagram of the cylinder (the grey region) with a cylindrical hole drilled into it (white).



Any plane *P*x that passes through *x* and is perpendicular to the *x*-axis intersects the hole in a rectangle. Use the diagram to find the length and width of the rectangle as *x* goes from *r* to *r*.

$$\text{length} = 2\sqrt{R^2 - x^2}$$
$$\text{width} = 2\sqrt{r^2 - x^2}$$

Then find the cross-sectional area by multiplying length by width.

$$A(x) = 4\sqrt{R^2 - x^2}\sqrt{r^2 - x^2}$$

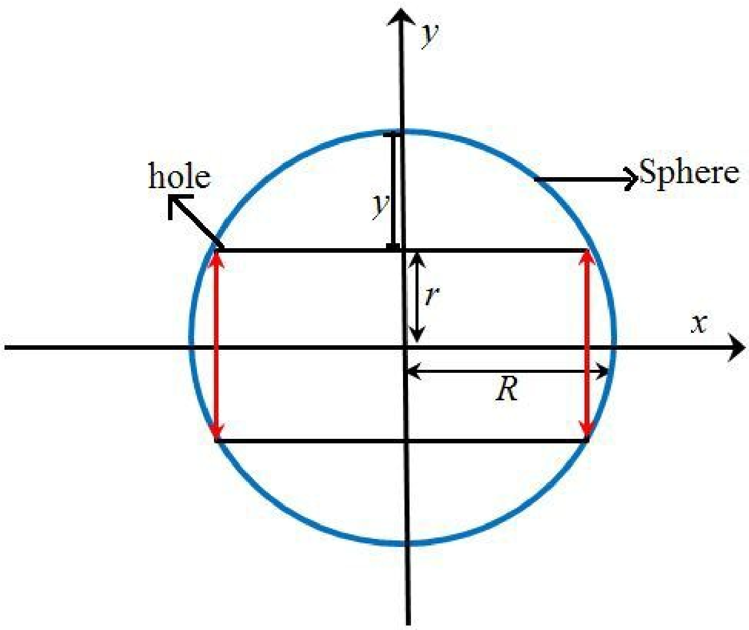
The hole lies between $x = -r$ and $x = r$. Write an integral for the volume of the hole.

$$V = \int_{-r}^{r} 4\sqrt{R^2 - x^2}\sqrt{r^2 - x^2}\,dx$$
$$= \boxed{8\int_{0}^{r} \sqrt{R^2 - x^2}\sqrt{r^2 - x^2}\,dx}$$

Answer 68E.

Consider that the radius of a hole is *r*, and the radius of a sphere is $R > r$.

Sketch the sphere and the hole (which is bored horizontally along the *x* –axis into the sphere) as shown below:



From the figure, observe that the radius of the semicircle is *y*, and the line, $y = r$ intersects the semicircle.

Also, each cross section is a ring.

The standard form of the equation of a circle is $(x-h)^2 + (y-k)^2 = r^2$.

Here, *r* is the radius, and (h,k) is the center of the circle.

So, the equation of a circle with radius *R* center at $(0,0)$ is $x^2 + y^2 = R^2$.

Solve the equation $x^2 + y^2 = R^2$ for *y*:

$$\begin{aligned} x^2 + y^2 &= R^2 \\ y^2 &= R^2 - x^2 \\ y &= \sqrt{R^2 - x^2} \end{aligned}$$

Therefore, the radius of the semicircle is $y = \sqrt{R^2 - x^2}$ that is the outer radius of the cross-section.

Clearly, the inner radius of the cross-section is *r*.

Use the following formula to find the volume of a solid *S* that rotated about the *x* – axis:

$$V = \int_a^b A(x)\,dx.$$

Here, $A(x)$ is the cross-sectional area of *S*, and the solid lies between the lines, $x = a,$ and $x = b.$

Also, the cross section is a washer.

So, the formula to find area *A* is given by $A(x) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$.

Here, the cross section is a washer with inner radius r, and the outer radius $\sqrt{R^2 - x^2}$.

So, calculate the area of the cross section as follows:

$$\begin{aligned} A(x) &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\ &= \pi\left(\sqrt{R^2 - x^2}\right)^2 - \pi(r)^2 \\ &= \pi\left(R^2 - x^2\right) - \pi r^2 \\ A(x) &= \pi\left(R^2 - x^2 - r^2\right) \end{aligned}$$

From the figure, observe that the solid is lies between the intersecting points of

$y=\sqrt{R^2-x^2}$ and $y=r$.

Solve $y=\sqrt{R^2-x^2}$ and $y=r$ for x:

$$\sqrt{R^2-x^2}=r$$
$$R^2-x^2=r^2$$
$$R^2-x^2-r^2=0$$
$$R^2-r^2=x^2$$
$$\pm\sqrt{R^2-r^2}=x$$

Therefore, the solid is lies between the lines $x=-\sqrt{R^2-r^2}$ and $x=\sqrt{R^2-r^2}$.

