1 Linear Algebra

1.1 Introduction

Linear Algebra is a branch of mathematics concerned with the study of vectors, with families of vectors called vector spaces or linear spaces and with functions that input one vector and output another, according to certain rules. These functions are called linear maps or linear transformations and are often represented by matrices. Matrices are rectangular arrays of numbers or symbols and matrix algebra or linear algebra provides the rules defining the operations that can be formed on such an object.

Linear Algebra and matrix theory occupy an important place in modern mathematics and has applications in almost all branches of engineering and physical sciences. An elementary application of linear algebra is to the solution of a system of linear equations in several unknowns, which often result when linear mathematical models are constructed to represent physical problems. Nonlinear models can often be approximated by linear ones. Other applications can be found in computer graphics and in numerical methods.

In this chapter, we shall discuss matrix algebra and its use in solving linear system of algebraic equations $AX = B$ and in solving the Eigen value problem $AX = \lambda X$.

1.2 Algebra of Matrices

1.2.1 Definition of Matrix

A system of $m \times n$ numbers arranged in the form of a rectangular array having m rows and n columns is called an matrix of order $m \times n$.

If $A = [a_{ij}]_{m\times n}$ be any matrix of order $m \times n$ then it is written in the form:

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12}\ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}\ldots\ldots\ldots\ldots a_{2n} \\ \ldots & \ldots\ldots\ldots\ldots\ldots \\ \ldots & \ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2}\ldots\ldots\ldots\ldots a_{mn} \end{bmatrix}$$

Horizontal lines are called rows and vertical lines are called columns.

1.2.2 Special Types of Matrices

1. **Square Matrix:** An $m\times n$ matrix for which $m = n$ (The number of rows is equal to number of columns) is called square matrix. It is also called an n-rowed square matrix. i.e. $A = [a_{ij}]_{n\times n}$. The elements $a_{ij} \mid i = j$, i.e. a_{11}, a_{22}.... are called DIAGONAL ELEMENTS and the line along which they lie is called PRINCIPLE DIAGONAL of matrix. Elements other than a_{11}, a_{22}, etc are called off-diagonal elements i.e. $a_{ij} \mid i \neq j$.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 8 & 3 \end{bmatrix}_{3\times 3}$ is a square Matrix

NOTE

A square sub-matrix of a square matrix A is called a **"principle sub-matrix"** if its diagonal elements are also the diagonal elements of the matrix A. So $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ is a principle sub matrix of the matrix A given above, but $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$ is not.

2. **Diagonal Matrix:** A square matrix in which all off-diagonal elements are zero is called a diagonal matrix. The diagonal elements may or may not be zero. $\begin{cases} a_{ij} = 0 & \text{if} \quad i \neq j \\ a_{ij} & \text{if} \quad i = j \end{cases}$

Example: $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ is a diagonal matrix

The above matrix can also be written as A = diag [3, 5, 9]

Properties of Diagonal Matrix:

diag $[x, y, z]$ + diag $[p, q, r]$ = diag $[x + p, y + q, z + r]$

diag $[x, y, z] \times$ diag $[p, q, r]$ = diag $[xp, yq, zr]$

(diag $[x, y, z])^{-1}$ = diag $[1/x, 1/y, 1/z]$

(diag $[x, y, z])^{T}$ = diag $[x, y, z]$

(diag $[x, y, z])^{n}$ = diag $[x^n, y^n, z^n]$

Eigen values of diag $[x, y, z] = x$, y and z.

Determinant of diag $[x, y, z] = |\text{diag } [x, y, z]| = xyz$

3. **Scalar Matrix:** A scalar matrix is a diagonal matrix with all diagonal elements being equal.

$\begin{cases} a_{ij} = 0 & \text{if} \quad i \neq j \\ a_{ij} = k & \text{if} \quad i = j \end{cases}$

Example: $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a scalar matrix.

4. **Unit Matrix or Identity Matrix:** A square matrix each of whose diagonal elements is 1 and each of whose non-diagonal elements are zero is called unit matrix or an identity matrix which is denoted by I. Identity matrix is always square.

Thus a square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$. $\begin{cases} a_{ij} = 0 & \text{if} \quad i \neq j \\ a_{ij} = 1 & \text{if} \quad i = j \end{cases}$

Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is unit matrix, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Properties of Identity Matrix:

(a) I is Identity element for multiplication, so it is called multiplicative identity.

(b) $AI = IA = A$

(c) $I^n = I$

(d) $I^{-1} = I$

(e) $|I| = 1$

5. **Null Matrix:** The $m \times n$ matrix whose elements are all zero is called null matrix. Null matrix is denoted by O. Null matrix need not be square. $a_{ij} = 0 \ \forall \ i, j$

Example: $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Properties of Null Matrix:

(a) $A + O = O + A = A$

So, O is additive identity.

(b) $A + (-A) = O$

6. **Upper Triangular Matrix:** An upper triangular matrix is a square matrix whose lower off-diagonal elements are zero, i.e. $a_{ij} = 0$ whenever $i > j$

It is denoted by U.

The diagonal and upper off diagonal elements may or may not be zero. $\begin{cases} a_{ij} = 0 & \text{if } i > j \\ a_{ij} & \text{if } i < j \end{cases}$

Example: $U = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 5 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

7. **Lower Triangular Matrix:** A lower triangular matrix is a square matrix whose upper off-diagonal triangular elements are zero, i.e. $a_{ij} = 0$ whenever $i < j$. The diagonal and lower off-diagonal elements may or may not be zero. $\begin{cases} a_{ij} = 0 & \text{if } i < j \\ a_{ij} & \text{if } i > j \end{cases}$

It is denoted by L,

Example: $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 5 & 0 \\ 2 & 3 & 6 \end{bmatrix}$

8. **Idempotent Matrix:** A matrix A is called Idempotent if $A^2 = A$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ are examples of Idempotent matrices.

9. **Involuntary Matrix:** A matrix A is called Involutory if $A^2 = I$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is involuntary. Also $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ is involuntary since $A^2 = I$.

10. **Nilpotent Matrix:** A matrix A is said to be nilpotent of class x or index x if $A^x = O$ and $A^{x-1} \neq O$ i.e. x is the smallest index which makes $A^x = O$.

Example: The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent class 3, since $A \neq 0$ and $A^2 \neq 0$, but $A^3 = 0$.

11. **Singular matrix:** If the determinant of a matrix is zero, then matrix is called as singular matrix.

$|A| = 0$ e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

**If determinant is not zero, then matrix is known as non-singular matrix.*

If matrix is singular then its inverse doesn't exist 3.

1.2.3 Equality of Two Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if,

1. They are of same size.
2. The elements in the corresponding places of two matrices are the same i.e., $a_{ij} = b_{ij}$ for each pair of subscripts i and j.

Example: Let $\begin{bmatrix} x-y & p+q \\ p-q & x+y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$

Then $x - y = 2$, $p + q = 5$, $p - q = 1$ and $x + y = 10$

$\Rightarrow$ $x = 6$, $y = 4$, $p = 3$ and $q = 2$.

1.2.4 Addition of Matrices

Two matrices A and B are compatible for addition only if they both have exactly the same size say $m \times n$. Then their sum is defined to be the matrix of the type $m \times n$ obtained by adding corresponding elements of A and B. Thus if, $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix}$;

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 10 & 13 \end{bmatrix}$$

Properties of Matrix Addition:

1. Matrix addition is commutative $A + B = B + A$.
2. Matrix addition is associative $(A + B) + C = A + (B + C)$
3. Existence of additive identity: If O be $m \times n$ matrix each of whose elements are zero. Then, $A + O = A = O + A$ for every $m \times n$ matrix A.
4. Existence of additive inverse: Let $A = [a_{ij}]_{m \times n}$.

   Then the negative of matrix A is defined as matrix $[\ a_{ij}]_{m \times n}$ and is denoted by $-A$.

   $\Rightarrow$ Matrix $-A$ is additive inverse of A. Because $(-A) + A = O = A + (-A)$. Here O is null matrix of order $m \times n$.
5. Cancellation laws holds good in case of addition of matrices, which is $X = -A$.

   $A + X = B + X \Rightarrow A = B$

   $X + A = X + B \Rightarrow A = B$
6. The equation $A + X = 0$ has a unique solution in the set of all $m \times n$ matrices.

1.2.5 Substraction of Two Matrices

If A and B are two $m \times n$ matrices, then we define, $A - B = A + (-B)$.

Thus the difference $A - B$ is obtained by subtracting from each element of A corresponding elements of B.

NOTE: Subtraction of matrices is neither commutative nor associative.

1.2.6 Multiplication of a Matrix by a Scalar

Let A be any $m \times n$ matrix and k be any real number called scalar. The $m \times n$ matrix obtained by multiplying every element of the matrix A by k is called scalar multiple of A by k and is denoted by kA.

$\Rightarrow$ If $A = [a_{ij}]_{m \times n}$ then $Ak = kA = [kA]_{m \times n}$.

$$\text{If } A = \begin{bmatrix} 5 & 2 & 1 \\ 6 & -5 & 2 \\ 1 & 3 & 6 \end{bmatrix} \text{ then, } 3A = \begin{bmatrix} 15 & 6 & 3 \\ 18 & -15 & 6 \\ 3 & 9 & 18 \end{bmatrix}$$

Properties of Multiplication of a Matrix by a Scalar:

1. Scalar multiplication of matrices distributes over the addition of matrices i.e., $k(A + B) = kA + kB$.
2. If p and q are two scalars and A is any $m \times n$ matrix then, $(p + q)A = pA + qA$.
3. If p and q are two scalers and $A = [a_{ij}]_{m \times n}$ then, $p(qA) = (pq)A$.
4. If $A = [a_{ij}]_{m \times n}$ be a matrix and k be any scalar then, $(-k)A = -(kA) = k(-A)$.

1.2.7 Multiplication of Two Matrices

Let $A = [a_{ij}]_{m \times n}$; $B = [b_{jk}]_{n \times p}$ be two matrices such that the number of columns in A is equal to the number of rows in B.

Then the matrix $C = [c_{ik}]_{m \times p}$ such that $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ is called the product of matrices A and B in that order and we write $C = AB$.

Properties of Matrix Multiplication:

1. Multiplication of matrices is not commutative. In fact, if the product of AB exists, then it is not necessary that the product of BA will also exist. For example, $A_{3 \times 2} \times B_{2 \times 4} = C_{3 \times 4}$ but $B_{2 \times 4} \times A_{3 \times 2}$ does not exist since these are not compatible for multiplication.
2. Matrix multiplication is associative, if conformability is assured. i.e., $A(BC) = (AB)C$ where A, B, C are $m \times n$, $n \times p$, $p \times q$ matrices respectively.
3. Multiplication of matrices is distributive with respect to addition of matrices. i.e., $A(B + C) = AB + AC$.
4. The equation $AB = O$ does not necessarily imply that at least one of matrices A and B must be a zero matrix. For example, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
5. In the case of matrix multiplication if $AB = O$ then it is not necessarily imply that $BA = O$. In fact, BA may not even exist.
6. Both left and right cancellation laws hold for matrix multiplication as shown below:

   $AB = AC \Rightarrow B = C$ (if A is non-singular matrix) and

   $BA = CA \Rightarrow B = C$ (if A is non-singular matrix).

1.2.8 Trace of a Matrix

Let A be a square matrix of order n. The sum of the elements lying along principal diagonal is called the trace of A denoted by $Tr(A)$.

Thus if $A = [a_{ij}]_{n \times n}$ then, $Tr(A) = \sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + \ldots a_{nn}$.

Let
$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 6 & 5 \end{bmatrix}$$

Then, Trace $(A) = Tr(A) = 1+(-3) + 5 = 3$

Properties of Trace of a Matrix:

Let A and B be two square matrices of order n and λ be a scalar. Then,

1. $Tr(\lambda A) = \lambda\, Tr\, A$
2. $Tr(A + B) = Tr\, A + Tr\, B$
3. $Tr(AB) = Tr(BA)$ [If both AB and BA are defined]

1.2.9 Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$. Then the $n \times m$ matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by A' or A^T.

Let
$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{bmatrix} \text{ then, } A^T = A' = \begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{bmatrix}$$

If
$$B = [1\ 2\ 3]$$

Then
$$B' = [1\ \ 2\ \ 3]' = [1\ \ 2\ \ 3]^t = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Properties of Transpose of a Matrix:

If A^T and B^T be transposes of A and B respectively then,

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$, k being any complex number
4. $(AB)^T = B^T A^T$
5. $(ABC)^T = C^T B^T A^T$

1.2.10 Conjugate of a Matrix

The matrix obtained from given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$.

Example: If $A = \begin{bmatrix} 2+3i & 4-7i & 8 \\ -i & 6 & 9+i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 2-3i & 4+7i & 8 \\ +i & 6 & 9-i \end{bmatrix}$$

Properties of Conjugate of a Matrix:

If $\bar{A}$ and $\bar{B}$ be the conjugates of A and B respectively. Then,

1. $\overline{(\bar{A})} = A$
2. $\overline{(A+B)} = \bar{A}+\bar{B}$
3. $\overline{(kA)} = \bar{k}\bar{A}$, k being any complex number
4. $\overline{(AB)} = \bar{A}\bar{B}$, A and B being conformable to multiplication
5. $\bar{A} = A$ if A is real matrix

   $\bar{A} = -A$ if A is purely imaginary matrix

1.2.11 Transposed Conjugate of Matrix

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^θ or A^* or $(\bar{A})^T$. It is also called conjugate transpose of A.

Example: If $A = \begin{bmatrix} 2+i & 3-i \\ 4 & 1-i \end{bmatrix}$

To find A^θ, we first find $\bar{A} = \begin{bmatrix} 2-i & 3+i \\ 4 & 1+i \end{bmatrix}$

Then $A^\theta = (\bar{A})^T = \begin{bmatrix} 2-i & 4 \\ 3+i & 1+i \end{bmatrix}$

Some properties: If A^θ & B^θ be the transposed conjugates of A and B respectively then,

1. $(A^\theta)^\theta = A$
2. $(A + B)^\theta = A^\theta + B^\theta$
3. $(kA)^\theta = \bar{k}A^\theta$, $k \rightarrow$ complex number
4. $(AB)^\theta = B^\theta A^\theta$

1.2.12 Classification of Real Matrices

Real matrices can be classified into the following three types based on the relationship between A^T and A.

1. Multip

1. Symmetric Matrices ($A^T = A$)
2. Skew Symmetric Matrices ($A^T = -A$)
3. Orthogonal Matrices ($A^T = A^{-1}$ or $AA^T = I$)

1. **Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is said to be symmetric if its $(i, j)^{th}$ elements is same as its $(j, i)^{th}$ element i.e., $a_{ij} = a_{ji}$ for all i & j.

   In a symmetric matrix, $A^T = A$

   Example: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix, since $A^T = A$.

   Note: For any matrix A,

   (a) AA^T is always a symmetric matrix.

(b) $\dfrac{A+A^T}{2}$ is always symmetric matrix.

Note: If A and B an symmetric, then

(a) $A + B$ and $A - B$ are also symmetric.

(b) AB, BA may or may not be symmetric.

2. **Skew Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $(i, j)^{th}$ elements of A is the negative of the $(j, i)^{th}$ elements of A if $a_{ij} = -a_{ji}\ \forall\ i, j$.

In a skew symmetric matrix $A^T = -A$.

A skew symmetric matrix must have all 0's in the diagonal.

Example: $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Note: For any matrix A, the matrix $\dfrac{A-A^T}{2}$ is always skew symmetric.

3. **Orthogonal Matrix:** A square matrix A is said be orthogonal if:

$A^T = A^{-1} \Rightarrow AA^T = AA^{-1} = I$. Thus A will be an orthogonal matrix if, $AA^T = I = A^TA$.

Example: The identity matrix is orthogonal since $I^T = I^{-1} = I$.

Note: Since for an orthogonal matrix A,

$$AA^T = I$$
$$\Rightarrow \quad |AA^T| = |I| = 1$$
$$\Rightarrow \quad |A|\,|A^T| = 1$$
$$\Rightarrow \quad (|A|)^2 = 1$$
$$\Rightarrow \quad |A| = \pm 1$$

So the determinant of an orthogonal matrix always has a modulus of 1.

1.2.13 Classification of Complex Matrices

Complex matrices can be classified into the following three types based on relationship between A^θ and A.

1. Hermitian Matrix ($A^\theta = A$)
2. Skew-Hermitian Matrix ($A^\theta = -A$)
3. Unitary Matrix ($A^\theta = A^{-1}$ or $AA^\theta = I$)

1. **Hermitian Matrix:** A necessary and sufficient condition for a matrix A to be Hermitian is that $A^\theta = A$.

Example: $A = \begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$ is a Hermitian matrix.

2. **Skew-Hermitian Matrix:** A necessary and sufficient condition for a matrix to be skew-Hermitian if $A^\theta = -A$.

Example: $A = \begin{bmatrix} 0 & -2-i \\ 2-i & 0 \end{bmatrix}$ is skew-Hermitian.

3. **Unitary Matrix:** A square matrix A is said to be unitary if:

$$A^\theta = A^{-1}$$

Multiplying both sides by A, we get an alternate definition of unitary matrix as given below:
A square matrix A is said to be unitary if:

$$AA^{\theta} = I = A^{\theta} A$$

Example: $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is an example of a unitary matrix.

1.3 Determinants

1.3.1 Definition

Let $a_{11}, a_{12}, a_{21}, a_{22}$ be any four numbers. The symbol $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ represents the number $a_{11}a_{22} - a_{12}a_{21}$ and is called determinants of order 2. The number $a_{11}, a_{12}, a_{21}, a_{22}$ are called elements of the determinant and the number $a_{11}a_{22} - a_{21}a_{12}$ is called the value of determinant.

1.3.2 Minors, Cofactors and Adjoint

Consider the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Leaving the row and column passing through the elements a_{ij}, then the second order determinant thus obtained is called the minor of element a_{ij} and we will be denoted by M_{ij}.

Example: The Minor of element $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$

Similarly Minor of element $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$

1.3.3 Cofactors

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of element a_{ij}. We shall denote the cofactor of an element by corresponding capital letter.

Example: Cofactor of $a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$.

Cofactor of element $a_{21} = A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

by cofactor of element $a_{32} = A_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

We define for any matrix, the sum of the products of the elements of any row or column with corresponding cofactors is equal to the determinant of the matrix.

Example: If

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 6 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

then,

$$\text{cof}(A) = \begin{bmatrix} 12 & 4 & -12 \\ -4 & 2 & 4 \\ 2 & -1 & 8 \end{bmatrix}$$

$$\begin{aligned} |A| &= (1 \times 12) + (2 \times 4) + (0 \times -12) \\ &= (-1 \times -4) + (6 \times 2) + (1 \times 4) \\ &= (2 \times 2) + (0 \times -1) + (2 \times 8) = 20 \end{aligned}$$

1.3.4 Adjoint

When all the elements of a matrix 'A' are replaced by its co-factor, then the transpose of that matrix is known as adjoint of matrix 'A'.

$$a_{ij} \rightarrow C_{ij}$$

$$\text{Adj}A = [C_{ij}]^T$$

Properties of adjoint matrix A(Adj A)

1. $A \times \text{Adj}A = |A| \times I$
2. $A^{-1} = \frac{1}{|A|} \times (\text{Adj}A)$

1.3.5 Determinant of order *n*

A determinant of order n has n-row and n-columns. It has $n \times n$ elements.

A determinant of order n is a square array of $n \times n$ quantities enclosed between vertical bars.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \ldots\ldots\ldots & a_{1n} \\ a_{21} & \ldots\ldots\ldots\ldots & a_{2n} \\ a_{n1} & a_{n2} \ldots\ldots\ldots & a_{nn} \end{vmatrix}$$

Cofactor of A_{ij} of elements a_{ij} in D is equal to $(-1)^{i+j}$ times the determinants of order $(n-1)$ obtained from D by leaving the row and column passing through element a_{ij}.

If A is a 3 × 3 matrix, then $|A| = \sum_{j=1}^{3} A_{1j}\,\text{cof}(A_{1j}) = \sum_{j=1}^{3} A_{2j}\,\text{cof}(A_{2j}) = \sum_{j=1}^{3} A_{3j}\,\text{cof}(A_{3j}) = \sum_{i=1}^{3} A_{i1}\,\text{cof}(A_{i1})$, etc.

Therefore, determinant can be expanded using any row or column.

1.3.6 Properties of Determinants

1. The value of a determinant does not change when rows and columns are interchanged. i.e. $|A^T| = |A|$
2. If any row (or column) of a matrix A is completely zero, then $|A| = 0$.
   Such a row (or column) is called a zero row (or column).
   Also if any two rows (or columns) of a matrix A are identical, then $|A| = 0$.
3. If any two rows or two columns of a determinant are interchanged the value of determinant is multiplied by -1.

4. If all elements of the one row (or one column) of a determinant are multiplied by same number k the value of determinant is k times the value of given determinant.

5. If A be n-rowed square matrix, and k be any scalar, then $|kA| = k^n|A|$

6. (a) In a determinant the sum of the products of the elements of any row (or column) with the cofactors of corresponding elements of any row or column is equal to the determinant value.

   (b) In determinant the sum of the products of the elements of any row (or column) with the cofactors of some other row or column is zero.

Example:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then
$$a_1A_1 + b_1B_1 + c_1C_1 = \Delta$$
$$a_1A_2 + b_1B_2 + c_1C_2 = 0$$
$$a_1A_3 + b_1B_3 + c_1C_3 = 0$$
$$a_2A_2 + b_2B_2 + c_2C_2 = \Delta$$
$$a_2A_1 + b_2B_1 + c_2C_1 = 0 \text{ etc}$$

where A_1, B_1, C_1 etc., be cofactors of the elements a_1, b_1, c_1 in D.

7. If to the elements of a row (or column) of a determinant are added m times the corresponding elements of another row (or column) the value of determinant thus obtained is equal to the value of original determinant.

$$\text{i.e., } A \xrightarrow{R_i + kR_j} B \text{ then } |A| = |B|$$

$$\text{and } A \xrightarrow{C_i + kC_j} B \text{ then } |A| = |B|$$

8. $|AB| = |A| * |B|$ and based on this we can prove the following:

   (a) $|A^n| = (|A|)^n$

   (b) $|A^{-1}| = \dfrac{1}{|A|}$

Proof of a:
$$|A^n| = |A * A * A \ldots n \text{ times}|$$
$$= |A| * |A| * |A| \ldots n \text{ times}$$
$$= (|A|)^n$$

Proof of b:
$$|A\,A^{-1}| = |I|$$
$$= 1$$

Now since, $|A\,A^{-1}| = |A|\,|A^{-1}|$

$\therefore \quad |A|\,|A^{-1}| = 1$

$$\Rightarrow \quad |A^{-1}| = \frac{1}{|A|}$$

9. Using the fact that $A \,.\, \text{Adj } A = |A| \,.\, I$, the following can be proved for $A_{n \times n}$.

   (a $|\text{Adj } A| = |A|^{n-1}$

   (b) $|\text{Adj (Adj } (A))| = |A|^{(n-1)^2}$

1.4 Inverse of Matrix

The inverse of a matrix A, exists if A is non-singular (i.e. $|A| \neq 0$) and is given by the formula

$$A^{-1} = \frac{\text{Adj}(A)}{|A|}.$$

Inverse of a matrix is always unique.

1.4.1 Adjoint of a Square Matrix

Let $A=[a_{ij}]$ be any $n \times n$ matrix. The transpose B *of the matrix* $B = [A_{ij}]_{n \times n}$ *where* A_{ij} *denotes the cofactor of element* a_{ij} *is called the adjoint of matrix* A and is denoted by symbol Adj A.

$$\therefore \quad \text{Adj}(A) = [\text{cof}(A)]^T$$

Properties of Adjoint:

If A be any n-rowed square matrix, then $(\text{Adj } A)\, A = A\,(\text{Adj } A) = |A|\, I_n$

where I_n is the $n \times n$ Identity matrix.

1.4.2 Properties of Inverse

1. $AA^{-1} = A^{-1}A = I$
2. A and B are inverse of each other if $AB = BA = I$
3. $(AB)^{-1} = B^{-1}\, A^{-1}$
4. $(ABC)^{-1} = C^{-1}\, B^{-1}\, A^{-1}$
5. If A be an $n \times n$ non-singular matrix, then $(A^T)^{-1} = (A^{-1})^T$.
6. If A be an $n \times n$ non-singular matrix then $(A^{-1})^\theta = (A^\theta)^{-1}$.
7. For a 2×2 matrix there is a shortcut formula for inverse as given below

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

1.5 Rank of A Matrix

Rank is defined for any matrix $A_{m \times n}$ (need not be square)

Some important concepts:

1. **Submatrix of a Matrix:** Suppose A is any matrix of the type $m \times n$. Then a matrix obtained by leaving some rows and some columns from A is called sub-matrix of A.
2. **Rank of a Matrix:** A number r is said to be the rank of a matrix A, if it possesses the following properties:
   (a) There is at least one square sub-matrix of A of order r whose determinant is not equal to zero.
   (b) If the matrix A contains any square sub-matrix of order $(r + 1)$ and above, then the determinant of such a matrix should be zero.
   Put together property (a) and (b) give the definition of the rank of a matrix as the "size of the largest non-zero minor".

