3.6 Oscillating systems

Free oscillations

Differential equation	$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=0$$	(3.196)	x t γ ω_0	oscillating variable time damping factor (per unit mass) undamped angular frequency
Underdamped solution ($\gamma<\omega_0$)	$x=Ae^{-\gamma t}\cos(\omega t+\phi)$ where $\omega=(\omega_0^2-\gamma^2)^{1/2}$	(3.197) (3.198)	A ϕ ω	amplitude constant phase constant angular eigenfrequency
Critically damped solution ($\gamma=\omega_0$)	$x=e^{-\gamma t}(A_1+A_2t)$	(3.199)	A_i	amplitude constants
Overdamped solution ($\gamma>\omega_0$)	$x=e^{-\gamma t}(A_1e^{qt}+A_2e^{-qt})$ where $q=(\gamma^2-\omega_0^2)^{1/2}$	(3.200) (3.201)		
Logarithmic decrement[a]	$$\Delta=\ln\frac{a_n}{a_{n+1}}=\frac{2\pi\gamma}{\omega}$$	(3.202)	Δ a_n	logarithmic decrement nth displacement maximum
Quality factor	$$Q=\frac{\omega_0}{2\gamma}\quad\left[\simeq\frac{\pi}{\Delta}\quad\text{if}\quad Q\gg 1\right]$$	(3.203)	Q	quality factor

[a]The *decrement* is usually the ratio of successive displacement *maxima* but is sometimes taken as the ratio of successive displacement *extrema*, reducing Δ by a factor of 2. Logarithms are sometimes taken to base 10, introducing a further factor of $\log_{10}e$.

Forced oscillations

Differential equation	$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=F_0e^{\mathbf{i}\omega_f t}$$	(3.204)	x t γ	oscillating variable time damping factor (per unit mass)
Steady-state solution[a]	$x=Ae^{\mathbf{i}(\omega_f t-\phi)}$, where $A=F_0[(\omega_0^2-\omega_f^2)^2+(2\gamma\omega_f)^2]^{-1/2}$ $\simeq\dfrac{F_0/(2\omega_0)}{[(\omega_0-\omega_f)^2+\gamma^2]^{1/2}}\quad(\gamma\ll\omega_f)$ $\tan\phi=\dfrac{2\gamma\omega_f}{\omega_0^2-\omega_f^2}$	(3.205) (3.206) (3.207) (3.208)	ω_0 F_0 ω_f A ϕ	undamped angular frequency force amplitude (per unit mass) forcing angular frequency amplitude phase lag of response behind driving force
Amplitude resonance[b]	$\omega_{ar}^2=\omega_0^2-2\gamma^2$	(3.209)	ω_{ar}	amplitude resonant forcing angular frequency
Velocity resonance[c]	$\omega_{vr}=\omega_0$	(3.210)	ω_{vr}	velocity resonant forcing angular frequency
Quality factor	$$Q=\frac{\omega_0}{2\gamma}$$	(3.211)	Q	quality factor
Impedance	$$Z=2\gamma+\mathbf{i}\frac{\omega_f^2-\omega_0^2}{\omega_f}$$	(3.212)	Z	impedance (per unit mass)

[a]Excluding the free oscillation terms.
[b]Forcing frequency for maximum displacement.
[c]Forcing frequency for maximum velocity. Note $\phi=\pi/2$ at this frequency.