Evaluate the volume of the solid as follows:

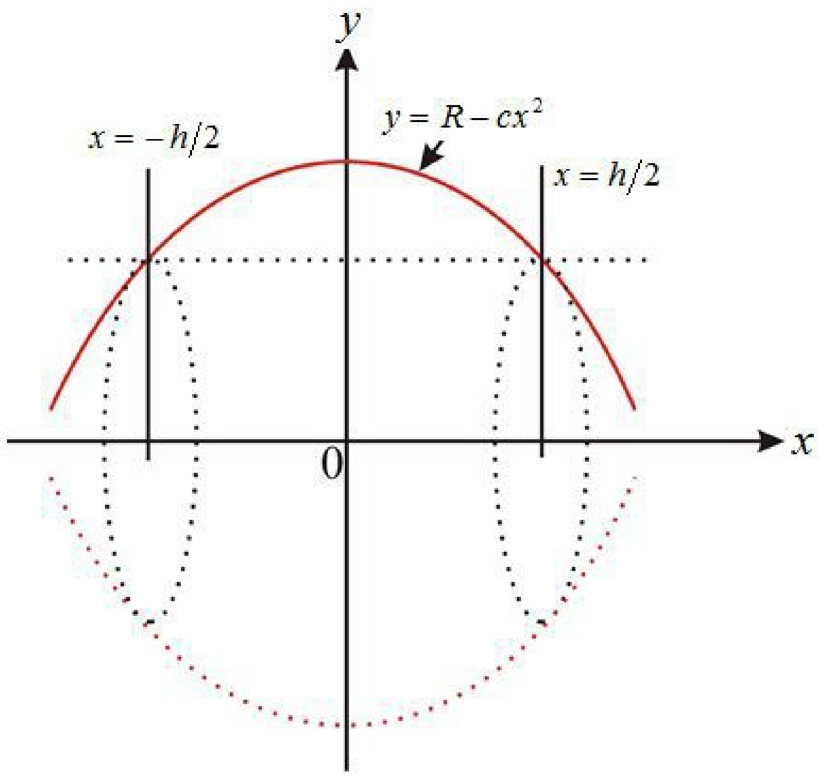
$$V=\int_a^b A(x)\,dx$$
$$=\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}\left[\pi\left(R^2-x^2-r^2\right)\right]dx$$
$$=2\int_0^{\sqrt{R^2-r^2}}\left[\pi\left(R^2-x^2-r^2\right)\right]dx \qquad \text{use } \int_{-a}^{a} f(x)\,dx=2\int_0^a f(x)\,dx$$
$$=2\pi\int_0^{\sqrt{R^2-r^2}}\left(R^2-r^2\right)-x^2\,dx$$
$$=2\pi\left[\int_0^{\sqrt{R^2-r^2}}\left(R^2-r^2\right)dx-\int_0^{\sqrt{R^2-r^2}}x^2\,dx\right]$$

$$=2\pi\left[\left(R^2-r^2\right)[x]_0^{\sqrt{R^2-r^2}}-\left[\frac{x^3}{3}\right]_0^{\sqrt{R^2-r^2}}\right]$$
$$=2\pi\left[\left(R^2-r^2\right)\left[\sqrt{R^2-r^2}-0\right]-\left[\frac{\left(\sqrt{R^2-r^2}\right)^3}{3}-0\right]\right]$$
$$=2\pi\left[\left(R^2-r^2\right)^{3/2}-\frac{\left(R^2-r^2\right)^{3/2}}{3}\right]$$
$$=2\pi\left[\frac{2\left(R^2-r^2\right)^{3/2}}{3}\right]$$
$$=\frac{4\pi\left(R^2-r^2\right)^{3/2}}{3}$$

Therefore, the volume of the remaining portion of the sphere is $\boxed{\frac{4\pi\left(R^2-r^2\right)^{3/2}}{3}}$.

Answer 69E.

Consider the graph



(a) The function $y = R - cx^2$ is an even function so its graph will be symmetric about y – axis. So radius of both the ends will be equal.

If rotate the region bounded by the curve $y = R - cx^2$ from $x = -h/2$ to $x = h/2$ about x – axis, then get a solid that is a barrel.

Consider a vertical strip in this region then after rotation get a circular disk with radius

$$r = y = R - cx^2$$

When $x = h/2$ then

$$r(h/2) = R - c(h/2)^2$$
$$r(h/2) = R - ch^2/4$$

Let $d = ch^2/4$

Therefore, the radius of each ends of the barrel is $\boxed{r = R - d}$.