Note:

(a) The rank of a matrix is $\leq r$, if all $(r + 1)$ - rowed minors of the matrix vanish.

(b) The rank of a matrix is $\geq r$, if there is at least one r-rowed minor of the matrix which is not equal to zero.

(c) The rank of transpose of a matrix is same as that of original matrix. i.e. $r(A^T) = r(A)$.

(d) Rank of a matrix is same as the number of linearly independent row vectors in the matrix as well as the number of linearly independent column vectors in the matrix.

(e) For any matrix A, rank $(A) \leq \min(m,n)$

i.e., maximum rank of $A_{m \times n} = \min(m, n)$

(f) Rank $(AB) \leq$ Rank A

Rank $(AB) \leq$ Rank B

So, Rank $(AB) \leq \min$(Rank A, Rank B)

(g) Rank (A^t) = Rank (A)

(h) Rank of a matrix is the number of non-zero rows in its echelon form.

Echelon form: A matrix is in echelon form if only if

1. Leading non-zero element in every row is behind leading non-zero element in previous row. This means below the leading non-zero element in every row all the elements must be zero.
2. All the zero rows should be below all the non-zero rows.

This definition gives an alternate way of calculating the rank of larger matrices (larger than 3×3) more easily. To reduce a matrix to its echelon form use gauss elimination method on the matrix and convert it into an upper triangular matrix, which will be in echelon form. Then count the number of non-zero rows in the upper triangular matrix to get the rank of the matrix.

(i) Elementary transformations do not alter the rank of a matrix.

(j) Only null matrix can have a rank of zero. All other matrices have rank of atleast one.

1.5.1 Elementary Matrices

A matrix obtained from a unit matrix by a single elementary transformation is called an elementary matrix.

1.5.2 Results

1. Elementary transformations do not change the rank of a matrix.
2. Two matrices are equivalent if one can be obtained from another by elementary row or column transformations. Equivalent matrices have same rank, since elementary transformations do not change the rank.
3. The rank of a product of two matrices cannot exceed the rank of either matrix. i.e. $r(AB) \leq r(A)$ and $r(AB) \leq r(B)$.
4. Rank of sum of two matrices cannot exceed the sum of their ranks. $r(A+B) \leq r(A) + r(B)$.
5. If A, B are two n-rowed square matrices then Rank $(AB) \geq$ (Rank A) + (Rank B) $- n$.

1.6 Sub-Spaces : Basis and Dimension

1.6.1 Introduction

A matrix can be thought of as an array of its rows as also an array of its columns. Further a row as well as a column is an ordered set of numbers. This view of matrix as an array of ordered sets of rows and columns is very useful in dealing with various linear problems. This chapter will be devoted to consideration of such ordered sets of numbers.

1.6.2 Vector

Definition: An ordered n-tuple of numbers is called an n-vector. The n numbers which are called components of the n-vector may be written in a horizontal or in a vertical line, and thus a vector will appear either as a row or a column matrix. A vector whose components belong to a field F is said to be over F. A vector over the field of real numbers is called a Real vector and that over the complex field is called a complex vector.

The n-vector space: The set of all n-vectors over a field F, to be denoted by $V_n(F)$, is called the n-vector space over F. The elements of the field F will be known as scalars relatively to the vector space.

1.6.3 Linearly dependent and Linearly Independent Sets of Vectors

1.6.3.1 Linear dependence and independence of vector

Vectors (matrices) $X_1, X_2, X_3, \ldots X_n$ are said to be dependent if,

1. All the vectors (row or column matrices) are of same order.
2. n scalars $\lambda_1, \lambda_2, \ldots \lambda_n$ (not all zero) exists such that $\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_n X_n = 0$ otherwise they are linearly independent.

1.6.3.2 Dependence / Independency of vector by matrix method

1. If the rank of the matrix of the given vectors is equal to number of vectors, then the vectors are linearly independent.
2. If the rank of the matrix of the given vectors is less than number of vectors, than the vectors are linearly dependent.

1.6.3.3 A vector as a Linear Combination of a Set of Vectors

Definition: A vector ξ which can be expressed in the form $\{\xi = k_1\xi_1 + \ldots + k_r\xi_r\}$ is said to be a linear combination of the set $\{\xi_1, \xi_2, \ldots, \xi_r\}$ of vectors.

Example: Given a linearly dependent set of vectors, show that at least one member of the set is a linear combination of the remaining members of the set.

Example:

1. ~~Show that the vectors [1 2 3], [2 – 2 0] from a linearly independent set.~~
2. Show that the set consisting only of the zero vector, O, is linearly dependent.

Solution:

1. Consider the relation

   $k_1[1\ 2\ 3] + k_2[2\ {-2}\ 0]$ – zero

   This relation is equivalent to the ordinary system of liner equations

   $$k_1 + 2k_2 = 0,\ 2k_1 - 2k_2 = 0,\ 3k_1 = 0$$

   As $k_1 = 0$, $k_2 = 0$ are the only values of k_1, k_2 which satisfy these three equations, we see that the given set is linearly independent.

2. Let $X = (0, 0, 0, \ldots, 0)$ be an n-vector whose components are all zero. Then the relation $kX = 0$ is true for some non-zero value of the number k, For example $2x = 0$ and $2 \neq 0$.

   Hence the vector 0 is linearly dependent.

1.6.4 Some properties of linearly Independent and Dependent Sets of Vectors

In the following, it is understood that the vectors belong to a given vector space $V_n(F)$.

1. If η is a linear combination of the set $\{\xi_1, \ldots\ldots \xi_r\}$, then the set $\{\eta, \xi_1, \xi_2, \ldots, \xi_r\}$ is linearly dependent we have

$$\eta = k_1\xi_1 + k_2\xi_2 + \ldots + k_r\xi_r$$

$$\Rightarrow \quad \eta - k_1\xi_1 - k_2\xi_2 - \ldots - k_r\xi_r = 0$$

As at least one of the coefficients, viz., that of η, in this latter relation is not zero, we establish the linear dependence of the set

$$\{\eta, \xi_1, \dots, \xi_r\}$$

2. Also, If $\{\xi_1, \dots, \xi_r\}$ is a linearly independent and $\{\xi_1, \dots \xi_r, \eta\}$ is a linearly dependent set, then η is a linear combination of the set $\{\xi_1, \dots \xi_r\}$.
3. Every super-set of a linearly dependent set is linearly dependent.
4. It may also be easily shown that every sub-set of a linearly independent set is linearly independent.

1.6.5 Subspaces of an N-vector space V_n

Definition: Any non-empty set S, of vectors of $V_n(F)$ is called a subspace of $V_n(F)$, if when

1. ξ_1, ξ_2 are any two members of S, then $\xi_1 + \xi_2$ is also a member of S; and
2. ξ is a member of S, and k is a scalar, then $k\xi$ is also a member of S.

Briefly, we may say that a set S of vectors of $V_n(F)$ is a subspace of $V_n(F)$ it closed w.r.t. the compositions of "addition" and "multiplication with scalars".

Every subspace of V_n contains the zero vector; being the product of any vector with the scalar zero.

Example: $\xi = [a, b, c]$ is a non-zero vector of V_3. Show that the set of vectors $k\xi$ is a subspace of V_3; k being variable.

1.6.5.1 Construction of Subspaces

Theorem 1: The set S, of all linear combinations of a given set of r fixed vectors of V_n is a subspace of V_n.

Def.1 A subspace Spanned by a Set of Vectors. A subspace which arises as a set of **all** linear combinations of any given set of vectors, is said to be spanned by the given set of vectors.

Def. 2. Basis of a Subspace. A set of vectors is said to be a basis of a subspace, if

1. the subspace is spanned by the set, and
2. the set is linearly independent.

It is important to notice that the set of vectors

$e_1 = [1\ 0\ 0 \dots 0]$, $e_2 = [0\ 1\ 0 .. 0]$, ..., $e_n = [0\ 0 \dots 0\ 1]$

is a basis of the vector space V_n, for, if

$k_1 e_1 + k_2 e_2 + \dots + k_n e_n = 0$

then, $k_1 = 0, \dots, k_n = 0$ so that the set is linearly independent and any vector

$$\xi = [a_1, a_2, \dots, a_n]$$

of V_n is expressible as

$$\xi = a_1 e_1 + a_2 e_2 + \dots + a_n e_n$$

Theorem 2: A basis of a subspace, S, can always be selected from a set of vectors which span S.

Let $\{\xi_1, \dots, \xi_r\}$

be a set of vectors which span a subspace S.

If this set is linearly independent, then it is already a basis. In case it is linearly dependent, then some member of the set is a linear combination of the preceding members. Deleting this member, we obtain another set which also spans S.

Continuing in this manner, we shall ultimately, in a finite number of steps, arrive at a basis of S.

NOTE: It has yet to be shown that every subspace, S, of V_n possesses a basis and that the number of vectors in every basis of S, is the same.

1.6.6 Row and column spaces of a matrix. Row and column ranks of a Matrix

Let A, be any $m \times n$ matrix over a field F.

Each of the m rows of A, consisting of n elements, is an n-vector and is as such a member of $V_n(F)$.

The space spanned by the m rows which is a subspace of V_n is called the Row space of the $m \times n$ matrix A.

Again each of the n columns consisting of m elements is an m-vector and is a member of $V_m(F)$.

The space spanned by the n columns which is a subspace of V_m is called the Column space of the $m \times n$ matrix A.

The dimensions of these row and column spaces of matrix are respectively called the Row rank and the Column rank of the matrix.

Theorem 1: Pre-multiplication by a non-singular matrix does not alter the rank of a matrix.

In a similar manner, we may prove that post-multiplication with a non-singular matrix does not alter the column rank of a matrix.

1.6.6.1 Equality of row rank, column rank and rank

Theorem 2: The row rank of a matrix is the same as its rank.

Theorem 3: The column rank of a matrix is the same as its rank.

Corollary 1: The rank of a matrix is equal to the maximum number of its linearly independent rows and also to the maximum number of its linearly independent columns. Thus a matrix of rank r, has a set of r linearly independent rows (columns), such that each of the other rows (columns), is a linear combination of the same.

Corollary 2: The rows and columns of an n-rowed non-singular square matrix form linearly independent sets and are as such bases of V_n.

1.6.6.2 Connection between Rank and Span

A set of n vectors $X_1, X_2, X_3 \ldots X_n$ spans R^n if they are linearly independent which can be checked by constructing a matrix with $X_1, X_2, X_3 \ldots X_n$ as its rows (or columns) and checking that the rank of such a matrix is indeed n. If however the rank is less than n, say m, then the vectors span only a subspace of R^n.

Example: Check if the vectors $[1 \;\; 2 \;\; -1]$, $[2 \;\; 3 \;\; 0]$, $[-1 \;\; 2 \;\; 5]$ span R^3.

Solution:

Step 1: Construct a matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$

Step 2: Find its rank

Since $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 2 & 5 \end{vmatrix} = 1(15-0) - 2(10-0) - 1(4+3)$

$$= 15 - 20 - 7 = -12$$

$$\neq 0$$

So, rank $= 3$

$\therefore$ The vectors are linearly independent and hence span R^3.

Example: Cheek if the vectors $[1 \;\; 2 \;\; 3]$, $[4 \;\; 5 \;\; 6]$ and $[7 \;\; 8 \;\; 9]$ span R^3.

Solution:

Since, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

has a

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-45)$$

$$= 0$$

So its rank $\neq 3$

Since, $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$

$\therefore$ Rank $= 2$

So the vectors $[1 \quad 2 \quad 3]$, $[4 \quad 5 \quad 6]$ and $[7 \quad 8 \quad 9]$ span a subspace of R^3 but do not span R^3.

1.6.7 Orthogonality of Vectors

1. Two vectors X_1 and X_2 are orthogonal if each is non zero and the dot product $X_1' X_2 = 0$.

   Example: The vectors $[a \quad b \quad c]$ and $[d \quad e \quad f]$ are orthogonal if

   $$[a \quad b \quad c]^t \times [d \quad e \quad f] = 0$$

   i.e. $$ad + be + cf = 0$$

   Example: The vectors $[1 \quad 2]$ and $[-2 \quad 1]$ are orthogonal since

   $$[1 \quad 2]^t \times [-2 \quad 1] = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \times [-2 \quad 1]$$

   $$= (1 \times -2) + (2 \times 1)$$

   $$= 0$$

   Example: The vectors $[1 \quad 2 \quad 3]$ and $[-1 \quad 2 \quad 5]$ are not orthogonal since

   $$(1 \times -1) + (2 \times 2) + (3 \times 5) = 18 \neq 0$$

2. Three vectors X_1, X_2 and X_3 are orthogonal if each is non zero and they are pairwise orthogonal.

   i.e. $X_1' X_2 = 0$

   and $X_1' X_3 = 0$

   and $X_2' X_3 = 0$

   Example: The vectors $[1 \quad 0 \quad 0]$, $[0 \quad 1 \quad 0]$ and $[0 \quad 0 \quad 1]$ are orthogonal since

   $$[1 \quad 0 \quad 0]^t [0 \quad 1 \quad 0] = [0 \quad 0 \quad 0]$$

   and $$[0 \quad 1 \quad 0]^t [0 \quad 0 \quad 1] = [0 \quad 0 \quad 0]$$

   and $$[1 \quad 0 \quad 0]^t [0 \quad 0 \quad 1] = [0 \quad 0 \quad 0]$$

3. If n vectors $X_1, X_2, X_3, \ldots X_n$ each of which is in R^n, are orthogonal, then they are surely linearly independent and hence span R^n and therefore form a basis for R^n.

   Example: The vectors $[1 \quad 0 \quad 0]$, $[0 \quad 1 \quad 0]$ and $[0 \quad 0 \quad 1]$ are orthogonal and hence are linearly independent and hence span R^3. They form a basis for R^3.

   The vectors $[0 \quad -2]$, $[-2 \quad 0]$ are orthogonal and hence are linearly independent and span R^2 and form a basis of R^2.

4. The set of n vectors $X_1, X_2, X_3, \ldots X_n$ are called orthonormal if they are
(a) orthogonal and
(b) if each vector has unit length.
The two conditions together can be written as

$$X_i' \cdot X_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

A set of orthogonal vectors X can be converted to a set of orthonormal vectors by devising each vector in the orthogonal set by its length (Euclidean norm $\|X\|$).

Example: The set [1, 2, 1], [2, 1, –4] and [3, –2, 1] is an orthogonal basis of vectors for R^3, since these are pairwise orthogonal and hence are linearly independent and hence span R^3.
To convert this set to an orthonormal basis of R^3, we need to divide each vector by its length

$$\|u_1\| = \sqrt{1+4+1} = \sqrt{6}$$

$$\|u_2\| = \sqrt{4+1+16} = \sqrt{21}$$

$$\|u_3\| = \sqrt{9+4+1} = \sqrt{14}$$

So an orthonormal basis of R^3 is $\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, $\left(\frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{-4}{\sqrt{21}}\right)$ and $\left(\frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$.

1.7 System of Equations

1.7.1 Homogenous Linear Equations

Suppose,

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n = 0 \end{array}\right\} \quad \ldots \text{(i)}$$

is a system of m homogenous equations in n unknowns $x_1, x_2, \ldots x_n$.

Let

$$A = \begin{bmatrix} a_{11}\ a_{12} \ldots\ldots\ldots a_{1n} \\ a_{21}\ a_{22} \ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}\ a_{m2} \ldots\ldots\ldots a_{mn} \end{bmatrix}_{m \times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

where A, X, O are $m \times n$, $n \times 1$, $m \times 1$ matrices respectively. Then obviously we can write the system of equations in the form of a single matrix equation

$$AX = 0 \quad \text{... (ii)}$$

The matrix A is called coefficient matrix of the system of equation (i).

The set $S = \{x_1 = 0, x_2 = 0, \ldots x_n = 0\}$ i.e., $X = 0$ is always a solution of equation (i).

But in general there may be infinite number of solutions to equation (ii).

Again suppose X_1 and X_2 are two solutions of (ii). Then their linear combination, $R_1X_1 + R_2X_2$ when R_1 and R_2 are any arbitrary numbers, is also solution of (ii).

1.7.1.1 Important Results

The number of linearly independent infinite solutions of m homogenous linear equations in n variables, $AX = 0$, is $(n - r)$, where r is rank of matrix A.

$n - r$ is also the number of parameters in the infinite solution.

1.7.1.2 Some important results regarding nature of solutions of equation $AX = O$

Suppose there are m equations in n unknowns. Then the coefficient matrix A will be of the type $m \times n$. Let r be rank of matrix A. Obviously r cannot greater than n. Therefore we have either $r = n$ or $r < n$.

Case 1: Inconsistency: This is not possible in a homogeneous system since such a system is always consistent (since the trivial solution $X = [0, 0, 0\ldots]^t$ always exists for a homogeneous system).

Case 2: Consistent Unique Solution: If $r = n$; the equation $AX = O$ will have only the trivial unique solution $X = [0, 0, 0\ldots]^t$.

Note: That $r = n \Rightarrow |A| \neq 0$ i.e. A is non-singular.

Case 3: Consistent Infinite Solution: If $r < n$ we shall have $n - r$ linearly independent non-trivial infinite solutions. Any linear combination of these $(n - r)$ solutions will also be a solution of $AX = O$. Thus in this case, the equation $AX = O$ will have infinite solutions.

Note: That $r < n \Rightarrow |A| = 0$ i.e. A is a singular matrix.

1.7.2 System of Linear Non-Homogeneous Equations

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n = b_m \end{array}\right\} \quad \text{... (i)}$$

be a system of m non-homogenous equations in n unknown, $x_1, x_2 \ldots x_n$.

If we write

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n} \\ a_{21} & \ldots\ldots\ldots\ldots a_{2n} \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2} \ldots\ldots\ldots a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{m\times 1}$$

where A, X, B are $m \times n$, $n \times 1$, and $m \times 1$ matrices respectively. The above equations can be written in the form of a single matrix equation $AX = B$.

"Any set of values of x_1, x_2 .... x_n which simultaneously satisfy all these equation is called a solutions of the system. When the system of equations has one or more solutions, the equation are said to be consistent otherwise they are said to be inconsistent".

$$\text{The matrix } [A \;\; B] = \begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} & b_1 \\ a_{21} & a_{22} \ldots a_{2n} & b_2 \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \\ a_{m1} & a_{m2} \ldots a_{mn} & b_m \end{bmatrix}$$

is called augmented matrix of the given system of equations.

Condition for Consistency: The system of equations $AX = B$ is consistent i.e., possess a solution if the coefficient matrix A and the augmented matrix $[A \;\; B]$ are of the same rank. i.e. $r(A) = r(A, B)$.

Case 1: Inconsistency: If $r(A) \neq r(A|B)$ the system $AX = B$, has no solution. We say that such a system is inconsistent.

Cases 2 and 3: Consistent systems: Now, when $r(A) = r(A|B) = r$. The system is consistent and has solution. We say, that the rank of the system is r. Now two cases arise.

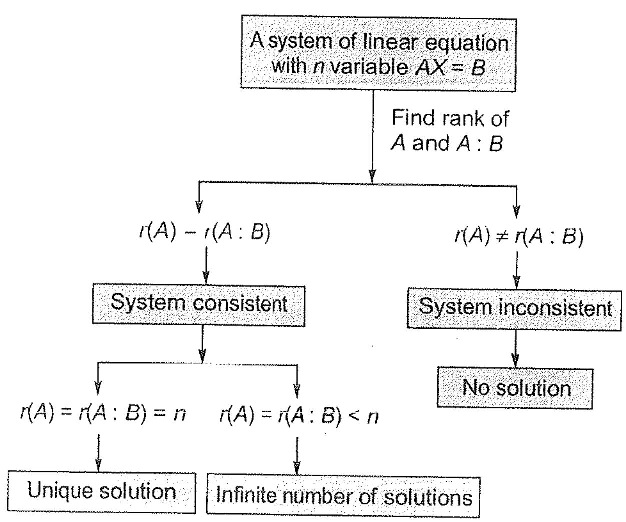
Case 2: Consistent Unique Solution: If $r(A) = r(A|B) = r = n$ (where n is the number of unknown variables of the system), then the system is not only consistent but also has a unique solution.

Case 3: Consistent Infinite solution: If $r(A) = r(A|B) = r < n$, then the system is consistent, but has infinite number of solutions.

In summary we can say the following:

1. If $r(A) \neq r(A|B)$ (Inconsistent and hence, no solution)
2. If $r(A) = r(A|B) = r = n$ (consistent and unique solution)
3. If $r(A) = r(A|B) = r < n$ (consistent and infinite solution)

The rank of a system of equations as well as its solution (if it exists) can be obtained by a procedure called Gauss - Elimination method, which reduces the matrix A to its Echelon form and then by counting the number of non-zero rows in that matrix we get the rank of A.



1.7.3 Homogenous Polynomial

The quadratic forms are defined as a homogenous polynomial of second degree in any number of variables.

Two variables : $ax^2+2hxy + by^2 = Q(x, y)$

Three variables : $ax^2 + by^2 + cz^2 + 2hxy + 2gy^2 + 2fzx = Q(x, y, z)$

n-variable $= Q(x_1, x_2, \ldots x_n)$

Quadratic form can be expressed as a product of matrices.

$$Q(x) = X^TAX$$

$$= [x_1, x_2, x_3]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12} + a_{21})x_1x_2 + (a_{23} + a_{32})x_2x_3 + (a_{31} + a_{13})x_3x_1$$

from here, coefficient of $x_1x_2 = a_{12}$ and a_{21}

coefficient of $x_2x_3 = a_{23}$ and a_{32}

coefficient of $x_1x_3 = a_{13}$ and a_{31}

In general, the coefficient of $x_ix_j = a_{ij}$ and a_{ji}

Let $\quad c_{ij} = c_{ji} = \frac{1}{2}(a_{ij} + a_{ji})$ new coefficient of x_ix_j

$$C = \frac{1}{2}(A + A^T) = \text{Symmetric matrix}$$

If the coefficient matrix C in quadratic form is always symmetric matrix without loss of generality then,

$$X^TAX = c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 + 2c_{12}x_1x_2 + 2c_{23}x_2x_3 + 2c_{31}x_3x_1$$

Matrix A is coefficient matrix.

1.8 Eigenvalues and Eigenvectors

Let $A = [a_{ij}]_{n\times n}$ be any n-rowed square matrix and λ is a scalar. The equation $AX = \lambda X$ is called eigen value problem. We wish to find non zero solutions to X satisfying the eigen value problem, and these non zero solution to X are called as the **eigen vectors** of A. The corresponding λ values are called **eigen values** of A.

1.8.1 Definitions

The matrix $A - \lambda I$ is called **characteristic matrix** of A, where I is the unit matrix of order n. Also the determinant

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12}\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}-\lambda\ldots\ldots\ldots a_{2n} \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ a_{n1} & a_{n2}\ldots\ldots\ldots a_{nn}-\lambda \end{vmatrix}$$

which is ordinary polynomial in λ of degree n is called "**characteristic polynomial of** A". The equation $|A - \lambda I| = 0$ is called "**characteristic equation of** A".

Characteristic Roots: The roots of the characteristic equation are called "**characteristic roots or characteristic values or latent roots or proper values or eigen values**" of the matrix A. The set of eigenvalues of A is called the "**spectrum of** A".

If l is a characteristic root of the matrix A, then if $|A-\lambda I| = 0$, then the matrix $A-\lambda I$ is singular. Therefore there exist a non-zero vector X such that $(A-\lambda I)X = 0$ or $AX = \lambda X$, which is the eigen value problem.

Characteristic Vectors: If λ is a characteristic root of an $n \times n$ matrix A, then a non-zero vector X such that $AX = \lambda X$ is called characteristic vector or eigenvector of A corresponding to characteristic root λ.

1.8.2 Some Results Regarding Characteristic Roots and Characteristic Vectors

1. λ is a characteristic root of a matrix A if there exist a non-zero vector X such that $AX = \lambda X$.
2. If X is a characteristic vector of matrix A corresponding to characteristic value λ, then kX is also a characteristic vector of A corresponding to the same characteristic value λ where k is non-zero vector.
3. If X is a characteristic vector of a matrix A, then X cannot correspond to more than one characteristic values of A.
4. If a matrix A is of size $n \times n$, and if it has n distinct eigen values, then there will be n linearly independent eigen vectors. However, if the n eigen values are not distinct, then there may or may not be n linearly independent eigen vectors.
5. The characteristic roots (Eigen values) of a Hermitian matrix are real.
6. The characteristic roots (Eigen values) of a real symmetric matrix are all real, since every such matrix is Hermitian.
7. Characteristic roots (Eigen values) of a skew Hermitian matrix are either pure imaginary or zero.
8. The characteristic roots (Eigen values) of a real skew symmetric matrix are either pure imaginary or zero, for every such matrix is skew Hermitian.
9. The characteristic roots (Eigen values) of a unitary matrix are of unit modulus. i.e., $|\lambda| = 1$.
10. The characteristic roots (Eigen values) of an orthogonal matrix is also of unit modulus, since every such matrix is unitary.

1.8.3 Process of Finding the Eigenvalues and Eigenvectors of a Matrix

Let $A = [a_{ij}]_{n \times n}$ be a square matrix of order n, first we should write the characteristic equation of the matrix A. i.e., the equation $|A-\lambda I| = 0$. This equation will be of degree n in λ. So it will have n roots. These n roots will be the n eigenvalues of the matrix A.

If λ_1 is an eigenvalue of A, the corresponding eigenvectors of A will be given by the non-zero vectors $X_1 = [x_1, x_2, \ldots\ldots\ldots\ldots\ldots x_n]'$ satisfying the equations $AX_1 = \lambda_1 X_1$ or $[A - \lambda_1 I]X_1 = 0$

1.8.4 Properties of Eigen Values

1. If $\lambda_1, \lambda_2 \ldots\ldots\ldots\ldots \lambda_n$ are the eigenvalues of A, then $k\lambda_1, k\lambda_2 \ldots\ldots\ldots\ldots k\lambda_n$ are eigenvalues of kA.
2. the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

   i.e. if $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ are two eigen value of A, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ are the eigen value of A^{-1}.

3. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A, then $\lambda_1^k, \lambda_2^k, \ldots \lambda_n^k$ are the eigen values of A^k.

4. If $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are the eigen values of a non-singular matrix A, then $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2} \ldots \frac{|A|}{\lambda_n}$ are the eigen values of Adj A.
5. Eigen values of A = Eigen values of A^T.
6. Maximum no. of distinct eigen values = size of A.

7. Sum of eigen values = Trace of A = Sum of diagonal elements.
8. Product of eigen values = $|A|$ (i.e. At least one eigen value is zero if A is singular).
9. In a triangular and diagonal matrix, eigen values are diagonal elements themselves.
10. Similar matrices have same eigen values. Two matrices A and B are said to be similar if there exists a non singular matrix P such that $B = P^{-1} AP$.
11. If A and B are two matrices of same order then the matrix AB and BA will have same characteristic roots.

1.8.5 The Cayley-Hamilton Theorem

This theorem is an interesting one that provides an alternative method for finding the inverse of a matrix A. Also any positive integral power of A can be expressed, using this theorem, as a linear combination of those of lower degree. We give below the statement of the theorem without proof.

Statement of the Theorem: Every square matrix satisfies its own characteristic equation.

This means that, if $c_0 \lambda^n + c_1 \lambda^{n-1} + ... + c_{n-1} \lambda + c_n = 0$ is the characteristic equation of a square matrix A of order n, then

$$c_0 A^n + c_1 A^{n-1} + ... + c_{n-1} A + c_n I = 0 \qquad ...(i)$$

NOTE: When λ is replaced by A in the characteristic equation, the constant term c_n should be replaced by $c_n I$ to get the result of Cayley-Hamilton theorem, where I is the unit matrix of order n.
Also 0 in the R.H.S. of (i) is a null matrix of order n.

1.8.5.1 Finding Inverse off a Matrix by using Cayley-Hamilton Theorem

Example: Find A^{-1} by Cayley-Hamilton theorem, if

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(2-\lambda) - 12 = 0$$

$$\Rightarrow \quad \lambda^2 - 3\lambda - 10 = 0$$

By Cayley-Hamilton theorem

$$A^2 - 3A - 10 I = 0$$

$$\Rightarrow \quad I = \frac{1}{10}[A^2 - 3A]$$

Pre-multiplying by A^{-1} we get

$$A^{-1} = \frac{1}{10}[A - 3I] = \frac{1}{10}\left(\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{10}\begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{3}{10} \\ \frac{2}{5} & -\frac{1}{10} \end{bmatrix}$$

1.8.5.2 Finding Higher Powers of a Matrix in Terms of its Lower Powers

Example: If $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, express A^5 as a linear polynomial in A.

Characteristic equation is

$$\lambda^2 - 3\lambda - 10 = 0$$

By Cayley-Hamilton theorem,

$$A^2 - 3A - 10I = 0$$

$$\Rightarrow \qquad A^2 = 3A + 10I$$

If A is $n \times n$ matrix, any power of A can be written as a polynomial of maximum degree $n - 1$.

Here, since A is 2×2, we can write any power of A as a polynomial of degree 1, i.e., a linear polynomial of A, as shown below.

$$A^2 = 3A + 10I \qquad \dots \text{(i)}$$

$$A^3 = 3A^2 + 10A \qquad \dots \text{(ii)}$$

Substituting (i), again in (ii), we get

$$A^3 = (3A + 10I) + 10A = 19A + 30I$$

Now $$A^4 = 19A^2 + 30A \qquad \dots \text{(iii)}$$

Again we substitute equation (i) in equation (iii) to get,

$$A^4 = 19(A + 10I) + 30A = 87A + 190I$$

Now $$A^5 = 87A^2 + 190A \qquad \dots \text{(iv)}$$

Again substituting equation (i) in equation (iv) we get,

$$A^5 = 87(3A + 10I) + 190A = 451A + 870I$$

Which is the desired result.

1.8.5.3 Expressing Any Matrix Polynomial in A of size $n \times n$ as a Polynomial of Degree $n - 1$ in A by using Cayley-Hamilton Theorem

Example: Process to express a polynomial of a 2×2 Matrix as a linear polynomial in A:

Example: Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ Express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A.

Step 1: First of all write the characteristic equation of A.

In this case.

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda) + 1$$

$$= \lambda^2 - 5\lambda + 7$$

Thus the characteristic equation of A is $|A - \lambda I| = 0$

i.e., is $$\lambda^2 - 5\lambda + 7 = 0 \qquad \dots \text{(i)}$$

Step 2: By Cayley Hamilton theorem, matrix A satisfies the equation (i). Therefore, putting $A = \lambda$ in (i) we get

$$A^2 - 5A + 7 = 0$$

$$\Rightarrow \qquad A^2 = 5A - 7I \qquad \dots \text{(ii)}$$

Step 3: Find the A^5, A^4, A^3 with the help of (ii). In this case

$$A^3 = 5A^2 - 7A$$

$$\Rightarrow \quad A^4 = 5A^3 - 7A^2$$

$$\Rightarrow \quad A^4 = 5A^4 - 7A^3$$

$$2A^5 - 3A^4 + A^2 - 4I = 2(5A^4 - 7A^3) - 3A^4 + A^2 - 4I$$

$$= 7A^4 - 14A^3 + A^2 - 4I = 7[5A^3 - 7A^2) - 14A^3 + A^2 - 4I$$

$$= 21A^3 - 48A^2 - 4I = 21(5A^2 - 7A) - 48A^2 - 4I$$

$$= 57A^2 - 147A - 4I = 57[5A - 7I] - 147A - 4I = 138A - 403I$$

$\Rightarrow$ which is a linear polynomial in A.

1.8.6 Similar Matrices

Two matrices A and B are said to be similar, if there exists a non-singular matrix P such that $B = P^{-1}AP$.

1.8.6.1 Properties of Similar Matrices

1. A is always similar to A.
   Proof: Since $A = I^{-1}AI$ and I is always non-singular, therefore A is similar to A.
2. If A is similar to B then B is also similar to A.
   Proof: If A is similar to B then $B = P^{-1}AP$ (where P is non-singular)
   Pre-multiplying above equation by P and post-multiplying by P^{-1}, we get $PBP^{-1} = PP^{-1}APP^{-1} = A$.
   i.e., $A = PBP^{-1}$
   So B is also similar to A.
3. If A is similar to B and B is similar to C then A is similar to C.
   Proof: A is similar to $B \Rightarrow B = P^{-1}AP$ ...(i)
   B is similar to $C \Rightarrow C = Q^{-1}BQ$ ...(ii)
   Substituting eq. (i) and (ii) we get
   $C = Q^{-1}P^{-1}APQ$
   Now putting $PQ = D$, we get $C = D^{-1}AD$, which proves that A is similar to C.
4. Combining properties 1, 2 and 3 above we can say that the similarity relation between matrices is reflexive, symmetric and transitive and hence an equivalence relation.
5. Similar matrices have the same eigenvalues.

1.8.7 Diagonalisation of a Matrix

Finding the a matrix D which is a diagonal matrix and which is similar to A is called diagonalisation i.e., we wish to find a non-singular matrix M such that

$$A = M^{-1}DM$$

where D is a diagonal matrix.

Condition for a Matrix to be Diagonalisable:

1. A necessary and sufficient condition for a matrix $A_{n \times n}$ to be diagonalisable is that the matrix must have n linearly independent eigen vectors.
2. A sufficient (but not necessary) condition for a matrix $A_{n \times n}$ to be diagonalisable is that the matrix must have n linearly independent eigen values.
   This is because if a matrix has n linearly independent eigen values then it surely has n linearly independent eigen vectors (although the converse of this is not true).

When A is diagonalisable $A = M^{-1}DM$, where the matrix D is a diagonal matrix constructed using the eigen values of A as its diagonal elements. Also the corresponding matrix M can be obtained by constructing a $n \times n$ matrix whose columns are the eigen vectors of A.

Practical application of Diagonalisation:

One of the uses of diagonalisation is for computing higher powers of a matrix efficiently.

If $A = M^{-1}DM$ then $A^n = M^{-1}D^n M$

The above property makes it easy to compute higher powers of a matrix A, since computing D^n is much more easy compared with computing A^n.

■■■■

Previous GATE and ESE Questions

Q.1 Given Matrix $[A] = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, the rank of the matrix is

(a) 4 (b) 3
(c) 2 (d) 1

[CE, GATE-2003, 1 mark]

Q.2 Consider the system of simultaneous equations

$$x + 2y + z = 6$$
$$2x + y + 2z = 6$$
$$x + y + z = 5$$

This system has

(a) unique solution
(b) infinite number of solutions
(c) no solution
(d) exactly two solutions

[ME, GATE-2003, 2 marks]

Q.3 Consider the following system of linear equations

$$\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$

Notice that the second and the third columns of the coefficient matrix are linearly dependent. For how many values of α, does this system of equations have infinitely many solutions?

(a) 0 (b) 1
(c) 2 (d) infinitely many

[CS, GATE-2003, 2 marks]

Q.4 For the matrix $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ the eigen values are

(a) 3 and –3 (b) –3 and – 5
(c) 3 and 5 (d) 5 and 0

[ME, GATE-2003, 1 mark]

Q.5 For which value of x will the matrix given below become singular?

$$\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

(a) 4 (b) 6
(c) 8 (d) 12

[ME, GATE-2004, 2 marks]

Q.6 Let A, B, C, D be $n \times n$ matrices, each with non-zero determinant, If $ABCD = I$, then B^{-1} is

(a) $D^{-1} C^{-1} A^{-1}$
(b) CDA
(c) ADC
(d) does not necessarily exist

[CS, GATE-2004, 1 mark]

Q.7 How many solutions does the following system of linear equations have?

$-x + 5y = -1$; $x - y = 2$; $x + 3y = 3$

(a) infinitely many (b) two distinct solutions
(c) unique (d) none

[CS, GATE-2004, 2 marks]

Q.8 The eigen values of the matrix $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$

(a) are 1 and 4
(b) are −1 and 2
(c) are 0 and 5
(d) cannot be determined

[CE, GATE-2004, 2 marks]

Q.9 The sum of the eigen values of the matrix given below is $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

(a) 5
(b) 7
(c) 9
(d) 18

[ME, GATE-2004, 1 mark]

Q.10 Consider the matrices $X_{(4 \times 3)}$, $Y_{(4 \times 3)}$ and $P_{(2 \times 3)}$. The order of $[P(X^TY)^{-1} P^T]^T$ will be

(a) (2×2)
(b) (3×3)
(c) (4×3)
(d) (3×4)

[CE, GATE-2005, 1 mark]

Q.11 Given an orthogonal matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \ \left[AA^T\right]^{-1} \text{ is}$$

(a) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$
(b) $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
(d) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

[EC, GATE-2005, 2 marks]

Q.12 If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, then top row of R^{-1} is

(a) [5 6 4]
(b) [5 −3 1]
(c) [2 0 −1]
(d) [2 −1 1/2]

[EE, GATE-2005, 2 marks]

Q.13 Let, $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix}$.

Then $(a + b) =$

(a) $\frac{7}{20}$
(b) $\frac{3}{20}$
(c) $\frac{19}{60}$
(d) $\frac{11}{20}$

[EC, GATE-2005, 2 marks]

Q.14 Consider a non-homogeneous system of linear equations representing mathematically an over-determined system. Such a system will be

(a) consistent having a unique solution
(b) consistent having many solutions
(c) inconsistent having a unique solution
(d) inconsistent having no solution

[CE, GATE-2005, 1 mark]

Q.15 A is a 3 × 4 real matrix and $Ax = b$ is an inconsistent system of equations. The highest possible rank of A is

(a) 1
(b) 2
(c) 3
(d) 4

[ME, GATE-2005, 1 mark]

Q.16 In the matrix equation $Px = q$, which of the following is a necessary condition for the existence of at least one solution for the unknown vector x

(a) Augmented matrix $[Pq]$ must have the same rank as matrix P
(b) Vector q must have only non-zero elements
(c) Matrix P must be singular
(d) Matrix P must be square

[EE, GATE-2005, 1 mark]

Q.17 Consider the following system of equations in three real variables x_1, x_2 and x_3

$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 - 2x_2 + 5x_3 = 2$$
$$-x_1 - 4x_2 + x_3 = 3$$

This system of equations has

(a) no solution
(b) a unique solution
(c) more than one but a finite number of solutions
(d) an infinite number of solutions

[CS, GATE-2005, 2 marks]

Q.18 Which one of the following is an eigen vector of the matrix $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

[ME, GATE-2005, 2 marks]

~~**Q.19** For the matrix $A = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigen~~ values is equal to –2. Which of the following is an eigen vector?

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

[EE, GATE-2005, 2 marks]

Q.20 Given the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$, the eigen vector is

(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

[EC, GATE-2005, 2 marks]

Q.21 What are the eigen values of the following 2×2 matrix?

$$\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

(a) –1 and 1 (b) 1 and 6
(c) 2 and 5 (d) 4 and –1

[CS, GATE-2005, 2 marks]

Q.22 Consider the system of equations $A_{(n \times n)}\, x_{(n \times l)} = \lambda_{(n \times l)}$ where, λ is a scalar. Let (λ_i, x_i) be an eigen-pair of an eigen value and its corresponding eigen vector for real matrix A. Let I be a $(n \times n)$ unit matrix. Which one of the following statement is NOT correct?

(a) For a homogeneous $n \times n$ system of linear equations, $(A - \lambda I)x = 0$ having a nontrivial solution, the rank of $(A - \lambda I)$ is less than n
(b) For matrix A^m, m being a positive integer, (λ_i^m, x_i^m) will be the eigen-pair for all i
(c) If $A^T = A^{-1}$, then $|\lambda_i| = 1$ for all i
(d) If $A^T = A$, then λ_i is real for all i

[CE, GATE-2005, 2 marks]

Q.23 Multiplication of matrices E and F is G. Matrices E and G are

$$E \equiv \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

What is the matrix F?

(a) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} \cos\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

[ME, GATE-2006, 2 marks]

Q.24 Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I

A. Singular matrix
B. Non-square matrix
C. Real symmetric
D. Orthogonal matrix

List-II

1. Determinant is not defined
2. Determinant is always one
3. Determinant is zero
4. Eigen values are always real
5. Eigen values are not defined

Codes:

	A	B	C	D
(a)	3	1	4	2
(b)	2	3	4	1
(c)	3	2	5	4
(d)	3	4	2	1

[ME, GATE-2006, 2 marks]

Q.25 The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is

(a) 0
(b) 1
(c) 2
(d) 3

[EC, GATE-2006, 1 mark]

Q.26 $P = \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}^T$, $Q = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}^T$ and $R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T$ are three vectors. An orthogonal set of vectors having a span that contains P, Q, R is

(a) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ -11 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$

(c) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$

(d) $\begin{bmatrix} 4 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} 1 \\ 31 \\ 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$

[EE, GATE-2006, 2 marks]

Q.27 The following vector is linearly dependent upon the solution to the previous problem

(a) $\begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix}$
(b) $\begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix}$
(c) $\begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}$
(d) $\begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix}$

[EE, GATE-2006, 2 marks]

Q.28 Solution for the system defined by the set of equations $4y + 3z = 8$; $2x - z = 2$ and $3x + 2y = 5$ is

(a) $x = 0$; $y = 1$; $z = 4/3$
(b) $x = 0$; $y = 1/2$; $z = 2$
(c) $x = 1$; $y = 1/2$; $z = 2$
(d) non-existent

[CE, GATE-2006, 1 mark]

Q.29 For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ the eigen value corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is

(a) 2
(b) 4
(c) 6
(d) 8

[EC, GATE-2006, 2 marks]

Q.30 For a given matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$, one of the eigen values is 3. The other two eigen values are

(a) 2, –5
(b) 3, –5
(c) 2, 5
(d) 3, 5

[CE, GATE-2006, 2 marks]

Q.31 Eigen values of a matrix $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5 and 1.

What are the eigen values of the matrix $S^2 = SS$?

(a) 1 and 25
(b) 6 and 4
(c) 5 and 1
(d) 2 and 10

[ME, GATE-2006, 2 marks]

Q.32 The eigen values and the corresponding eigen vectors of a 2 × 2 matrix are given by

Eigen value	Eigen vector
$\lambda_1 = 8$	$v_1 = \begin{bmatrix}1\\1\end{bmatrix}$
$\lambda_2 = 4$	$v_2 = \begin{bmatrix}1\\-1\end{bmatrix}$

The matrix is

(a) $\begin{bmatrix}6 & 2\\2 & 6\end{bmatrix}$ (b) $\begin{bmatrix}4 & 6\\6 & 4\end{bmatrix}$

(c) $\begin{bmatrix}2 & 4\\4 & 2\end{bmatrix}$ (d) $\begin{bmatrix}4 & 8\\8 & 4\end{bmatrix}$

[EC, GATE-2006, 2 marks]

Q.33 $[A]$ is square matrix which is neither symmetric nor skew-symmetric and $[A]^T$ is its transpose. The sum and difference of these matrices are defined as $[S] = [A] + [A]^T$ and $[D] = [A] - [A]^T$, respectively. Which of the following statements is TRUE?

(a) Both $[S]$ and $[D]$ are symmetric
(b) Both $[S]$ and $[D]$ are skew-symmetric
(c) [S] is skew-symmetric and $[D]$ is symmetric
(d) [S] is symmetric and $[D]$ is skew-symmetric

[CE, GATE-2007, 1 mark]

Q.34 The inverse of the 2 × 2 matrix $\begin{bmatrix}1 & 2\\5 & 7\end{bmatrix}$ is

(a) $\frac{1}{3}\begin{bmatrix}-7 & 2\\5 & -1\end{bmatrix}$ (b) $\frac{1}{3}\begin{bmatrix}7 & 2\\5 & 1\end{bmatrix}$

(c) $\frac{1}{3}\begin{bmatrix}7 & -2\\-5 & 1\end{bmatrix}$ (d) $\frac{1}{3}\begin{bmatrix}-7 & -2\\-5 & -1\end{bmatrix}$

[CE, GATE-2007, 2 marks]

Q.35 $X = [x_1, x_2, \ldots x_n]^T$ is an n-tuple nonzero vector. The n × n matrix $V = XX^T$

(a) has rank zero (b) has rank 1
(c) is orthogonal (d) has rank n

[EE, GATE-2007, 1 mark]

Q.36 It is given that $X_1, X_2, \ldots X_M$ are M non-zero, orthogonal vectors. The dimension of the vector space spanned by the 2M vectors $X_1, X_2, \ldots X_M, -X_1, -X_2, \ldots -X_M$ is

(a) $2M$
(b) $M + 1$
(c) M
(d) dependent on the choice of $X_1, X_2, \ldots X_M$

[EC, GATE-2007, 2 marks]

Q.37 Consider the set of (column) vectors defined by $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0$, where $x^T = [x_1, x_2, x_3]^T\}$. Which of the following is TRUE?

(a) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a basis for the subspace X.
(b) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a linearly independent set, but it does not span X and therefore is not a basis of X
(c) X is not a subspace for R^3
(d) None of the above

[CS, GATE-2007, 2 marks]

Q.38 For what values of α and β, the following simultaneous equations have an infinite number of solutions?

$$x + y + z = 5$$
$$x + 3y + 3z = 9$$
$$x + 2y + \alpha z = \beta$$

(a) 2, 7 (b) 3, 8
(c) 8, 3 (d) 7, 2

[CE, GATE-2007, 2 marks]

Q.39 The number of linearly independent eigen vectors of $\begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) infinite

[ME, GATE-2007, 2 marks]

Q.40 The linear operation $L(x)$ is defined by the cross product $L(x) = b \times X$, where $b = [0\ 1\ 0]^T$ and $X = [x_1 x_2 x_3]^T$ are three dimensional vectors. The 3 × 3 matrix M of this operation satisfies

$$L(x) = M\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Then the eigen values of M are

(a) 0, +1, −1 (b) 1, −1, 1
(c) $i, -i$, 1 (d) $i, -i$, 0

[EE, GATE-2007, 2 marks]

Q.41 The minimum and the maximum eigen values of the matrix $\begin{bmatrix}1 & 1 & 3\\1 & 5 & 1\\3 & 1 & 1\end{bmatrix}$ are −2 and 6, respectively. What is the other eigen value?

(a) 5 (b) 3
(c) 1 (d) −1

[CE, GATE-2007, 1 mark]

Q.42 If a square matrix A is real and symmetric, then the eigen values
(a) are always real
(b) are always real and positive
(c) are always real and non-negative
(d) occur in complex conjugate pairs

[ME, GATE-2007, 1 mark]

Statement for Linked Answer Question 43 and 44.

Cayley-Hamilton Theorem states that a square matrix satisfies its own characteristic equation. Consider a matrix

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

Q.43 A satisfies the relation
(a) $A + 3I + 2A^{-1} = 0$ (b) $A^2 + 2A + 2I = 0$
(c) $(A + I)(A + 2I) = I$ (d) $\exp(A) = 0$

[EE, GATE-2007, 2 marks]

Q.44 A^9 equals
(a) $511A + 510I$ (b) $309A + 104I$
(c) $154A + 155I$ (d) $\exp(9A)$

[EE, GATE-2007, 2 marks]

Q.45 The product of matrices $(PQ)^{-1}P$ is
(a) P^{-1} (b) Q^{-1}
(c) $P^{-1}Q^{-1}P$ (d) PQP^{-1}

[CE, GATE-2008, 1 mark]

Q.46 A is $m \times n$ full rank matrix with $m > n$ and I is an identity matrix. Let matrix $A' = (A^TA)^{-1}A^T$, Then, which one of the following statement is TRUE?
(a) $AA'A = A$ (b) $(AA')^2 = A$
(c) $AA'A = I$ (d) $AA'A = A'$

[EE, GATE-2008, 2 marks]

Q.47 If the rank of a (5×6) matrix Q is 4, then which one of the following statements is correct?
(a) Q will have four linearly independent rows and four linearly independent columns
(b) Q will have four linearly independent rows and five linearly independent columns
(c) QQ^T will be invertible
(d) Q^TQ will be invertible

[EE, GATE-2008, 1 mark]

Q.48 The following simultaneous equations

$x + y + z = 3$

$x + 2y + 3z = 4$

$x + 4y + kz = 6$

will NOT have a unique solution for k equal to
(a) 0 (b) 5
(c) 6 (d) 7

[CE, GATE-2008, 2 marks]

Q.49 For what value of a, if any, will the following system of equations in x, y and z have a solution?

$$2x + 3y = 4 \,;\; x + y + z = 4 \,;\; x + 2y - z = a$$

(a) Any real number (b) 0
(c) 1 (d) There is no such value

[ME, GATE-2008, 2 marks]

Q.50 The system of linear equations

$4x + 2y = 7$

$2x + y = 6$

has
(a) a unique solution
(b) no solution
(c) an infinite number of solutions
(d) exactly two distinct solutions

[EC, GATE-2008, 1 mark]

Q.51 The following system of equations

$$x_1 + x_2 + 2x_3 = 1$$
$$x_1 + 2x_3 + 3x_3 = 2$$
$$x_1 + 4x_2 + ax_3 = 4$$

has a unique solution. The only possible value(s) for a is/are
(a) 0
(b) either 0 or 1
(c) one of 0,1 or –1
(d) any real number other than 5

[CS, GATE-2008, 1 mark]

Q.52 The Eigen values of the matrix $[P] = \begin{bmatrix} 4 & 5 \\ 2 & -5 \end{bmatrix}$ are
(a) –7 and 8 (b) –6 and 5
(c) 3 and 4 (d) 1 and 2

[CE, GATE-2008, 2 marks]

Q.53 The eigen vectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is $a + b$?
(a) 0 (b) 1/2
(c) 1 (d) 2

[ME, GATE-2008, 2 marks]

Q.54 How many of the following matrices have an eigen value 1?

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$

(a) one (b) two
(c) three (d) four

[CS, GATE-2008, 2 marks]

Q.55 The matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & p \end{bmatrix}$ has one eigen value equal to 3. The sum of the other two eigen values is

(a) p (b) $p-1$
(c) $p-2$ (d) $p-3$

[ME, GATE-2008, 1 mark]

Q.56 All the four entries of the 2 × 2 matrix $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ are nonzero, and one of its eigen values is zero. Which of the following statements is true?