(b) Volume of the barrel is

$$V = 2\int_0^{h/2} \pi\left(R - cx^2\right)^2 dx$$
$$= 2\pi\int_0^{h/2}\left(R^2 + c^2x^4 - 2Rcx^2\right)dx$$
$$= 2\pi\left[R^2x + \frac{1}{5}c^2x^5 - \frac{2}{3}Rcx^3\right]_0^{h/2}$$
$$= 2\pi\left[\frac{R^2h}{2} + \frac{1}{160}c^2h^5 - \frac{2}{24}Rch^3\right]$$

Multiply numerator and denominator with 3

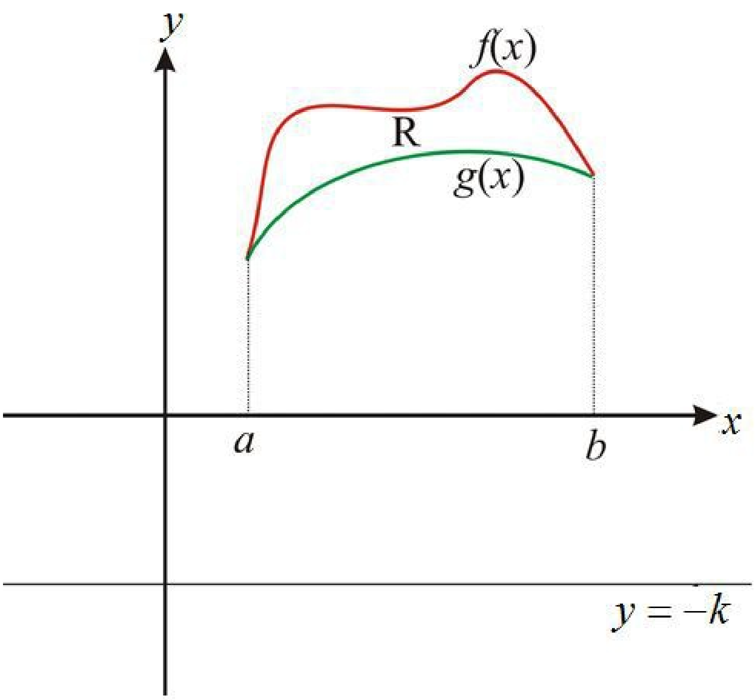
Thus

$$V=\frac{1}{3}\pi h\left[3R^2+\frac{3}{80}c^2h^4-\frac{1}{2}Rch^2\right]$$
$$=\frac{1}{3}\pi h\left[2R^2+R^2+\frac{5}{80}c^2h^4-\frac{2}{80}c^2h^4-\frac{1}{2}Rch^2\right]$$
$$=\frac{1}{3}\pi h\left[2R^2+R^2+\frac{1}{16}c^2h^4-2R\left(\frac{ch^2}{4}\right)-\frac{1}{40}c^2h^4\right]$$
$$=\frac{1}{3}\pi h\left[2R^2+\left[R-\left(\frac{ch^2}{4}\right)\right]^2-\frac{1}{40}c^2h^4\right]$$
$$=\frac{1}{3}\pi h\left[2R^2+[R-d]^2-\frac{16}{40}\left(d^2\right)\right] \qquad \text{From part } (a)$$
$$=\frac{1}{3}\pi h\left[2R^2+r^2-\frac{2}{5}d^2\right]$$

Therefore, volume enclosed by the barrel is $\boxed{V=\frac{1}{3}\pi h\left[2R^2+r^2-\frac{2}{5}d^2\right]}$

Answer 70E.

Consider the graph



Let the region is bounded by the curve $f(x)$ and $g(x)$ where $g(x)<f(x)$ and $a<x<b$.

Given that the area of the region $=A$

Then $\int_a^b [f(x)-g(x)]dx=A$ ...... (1)

Use washer method for finding the volume of the solid obtained by rotating the region R about x –axis.

Outer radius of the washer is $=f(x)$

Inner radius of the washer is $=g(x)$

Then the volume of the solid obtained by rotating the region R about x-axis is

$$V_1=\pi\int_a^b\left[\{f(x)\}^2-\{g(x)\}^2\right]dx \quad ...... (2)$$

If want to rotate the region R about the line $y=-k$ then

Outer radius of the washer is $=f(x)+k$

Inner radius of the washer is $=g(x)+k$

Then the volume of the solid obtained by rotating the region R about the line $y=-k$ is

$$V_2=\pi\int_a^b\left[\{f(x)+k\}^2-\{g(x)+k\}^2\right]dx$$

$$=\pi\int_a^b\left[\{f(x)\}^2+k^2+2kf(x)-\{g(x)\}^2-k^2-2kg(x)\right]dx$$

$$=\pi\int_a^b\left[\{f(x)\}^2-\{g(x)\}^2+2k\{f(x)-g(x)\}\right]dx$$

$$=\pi\int_a^b\left[\{f(x)\}^2-\{g(x)\}^2\right]dx+2\pi k\int_a^b\{f(x)-g(x)\}dx$$

From (1) and (2)

$$\boxed{V_2=V_1+2\pi kA}$$