(a) $p_{11}p_{22} - p_{12}p_{21} = 1$
(b) $p_{11}p_{22} - p_{12}p_{21} = -1$
(c) $p_{11}p_{22} - p_{12}p_{21} = 0$
(d) $p_{11}p_{22} + p_{12}p_{21} = 0$

[EC, GATE-2008, 1 mark]

Q.57 The characteristic equation of a (3 × 3) matrix P is defined as

$a(\lambda) = |P - \lambda I| = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$

If I denotes identity matrix, then the inverse of matrix P will be

(a) $(P^2 + P + 2I)$ (b) $(P^2 + P + 1)$
(c) $-(P^2 + P + 1)$ (d) $-(P^2 + P + 2I)$

[EE, GATE-2008, 1 mark]

Q.58 A square matrix B is skew-symmetric if

(a) $B^T = -B$ (b) $B^T = B$
(c) $B^{-1} = B$ (d) $B^{-1} = B^T$

[CE, GATE-2009, 1 mark]

Q.59 For a matrix $[M] = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$, the transpose of the matrix is equal to the inverse of the matrix, $[M]^T = [M]^{-1}$. The value of x is given by

(a) $-\frac{4}{5}$ (b) $-\frac{3}{5}$
(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

[ME, GATE-2009, 1 mark]

Q.60 The trace and determinant of a 2 × 2 matrix are known to be –2 and –35 respectively. It eigen values are

(a) –30 and –5 (b) –37 and –1
(c) –7 and 5 (d) 17.5 and –2

[EE, GATE-2009, 1 mark]

Q.61 The eigen values of the following matrix are

$\begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$

(a) $3, 3 + 5j, 6 - j$ (b) $-6 + 5j, 3 + j, 3 - j$
(c) $3 + j, 3 - j, 5 + j$ (d) $3, -1 + 3j, -1-3j$

[EC, GATE-2009, 2 marks]

Q.62 The inverse of the matrix $\begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$ is

(a) $\frac{1}{12}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$

(b) $\frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

(c) $\frac{1}{14}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$

(d) $\frac{1}{14}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

[CE, GATE-2010, 2 marks]

Q.63 For the set of equations

$x_1 + 2x_2 + x_3 + 4x_4 = 2$
$3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$

the following statement is true:

(a) Only the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$ exists
(b) There are no solution
(c) A unique non-trivial solution exists
(d) Multiple non-trivial solutions exist

[EE, GATE-2010, 2 marks]

Q.64 One of the eigen vectors of the matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ is

(a) $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ (b) $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$

(c) $\begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$ (d) $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

[ME, GATE-2010, 2 marks]

Q.65 An eigen vector of $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is

(a) $[-1 \quad 1 \quad 1]^T$ (b) $[1 \quad 2 \quad 1]^T$

(c) $[1 \quad -1 \quad 2]^T$ (d) $[2 \quad 1 \quad -1]^T$

[EE, GATE-2010, 2 marks]

Q.66 The eigen values of a skew-symmetric matrix are

(a) always zero

(b) always pure imaginary

(c) either zero or pure imaginary

(d) always real **[EC, GATE-2010, 1 mark]**

Q.67 Consider the following matrix.

$$A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$$

If the eigen values of A are 4 and 8, then

(a) $x = 4,\ y = 10$ (b) $x = 5,\ y = 8$

(c) $x = -3,\ y = 9$ (d) $x = -4,\ y = 10$

[CS, GATE-2010, 2 marks]

Q.68 Consider the following system of equations

$2x_1 + x_2 + x_3 = 0$

$x_2 - x_3 = 0$

$x_1 + x_2 = 0$

This system has

(a) a unique solution

(b) no solution

(c) infinite number of solutions

(d) five solutions **[ME, GATE-2011, 2 marks]**

Q.69 The system of equations

$$x + y + z = 6$$
$$x + 4y + 6z = 20$$
$$x + 4y + \lambda z = \mu$$

has **NO** solution for values of λ and μ given by

(a) $\lambda = 6,\ \mu = 20$ (b) $\lambda = 6,\ \mu \neq 20$

(c) $\lambda \neq 6,\ \mu = 20$ (d) $\lambda \neq 6,\ \mu \neq 20$

[EC, GATE-2011, 2 mark]

Q.70 Eigen values of a real symmetric matrix are always

(a) positive (b) negative

(c) real (d) complex

[ME, GATE-2011, 1 mark]

Q.71 Consider the matrix as given below:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix}$$

Which one of the following options provides the CORRECT values of the eigen values of the matrix?

(a) 1, 4, 3 (b) 3, 7, 3

(c) 7, 3, 2 (d) 1, 2, 3

[CS, GATE-2011, 2 marks]

Q.72 The eigen values of matrix $\begin{bmatrix} 9 & 5 \\ 5 & 8 \end{bmatrix}$ are

(a) –2.42 and 6.86 (b) 3.48 and 13.53

(c) 4.70 and 6.86 (d) 6.86 and 9.50

[CE, GATE-2012, 2 marks]

Q.73 $x + 2y + z = 4$

$2x + y + 2z = 5$

$x - y + z = 1$

The system of algebraic given below has

(a) A unique solution of $x = 1$, $y = 1$ and $z = 1$

(b) only the two solutions of ($x = 1$, $y = 1$, $z = 1$) and ($x = 2$, $y = 1$, $z = 0$)

(c) infinite number of solutions

(d) no feasible solution

[ME, GATE-2012, 2 marks]

Q.74 Let A be the 2×2 matrix with elements

$$a_{11} = a_{12} = a_{21} = +1 \text{ and } a_{22} = -1$$

Then the eigen values of the matrix A^{19} are

(a) 1024 and – 1024

(b) $1024\sqrt{2}$ and $-1024\sqrt{2}$

(c) $4\sqrt{2}$ and $-4\sqrt{2}$

(d) $512\sqrt{2}$ and $-512\sqrt{2}$

[CS, GATE-2012, 1 marks]

Q.75 For the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, ONE of the normalized eigen vectors is given as

(a) $\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$

(c) $\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

[ME, GATE-2012, 2 marks]

Q.76 Given that

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

the value A^3 is

(a) $15A + 12I$ (b) $19A + 30I$

(c) $17A + 15I$ (d) $17A + 21I$

[EC, EE, IN GATE-2012, 2 marks]

Q.77 There are three matrixes $P(4 \times 2)$, $Q(2 \times 4)$ and $R(4 \times 1)$. The minimum of multiplication required to compute the matrix PQR is

[CE, GATE-2013, 1 Mark]

Q.78 Let A be an $m \times n$ matrix and B an $n \times m$ matrix. It is given that determinant $(I_m + AB)$ = determinant $(I_n + BA)$, where I_k is the $k \times k$ identity matrix. Using the above property, the determinant of the matrix given below is

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(a) 2 (b) 5

(c) 8 (d) 16

[EC, GATE-2013, 2 Marks]

Q.79 Which one of the following does NOT equal

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} ?$$

(a) $\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$

(d) $\begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$

[CS, GATE-2013, 1 Mark]

Q.80 The dimension of the null space of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$ is

(a) 0 (b) 1

(c) 2 (d) 3

[IN, GATE-2013 : 1 mark]

Q.81 Choose the **CORRECT** set of functions, which are linearly dependent.

(a) $\sin x$, $\sin^2 x$ and $\cos^2 x$

(b) $\cos x$, $\sin x$ and $\tan x$

(c) $\cos 2x$, $\sin^2 x$ and $\cos^2 x$

(d) $\cos 2x$, $\sin x$ and $\cos x$

[ME, GATE-2013, 1 Mark]

Q.82 The equation $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has

(a) no solution

(b) only one solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(c) non-zero unique solution

(d) multiple solutions **[EE, GATE-2013, 1 Mark]**

Q.83 One pair of eigen vectors corresponding to the two eigen values of the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 \\ -j \end{bmatrix}, \begin{bmatrix} j \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} j \\ 1 \end{bmatrix}$

[IN, GATE-2013 : 2 marks]

Q.84 The eigen values of a symmetric matrix are all

(a) complex with non-zero positive imaginary part

(b) complex with non-zero negative imaginary part

(c) real

(d) pure imaginary

[ME, GATE-2013, 1 Mark]

Q.85 A matrix has eigen values –1 and –2. The corresponding eigen vectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively. The matrix is

(a) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

[EE, GATE-2013, 2 Marks]

Q.86 The minimum eigen value of the following matrix is

$$\begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}$$

(a) 0 (b) 1
(c) 2 (d) 3

[EC, GATE-2013, 1 Mark]

Q.87 Real matrices $[A]_{3\times1}$, $[B]_{3\times3}$, $[C]_{3\times5}$, $[D]_{5\times3}$, $[E]_{5\times5}$ and $[F]_{5\times1}$ are given. Matrices $[B]$ and $[E]$ are symmetric.

Following statements are made with respect to these matrices.

1. Matrix product $[F]^T[C]^T[B][C][F]$ is a scalar.
2. Matrix product $[D]^T[F][D]$ is always symmetric.

With reference to above statements, which of the following applies?

(a) Statement 1 is true but 2 is false
(b) Statement 1 is false but 2 is true
(c) Both the statements are true
(d) Both the statements are false

[CE, GATE-2004, 1 mark]

Q.88 Given the matrices $J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ and

$K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, the product $K^T JK$ is ______.

[CE, GATE-2014 : 1 Mark]

Q.89 With reference to the conventional Cartesian (x, y) coordinate system, the vertices of a triangle have the following coordinates; $(x_1, y_1) = (1, 0)$; $(x_2, y_2) = (2, 2)$; $(x_3, y_3) = (4, 3)$. The area of the triangle is equal to

(a) $\frac{3}{2}$ (b) $\frac{3}{4}$

(c) $\frac{4}{5}$ (d) $\frac{5}{2}$

[CE, GATE-2014 : 1 Mark]

Q.90 Which one of the following equations is a correct identity for arbitrary 3×3 real matrices P, Q and R?

(a) $P(Q + R) = PQ + RP$
(b) $(P - Q)^2 = P^2 - 2PQ + Q^2$
(c) $\det(P + Q) = \det P + \det Q$
(d) $(P + Q)^2 = P^2 + PQ + QP + Q^2$

[ME, GATE-2014 : 1 Mark]

Q.91 Which one of the following statements is true for all real symmetric matrices?

(a) All the eigen values are real
(b) All the eigen values are positive.
(c) All the eigen values are distinct
(d) Sum of all the eigen values is zero.

[EE, GATE-2014 : 1 Mark]

Q.92 For matrices of same dimension M, N and scalar c, which one of these properties DOES NOT ALWAYS hold?

(a) $(M^T)^T = M$
(b) $(cM)^T = c(M)^T$
(c) $(M + N)^T = M^T + N^T$
(d) $MN = NM$

[EC, GATE-2014 : 1 Mark]

Q.93 Which one of the following statements is NOT true for a square matrix A?

(a) If A is upper triangular, the eigen values of A are the diagonal elements of it
(b) If A is real symmetric, the eigen values of A are always real and positive
(c) If A is real, the eigen values of A and A^T are always the same
(d) If all the principal minors of A are positive, all the eigen values of A are also positive

[EC, GATE-2014 : 2 Marks]

Q.94 The determinant of matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ is ______.

[CE, GATE-2014 : 1 Mark]

Q.95 Given that the determinant of the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$ is – 12, the determinant of the matrix $\begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix}$ is

(a) – 96 (b) – 24
(c) 24 (d) 96

[ME, GATE-2014 : 1 Mark]

Q.96 The determinant of matrix A is 5 and the determinant of matrix B is 40. The determinant of matrix AB is

[EC, GATE-2014 : 1 Mark]

Q.97 The maximum value of the determinant among all 2×2 real symmetric matrices with trace 14 is ____.

[EC, GATE-2014 : 2 Marks]

Q.98 If the matrix A is such that

$$A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} [1 \; 9 \; 5]$$

then the determinant of A is equal to ____.

[CS, GATE-2014 : 1 Mark]

Q.99 The rank of the matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$ is ____.

[CE, GATE-2014 : 2 Marks]

Q.100 Two matrices A and B are given below:

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}; \; B = \begin{bmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{bmatrix}$$

If the rank of matrix A is N, then the rank of matrix B is

(a) $\frac{N}{2}$ (b) $N-1$
(c) N (d) $2N$

[EE, GATE-2014 : 1 Mark]

Q.101 Given a system of equations:

$x + 2y + 2z = b_1$
$5x + y + 3z = b_2$

Which of the following is true regarding its solution?

(a) The system has a unique solution for any given b_1 and b_2
(b) The system will have infinitely many solutions for any given b_1 and b_2
(c) Whether or not a solution exists depends on the given b_1 and b_2
(d) The system would have no solution for any values of b_1 and b_2

[EE, GATE-2014 : 1 Mark]

Q.102 The system of linear equations

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 14 \end{bmatrix} \text{ has}$$

(a) a unique solution
(b) infinitely many solutions
(c) no solution
(d) exactly two solutions

[EC, GATE-2014 : 2 Marks]

Q.103 Consider the following system of equations:

$3x + 2y = 1$
$4x + 7z = 1$
$x + y + z = 3$
$x - 2y + 7z = 0$

The number of solutions for this system is ____.

[CS, GATE-2014 : 1 Mark]

Q.104 The sum of Eigen values of matrix, $[M]$ is

$$\text{where } [M] = \begin{bmatrix} 215 & 650 & 795 \\ 655 & 150 & 835 \\ 485 & 355 & 550 \end{bmatrix}$$

(a) 915 (b) 1355
(c) 1640 (d) 2180

[CE, GATE-2014 : 1 Mark]

Q.105 Consider a 3×3 real symmetric matrix S such that two of its eigen values are $a \neq 0$, $b \neq 0$ with respective eigen vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. If $a \neq b$ then $x_1y_1 + x_2y_2 + x_3y_3$ equals

(a) a (b) b
(c) ab (d) 0

[ME, GATE-2014 : 1 Mark]

Q.106 One of the eigen vectors of matrix $\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ is

(a) $\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$ (b) $\begin{Bmatrix} -2 \\ 9 \end{Bmatrix}$

(c) $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ (d) $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

[ME, GATE-2014 : 1 Mark]

Q.107 A system matrix is given as follows.

$$A\begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$$

The absolute value of the ratio of the maximum eigen value to the minimum eigen value is ______.

[EE, GATE-2014 : 2 Marks]

Q.108 A real (4×4) matrix A satisfies the equation $A^2 = I$, where I is the (4×4) identity matrix. The positive eigen value of A is ______.

[EC, GATE-2014 : 1 Mark]

Q.109 The value of the dot product of the eigen vectors corresponding to any pair of different eigen values of a 4×4 symmetric positive definite matrix is ________.

[CS, GATE-2014 : 1 Mark]

Q.110 The product of the non-zero eigen values of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is ________.

[CS, GATE-2014 : 2 Marks]

Q.111 Which one of the following statements is TRUE about every $n \times n$ matrix with only real eigen values?

(a) If the trace of the matrix is positive and the determinant of the matrix is negative, at least one of its eigen values is negative.

(b) If the trace of the matrix is positive, all its eigen values are positive.

(c) If the determinant of the matrix is positive, all its eigen values are positive.

(d) If the product of the trace and determinant of the matrix is positive, all its eigen values are positive.

[CS, GATE-2014 : 1 Mark]

Q.112 If any two columns of a determinant $P = \begin{vmatrix} 4 & 7 & 8 \\ 3 & 1 & 5 \\ 9 & 6 & 2 \end{vmatrix}$ are interchanged, which one of the following statements regarding the value of the determinant is CORRECT?

(a) Absolute value remains unchanged but sign will change

(b) Both absolute value and sign will change

(c) Absolute value will change but sign will not change

(d) Both absolute value and sign will remain unchanged

[ME, GATE-2015 : 1 Mark]

Q.113 Perform the following operations on the matrix

$$\begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{bmatrix}.$$

1. Add the third row to the second row.
2. Subtract the third column from the first column.

The determinant of the resultant matrix is ________.

[CS, GATE-2015 : 2 Marks]

Q.114 For $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, the determinant of $A^T A^{-1}$ is

(a) $\sec^2 x$ (b) $\cos 4x$

(c) 1 (d) 0

[EC, GATE-2015 : 1 Mark]

Q.115 For given matrix $P = \begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$ where $i = \sqrt{-1}$, the inverse of matrix P is

(a) $\frac{1}{24}\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$ (b) $\frac{1}{25}\begin{bmatrix} i & 4-i \\ 4+3i & -i \end{bmatrix}$

(c) $\frac{1}{24}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$ (d) $\frac{1}{25}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$

[ME, GATE-2015 : 2 Marks]

Q.116 Let $A=\left[a_{ij}\right], 1\leq i,j\leq n$ with $n\geq 3$ and $a_{ij}=i.j$. The rank of A is

(a) 0 (b) 1
(c) $n-1$ (d) n

[CE, GATE-2015 : 1 Mark]

Q.117 For what value of p the following set of equations will have no solution?

$$2x+3y=5$$
$$3x+py=10$$

[CE, GATE-2015 : 1 Mark]

Q.118 We have a set of 3 linear equations in 3 unknowns. '$X\equiv Y$' means X and Y are equivalent statements and '$X\not\equiv Y$' means X and Y are not equivalent statements.

P : There is a unique solution.
Q : The equations are linearly independent.
R : All eigen values of the coefficient matrix are nonzero.
S : The determinant of the coefficient matrix is nonzero.

Which one of the following is TRUE?

(a) $P\equiv Q\equiv R\equiv S$ (b) $P\equiv R\not\equiv Q\equiv S$
(c) $P\equiv Q\not\equiv R\equiv S$ (d) $P\not\equiv Q\not\equiv R\not\equiv S$

[EE, GATE-2015 : 1 Mark]

Q.119 Consider a system of linear equations:

$$x-2y+3z=-1,$$
$$x-3y+4z=1, \text{ and}$$
$$-2x+4y-6z=k$$

The value of k for which the system has infinitely many solution is ________.

[EC, GATE-2015 : 1 Mark]

Q.120 If the following system has non-trivial solution,

$$px+qy+rz=0$$
$$qx+ry+pz=0$$
$$rx+py+qz=0$$

then which one of the following options is TRUE?

(a) $p-q+r=0$ or $p=q=-r$
(b) $p+q-r=0$ or $p=-q=r$
(c) $p+q+r=0$ or $p=q=r$
(d) $p-q+r=0$ or $p=-q=-r$

[CS, GATE-2015 : 2 Marks]

Q.121 Let A be an $n\times n$ matrix with rank $r(0<r<n)$. Then $AX=0$ has p independent solutions, where p is

(a) r (b) n
(c) $n-r$ (d) $n+r$

[IN, GATE-2015 : 1 Mark]

Q.122 The smallest and largest Eigen values of the following matrix are

$$\begin{bmatrix}3 & -2 & 2\\ 4 & -4 & 6\\ 2 & -3 & 5\end{bmatrix}$$

(a) 1.5 and 2.5 (b) 0.5 and 2.5
(c) 1.0 and 3.0 (d) 1.0 and 2.0

[CE, GATE-2015 : 2 Marks]

Q.123 The lowest eigen value of the 2×2 matrix $\begin{bmatrix}4 & 2\\ 1 & 3\end{bmatrix}$ is ________.

[ME, GATE-2015 : 1 Mark]

Q.124 The value of p such that the vector $\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$ is an eigen vector of the matrix $\begin{bmatrix}4 & 1 & 2\\ p & 2 & 1\\ 14 & -4 & 10\end{bmatrix}$ is ________

[EC, GATE-2015 : 1 Mark]

Q.125 The larger of the two eigen values of the matrix $\begin{bmatrix}4 & 5\\ 2 & 1\end{bmatrix}$ is ________.

[CS, GATE-2015 : 1 Mark]

Q.126 In the given matrix $\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & 0\\ 1 & 2 & 1\end{bmatrix}$, one of the eigen values is 1. The eigen vectors corresponding to the eigen value 1 are

(a) $\{\alpha(4,2,1)\mid\alpha\neq 0,\alpha\in R\}$

(b) $\{\alpha(-4,2,1)\mid\alpha\neq 0,\alpha\in R\}$

(c) $\{\alpha(\sqrt{2},0,1)\mid\alpha\neq 0,\alpha\in R\}$

(d) $\{\alpha(-\sqrt{2},0,1)\mid\alpha\neq 0,\alpha\in R\}$

[CS, GATE-2015 : 1 Mark]

Q.127 The two Eigen values of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$ have a ratio of 3 : 1 for $p = 2$. What is another value of p for which the Eigen values have the same ratio of 3 : 1?

(a) −2 (b) 1
(c) 7/3 (d) 14/3

[CE, GATE-2015 : 2 Marks]

Q.128 At least one eigen value of a singular matrix is

(a) positive (b) zero
(c) negative (d) imaginary

[ME, GATE-2015 : 1 Mark]

Q.129 The maximum value of "a" such that the matrix $\begin{pmatrix} -3 & 0 & -2 \\ 1 & -1 & 0 \\ 0 & a & -2 \end{pmatrix}$ has three linearly independent real eigen vectors is

(a) $\frac{2}{3\sqrt{3}}$ (b) $\frac{1}{3\sqrt{3}}$
(c) $\frac{1+2\sqrt{3}}{3\sqrt{3}}$ (d) $\frac{1+\sqrt{3}}{3\sqrt{3}}$

[EE, GATE-2015 : 2 Marks]

Q.130 The value of x for which all the eigen-values of the matrix given below are real is

$$\begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

(a) $5 + j$ (b) $5 - j$
(c) $1 - 5j$ (d) $1 + 5j$

[EC, GATE-2015 : 1 Mark]

Q.131 Consider the following 2 × 2 matrix A where two elements are unknown and are marked by a and b. The eigen values of this matrix are −1 and 7. What are the values of a and b?

$$A = \begin{pmatrix} 1 & 4 \\ b & a \end{pmatrix}$$

(a) $a = 6, b = 4$ (b) $a = 4, b = 6$
(c) $a = 3, b = 5$ (d) $a = 5, b = 3$

[CS, GATE-2015 : 2 Marks]

Q.132 A real square matrix A is called skew-symmetric if

(a) $A^T = A$ (b) $A^T = A^{-1}$
(c) $A^T = -A$ (d) $A^T = A + A^{-1}$

[ME, GATE-2016 : 1 Mark]

Q.133 Let $M^4 = I$, (where I denotes the identity matrix) and $M \neq I$, $M^2 \neq I$ and $M^3 \neq I$. Then, for any natural number k, M^{-1} equals:

(a) M^{4k+1} (b) M^{4k+2}
(c) M^{4k+3} (d) M^{4k}

[EC, GATE-2016 : 1 Mark]

Q.134 The matrix $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix}$ has det(A)=100 and trace (A)=14. The value of $|a - b|$ is ________.

[EC, GATE-2016 : 2 Marks]

Q.135 Let A be a 4 × 3 real matrix with rank 2. Which one of the following statement is TRUE?

(a) Rank of A^TA is less than 2.
(b) Rank of A^TA is equal to 2.
(c) Rank of A^TA is greater than 2.
(d) Rank of A^TA can be any number between 1 and 3.

[EE, GATE-2016 : 2 Marks]

Q.136 Let $P = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Consider the set S of all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $a^2 + b^2 = 1$ where $\begin{pmatrix} a \\ b \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix}$. Then S is

(a) a circle of radius $\sqrt{10}$

(b) a circle of radius $\frac{1}{\sqrt{10}}$

(c) an ellipse with major axis along $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(d) an ellipse with minor axis along $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

[EE, GATE-2016 : 2 Marks]

Q.137 If the vectors $e_1 = (1, 0, 2)$, $e_2 = (0, 1, 0)$ and $e_3 = (-2, 0, 1)$ form an orthogonal basis of the three-dimensional real space R^3, then the vector $u = (4, 3, -3) \in R^3$ can be expressed as

(a) $u = -\frac{2}{5}e_1 - 3e_2 - \frac{11}{5}e_3$

(b) $u = -\frac{2}{5}e_1 - 3e_2 + \frac{11}{5}e_3$

(c) $u = -\frac{2}{5}e_1 + 3e_2 + \frac{11}{5}e_3$

(d) $u = -\frac{2}{5}e_1 + 3e_2 - \frac{11}{5}e_3$

[EC, GATE-2016 : 2 Marks]

Q.138 Consider the following linear system.

$x + 2y - 3z = a$

$2x + 3y + 3x = b$

$5x + 9y - 6z = c$

This system is consistent if a, b and c satisfy the equation

(a) $7a - b - c = 0$ (b) $3a + b - c = 0$

(c) $3a - b + c = 0$ (d) $7a - b + c = 0$

[CE, GATE-2016 : 2 Marks]

Q.139 The solution to the system of equations is

$$\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -30 \end{bmatrix}$$

(a) 6, 2 (b) –6, 2

(c) –6, –2 (d) 6, – 2

[ME, GATE-2016 : 1 Mark]

Q.140 Consider the systems, each consisting of m linear equations in n variables.

I. If $m < n$, then all such systems have a solution.

II. If $m > n$, then none of these sytems has a solution.

III. If $m = n$, then there exists a system which has a solution

Which one of the following is **CORRECT**?

(a) I, II and III are true

(b) Only II and III are true

(c) Only III is true

(d) None of them is true

[CS, GATE-2016 : 1 Mark]

Q.141 If the entries in each column of a square matrix M add up to 1, then an eigen value of M is

(a) 4 (b) 3

(c) 2 (d) 1

[CE, GATE-2016 : 1 Mark]

Q.142 Consider a 3×3 matrix with every element being equal to 1. Its only non-zero eigen value is ________.

[EE, GATE-2016 : 1 Mark]

Q.143 The condition for which the eigen values of the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$ are positive, is

(a) $k > \frac{1}{2}$ (b) $k > -2$

(c) $k > 0$ (d) $k < -\frac{1}{2}$

[ME, GATE-2016 : 1 Mark]

Q.144 Consider a 2×2 square matrix

$$A = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$$

where x is unknown. If the eigen values of the matrix A are $(\sigma + j\omega)$ and $(\sigma - j\omega)$, then x is equal to

(a) $+j\omega$ (b) $-j\omega$

(c) $+\omega$ (d) $-\omega$

[EC, GATE-2016 : 1 Mark]

Q.145 Two eigen values of a 3×3 real matrix P are $(2 + \sqrt{-1})$ and 3. The determinant of P is ______.

[CS, GATE-2016 : 1 Mark]

Q.146 Consider a linear time invariant system $\dot{x} = Ax$, with initial conditions $x(0)$ at $t = 0$. Suppose a and b are eigen vectors of (2×2) matrix A corresponding to distinct eigen values λ_1 and λ_2 respectively. Then the response $x(t)$ of the system due to initial condition $x(0) = \alpha$ is

(a) $\alpha e^{\lambda_1 t}$ (b) $e^{\lambda_2 t}\alpha\beta$

(c) $e^{\lambda_2 t}\alpha$ (d) $e^{\lambda_1 t}\alpha + e^{\lambda_2 t}\beta$

[EE, GATE-2016 : 2 Marks]

Q.147 Let the eigen values of a 2 × 2 matrix A be 1, −2 with eigen vectors x_1 and x_2 respectively. Then the eigen values and eigen vectors of the matrix $A^2 - 3A + 4I$ would, respectively, be

(a) 2, 14; x_1, x_2
(b) 2, 14; $x_1 + x_2$, $x_1 - x_2$
(c) 2, 0; x_1, x_2
(d) 2, 0; $x_1 + x_2$, $x_1 - x_2$

[EE, GATE-2016 : 2 Marks]

Q.148 The number of linearly independent eigen vectors of matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is________.

[ME, GATE-2016 : 2 Marks]

Q.149 The value of x for which the matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{bmatrix}$$ has zero as an eigen value is ______.

[EC, GATE-2016 : 1 Mark]

Q.150 Consider the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$ whose eigen values are 1, −1 and 3. Then Trace of $(A^3 - 3A^2)$ is______.

[IN, GATE-2016 : 2 Marks]

Q.151 Suppose that the eigen values of matrix A are 1, 2, 4. The determinant of $(A^{-1})^T$ is______.

[CS, GATE-2016 : 1 Mark]

Q.152 A 3 × 3 matrix P is such that, $P^3 = P$. Then the eigen values of P are

(a) 1, 1, −1
(b) 1, 0.5 + j0.866, 0.5 − j0.866
(c) 1, −0.5 + j0.866, − 0.5 − j0.866
(d) 0, 1, −1

[EE, GATE-2016 : 1 Mark]

Q.153 A sequence $x[n]$ is specified as

$$\begin{bmatrix} x[n] \\ x[n-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ for } n \geq 2.$$

The initial conditions are $x[0] = 1$, $x[1] = 1$, and $x[n] = 0$ for $n < 0$. The value of $x[12]$ is _____

[EC, GATE-2016 : 2 Marks]

Q.154 Let A be $n \times n$ real valued square symmetric matrix of rank 2 with $\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^2 = 50$. Consider the following statements.

I. One eigen value must be in [−5, 5]
II. The eigen value with the largest magnitude must be strictly greater than 5.

Which of the above statements about eigen values of A is/are necessarily CORRECT?

(a) Both I and II
(b) I only
(c) II only
(d) Neither I nor II

[CS, GATE-2017 : 2 Marks]

Q.155 The determinant of a 2 × 2 matrix is 50. If one eigen value of the matrix is 10, the other eigen value is ___________.

[ME, GATE-2017 : 1 Mark]

Q.156 Consider the matrix $A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$ whose eigen vectors corresponding to eigen values λ_1 and λ_2 are $x_1 = \begin{bmatrix} 70 \\ \lambda_1 - 50 \end{bmatrix}$ and $x_2 = \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}$, respectively. The value of $x_1^T x_2$ is ___________.

[ME, GATE-2017 : 2 Marks]

Q.157 The product of eigen values of the matrix P is

$$P = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -3 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$

(a) − 6
(b) 2
(c) 6
(d) − 2

[ME, GATE-2017 : 1 Mark]

Q.158 Consider the matrix $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

Which one of the following statements about P is INCORRECT?

(a) Determinant of P is equal to 1.
(b) P is orthogonal.
(c) Inverse of P is equal to its transpose.
(d) All eigen values of P are real numbers.

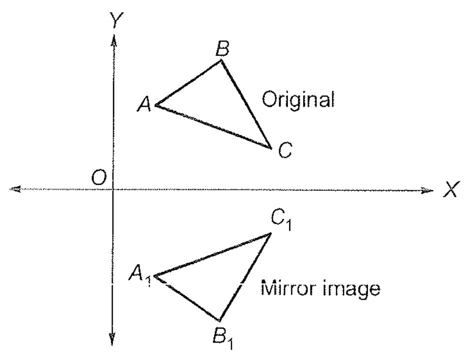
[ME, GATE-2017 : 2 Marks]

Q.159 The eigen values of the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & 6 \\ 0 & -6 & 5 \end{bmatrix}$ are

(a) −1, 5, 6 (b) 1, $-5 \pm j6$
(c) 1, $5 \pm j6$ (d) 1, 5, 5

[IN, GATE-2017 : 1 Mark]

Q.160 The figure shows a shape ABC and its mirror image $A_1 B_1 C_1$ across the horizontal axis (X-axis). The coordinate transformation matrix that maps ABC to $A_1B_1C_1$ is



(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

[IN, GATE-2017 : 1 Mark]

Q.161 The eigen values of the matrix given below are

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

(a) (0, −1, −3) (b) (0, −2, −3)
(c) (0, 2, 3) (d) (0, 1, 3)

[EE, GATE-2017 : 2 Marks]

Q.162 The matrix $A = \begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}$ has three distinct eigen values and one of its eigen vectors is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Which one of the following can be another eigen vector of A?

(a) $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

[EE, GATE-2017 : 1 Mark]

Q.163 The rank of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

is ______.

[EC, GATE-2017 : 1 Mark]

Q.164 Consider the 5 × 5 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$$

It is given that A has only one real eigen value Then the real eigen value of A is

(a) −2.5 (b) 0
(c) 15 (d) 25

[EC, GATE-2017 : 1 Mark]

Q.165 The rank of the matrix $M = \begin{bmatrix} 5 & 10 & 10 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 3

[EC, GATE-2017 : 1 Mark]

Q.166 Let $P = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ be two matrices. Then the rank of $P + Q$ is ______.

[CS, GATE-2017 : 1 Mark]

Q.167 If the characteristic polynomial of a 3 × 3 matrix M over R (the set of real numbers) is $\lambda^3 - 4\lambda^2 + a\lambda + 30$. $a \in R$ and one eigen value of M is 2. then the largest among the absolute values of the eigen values of M is ________.

[CS, GATE-2017 : 2 Marks]

Q.168 Let $c_1 \ldots c_n$ be scalars, not all zero, such that

$$\sum_{i=1}^{n} c_i a_i = 0$$ where a_i are column vectors in R^n.

Consider the set of linear equations

$Ax = b$

where $A = [a_1 \ldots a_n]$ and $b = \sum_{i=1}^{n} a_i$. The set of equations has

(a) a unique solution at $x = J_n$ where J_n denotes a n-dimensional vector of all 1

(b) no solution

(c) infinitely many solutions

(d) finitely many solutions

[CS, GATE-2017 : 1 Mark]

Q.169 Consider the following simultaneous equations (with c_1 and c_2 being constants):

$3x_1 + 2x_2 = c_1$

$4x_1 + x_2 = c_2$

The characteristics equation for these simultaneous equations is

(a) $\lambda^2 - 4\lambda - 5 = 0$ (b) $\lambda^2 - 4\lambda + 5 = 0$

(c) $\lambda^2 + 4\lambda - 5 = 0$ (d) $\lambda^2 + 4\lambda + 5 = 0$

[CE, GATE-2017 : 1 Mark]

Q.170 If $A = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 \\ 8 & 4 \end{bmatrix}$, AB^T is equal to

(a) $\begin{bmatrix} 38 & 28 \\ 32 & 56 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 40 \\ 42 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} 43 & 27 \\ 34 & 50 \end{bmatrix}$ (d) $\begin{bmatrix} 38 & 32 \\ 28 & 56 \end{bmatrix}$

[CE, GATE-2017 : 2 Marks]

Q.171 The matrix P is the inverse of a matrix Q. If I denotes the identity matrix, which one of the following options is correct?

(a) $PQ = I$ but $QP \neq I$ (b) $QP = I$ but $PQ \neq I$

(c) $PQ = I$ and $QP = I$ (d) $PQ - QP = I$

[CE, GATE-2017 : 1 Mark]

Q.172 Consider the matrix $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$. Which one of the following statements is TRUE for the eigen values and eigen vectors of this matrix?

(a) Eigen value 3 has a multiplicity of 2, and only one independent eigen vector exists.

(b) Eigen value 3 has a multiplicity of 2, and two independent eigen vector exists.

(c) Eigen value 3 has a multiplicity of 2, and no independent eigen vector exists.

(d) Eigen value are 3 and –3, and two indepedent eigen vectors exist.

[CE, GATE-2017 : 2 Marks]

Q.173 The solution of the system of equations

$x + y + z = 4$, $x - y + z = 0$, $2x + y + z = 5$ is

(a) $x = 2, y = 2, z = 0$

(b) $x = 1, y = 4, z = 1$

(c) $x = 2, y = 4, z = 3$

(d) $x = 1, y = 2, z = 1$ **[ESE Prelims-2017]**

Q.174 If a square matrix of order 100 has exactly 15 distinct eigen values, the degree of the minimal polynomial is

(a) At least 15 (b) At most 15

(c) Always 15 (d) Exactly 100

[EE, ESE-2017]

Q.175 For the given orthogonal matrix Q,

$$Q = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

The inverse is

(a) $\begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{3}{7} & -\frac{2}{7} & -\frac{6}{7} \\ \frac{6}{7} & -\frac{3}{7} & -\frac{2}{7} \\ -\frac{2}{7} & -\frac{6}{7} & \frac{3}{7} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$ (d) $\begin{bmatrix} -\frac{3}{7} & \frac{6}{7} & -\frac{2}{7} \\ -\frac{2}{7} & -\frac{3}{7} & -\frac{6}{7} \\ -\frac{6}{7} & -\frac{2}{7} & \frac{3}{7} \end{bmatrix}$

[CE, GATE-2018 : 1 Mark]

Q.176 Which one of the following matrices is singular?

(a) $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix}$

[CE, GATE-2018 : 1 Mark]

Q.177 The matrix $\begin{pmatrix} 2 & -4 \\ 4 & -2 \end{pmatrix}$ has

(a) real eigenvalues and eigenvectors
(b) real eigenvalues but complex eigenvectors
(c) complex eigenvalues but real eigenvectors
(d) complex eigenvalues and eigenvectors

[CE, GATE-2018 : 2 Marks]

Q.178 The rank of the following matrix is

$$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{pmatrix}$$

(a) 1 (b) 2
(c) 3 (d) 4

[CE, GATE-2018 : 2 Marks]

~~**Q.179** The rank of the matrix~~ $\begin{bmatrix} -4 & 1 & -1 \\ -1 & -1 & -1 \\ 7 & -3 & 1 \end{bmatrix}$ is

(a) 1 (b) 2
(c) 3 (d) 4

[ME, GATE-2018 : 1 Mark]

Q.180 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ then $\det(A^{-1})$ is ________ (correct to two decimal places).

[ME, GATE-2018 : 1 Mark]

Q.181 Let M be a real 4×4 matrix. Consider the following statements:

S1: M has 4 linearly independent eigenvectors.
S2: M has 4 distinct eigenvalues.
S3: M is non-singular (invertible).

Which one among the following is TRUE?

(a) S1 implies S2 (b) S1 implies S3
(c) S2 implies S1 (d) S3 implies S2

[EC, GATE-2018 : 1 Mark]

Q.182 Consider matrix $A = \begin{bmatrix} k & 2k \\ k^2 - k & k^2 \end{bmatrix}$ and vecto $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The number of distinct real values o k for which the equation $AX = 0$ has infinitel many solutions is ________.

[EC, GATE-2018 : 1 Mark

Q.183 Consider a non-singular 2×2 square matrix A If trace $(A) = 4$ and trace $(A^2) = 5$, the determinan of the matrix A is ______ (upto 1 decimal place

[EE, GATE-2018 : 1 Mark

Q.184 Let $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $B = A^3 - A^2 - 4A + 5I$ where I is the 3×3 identity matrix. The determinan of B is ______ (upto 1 decimal place).

[EE, GATE-2018 : 2 Marks

Q.185 Let N be a 3 by 3 matrix with real number entries The matrix N is such that $N^2 = 0$. The eige values of N are

(a) 0, 0, 0 (b) 0, 0, 1
(c) 0, 1, 1 (d) 1, 1, 1

[IN, GATE-2018 : 1 Mar

Q.186 Consider two functions $f(x) = (x - 2)^2$ an $g(x) = 2x - 1$, where x is real. The smallest valu of x for which $f(x) = g(x)$ is ________.

[IN, GATE-2018 : 1 Mar

Q.187 Consider the following system of line equations:

$$3x + 2ky = -2$$
$$kx + 6y = 2$$

Here, x and y are the unknown and k is a re constant. The value of k for which there are infini number of solutions is

(a) 3 (b) 1
(c) -3 (d) -6

[IN, GATE-2018 : 2 Mark

Q.188 Consider a matrix $A = uv^T$ where $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Note that v^T denotes the transpose v. The largest eigenvalue of A is ______

[CS, GATE-2018 : 1 Mar

Q.189 Consider a matrix P whose only eigenvectors are the multiples of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Consider the following statements:

I. P does not have an inverse.

II. P has a repeated eigenvalue.

III. P cannot be diagonalized.

Which one of the following options is correct?

(a) Only I and III are necessarily true

(b) Only II is necessarily true

(c) Only I and II are necessarily true

(d) Only II and III are necessarily true

[CS, GATE-2018 : 2 Marks]

Q.190 Let the Eigen vector of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ be written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is the value of $(a + b)$?

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) 2

[ESE Prelims-2018]

Q.191 Eigen values of the matrix $\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ are

(a) 1, 1, 1 (b) 1, 1, 2

(c) 1, 4, 4 (d) 1, 2, 4

[EE, ESE-2018]

Q.192 If the system

$$2x - y + 3z = 2$$
$$x + y + 2z = 2$$
$$5x - y + az = b$$

has infinitely many solutions, then the values of a and b, respectively, are

(a) –8 and 6 (b) 8 and 6

(c) –8 and –6 (d) 8 and –6

[EE, ESE-2018]

■■■■

Answers Linear Algebra

1.	(c)	2.	(c)	3.	(b)	4.	(c)	5.	(a)	6.	(b)	7.	(c)	8.	(c)	9.	(b)
10.	(a)	11.	(c)	12.	(b)	13.	(a)	14.	(a), (b) and (d)			15.	(b)	16.	(a)	17.	(b)
18.	(a)	19.	(d)	20.	(c)	21.	(b)	22.	(b)	23.	(c)	24.	(a)	25.	(c)	26.	(a)
27.	(b)	28.	(d)	29.	(c)	30.	(b)	31.	(a)	32.	(a)	33.	(d)	34.	(a)	35.	(b)
36.	(c)	37.	(a)	38.	(a)	39.	(b)	40.	(d)	41.	(b)	42.	(a)	43.	(a)	44.	(a)
45.	(b)	46.	(a)	47.	(a)	48.	(d)	49.	(b)	50.	(b)	51.	(d)	52.	(b)	53.	(b)
54.	(a)	55.	(c)	56.	(c)	57.	(d)	58.	(a)	59.	(a)	60.	(c)	61.	(d)	62.	(b)
63.	(d)	64.	(a)	65.	(b)	66.	(c)	67.	(d)	68.	(c)	69.	(b)	70.	(c)	71.	(a)
72.	(b)	73.	(c)	74.	(d)	75.	(b)	76.	(b)	77.	(16)	78.	(b)	79.	(a)	80.	(b)
81.	(c)	82.	(d)	83.	(a, d)	84.	(c)	85.	(d)	86.	(a)	87.	(a)	88.	(23)	89.	(a)
90.	(d)	91.	(a)	92.	(d)	93.	(b)	94.	(88)	95.	(a)	96.	(200)	97.	(49)	98.	(0)
99.	(2)	100.	(c)	101.	(b)	102.	(b)	103.	(1)	104.	(a)	105.	(d)	106.	(d)	107.	(3)
108.	(1)	109.	(0)	110.	(6)	111.	(a)	112.	(a)	113.	(0)	114.	(c)	115.	(a)	116.	(b)
117.	(4.5)	118.	(a)	119.	(2)	120.	(c)	121.	(c)	122.	(d)	123.	(2)	124.	(17)	125.	(6)
126.	(b)	127.	(d)	128.	(b)	129.	(b)	130.	(b)	131.	(d)	132.	(c)	133.	(c)	134.	(3)
135.	(b)	136.	(c)	137.	(d)	138.	(b)	139.	(d)	140.	(c)	141.	(d)	142.	(3)	143.	(a)
144.	(d)	145.	(15)	146.	(a)	147.	(a)	148.	(2)	149.	(1)	150.	(–6)	151.	(0.125)	152.	(d)
153.	(233)	154.	(b)	155.	(5)	156.	(0)	157.	(b)	158.	(d)	159.	(c)	160.	(d)	161.	(a)
162.	(c)	163.	(4)	164.	(c)	165.	(c)	166.	(2)	167.	(5)	168.	(c)	169.	(a)	170.	(a)
171.	(c)	172.	(a)	173.	(d)	174.	(a)	175.	(c)	176.	(c)	177.	(d)	178.	(b)	179.	(b)
180.	(0.25)	181.	(c)	182.	(2)	183.	(5.5)	184.	(1)	185.	(a)	186.	(1)	187.	(c)	188.	(3)
189.	(d)	190.	(b)	191.	(c)	192.	(b)										

1. (c)

Consider first 3 × 3 minors, since maximum possible rank is 3

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 1 & 3 \\ 6 & 4 & 7 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

and $$\begin{vmatrix} 4 & 2 & 3 \\ 6 & 3 & 7 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

Since all 3 × 3 minors are zero, now try 2 × 2 minors.

$$\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5 \neq 0$$

So rank = 2

2. (c)

Given equation are

$$x + 2y + z = 6$$
$$2x + y + 2z = 6$$
$$x + y + z = 5$$

Given system can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}$$

Augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right]$

By gauss elimination

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & -1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$r(A) = 2$$
$$r(A \mid B) = 3$$

Since the rank of coefficient matrix is 2 and rank of argument matrix is 3, which is not equal. Hence system has no solution i.e. system is inconsistent.

3. (b)

The augmented matrix for the given system is

$$\left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 4 & 3 & -12 & 5 \\ 1 & 2 & -8 & 7 \end{array}\right].$$

Performing Gauss-Elimination on the above matrix

$$\left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 4 & 3 & -12 & 5 \\ 1 & 2 & -8 & 7 \end{array}\right] \xrightarrow[R_3 - 1/2\,R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 0 & 1 & -4 & 5-2\alpha \\ 0 & 3/2 & -6 & 7-\alpha/2 \end{array}\right]$$

$$\xrightarrow{R_3 - 3/2\,R_2} \left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 0 & 1 & -4 & 5-2\alpha \\ 0 & 0 & 0 & \frac{5\alpha - 1}{2} \end{array}\right]$$

Now for infinite solution it is necessary that at least one row must be completely zero.

$$\therefore \quad \frac{5\alpha - 1}{2} = 0$$

$\alpha = 1/5$ is the solution

∴ There is only one value of α for which infinite solution exists.

4. (c)

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Now, $A - \lambda I = 0$

Where λ = eigen value

$$\therefore \quad \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)^2 - 1 = 0$$

or, $(4-\lambda)^2 - (1)^2 = 0$

or, $(4 - \lambda + 1)(4 - \lambda - 1) = 0$

or, $(5-\lambda)(3-\lambda) = 0$

∴ $\lambda = 3, \lambda = 5$

5. (a)

For singularity of matrix $= \begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix} = 0$

$\Rightarrow 8(0 - 12) - x(0 - 2 \times 12) = 0$

∴ $x = 4$

6. (b)

A, B, C, D is $n \times n$ matrix.

Given $ABCD = I$

$\Rightarrow \quad ABCDD^{-1}C^{-1} = D^{-1}C^{-1}$

$\Rightarrow \quad AB = D^{-1}C^{-1}$

$\Rightarrow \quad A^{-1}AB = A^{-1}D^{-1}C^{-1}$

$\Rightarrow \quad B = A^{-1}D^{-1}C^{-1}$

$$B^{-1} = (A^{-1}D^{-1}C^{-1})^{-1}$$
$$= (C^{-1})^{-1}\cdot(D^{-1})^{-1}\cdot(A^{-1})^{-1}$$
$$= CDA$$

7. (c)

$$-x + 5y = -1$$
$$x - y = 2$$
$$x + 3y = 3$$

The avg mented matrix is $\left[\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right]$.

Using gauss-elimination on above matrix we get,

$$\left[\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right] \xrightarrow[R_3+R_1]{R_2+R_1} \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2} \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Rank $[A \mid B] = 2$ (number of non zero rows in $[A \mid B]$)

Rank $[A] = 2$ (number of non zero rows in [A])

Rank $[A \mid B]$ = Rank $[A]$

= 2 = number of variables

∴ Unique solution exists. Correct choice is (c).

8. (c)

Characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\times(1-\lambda) - [(-2)\times(-2)] = 0$$
$$\lambda^2 - 5\lambda = 0$$
$$\Rightarrow \quad \lambda(\lambda - 5) = 0$$

Hence, $\lambda = 0, 5$ are the eigen values.

9. (b)

Sum of eigen values of given matrix = sum of diagonal element of given matrix = $1 + 5 + 1 = 7$.

10. (a)

With the given order we can say that order of matrices are as follows:

$$X^T \to 3 \times 4$$
$$Y \to 4 \times 3$$
$$X^T Y \to 3 \times 3$$
$$(X^T Y)^{-1} \to 3 \times 3$$
$$P \to 2 \times 3$$
$$P^T \to 3 \times 2$$
$$P(X^T Y)^{-1} P^T \to (2\times3)(3\times3)(3\times2) \to 2 \times 2$$
$$\therefore \quad (P(X^T Y)^{-1} P^T)^T \to 2 \times 2$$

11. (c)

For orthogonal matrix

$AA^T = I$ i.e. Identity matrix.

$\therefore \quad (AA^T)^{-1} = I^{-1} = I$

12. (b)

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$R^{-1} = \frac{\text{adj}(R)}{|R|} = \frac{[\text{cofactor }(R)]^T}{|R|}$$

$$|R| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{vmatrix}$$
$$= 1(2+3) - 0(4+2) - 1(6-2)$$
$$= 5 - 4 = 1$$

Since we need only the top row of R^{-1}, we need to find only first column of cof (R) which after transpose will become first row of adj (R).

$$\text{cof. }(1,1) = +\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 2 + 3 = 5$$

$$\text{cof. }(2,1) = -\begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = -3$$

$$\text{cof. }(3,1) = +\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = +1$$

$$\therefore \quad \text{cof. }(A) = \begin{bmatrix} 5 & - & - \\ -3 & - & - \\ 1 & - & - \end{bmatrix}$$

$$\text{Adj }(A) = [\text{cof. }(A)]^T = \begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

Dividing by $|R| = 1$ gives

$$R^{-1} = \begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

∴ Top row of R^{-1} = [5 −3 1]

13. (a)

$$[AA^{-1}] = I$$

$$\Rightarrow \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2a-0.1b \\ 0 & 3b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 2a - 0.1b = 0 \Rightarrow a = \frac{0.1b}{2} \quad \text{...(i)}$$

$$3b = 1 \Rightarrow b = \frac{1}{3}$$

Now substitute b in equation (i), we get

$$a = \frac{1}{60}$$

So,
$$a + b = \frac{1}{60} + \frac{1}{3}$$

$$= \frac{1+20}{60} = \frac{21}{60} = \frac{7}{20}$$

14. (a), (b) and (d) all possible.

In an over determined system having more equations than variables, all three possibilities still exist (a) consistent unique (b) consistent infinite and (d) in consistent with no situation.

15. (b)

~~$r(A_{m \times n}) \le \min(m, n)$~~

So, Highest possible rank = Least value of 3 and 4.

i.e. highest possible rank (based on size of A) = 3

However if the rank of A = 3 then rank of $[A \mid B]$ also would be 3, which means the system would become consistent. But it is given that the system is inconsistent. So the maximum rank of A could only be 2.

16. (a)

Rank $[Pq]$ = Rank $[P]$ is necessary for existence of at least one solution to $Px = q$.

17. (b)

The augmented matrix for the given system is

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 3 & -2 & 5 & 2 \\ -1 & -4 & 1 & 3 \end{array}\right].$$

Using gauss-elimination method on above matrix we get,

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 3 & -2 & 5 & 2 \\ -1 & -4 & 1 & 3 \end{array}\right] \xrightarrow[R_3 + \frac{1}{2}R_1]{R_2 - \frac{3}{2}R_1} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & -1/2 & 1/2 & 1/2 \\ 0 & -9/2 & 5/2 & 7/2 \end{array}\right]$$

$$\xrightarrow{R_3 - 9R_2} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & -2 & -1 \end{array}\right]$$

Rank $([A \mid B]) = 3$

Rank $([A]) = 3$

Since Rank $([A \mid B])$ = Rank $([A])$ = number of variables. The system has unique solution.

18. (a)

First solve for eigen values by solving characteristic equation $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 5 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$= (5-\lambda)(5-\lambda)[(2-\lambda)(1-\lambda)-3]$$

$$= 0$$

$$= (5-\lambda)(5-\lambda)(\lambda^2 - 3\lambda - 1) = 0$$

$$\lambda = 5, 5, \frac{3 \pm \sqrt{13}}{2}$$

put $\lambda = 5$ in $[A - \lambda I]X = 0$

$$\begin{bmatrix} 5-5 & 0 & 0 & 0 \\ 0 & 5-5 & 5 & 0 \\ 0 & 0 & 2-5 & 1 \\ 0 & 0 & 3 & 1-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 5x_3 = 0 \; ; \; -3x_3 + x_4 = 0 \; ; \; 3x_3 - 4x_4 = 0$

Solving which we get $x_3 = 0$, $x_4 = 0$, x_1 and x_2 may be anything.

The eigen vector corresponding to $\lambda = 5$, may be written as

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \\ 0 \end{bmatrix}$$

where k_1, k_2 may be any real number. Since choice (a) is the only matrix in this form with both x_3 and $x_4 = 0$, so it is the correct answer.

Since, we already got a correct eigen vector, there is no need to derive the eigen vector corresponding to $\lambda = \frac{3 \pm \sqrt{13}}{2}$.

19. (d)

Since matrix is triangular, the eigen values are the diagonal elements themselves namely $\lambda = 3, -2$ and 1. Corresponding to eigen value, $\lambda = -2$ let us find the eigen vector

$$[A - \lambda I]x = 0$$

$$\begin{bmatrix} 3-\lambda & -2 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = -2$ in above equation we get,

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives the equations,

$$5x_1 - 2x_2 + 2x_3 = 0 \quad \text{... (i)}$$
$$x_3 = 0 \quad \text{... (ii)}$$
$$3x_3 = 0 \quad \text{... (iii)}$$

Since eq. (ii) and (iii) are same we have

$$5x_1 - 2x_2 + 2x_3 = 0 \quad \text{... (i)}$$
$$x_3 = 0 \quad \text{... (ii)}$$

Putting $x_2 = k$, in eq. (i) we get

$5x_1 - 2k + 2 \times 0 = 0$

$\Rightarrow \quad x_1 = 2/5\,k$

$\therefore$ Eigen vectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/5\,k \\ k \\ 0 \end{bmatrix}$$

i.e. $x_1 : x_2 : x_3 = 2/5\,k : k : 0 = 2/5 : 1 : 0 = 2 : 5 : 0$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$ is an eigen vector of matrix A.

20. (c)

First, find the eigen values of $A = \begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (-4-\lambda)(3-\lambda) - 8 = 0$

$\Rightarrow \quad \lambda^2 + \lambda - 20 = 0$

$\Rightarrow \quad (\lambda + 5)(\lambda - 4) = 0$

$\Rightarrow \quad \lambda_1 = -5$ and $\lambda_2 = 4$

Corresponding to $\lambda_1 = -5$ we need to find eigen vector:

The eigen value problem is $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = 0$$

Putting $\lambda = -5$

we get, $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x_1 + 2x_2 = 0 \quad \text{... (i)}$$
$$4x_1 + 8x_2 = 0 \quad \text{... (ii)}$$

Since (i) and (ii) are the same equation we take

$$x_1 + 2x_2 = 0$$
$$x_1 = -2x_2$$
$$x_1 : x_2 = -2 : 1$$

$$\Rightarrow \quad \frac{x_1}{x_2} = -2$$

Now from the answers given, we look for any vector in this ratio and we find choice (c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is in this ratio $\frac{x_1}{x_2} = \frac{2}{-1} = -2.$

So choice (c) is an eigen vector corresponding to $\lambda = -5$.

Since we already got an answer, there is no need to find the second eigen vector corresponding to $\lambda = 4$.

21. (b)

$$A = \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

The characteristic equation of this matrix is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -4 & 5-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(5-\lambda) - 4 = 0$$
$$\lambda^2 - 7\lambda + 6 = 0$$
$$\lambda = 1, 6$$

$\therefore$ The eigen values of A are 1 and 6.

22. (b)

Although λ_i^m will be the corresponding eigen values of A^m, x_i^m need not be corresponding eigen vectors.

23. (c)

Method 1:

$$E = \begin{bmatrix} \cos & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

According to problem

$$E \times F = G$$

or $$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence we see that product of $(E \times F)$ is unit matrix so F has to be the inverse of E.

$$F = E^{-1} = \frac{Adj(E)}{|E|}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Method 2:

An easier method for finding F is by multiplying E with each of the choices (a), (b), (c) and (d) and finding out which one gives the product as identity matrix G. Again the answer is (c).

24. (a)

A. Singular matrix $\rightarrow$ Determinant is zero
B. Non-square matrix $\rightarrow$ Determinant is not defined
C. Real symmetric $\rightarrow$ Eigen values are always real
D. Orthogonal matrix $\rightarrow$ Determinant is always one

25. (c)

Perform, Gauss elimination

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It is in row Echelon form

So its rank is the number of non-zero rows in this form.

i.e., rank = 2

26. (a)

We are looking for orthogonal vectors having a span that contain P, Q and R.

Take choice (a) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

Firstly these are orthogonal, as can be seen by taking their dot product

$= -6 \times 4 + -3 \times -2 + 6 \times 3 = 0$

The space spanned by these two vectors is

$$k_1 \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix} + k_2 \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \quad \text{... (i)}$$

The span of $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$ contains P, Q and R.

We can show this by successively setting equation (i) to P, Q and R one by one and solving for k_1 and k_2 uniquely.

Notice also that choices (b), (c) and (d) are wrong since none of them are orthogonal as can be seen by taking pairwise dot products.

27. (b)

The vector $[-2 \ \ -17 \ \ 30]^T$ is linearly dependent upon the solution obtained in previous question namely $[-6 \ \ -3 \ \ 6]^T$ and $[4 \ \ -2 \ \ 3]^T$.

This can be easily cheeked by finding determinant of $\begin{bmatrix} -6 & -3 & 6 \\ 4 & -2 & 3 \\ -2 & -17 & 30 \end{bmatrix}$.

$$\begin{vmatrix} -6 & -3 & 6 \\ 4 & -2 & 3 \\ -2 & -17 & 30 \end{vmatrix}$$

$= -6(-60 + 51) + 3(120 + 6) + 6(-68 - 4) = 0$

Hence, it is linearly dependent.

28. (d)

The augmented matrix for given system is

$$\left[\begin{array}{ccc|c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{array}\right] \xrightarrow{\text{Exchange 1st and 2nd row}} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

then by Gauss elimination procedure

$$\left[\begin{array}{ccc|c}2&0&-1&2\\0&4&3&8\\3&2&0&5\end{array}\right]\xrightarrow{R_3-\frac{3}{2}R_1}\left[\begin{array}{ccc|c}2&0&-1&2\\0&4&3&8\\0&2&3/2&2\end{array}\right]$$

$$\xrightarrow{R_3-\frac{2}{4}R_2}\left[\begin{array}{ccc|c}2&0&-1&8\\0&4&3&8\\0&0&0&-2\end{array}\right]$$

For last row we see $0 = -2$ which is inconsistent. Also notice that $r(A) = 2$, while $r(A \mid B) = 3$, ($r(A) \neq r(A \mid B)$ means inconsistent).

$\therefore$ Solution is non-existent for above system.

29. (c)

$$M = \begin{bmatrix}4&2\\2&4\end{bmatrix}, [M-\lambda I] = \begin{bmatrix}4-\lambda&2\\2&4-\lambda\end{bmatrix}$$

Given eigen vector $\begin{bmatrix}101\\101\end{bmatrix}$

$$[M-\lambda I]X = 0$$

$$\Rightarrow \begin{bmatrix}4-\lambda&2\\2&4-\lambda\end{bmatrix}\begin{bmatrix}101\\101\end{bmatrix} = 0$$

$\Rightarrow (4-\lambda)(101) + 2 \times 101 = 0$

$\Rightarrow \qquad \lambda = 6$

30. (b)

$$\Sigma\lambda_i = \text{Trace (A)}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{Trace (A)} = 2 + (-1) + 0 = 1$$

Now $\lambda_1 = 3$

$\therefore 3 + \lambda_2 + \lambda_3 = 1$

$\Rightarrow \quad \lambda_2 + \lambda_3 = -2$

Only choice (b) satisfies this condition.

31. (a)

If $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_4$ are the eigen values of A. Then the eigen values of

A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m \ldots$

Here, S matrix has eigen values 1 and 5.

So, S^2 matrix has eigen values 1^2 and 5^2 i.e. 1 and 25.

32. (a)

By property of eigen values, sum of diagonal elements should be equal to sum of values of λ.

So, $\Sigma\lambda_i = \lambda_1 + \lambda_2 = 8 + 4 = 12 = \text{Trace }(A)$

Only in choice (a), Trace (A) = 12.

33. (d)

Since $\qquad S^T = (A + A^T)^T$

$\qquad\qquad = A^T + (A^T)^T$

$\qquad\qquad = A^T + A = S$

i.e. $\qquad S^T = S$

$\therefore$ S is symmetric

Since $\qquad D^T = (A - A^T)^T = A^T - (A^T)^T$

$\qquad\qquad = A^T - A = -(A - A^T) = -D$

i.e. $\qquad D^T = -D$

So D is Skew-Symmetric.

34. (a)

Inverse of $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ is

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$$

$$\therefore \begin{bmatrix}1&2\\5&7\end{bmatrix}^{-1} = \frac{1}{(7-10)}\begin{bmatrix}7&-2\\-5&1\end{bmatrix}$$

$$= \frac{1}{-3}\begin{bmatrix}7&-2\\-5&1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-7&2\\5&-1\end{bmatrix}$$

35. (b)

If $\qquad X = (x_1, x_2, \ldots x_n)^T$

Rank $X = 1$, since it is non-zero n-tuple.

Rank X^T = Rank $X = 1$

Now Rank $(X^T) \leq \min(\text{Rank } X, \text{Rank } X^T)$

$\Rightarrow$ Rank $(XX^T) \leq \min(1, 1)$

$\Rightarrow$ Rank $(XX^T) \leq 1$.

So XX^T has a rank of a either 0 or 1.

But since both X and X^T are non-zero vectors, so neither of their ranks can be zero.

So XX^T has a rank 1.

36. (c)

Since $(X_1, X_2, \ldots X_M)$ are orthogonal, they span a vector space of dimension M.

Since $(-X_1, -X_2, \ldots X_M)$ are linearly dependent on $X_1, X_2, \ldots X_M$, the set $(X_1, X_2, X_3, \ldots X_M, -X_1, -X_2, \ldots -X_M)$ will also span a vector space of dimension M only.

37. (a)

To be basis for subspace X, two conditions are to be satisfied

1. The vectors have to be linearly independent.
2. They must span X.

Here, $\qquad X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0\}$

$\qquad\qquad x^T = [x_1, x_2, x_3]^T$

Step 1: Now, $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a linearly independent set because one cannot be obtained from another by scalar multiplication. The fact that it is independent can also be established by seeing that rank of $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ is 2.

Step 2: Next, we need to check if the set spans X. Here, $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0\}$

The general infinite solution of $X = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$

Choosing k_1, k_2 as $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix}$ and $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix}$, we get 2 linearly independent solutions, for X,

$$X = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} \text{ or } \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix}$$

Now since both of these can be generated by linear combinations of $[1, -1, 0]^T$ and $[1, 0, -1]^T$, the set spans X. Since we have shown that the set is not only linearly independent but also spans X, therefore by definition it is a basis for the subspace X.

38. (a)

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right]$$

Using Gauss-elimination method we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha-1 & \beta-5 \end{array}\right]$$

$$\xrightarrow{R_3-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & \alpha-2 & \beta-7 \end{array}\right]$$

Now, for infinite solution last row must be completely zero

i.e.$\alpha - 2$ and $\beta - 7 = 0$

$\Rightarrow \quad \alpha = 2$ and $\beta = 7$

39. (b)

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$[a - \lambda I] = 0$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)^2 = 0$$

$$\Rightarrow \quad \lambda = 2$$

Now, consider the eigen value problem

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

put $\lambda = 2$, we get,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = 0 \quad \ldots \text{(i)}$$

$$0 = 0 \quad \ldots \text{(ii)}$$

The solution is therefore $x_2 = 0$, x_1 = anything

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

40. (d)

The cross product of b = $[0 \quad 1 \quad 0]^T$ and $X = [x_1 \quad x_2 \quad x_3]^T$ can be written as

$$b \times X = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

$$= x_3 \hat{i} + 0 \hat{j} - x_1 \hat{k}$$

$$= [x_3 \quad 0 \quad -x_1]$$

Now $\quad L(x) = b \times X = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

where M is a 3×3 matrix

Let $\quad M = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$

Now $\quad M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b \times X$

$$\Rightarrow \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

By matching LHS and RHS we get

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

So, $M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Now we have to find the eigen values of M

$$|M - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda(\lambda^2 - 0) + 1(0 - \lambda) = 0$

$\Rightarrow \lambda^3 + \lambda = 0$

$\Rightarrow \lambda(\lambda^2 + 1) = 0$

$\Rightarrow \lambda = 0, \lambda = \pm i$

So, the eigen values of M are i, $-i$ and 0.

41. (b)

$$\Sigma\lambda_i = \text{Trace (A)}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 5 + 1 = 7$$

Now, $\lambda_1 = -2, \lambda_2 = 6$

$\therefore$ $-2 + 6 + \lambda_3 = 7$

$\lambda_3 = 3$

42. (a)

The eigen values of any symmetric matrix is always real.

43. (a)

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -3-\lambda & 2 \\ -1 & 0-\lambda \end{vmatrix} = 0$$

$(-3 - \lambda)(-\lambda) + 2 = 0$

$\lambda^2 + 3\lambda + 2 = 0$

A will satisfy this equation according to Caylay-Hamilton theorem

i.e. $A^2 + 3A + 2I = 0$

multiplying by A^{-1} on both sides we get

$A^{-1}A^2 + 3A^{-1}A + 2A^{-1}I = 0$

$A + 3I + 2A^{-1} = 0$

44. (a)

To calculate A^9

start from $A^2 + 3A + 2I = 0$ which has been derived above

$$\begin{aligned} \Rightarrow \quad A^2 &= -3A - 2I \\ A^4 &= A^2 \times A^2 = (-3A - 2I)(-3A - 2I) \\ &= 9A^2 + 12A + 4I \\ &= 9(-3A - 2I) + 12A + 4I \\ &= -15A - 14I \\ A^8 &= A^4 \times A^4 \\ &= (-15A - 14I)(-15A - 14I) \\ &= 225A^2 + 420A + 156I \\ &= 225(-3A - 2I) + 420A\ 196I \\ &= -255A - 254I \\ A^9 &= A \times A^8 \\ &= A(-255A - 254I) \\ &= -255A^2 - 254A \\ &= -255(-3A - 2I) - 254A \\ &= 511A + 510I \end{aligned}$$

45. (b)

$$\begin{aligned} (PQ)^{-1}P &= (Q^{-1}P^{-1})P \\ &= (Q^{-1})(P^{-1}P) = (Q^{-1})(I) \\ &= Q^{-1} \end{aligned}$$

46. (a)

Choice (a) $AA'A = A$ is correct

Since, $AA'A = A[(A^TA)^{-1}A^T]A$

$= A[(A^TA)^{-1}A^TA]$

Let, $A^TA = P$

Then, $= A[P^{-1}P] = A \cdot I = A$

47. (a)

If rank of (5×6) matrix is 4, then surely it must have exactly 4 linearly independent rows as well as 4 linearly independent columns, since rank = row rank = column rank.

48. (d)

The augmented matrix for given system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & k & 6 \end{array}\right].$$

Using Gauss elimination we reduce this to an upper triangular matrix to investigate its rank.

$$\left[\begin{array}{ccc|c}1&1&1&3\\1&2&3&4\\1&4&k&6\end{array}\right]\xrightarrow[R_3-R_1]{R_2-R_1}\left[\begin{array}{ccc|c}1&1&1&3\\0&1&2&1\\0&3&k-1&3\end{array}\right]$$

$$\xrightarrow{R_3-3R_2}\left[\begin{array}{ccc|c}1&1&1&3\\0&1&2&1\\0&0&k-7&0\end{array}\right]$$

Now if $k \neq 7$

rank (A) = rank $(A \mid B) = 3$

$\therefore$ unique solution

If $k = 7$, rank (A) = rank $(A \mid B) = 2$

which is less than number of variables

$\therefore$ when $k = 7$, unique solution is not possible and only infinite solution is possible.

49. (b)

Augmented matrix is $\left[\begin{array}{ccc|c}2&3&0&4\\1&1&1&4\\1&2&-1&a\end{array}\right]$.

Performing guess-elimination on this matrix, we get,

$$\left[\begin{array}{ccc|c}2&3&0&4\\1&1&1&4\\1&2&-1&a\end{array}\right]\xrightarrow[R_3-\frac{1}{2}R_1]{R_2-\frac{1}{2}R_1}\left[\begin{array}{ccc|c}2&3&0&4\\0&-1/2&1&2\\0&1/2&-1&a-2\end{array}\right]$$

$$\xrightarrow{R_3+R_2}\left[\begin{array}{ccc|c}2&3&0&4\\0&-1/2&1&2\\0&0&0&a\end{array}\right]$$

If $a \neq 0$, $r(A) = 2$ and $r(A \mid B) = 3$, hence system will have no solutions.

If $a = 0$, $r(A) = r(A \mid B) = 2$, then the system will be consistent and will have solution (Infinite solution).

50. (b)

The system can be written in matrix from as

$$\begin{bmatrix}4&2\\2&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}7\\6\end{bmatrix}$$

The Augmented matrix $[A \mid B]$ is given by

$$\left[\begin{array}{cc|c}4&2&7\\2&1&6\end{array}\right]$$

Performing Gauss elimination on this [AIB] as follows:

$$\left[\begin{array}{cc|c}4&2&7\\2&1&6\end{array}\right]\xrightarrow[=R_2-\frac{1}{2}R_1]{R_2-\frac{2}{4}R_1}\left[\begin{array}{cc|c}4&2&7\\0&0&5/2\end{array}\right]$$

Now, Rank $[A \mid B] = 2$

(The number of non-zero rows in $[A \mid B]$

Rank $[A] = 1$

(The number of non-zero rows in $[A]$)

Since, Rank $[A \mid B] \neq$ Rank[A],

The system has no solution.

51. (d)

The augmented matrix for above system is

$$\left[\begin{array}{ccc|c}1&1&2&1\\1&2&3&2\\1&4&a&4\end{array}\right]\xrightarrow[R_3-R_1]{R_2-R_1}\left[\begin{array}{ccc|c}1&1&2&1\\0&1&1&1\\0&3&a-2&3\end{array}\right]$$

$$\xrightarrow{R_3-3R_2}\left[\begin{array}{ccc|c}1&1&2&1\\0&1&1&1\\0&0&a-5&0\end{array}\right]$$

Now as long as $a-5 \neq 0$, rank (A) = rank $(A \mid B) = 3$

$\therefore$ A can take any real value except 5. Closest correct answer is (d).

52. (b)

$$A=\begin{bmatrix}4&5\\2&-5\end{bmatrix}$$

Characteristic equation of A is

$$\begin{vmatrix}4-\lambda&5\\2&-5-\lambda\end{vmatrix}=0$$

$\Rightarrow (4-\lambda)(-5-\lambda)-2\times 5=0$

$\Rightarrow \lambda^2+\lambda-30=0$

$\lambda = 5, -6$

53. (b)

$$\begin{vmatrix}(1-\lambda)&2\\0&(2-\lambda)\end{vmatrix}=0$$

$\Rightarrow (1-\lambda)(2-\lambda)=0$

$\therefore \lambda = 1, 2$

Now since the eigen value problem is

$$[A-\lambda I]X = O$$

$$\begin{bmatrix}1-\lambda&2\\0&2-\lambda\end{bmatrix}X=\begin{bmatrix}0\\0\end{bmatrix}$$

putting the value of $\lambda = 1$ and $X = X_1 = \begin{bmatrix}1\\a\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}0&2\\0&1\end{bmatrix}\begin{bmatrix}1\\a\end{bmatrix}=0$$

$\Rightarrow a = 0$ ...(i

putting the value of $\lambda = 2$ and $\hat{X} = \hat{X}_2 = \begin{bmatrix}1\\b\end{bmatrix}$

$$\begin{bmatrix}-1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}1\\b\end{bmatrix}=0$$

$\Rightarrow \quad -1+2b=0$

and $\quad 0=0$

$\Rightarrow \quad b=\dfrac{1}{2}$ ...(ii)

From equations (i) and (ii)

$$a+b=0+\frac{1}{2}=\frac{1}{2}$$

54. (a)

Eigen values of $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

$$\begin{vmatrix}1-\lambda & 0\\0 & 0-\lambda\end{vmatrix}=0$$

$$(1-\lambda)\times(-\lambda)=0$$

$$\lambda=0 \text{ or } \lambda=1$$

Eigen values of $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$

$$\begin{bmatrix}-\lambda & 1\\0 & -\lambda\end{bmatrix}=0$$

$$\lambda^2=0$$

$$\lambda=0, 0$$

Eigen values of $\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$

$$\begin{bmatrix}1-\lambda & -1\\1 & 1-\lambda\end{bmatrix}=0$$

$$(1-\lambda)^2+1=0$$

$$(1-\lambda)^2=-1$$

$$1-\lambda=i \text{ or } -i$$

$$\lambda=1-i \text{ or } 1+i$$

Eigen values of matrix are $\begin{vmatrix}-1-\lambda & 0\\1 & -1-\lambda\end{vmatrix}=0$

$$(-1-\lambda)(-1-\lambda)=0$$

$$(1+\lambda)^2=0$$

$$\lambda=-1, -1$$

So, only one matrix has an eigen value of 1 which is $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$.

Correct choice is (a).

55. (c)

Sum of the eigen values of matrix is = trace of matrix = sum of diagonal values present in the matrix

$\therefore \quad 1+0+p=3+\lambda_2+\lambda_3$

$\Rightarrow \quad p+1=3+\lambda_2+\lambda_3$

$\Rightarrow \quad \lambda_2+\lambda_3=p+1-3=p-2$

56. (c)

Since, $\quad \Pi\lambda_i=|A|$

and If one of the eigen values is zero, then

$$\Pi\lambda_i=|A|=0$$

Now, $\quad |A|=\begin{vmatrix}p_{11} & p_{12}\\p_{21} & p_{22}\end{vmatrix}=0$

$\Rightarrow \quad p_{11}p_{22}-p_{12}p_{21}=0$

Which is choice (c).

57. (d)

If characteristic equation is

$\lambda^3+\lambda^2+2\lambda+1=0$

Then by cayley - hamilton theorem,

$P^3+P^2+2P+I=0$

$$I=-P^3-P^2-2P$$

Multiplying by P^{-1} on both sides,

$$P^{-1}=-P^2-P-2I$$

$$=-(P^2+P+2I)$$

58. (a)

A square matrix B is defined as skew-symmetric if and only if $B^T=-B$, by definition.

59. (a)

Given, $\quad M^T=M^{-1}$.

So $\quad M^TM=I$

$$\Rightarrow \begin{bmatrix}\frac{3}{5} & x\\ \frac{4}{5} & \frac{3}{5}\end{bmatrix}\begin{bmatrix}\frac{3}{5} & \frac{4}{5}\\ x & \frac{3}{5}\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\left(\frac{3}{5}\right)^2+x^2 & \left(\frac{3}{5}\cdot\frac{4}{5}\right)+\frac{3}{5}x\\ \left(\frac{4}{5}\cdot\frac{3}{5}\right)+\frac{3}{5}\cdot x & \left(\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$\Rightarrow$ Compare both sides a_{12}

$$a_{12}=\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)+\left(\frac{3}{5}\right)x=0$$

$$\Rightarrow \quad \frac{3}{5}x=-\frac{3}{5}\cdot\frac{4}{5}$$

$$\Rightarrow \quad x=-\frac{4}{5}$$

60. (c)

$$\Sigma \lambda_i = \text{Trace}(A) = -2$$

$\Rightarrow \quad \lambda_1 + \lambda_2 = -2$ ... (i)

$$\Pi \lambda_i = |A| = -35$$

$\Rightarrow \quad \lambda_1 \lambda_2 = -35$ ... (ii)

Solving (i) and (ii) we get λ_1 and $\lambda_2 = 5, -7$.

61. (d)

Sum of eigen values = Tr(A) = –1 + –1 + 3 = 1

So, $\quad \Sigma \lambda_i = 1$

Only choice (d) (3, $-1+3j$, $-1-3j$) gives $\Sigma \lambda_i = 1$.

62. (b)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore \begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}^{-1} = \frac{1}{[(3+2i)(3-2i)+i^2]}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$

63. (d)

$x_1 + 2x_2 + x_3 + 4x_4 = 2$

$3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$

The augmented matrix is $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 3 & 6 & 3 & 12 & 6 \end{array}\right]$

Performing gauss-elimination on this we get

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 3 & 6 & 3 & 12 & 6 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rank (A) = rank (A | B) = 1

So, system is consistent.

Since, system's rank = 1 is less than the number of variables, only infinite (multiple) non-trivial solution exists.

64. (a)

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

Characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(3-\lambda) - 2 = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda = 1, 4$$

The eigen value problem is $|A - \lambda I|x = 0$

$$\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = 1$,

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$ ... (i)

$x_1 + 2x_2 = 0$ ... (ii)

Solution is $x_2 = k$, $x_1 = -2k$

$$X_1 = \begin{bmatrix} -2k \\ k \end{bmatrix}$$

i.e. $\quad x_1 : x_2 = -2 : 1$

Since, choice (A) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is in same ratio of x_1 to x_2.

$\therefore$ Choice (a) is an eigen vector.

65. (b)

Given, $\quad P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

P is triangular. So eigen values are the diagonal elements themselves. Eigen values are therefore, $\lambda_1 = 1$, $\lambda_2 = 2$. $\lambda_3 = 3$.

Now, the eigen value problem is $[A - \lambda I]\hat{x} = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda_1 = 1$, we get the eigen vector corresponding to this eigen value,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives the equations

$$x_2 = 0$$

$$x_2 + 2x_3 = 0$$

$$2x_3 = 0$$

The solution is $x_2 = 0$, $x_3 = 0$, $x_1 = k$

So, one eigen vector is $\hat{x}_1 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ i.e., $x_1 : x_2 : x_3$

$= k : 0 : 0$

Since, none of the eigen vectors given in choices matches with this, ratio we need to proceed further and find the other eigen vectors corresponding to the other Eigen values.

Now, corresponding to $\lambda_2 = 2$, we get by substituting $\lambda = 2$, in the eigen value problem the following set of equations,

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives the equations,

$$-x_1 + x_2 = 0$$
$$2x_3 = 0$$
$$x_3 = 0$$

Solution is $x_3 = 0, x_1 = k, x_2 = k$

$$\therefore \; X_2 = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} \quad \text{i.e., } x_1 : x_2 : x_3 = 1 : 1 : 0$$

Since none of the eigen vectors given in the choices is of this ratio, we need to proceed further and find 3rd eigen vector also.

By putting $\lambda = 3$ in the eigen valve problem, we get

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 = 0$$
$$-x_2 + 2x_3 = 0$$

putting $x_1 = k$, we get, $x_2 = 2k$ and $x_3 = x_2/2 = k$

$$\therefore \qquad \hat{X}_3 = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix}$$

i.e., $\quad x_1 : x_2 : x_3 = 1 : 2 : 1$

Only the eigen vector given in choice (b) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, is in this ratio. So, the correct answer is choice (b).

66. (c)

Eigen values of a skew symmetric matrix are either zero or pure imaginary.

67. (d)

Sum of eigen values = Trace $(A) = 2 + y$

Product of eigen values = $|A| = 2y - 3x$

$\therefore \quad 4 + 8 = 2 + y \qquad \ldots$ (i)

$4 \times 8 = 2y - 3x \qquad \ldots$ (ii)

$\therefore \quad 2 + y = 12 \qquad \ldots$ (i)

$2y - 3x = 32 \qquad \ldots$ (ii)

$\therefore$ Solving (i) and (ii) we get $x = -4$ and $y = 10$.

68. (c)

The Augmented matrix

$$[A \mid B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

Performing gauss elimination on $[A \mid B]$ we get

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_3 - \frac{1}{2}R_1}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \xrightarrow{R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank (A) = Rank $(A \mid B) = 2 < 3$

So infinite number of solutions are obtained.

69. (b)

The augmented matrix for the system of equations is

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right]$$

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 0 & 0 & \lambda-6 & \mu-20 \end{array}\right] \qquad [R_3 \to R_3 - R_2]$$

If $\lambda = 6$ and $\mu \neq 20$ then

Rank $(A \mid B) = 3$ and Rank $(A) = 2$

$\because$ Rank $(A \mid B) \neq$ Rank (A)

$\therefore$ Given system of equations has no solution for $\lambda = 6$ and $\mu \neq 20$.

70. (c)

Eigen values of symmetric matrix are always real.

71. (a)

Since the given matrix is upper triangular, its eigen values are the diagonal elements themselves, which are 1, 4 and 3.

72. (b)

We need eigen values of $A = \begin{bmatrix} 9 & 5 \\ 5 & 8 \end{bmatrix}$

The characteristic equation is

$$\begin{vmatrix} 9-\lambda & 5 \\ 5 & 8-\lambda \end{vmatrix} = 0$$

$$(9-\lambda)(8-\lambda) - 25 = 0$$

$$\Rightarrow \quad \lambda^2 - 17\lambda + 47 = 0$$

So eigen values are,

$$\lambda = 3.48, 13.53$$

73. (c)

The given system is

$$x + 2y + z = 4$$
$$2x + y + 2z = 5$$
$$x - y + z = 1$$

Use Gauss elimination method as follows:

Augmented matrix is

$$[A/B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 1 & 2 & 5 \\ 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & -3 \\ 0 & -3 & 0 & -3 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Rank}(A) = 2$$
$$\text{Rank}[A \mid B] = 2$$

So $\quad \text{Rank}(A) = \text{Rank}[A \mid B] = 2$

System is consistent

Now system rank $\quad r = 2$

Number of variables $\quad n = 3$

$$r < n$$

So we have infinite number of solutions.

74. (d)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Eigen (A) are the roots of the characteristic polynomial given below:

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 1 = 0$$
$$-(1-\lambda)(1+\lambda) - 1 = 0$$
$$\lambda^2 - 2 = 0$$
$$\lambda = \pm\sqrt{2}$$

Eigen values of A are $\sqrt{2}$ and $-\sqrt{2}$ respectively.

So eigen values of $A^{19} = (\sqrt{2})^{19}$ and $(-\sqrt{2})^{19}$

$$= 2^{19/2} \text{ and } -2^{19/2}$$
$$= 2^9.2^{1/2} \text{ and } -2^9.2^{1/2}$$
$$= 512\sqrt{2} \text{ and } -512\sqrt{2}.$$

75. (b)

$$A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$$

Characteristic equation is

$$\begin{vmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(3-\lambda) - 3 = 0$$
$$\lambda^2 - 8\lambda + 12 = 0$$
$$\lambda = 2, 6$$

Now, to find eigen vectors:

$$[A - \lambda I]\hat{x} = 0$$

Which is $\begin{bmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Put $\lambda = 2$ in above equation and we get

$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which gives us the equation,

$$3x_1 + 3x_2 = 0$$

and $\quad x_1 + x_2 = 0$

Which is only one equation,

$$x_1 + x_2 = 0$$

Whose solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -k \\ k \end{bmatrix}$$

So one eigen vector is $\hat{x}_1 = \begin{bmatrix} -k \\ k \end{bmatrix}$

~~Which after normalization is~~ $= \dfrac{\hat{x}_1}{|\hat{x}_1|}$

$$= \frac{1}{\sqrt{(-k)^2 + (k^2)}}\begin{bmatrix} -k \\ k \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \end{bmatrix}$$

The other eigen vector is obtained by putting the other eigen value

$\lambda = 6$ in eigen value problem

$$\begin{bmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which gives,

$$\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which gives the equation

$$-x_1 + 3x_2 = 0$$

and $\quad x_1 - 3x_2 = 0$

Which is only one equation

$$-x_1 + 3x_2 = 0$$

Whose solution is

$$\hat{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3k \\ k \end{bmatrix}$$

Which after normalization is

$$\frac{\hat{x}_2}{|\hat{x}_2|} = \frac{1}{\sqrt{\left((3k)^2 + k^2\right)}} \begin{bmatrix} 3k \\ k \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

Choice (b) $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ is the only correct choice, since it is a constant multiple of one the normalized vectors which is $\hat{x}_1$.

76. (b)

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Characteristic equation of A is

$$\begin{vmatrix} -5-\lambda & -3 \\ 2 & 0-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(-\lambda) + 6 = 0$$

$\Rightarrow \quad \lambda^2 + 5\lambda + 6 = 0$

So, $\quad A^2 + 5A + 6I = 0$

(by Cayley Hamilton theorem)

$\Rightarrow \quad A^2 = -5A - 6I$

Multiplying by A on both sides, we have,

$$A^3 = -5A^2 - 6A$$

$\Rightarrow \quad A^3 = -5(-5A - 6I) - 6A$

$$= 19A + 30I$$

77. Sol.

The minimum number of multiplications required to multiply $A_{m \times n}$ with $B_{n \times p}$ is mnp. To compute PQR if we multiply PQ first and then R the number of multiplications required would be $4 \times 2 \times 4$ to get PQ and then $4 \times 4 \times 1$ multiplications to mulitply PQ with R. So total multiplications required in this method is

$$4 \times 2 \times 4 + 4 \times 4 \times 1 = 32 + 16 = 48$$

To compute PQR if we multiply QR first and then P the number of multiplications required would be $2 \times 4 \times 1$ to get QR and then $4 \times 2 \times 1$ multiplications to mulitply P with QR.

So total multiplications required in this method is

$$2 \times 4 \times 1 + 4 \times 2 \times 1 = 8 + 8 = 16$$

Therfore, the minimum of multiplication required to compute the matrix PQR is = 16

78. (b)

Take the determinant of given matrix $|A|$

$$= 2[2(4-1) - 1(2-1) + 1(1-2)] - 1[1(4-1) - 1(2-1) + 1(1-2)]$$
$$+1[1(2-1) - 2(2-1) + 1(1-1)] - 1[1(1-2) - 2(1-2) + 1(1-1)]$$
$$= 2[6-1-1] - 1[3-1-1] + 1[1-2+0] - 1[-1+2+0]$$
$$= 2(4) - 1(1) + 1(-1) - 1(1) = 8 - 1 - 1 - 1$$
$$= 5$$

79. (a)

The given matrix can be transformed into the matrix given in options (b) (c) and (d) by elementry operations of the type of $R_i \pm kR_j$ or $C_i \pm kC_j$ only as shown below:

Option (b):

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow[C_3+C_1]{C_2+C_1} \begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix}$$

Option (c):

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow[R_2-R_3]{R_1-R_2} \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$$

Option (d):

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow[R_2+R_3]{R_1+R_2} \begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$$

Option (a): We can show the given matrix can not be converted into option (a) without doing a column exchange which will change the sign of the determinant as can be seen below:

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow[C_3+C_2]{C_2+C_1} \begin{vmatrix} 1 & x+1 & x(x+1) \\ 1 & y+1 & y(y+1) \\ 1 & z+1 & z(z+1) \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$$

80. (b)

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}_{3\times3}$$

order of matrix = 3

Rank = 2

$\therefore$ dimension of null space of $A = 3 - 2 = 1$.

81. (c)

Since, $\cos 2x = \cos^2 x - \sin^2 x$, therefore $\cos 2x$ is a linear combination of $\sin^2 x$ and $\cos^2 x$ and hence these are linearly dependent.

82. (d)

$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 - 2x_2 = 0$$

$$x_1 - x_2 = 0$$

$$\Rightarrow \quad x_1 = x_2$$

i.e. x_1 and x_2 are are having infinite number of solutions.

$\Rightarrow$ Multiple solutions are these.

83. (a, d)

Eigen values are

$$|A - \lambda I| = 0$$

$$\left|\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\therefore \quad \lambda = \pm i$$

to find eigen vector,

$\lambda = +i$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad -i\,x_1 - x_2 = 0$ and $x_1 - ix_2 = 0$

clearly, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -j \end{bmatrix}$ and $\begin{bmatrix} j \\ 1 \end{bmatrix}$, satisfy

$$\lambda = -i \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$ix_1 - x_2 = 0$ and $x_1 + ix_2 = 0$

clearly,

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} j \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ j \end{bmatrix}$, satisfy

Thus, the two eigen value of the given matrix are $\begin{bmatrix} 1 \\ -j \end{bmatrix}, \begin{bmatrix} j \\ -1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} j \\ 1 \end{bmatrix}$.

84. (c)

(i) The Eigen values of symmetric matrix $[A^T = A]$ are purely real.

(ii) The Eigen value of skew-symmetric matrix $[A^T = -A]$ are either purely imaginary or zeros.

85. (d)

$$AX = \lambda X$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$a - b = -1$ ...(i)

$c - d = 1$ ...(ii)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = (-2)\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\Rightarrow \quad a - 2b = -2$ ...(iii)

$c - 2d = 4$ ...(iv)

From equation (i) and (iii), $a = 0$ and $b = 1$

From equation (ii) and (iv), $c = -2$ and $d = -3$

$$\therefore \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

86. (a)

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 5 & 2 \\ 5 & 12-\lambda & 7 \\ 2 & 7 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)[(12-\lambda)(5-\lambda) - 49] - 5[5(5-\lambda) - 14] + 2[35 - 2(12-\lambda)] = 0$$

$$(3-\lambda)[60 - 17\lambda + \lambda^2 - 49] - 5(25 - 5\lambda - 14) + 2(35 - 24 + 2\lambda) = 0$$

$$(3-\lambda)(\lambda^2 - 17\lambda + 11) - 5(11 - 5\lambda) + 2(11 + 2\lambda) = 0$$

$$3\lambda^2 - 51\lambda + 33 - \lambda^3 + 17\lambda^2 - 11\lambda - 55 + 25\lambda + 22 + 4\lambda = 0$$

$$-\lambda^3 + 20\lambda^2 - 33\lambda = 0$$

$$\lambda^3 - 20\lambda^2 + 33\lambda = 0$$

$$\lambda(\lambda^2 - 20\lambda + 33) = 0$$

$$\lambda = 0, 1.82, 18.2$$

So minimum eigen value is 0.

87. (a)

Statement 1 is true as shown below.

$[F]^T$ has a size 1×5

$[C]^T$ has a size 5×3

$[B]$ has a size 3×3

$[C]$ has a size 3×5

$[F]$ has a size 5×1

So $[F]^T[C]^T[B][C][F]$ has a size 1×1. Therefore it is a scalar.

So, Statement 1 is true.

Consider Statement 2: $D^T F D$ is always symmetric.

Now $D' F D$ does not exist since $D'_{3\times5}$, $F_{5\times1}$ and $D_{5\times3}$ are not compatible for multiplication since, $D'_{3\times5} F_{5\times1} = X_{3\times1}$ and $X_{3\times1} D_{5\times3}$ does not exist.

So, Statement 2 is false.

88. Sol.

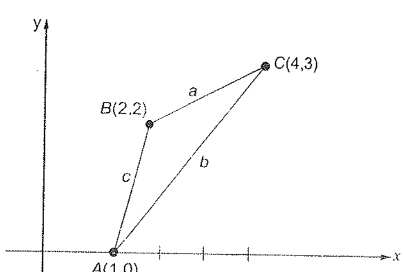
$$J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$K^T JK = [1\ 2\ -1]\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= [6\ 8\ -1]\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 6+16+1 = 23$$

89. (a)



Area of the triangle

$$= \frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

$$= \frac{1}{2}\left| 1(2-3) + 2(3-0) + 4(0-2)\right| = \frac{1}{2}\left| -1+6-8\right|$$

$$= \frac{3}{2}$$

90. (d)

$$(P+Q)^2 = P^2 + PQ + QP + Q^2$$
$$= P\cdot P + P\cdot Q + Q\cdot P + Q\cdot Q$$
$$= P^2 + PQ + QP + Q^2$$

92. (d)

Matrix multiplication is not commutative.

94. Sol.

$$\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$$

$R_4 \to R_4 - R_2 - R_3$

$$\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & -3 & -2 & 1 \end{vmatrix}$$

$R_4 \to R_4 + 3R_1$

$$\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & 4 & 10 \end{vmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 0 & -6 & -8 \\ 0 & 0 & 4 & 10 \end{vmatrix}$$

Interchanging column 1 and column 2 and taking transpose

$$\Delta = -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 3 & -6 & 4 \\ 3 & 0 & -8 & 10 \end{vmatrix}$$

$$= -1\times\begin{vmatrix} 1 & 2 & 0 \\ 3 & -6 & 4 \\ 0 & -8 & 10 \end{vmatrix}$$

$$= -1 \times \{1(-60+32)+2(0-30)\}$$
$$= -(-28 - 60) = 88$$

95. (a)

Let $D = -12$ for the given matrix

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix} = (2)^3\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$$

(Taking 2 common from each row)

$\therefore$ Det(A) $= (2)^3 \times D = 8 \times -12 = -96$

96. Sol.

Determinant of $A = 5$

Determinant of $B = 40$

Determinant of $AB = |A|\,|B|$

$= 5 \times 40 = 200$

97. Sol.

Let, $A = \begin{bmatrix} a & x \\ x & b \end{bmatrix}$

$\Rightarrow \quad |A| = ab - x^2$

Given trace $(A) = a + b = 14$

So, $|A| = a(14 - a) - x^2$

Since, x^2 is always positive maximum value of $a(14 - a) - x^2$ occurs only when $x = 0$.

So now, $|A| = a(14 - a) = 14a - a^2$.

Now maximizing this with respect to a,

$$\frac{d|A|}{da} = 14 - 2a = 0$$

$\Rightarrow \quad a = 7$

Since $\left.\dfrac{d^2|A|}{da^2}\right|_{a=7} = -2 < 0$

At $a = 7$, we have a maximum. The maximum value is $14 \times 7 - 7^2 = 49$

98. Sol.

$$A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} [1 \;\; 9 \;\; 5]$$

$$A = \begin{bmatrix} 2 & 18 & 10 \\ -4 & -36 & -20 \\ 7 & 63 & 35 \end{bmatrix} \Rightarrow |A| = 0$$

99. Sol.

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1 + R_2$

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14-2(6)+(-2) & -14-2(0)+(14) & 0-2(4)+8 & -10-2(4)+(18) \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Determinant of matrix $\begin{bmatrix} 6 & 0 \\ -2 & 14 \end{bmatrix}$ is not zero.

$\therefore$ Rank is 2.

100. (c)

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$B = \begin{bmatrix} p^2 + q^2 & pr + qs \\ pr + qs & r^2 + s^2 \end{bmatrix} = AA^t$$

There are three cases for the rank of A

Case I:

rank $(A) = 0$

$\Rightarrow$ A is null. So $B = AA^T$ also has to be null and hence rank (B) is also equal to 0. Therefore in this case rank (A) = rank (B).

Case II:

rank $(A) = 1$

$\Rightarrow$ A cannot be null. So B also cannot be null since $B = AA^T$

and $\quad |B| = |AA^T| = |A| \cdot |A^T| = |A|^2$

So rank $(B) \neq 0$. Now since rank $(A) \neq 2$ in this case, $|A| = 0$, which means that $|B| = |A|^2 = 0$

So rank (B) is also $\neq 2$. Now since rank $(B) \neq 0$ and $\neq 2$, therefore rank (B) must be equal to 1.

Therefore in this case also rank (A) = rank (B).

Case III:

rank $(A) = 2$

So A has to be non-singular. i.e. $|A| \neq 0$

Therefore, $|B| = |A|^2$ is also $\neq 0$. So rank $(B) = 2$.

Therefore in this case also rank (A) = rank (B).

Therefore, in all three cases rank (A) = rank (B). So rank of A is N, then the rank of matrix B is also N.

101. (b)

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 5 & 1 & 3 & b_2 \end{array}\right] \xrightarrow{R_2 - 5R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -9 & -7 & b_2 - 5b_1 \end{array}\right]$$

Now Gauss elimination is completed. We can see that the Rank(A) = 2.

Rank$[A \mid B]$ is also = 2 (does not depend on value of b_1 and b_2).

Rank(A) = Rank$[A \mid B]$ < Number of variables

Therefore the system is consistent and as infinitely many solutions.

102. (b)

$$\begin{bmatrix}2&1&3&5\\3&0&1&-4\\1&2&5&14\end{bmatrix} R_1 \longleftrightarrow R_3 \begin{bmatrix}1&2&5&14\\3&0&1&-4\\2&1&3&5\end{bmatrix}$$

$$\begin{matrix}R_2 \to R_2 - 3R_1\\R_3 \to R_2 - 2R_1\end{matrix}\begin{bmatrix}1&2&5&14\\0&-6&-14&-46\\0&-3&-7&-23\end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{2}R_2 \begin{bmatrix}1&2&5&14\\0&-6&-14&-46\\0&0&0&0\end{bmatrix}$$

Rank of $[A \mid B]$ = Rank of A < order of matrix

$\Rightarrow$ Infinite number of solutions are possible.

103. Sol.

Given: $3x + 2y = 1$

$4x + 7z = 1$

$x + y + z = 3$

$x - 2y + 7z = 0$

$x + y + z = 3$

$-x - 2y + 7z = 0$

$3y - 6z = 3$

$\Rightarrow \quad y - 2z = 1$

$\Rightarrow \quad 2y - 4z = 2$

$2y + 3x = 1$

$3x + 7z = -1$

$4x + 7z = 1$

$x = 2$

$x = 2$

$3x + 2y = 1$

$\Rightarrow \quad y = -5/2$

$\Rightarrow \quad 4x + 7z = 1$ (Put $x = 2$)

$8 + 7z = 1$

$z = -1$

$\therefore$ The number of solutions for this system is one.

$x = 2$, $y = -5/2$ and $z = -1$ is the only solution.

104. (a)

Sum of eigen values = trace of matrix

= 215 + 150 + 550 = 915

105. (d)

3 × 3 real symmetric matrix such that two of its eigen value are $a \neq 0$ $b \neq 0$ with respective eigen vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$ if $a \neq b$ then

$x_1y_1 + x_2y_2 + x_3y_3 = 0$ because they are orthogonal.

$\therefore \quad x^Ty = 0$ (since $a \neq b$)

$$[x_1 \; x_2 \; x_3]\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = 0$$

106. (d)

The characteristic equation $|A - \lambda I| = 0$

i.e. $\begin{vmatrix}-5-\lambda & 2\\-9 & 6-\lambda\end{vmatrix} = 0$

or $(\lambda - 6)(\lambda + 5) + 18 = 0$

or $\lambda^2 - 6\lambda + 5\lambda - 30 + 18 = 0$

or $\quad \lambda^2 - \lambda - 12 = 0$

or $\quad \lambda = \dfrac{1 \pm \sqrt{1+48}}{2} = \dfrac{1 \pm 7}{2} = 4,$ -3

Corresponding to $\lambda = 4$, we have

$$[A - \lambda I]x = \begin{bmatrix}-5-\lambda & 2\\-9 & 6-\lambda\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0$$

or, $\begin{bmatrix}-9 & 2\\-9 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0$

which gives only one independent equation, $-9x + 2y = 0$

$\therefore \dfrac{x}{2} = \dfrac{y}{9}$ gives eigen vector (2, 9)

Corresponding to $\lambda = -3$,

$$= \begin{bmatrix}-2 & 2\\-9 & 9\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0$$

which gives $-x + y = 0$ (only one independent equation)

$\therefore \quad \dfrac{x}{1} = \dfrac{y}{1}$ which gives (1,1)

So, the eigen vectors are $\begin{Bmatrix}2\\9\end{Bmatrix}$ and $\begin{Bmatrix}1\\1\end{Bmatrix}$.

107. Sol.

Characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix}-\lambda & 1 & -1\\-6 & -11-\lambda & 6\\-6 & -11 & 5-\lambda\end{vmatrix} = 0$$

$\Rightarrow -\lambda[-55 + 11\lambda - 5\lambda + \lambda^2 + 66] - 1[-30 + 6\lambda + 36]$
$-1[66 - 66 - \lambda 6] = 0$

$\Rightarrow -\lambda(\lambda^2 + 6\lambda + 11) - 1(6\lambda + 6) + 6\lambda = 0$

$\Rightarrow -\lambda^3 - 6\lambda^2 - 11\lambda - 6 = 0$

$\Rightarrow \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$\Rightarrow (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$

$\Rightarrow \lambda = -1, -2, -3$

Maximum eigen value is −1 of λ are $|\lambda| = 1, 2, 3$.

Ratio of maximum and minimum eigen value is

$$= 3 : 1 = \frac{3}{1} = 3$$

108. Sol.

Since, $A^2 = I$, $\text{eig}(A^2) = \text{eig}(I) = 1$

$\Rightarrow \text{eig}(A)^2 = 1$

$\Rightarrow \text{eig}(A) = \pm 1$

Therefore, the positive eigen value of A is +1.

109. Sol.

The value of the dot product of the eigenvectors corresponding to any pair of different eigen values of a 4 × 4 symmetric positive definite matrix is 0.

110. Sol.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ let } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$Ak = Xk$

$\rightarrow x_1 + x_5 = kx_1 = kx_5$

$\Rightarrow x_2 + x_3 + x_4 = kx_2 = kx_3 = kx_4$

(i) $k \neq 0$

say, $x_1 = x_5 = a$

$x_2 = x_3 = x_4 = b$

$\Rightarrow x_1 + x_5 = kx_1$

$\Rightarrow 2a = ka$

$\Rightarrow k = 2$

$\Rightarrow x_2 + x_3 + x_4 = kx_2$

$\Rightarrow 3b = kb$

$\Rightarrow k = 3$

(ii) $k = 0$

$\Rightarrow$ Eigen value $k = 0$

∴ There are 3 distinct eigen values: 0,2,3

Product of non-zero eigen values: 2 × 3 = 6

111. (a)

If either the trace or determinant is positive, there exist at least one positive eigen value.

Trace of the matrix is positive and the determinant of the matrix is negative, this is possible only when there is odd number of negative eigen values. Hence at least one eigen value is negative.

112. (a)

Property of determinant : If any two rows or columns are interchanged, then mangnitude of determinant remains same but sign changes.

113. Sol.

Since operations 1 and 2 are elementary operations of the type of $R_i \pm kR_j$ and $C_i \pm kC_j$ respectively, the determinant will be unchanged from the original determinant.

So the required determinant = $\begin{vmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{vmatrix}$

$$\begin{vmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{vmatrix} \xrightarrow{C_3 - 15C_1} \begin{vmatrix} 3 & 4 & 0 \\ 7 & 9 & 0 \\ 13 & 2 & 0 \end{vmatrix} = 0$$

So the required determinant = 0.

114. (c)

Long Method:

$$A = \begin{bmatrix} 1 & \tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}[adj\,(A)]^T$$

$$= \frac{1}{1 + \tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{\sec^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

Here, $A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \frac{1}{\sec^2 x} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$$= \frac{1}{\sec^2 x}\begin{bmatrix} 1 - \tan^2 x & -2\tan x \\ 2\tan x & 1 - \tan^2 x \end{bmatrix}$$

$$A^T A^{-1} = \begin{bmatrix} \frac{1-\tan^2 x}{\sec^2 x} & \frac{-2\tan x}{\sec^2 x} \\ \frac{2\tan x}{\sec^2 x} & \frac{1-\tan^2 x}{\sec^2 x} \end{bmatrix}$$

$$\left|A^T A^{-1}\right| = \left(\frac{1-\tan^2 x}{\sec^2 x}\right)^2 + \left(\frac{2\tan x}{\sec^2 x}\right)^2$$

$$= \frac{1+\tan^4 x - 2\tan^2 x + 4\tan^2 x}{\sec^4 x}$$

$$= 1$$

(or)

Short Method:

Since $|AB| = |A|\ |B|$

$$\left|A^T A^{-1}\right| = \left|A^T\right|\left|A^{-1}\right|$$

$$= |A| \times \frac{1}{|A|} = 1$$

$$\left(\text{Note} : \left|A^T\right| = |A| \text{ and } \left|A^{-1}\right| = \frac{1}{|A|}\right)$$

115. (a)

$$P = \begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$$

$$P^{-1} = \frac{\begin{bmatrix} 4-3i & -(-i) \\ -i & 4+3i \end{bmatrix}}{|A|}$$

$$= \frac{1}{24}\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$$

116. (b)

Rank of $A = 1$

Because each row will be scalar multiple of first row. So we will get only one non-zero row in row Echeleaon form of A.

Alternative:

Rank of $A = 1$

Because all the minors of order greater than 1 will be zero.

117. Sol.

Given system of equations has no solution if the lines are parallel i.e., their slopes are equal

$$\frac{2}{3} = \frac{3}{p}$$

$$\Rightarrow \quad p = 4.5$$

118. (a)

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $|A| \neq 0$ then $AX = B$ can be written as $X = A^{-1}B$. It leads unique solutions.

If $|A| \neq 0$ then $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \neq 0$ each λ_i is non-zero.

If $|A| \neq 0$ then all the row (column) vectors of A are linearly independent.

119. Sol.

$$x - 2y + 3z = -1,$$
$$x - 3y + 4z = 1, \text{ and}$$
$$-2x + 4y - 6z = k$$

$$[A : B] = \begin{bmatrix} 1 & -2 & 3 & : & -1 \\ 1 & -3 & 4 & : & 1 \\ -2 & 4 & 6 & : & k \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & -2 & 3 & : & -1 \\ 0 & -1 & 1 & : & 2 \\ 0 & 0 & 0 & : & k-2 \end{bmatrix}$$

For infinite may solution

$$\rho(A : B) = \rho(A)$$
$$= r < \text{number of variables}$$
$$\rho(A : B) = 2$$
$$k - 2 = 0$$
$$k = 2$$

120. (c)

$$px + qy + rz = 0$$
$$qx + ry + pz = 0$$
$$rx + py + qz = 0$$

Let $A = \begin{bmatrix} p & q & r \\ q & r & p \\ r & p & q \end{bmatrix}$. The system is $A\hat{x} = O$

This is a homogenous system. Such a system has non-trivial solution iff $|A|=0$.

So, $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

$p(qr - p^2) - q(q^2 - pr) + r(pq - r^2) = 0$

$p^3 + q^3 + r^3 - 3pqr = 0$

$p = q = r$ satisfies the above equation.

Also if $p + q + r = 0$ then a can be transformed into one of the row as completely 0's as shown below.

$$\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} \xrightarrow{R_1+R_2+R_3} \begin{vmatrix} p+q+r & p+q+r & p+q+r \\ q & r & p \\ r & p & q \end{vmatrix}$$

$$= (p+q+r)\cdot \begin{vmatrix} 1 & 1 & 1 \\ q & r & p \\ r & p & q \end{vmatrix} = 0$$

Therefore the correct option is (c) which is $p + q + r = 0$ or $p = q = r$.

121. (c)

Given $AX = 0$

$\rho(A_{n\times n}) = r\,(0 < r < n)$

p = Number of independent solutions = nullity

We know that

rank + nullity = n

$r + p = n$

$p = n - r$

122. (d)

For eigen values $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -2 & 2 \\ 4 & -4-\lambda & 6 \\ 2 & -3 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow (3 - \lambda)(-20 + 4\lambda - 5\lambda + \lambda^2 + 18)$

$+ 2(20 - 4\lambda - 12) + 2(-12 + 8 + 2\lambda) = 0$

$\Rightarrow \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$

Only 1 and 2 satisfy this equation.

$\lambda = 1, 1, 2$

Hence,Smallest eigen value = 1 and

Largest eigen value = 2

123. Sol.

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

$\therefore \quad |A - \lambda I| = 0$

$$\begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda) - 2 = 0$$

$(\lambda-4)(\lambda-3)-2 = 0$

$\lambda^2 - 7\lambda + 10 = 0$

$\Rightarrow \quad \lambda = 5, 2$

Miniumum value = 2

124. Sol.

$AX = \lambda X$

$$\begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 12 \\ p+7 \\ 36 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\frac{p+7}{12} = 2 \Rightarrow p = 17$$

125. Sol.

$|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 5 \\ 2 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) - 10 = 0$$

$\lambda^2 - 5\lambda - 6 = 0$

$(\lambda - 6)(\lambda + 1) = 0$

$\Rightarrow \quad \lambda = 6, -1$

$\therefore$ Maximum eigen value is '6'.

126. (b)

Let $$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Given eigen value $\lambda = 1$.

Lex X be the vector. Then $[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-\lambda & -1 & 2 \\ 0 & 1-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{bmatrix} X = 0$$

put $\lambda = 1$

$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -x_2 + 2x_3 \\ 0 \\ x_1 + 2x_2 \end{bmatrix} = 0$$

putting $x_1 = k$ we get $x_2 = -k/2$ and $x_3 = -k/4$

So the eigen vector $= k\begin{bmatrix} 1 \\ -1/2 \\ -1/4 \end{bmatrix}$

The ratios are $x_1/x_2 = \dfrac{-1}{-1/2} = -2$

and $x_2/x_3 = \dfrac{-1/2}{-1/4} = 2$

Only option (b) (–4, 2, 1) has the same ratios and therefore is a correct eigen vector.

127. (d)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$$

Let λ_1 and λ_2 be the eigen value of matrix A

$$\therefore \quad \frac{\lambda_1}{\lambda_2} = \frac{3}{1} \text{ for } p = 2$$

Sum of eigen value

$$= \lambda_1 + \lambda_2 = 2 + p \qquad \text{...(i)}$$

Product of eigen value

$$= \lambda_1\lambda_2 = 2p - 1 \qquad \text{...(ii)}$$

$$\frac{\lambda_1}{\lambda_2} = \frac{3}{1}$$

$$\Rightarrow \quad \lambda_1 = 3\lambda_2$$

From equation (i)

$$\Rightarrow \quad 3\lambda_2 + \lambda_2 = 2 + p$$

$$4\lambda_2 = 2 + p$$

$$\lambda_2 = \frac{p+2}{4}$$

From equation (ii)

$$\Rightarrow \quad 3\lambda_2^2 = 2p - 1$$

$$\Rightarrow \quad 3\left(\frac{p+2}{4}\right)^2 = 2p - 1$$

$$\Rightarrow \quad p = 2, \frac{14}{3}$$

OR

$$\begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$$

$$|A - I\lambda| = 0$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & p-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(p-\lambda) - 1 = 0$$

$$\lambda^2 - (p+2)\lambda + (2p-1) = 0$$

By putting values of p from options.

By putting option (d) $\dfrac{14}{3}$ in above equations gives value $5, \dfrac{5}{3}$.

Hence ratio of two eigen values $= \dfrac{5}{5/3} = 3:1$.

So option (d) is correct.

128. (b)

For singular matrix

$$|A| = 0$$

According to properties of eigen value

Product of eigen values $= |A| = 0$

$\Rightarrow$ Alleast one of the eigen value is zero.

129. (b)

Characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} -3-\lambda & 0 & -2 \\ 1 & -1-\lambda & 0 \\ 0 & a & -2-\lambda \end{vmatrix} = 0$$

$$-(3+\lambda)\left[(1+\lambda)(2+\lambda) - 0\right] - 2(a-0) = 0$$

$$2a = -(\lambda+1)(\lambda+2)(\lambda+3)$$

$$= -(\lambda+1)(\lambda^2+5\lambda+6)$$

$$2a = -(\lambda^3 + 6\lambda^2 + 11\lambda + 6)$$

$$\frac{2da}{d\lambda} = -(3\lambda^2 + 12\lambda + 11)$$

$$= 0 \quad \text{(for a maxima and minima)}$$

$$3\lambda^2 + 12\lambda + 11 = 0$$

$$\lambda = \frac{-12 \pm \sqrt{144 - 132}}{6} = -2 \pm \frac{1}{\sqrt{3}}$$

$$\lambda = -2 + \frac{1}{\sqrt{3}}$$

$$2a = -\left(-2 + \frac{1}{\sqrt{3}} + 1\right)\left(-2 + \frac{1}{\sqrt{3}} + 2\right)\left(-2 + \frac{1}{\sqrt{3}} + 3\right)$$

$$= -\left(\frac{1}{\sqrt{3}} - 1\right)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}} + 1\right) = -\left(\frac{1}{3} - 1\right)\left(\frac{1}{\sqrt{3}}\right)$$

$$2a = \frac{2}{3} \times \frac{1}{\sqrt{3}}$$

$$a = \frac{1}{3\sqrt{3}}$$

130. (b)

For a matrix containing complex number, eigen values are real if and only if

$$A = A^{\theta} = (\bar{A})^T$$

$$A = \begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

$$A^{\theta} = (\bar{A})^T = \begin{bmatrix} 10 & \bar{x} & 4 \\ 5-j & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

By comparing these,

$$x = 5 - j$$

131. (d)

$$\text{Trace} = \text{Sum of eigen values}$$

$$1 + a = 6$$

$$\Rightarrow \quad a = 5$$

Determinant = Product of eigen values

$$(a - 4b) = -7$$

$$5 - 4b = -7$$

$$-4b = -12$$

$$\Rightarrow \quad b = 3$$

$$\therefore \quad a = 5, \; b = 3$$

132. (c)

A is skew-symmetric

$$\therefore \quad A^T = -A$$

133. (c)

Given that $M^4 = I$ or $M^{4k} = I$ or $M^{4(k+1)} = I$

$$\therefore \quad M^{-1} \times I = M^{4(k+1)} \times M^{-1}$$

$$\therefore \quad M^{-1} = M^{4k+3}$$

134. Sol.

$$\text{Trace of } A = 14$$

$$a + 5 + 2 + b = 14$$

$$a + b = 7 \qquad \text{...(i)}$$

$$\det(A) = 100$$

$$5\begin{vmatrix} a & 3 & 7 \\ 0 & 2 & 4 \\ 0 & 0 & b \end{vmatrix} = 100$$

$$5 \times 2 \times a \times b = 100$$

$$10\,ab = 100$$

$$ab = 10 \qquad \text{... (ii)}$$

From equation (i) and (ii)

either $\quad a = 5, \; b = 2$

or $\quad a = 2 \; b = 5$

$$|a - b| = |5 - 2| = 3$$

135. (b)

Result, Rank $(A^T A)$ = Rank (A)

136. (c)

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$3x + y = a$$

$$x + 3y = b$$

$$a^2 + b^2 = 1$$

$$\Rightarrow \quad 10x^2 + 10y^2 + 12xy = 1$$

Ellipse with major axis along $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

137. (d)

$$\begin{bmatrix} 4 \\ 3 \\ -3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$a - 2c = 4$$

$$b = 3$$

$$2a + c = -3$$

from here $\quad a = -\dfrac{2}{5}$

$$b = 3$$

$$c = -\frac{11}{5}$$

$$u = -\frac{2}{5}e_1 + 3e_2 - \frac{11}{5}e_3$$

138. (b)

We can represent the system of equation in matrix form as

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 3 \\ 5 & 9 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[A : B] = \begin{bmatrix} 1 & 2 & -3 & : & a \\ 2 & 3 & 3 & : & b \\ 5 & 9 & -6 & : & c \end{bmatrix}$$

By elementary operation $R_3 \to R_3 - (3R_1 - R_2)$

$$[A : B] = \begin{bmatrix} 1 & 2 & -3 & : & a \\ 2 & 3 & 3 & : & b \\ 0 & 0 & 0 & : & c - 3a - b \end{bmatrix}$$

For consisting of system, $c - 3a - b = 0$

139. (d)

$$\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -30 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix}$$

$R_2 + 2R_1$

$$\begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$$

$$13y = -26$$

or $y = -2$

$$2x + 5y = 2$$

$$2x + 5(-2) = 2$$

$$2x = 2 + 10$$

$$2x = 12$$

or $x = 6$

140. (c)

I. $m < n$ (system may still be inconsistent so incorrect)

II. $m > n$ (rank may still be equal to n of hence solution may exist so incorrect).

III. $m = n$ (some system rank may be equal to n and hence may have solution so correct).

So only III is correct.

141. (d)

Consider the '2 × 2' square matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow \quad \lambda^2 - (a + d)\lambda + (ad - bc) = 0 \quad ...(i)$

Putting $\lambda = 1$, we get

$1 - (a + d) + ad - bc = 0$

$1 - a - d + ad - (1 - d)(1 - a) = 0$

$1 - a - d + ad - 1 + a + d - ad = 0$

$0 = 0$ which is true.

$\therefore \lambda = 1$ satisfied the eq. (i) but $\lambda = 2, 3, 4$ does not satisfy the eq. (i). For all possible values of a, d.

142. Sol.

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Eigen value are 0, 0, 3

143. (a)

All Eigen values of $A = \begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$ are positive

$$2 > 0$$

$\therefore$ 2 × 2 leading minor must be greater than zero

$$\begin{vmatrix} 2 & 1 \\ 1 & k \end{vmatrix} > 0$$

$$2k - 1 > 0$$

$$2k > 1$$

$$k > \frac{1}{2}$$

144. (d)

$$A = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$$

Trace = sum of eigen values

$2\sigma = \sigma + j\omega + \sigma - j\omega$

$|A|$ = product of eigens

$\sigma^2 - x\omega = (\sigma + j\omega)(\sigma - j\omega) = \sigma^2 + \omega^2$

which is possible only when $x = -\omega$

145. Sol.

Two eigen values are $2 + i$ and 3 of a 3 × 3 matrix. The third eigen value must be $2 - i$

Now, $\quad \Pi\lambda_i = |A|$

$\Rightarrow \quad |A| = (2 + i)(2 - i) \times 3$

$= (4 - i^2) \times 3$

$= 5 \times 3 = 15$

146. (a)

$$\dot{x} = Ax$$

Eigen values are λ_1 and λ_2

We can write,

$$\phi(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

Response due to initial conditions,

$$x(t) = \phi(t) \cdot x(0)$$

$$x(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \alpha\, e^{\lambda 1 t}$$

147. (a)

Eigen values of $A^2 - 3A + 4I$ are

$= (1)^2 - 3(1) + 4$ and $(-2)^2 - 3(-2) + 4 = 2, 14$

Note: $A^2X = \lambda^2 X$

$\Rightarrow X$ is eigen vector for A^2 corresponding to eighen value λ^2

X_1 and X_2 are eigen vector of A corresponding to 1, −2

Then X_1 and X_2 are eigen vector of $A^2 - 3 + 4I$ corresponding to 2, 14.

148. Sol.

Consider $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$(2 - \lambda)(2 - \lambda)(3 - \lambda) = 0$

$\lambda = 2, 2, 3$

$\lambda = 3$ there is one L.I. Eigen vector

$\lambda = 2$ Consider $(A - 2I)x = 0$

rank = 2 The equation are $x_2 = 0$

No. of variables = 3 $\quad x_3 = 0$

Let $x_1 = k$ be independent.

$\therefore$ Eigen vector is $\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Only one independent Eigen vector in the case of $\lambda = 2$

Hence finally no. of L.I. Eigen vectors = 2

149. Sol.

A has an eigen value as zero

$\therefore \quad |A| = 0$

$$\begin{vmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{vmatrix} = 0$$

$3(-63+7x+52)-2(-81+9x+78)+4(-36+42) = 0$

$3(7x-11)-2(9x-3)+4(6) = 0$

$21x - 33 - 18x + 6 + 24 = 0$

$3x - 3 = 0$

$x = 1$

150. Sol.

Eigen values of given matrix A are 1 −1, 3

Eigen values of A^3 are	1,	−1,	27
Eigen values of $3A^2$ are	3,	3,	27
Eigen values of $A^3 - 3A^2$ are	−2,	−4,	0

trace of $A^3 - 3A^2 = -2 - 4 + 0 = -6$

151. Sol.

$\text{Eigen}(A) = 1, 2, 4 \Rightarrow |A| = 1\times 2\times 4 = 8$

Now, $|(A^{-1})^T| = |A^{-1}| = \dfrac{1}{|A|} = \dfrac{1}{8} = 0.125$

152. (d)

By Calyey Hamilton theorem,

$\lambda^3 = \lambda$

$\lambda = 0, 1, -1$

153. Sol.

For $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

equation $\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$

$-\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - \lambda - 1 = 0$

By Cayley Hamilton Theorem

$A^2 - A - I = 0$

$A^2 = A + I$

$A^4 = A^2 + 2A + I = A + I + 2A + I$

$= 3A + 2I$

$A^8 = 9A^2 + 12A + 4I$

$= 9(A + I) + 12A + 4I = 21A + 13I$

$A^{12} = A^4 \cdot A^8 = 144A + 89I$

$= \begin{bmatrix} 233 & 144 \\ 144 & 89 \end{bmatrix}$

$\begin{bmatrix} x[12] \\ x[11] \end{bmatrix} = \begin{bmatrix} 233 & 144 \\ 144 & 89 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x[12] = 233$

154. (b)

Consider a random matrix which satisfy property of square symmetrix matrix contain real values

i.e., $A = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}_{2\times 2}$ whose eigen values +5 and −5. Since, $|A| = -25$ and sum of trace $|A| = 0$ i.e., $+5 - 5 = 0$.

Since, rank of matrix is 2, so atleast one eigen value would be zero for every square matrix size $n > 2$. For $n = 2$

$$\lambda_1^2 + \lambda_2^2 \le \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^2$$

$$\lambda_1^2 + \lambda_2^2 \le 50$$

Both λ_1 and λ_2 are real, since A is real symmetric matrix, which means atleast one eigen value would be in range [−5, 5]. But since in our example no eigen value greater than 5. So IInd statement is wrong.

So, option (b) is only correct.

155. Sol.

The product of eigen value of always equal to the determinant value of the matrix.

$\lambda_1 = 10 \quad \lambda_2 = \text{unknown} \quad |A| = 50$

$\lambda_1 \cdot \lambda_2 = 50$

$10(\lambda_2) = 50$

$\therefore \quad \lambda_2 = 5$

156. Sol.

$$A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$$

Eigen values of A are λ_1, λ_2

$$\lambda_1 + \lambda_2 = 130$$

$$\lambda_1\lambda_2 = -900$$

Given that $x_1 = \begin{bmatrix} 70 \\ \lambda_1 - 50 \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}$

$$x_1^T x_2 = [70 \quad \lambda_1 - 50] \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}$$

$$= 70\lambda_2 - 5600 + 70\lambda_1 - 3500$$

$$= 70(\lambda_1 + \lambda_2) - 9100$$

$$= 70(130) - 9100$$

$$= 9100 - 9100 = 0$$

157. (b)

Product of Eigen values = determinant value

$$= 2(3 - 6) + 1(8 - 0)$$

$$= 2(-3) + 8 = -6 + 8 = 2$$

158. (d)

(a) $|P| = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{2} + \frac{1}{2} = 1$

(b) For P to be orthogonal $P \times P^T = 1$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Since P is orthogonal, its inverse is equal to its transpose

(d) For eigen values

$$|P - \lambda I| = \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 - \lambda & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{1}{\sqrt{2}} - \lambda\right)^2 (1 - \lambda) + \frac{1}{\sqrt{2}}\left[0 - \left(-\frac{1}{\sqrt{2}}(1 - \lambda)\right)\right] = 0$$

$$\left(\frac{1}{\sqrt{2}} - \lambda\right) 2(1 - \lambda) + \frac{1}{2}(1 - \lambda) = 0$$

$$(1 - \lambda)\left(\frac{1}{2} + \lambda^2 - \sqrt{2}\lambda + \frac{1}{2}\right) = 0$$

$$\lambda = 1, \ \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i$$

159. (c)

Characteristics equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1 - \lambda & -1 & 5 \\ 0 & 5 - \lambda & 6 \\ 0 & -6 & 5 - \lambda \end{vmatrix} = 0$$

$$(1 - \lambda)((5 - \lambda)^2 + 36) = 0$$

$$(1 - \lambda)(\lambda^2 - 10\lambda + 61) = 0$$

$$\lambda = 1,$$

$$\lambda = \frac{10 \pm \sqrt{100 - 244}}{2} = \frac{10 \pm 12i}{2} = 5 \pm 6i$$

$$\lambda = 1, 5 \pm 6i$$

160. (d)

Coordinate transformation matrix

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Orthogonal, $\theta = 90°$

Coordinate transformation matrix of mirror image

$$= \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sin 90° & \cos 90° \\ \cos 90° & -\sin 90° \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

161. (a)

The characteristics equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 0 - \lambda & 1 & 0 \\ 0 & 0 - \lambda & 1 \\ 0 & -3 & -4 - \lambda \end{vmatrix} = 0$$

$$-\lambda(4\lambda + \lambda^2 + 3) - 1(0 - 0) = 0$$

$$-\lambda(\lambda^2 + 4\lambda + 3) = 0$$

$$\lambda = 0, (\lambda + 1)(\lambda + 3) = 0$$

$$\lambda = -1, -3$$

$$\lambda = (0, -1, -3)$$

162. (c)

The given matrix is symmetric and all its eigen values are distinct. Hence all its eigen vectors are orthogonal one of the eigen vector is $x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

The corresponding orthogonal vector in the given option is C. i.e. $x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$x_1^T x_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 + 0 - 1 = 0$$

163. Sol.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_4 \to R_4 + R_1$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_4 \to R_4 + R_2$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$ and $R_5 \to R_5 + R_4$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From here,

$\therefore \qquad \rho(A) = 4$

164. (c)

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 & 4 & 5 \\ 5 & 1-\lambda & 2 & 3 & 4 \\ 4 & 5 & 1-\lambda & 2 & 3 \\ 3 & 4 & 5 & 1-\lambda & 2 \\ 2 & 3 & 4 & 5 & 1-\lambda \end{vmatrix} = 0$$

Sum of all elements in any one row must be zero.

i.e., $15 - \lambda = 0$

$\lambda = 15$

165. (c)

$$M = \begin{bmatrix} 5 & 10 & 10 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$:

$$\begin{bmatrix} 1 & 0 & 2 \\ 5 & 10 & 10 \\ 3 & 6 & 6 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 5R_1$ and $R_3 \leftarrow R_3 - 3R_1$:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 10 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

$R_3 \leftarrow R_3 - \frac{6}{10}R_2$:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is in Echelon form

Rank of matrix M is,

$\rho(M) = 2$

166. Sol.

$$P + Q = \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix}$$

$|P + Q| = -16 + 16 = 0$

So, rank $\neq 3$

Take the 2×2 minor $\begin{bmatrix} 0 & -1 \\ 8 & 9 \end{bmatrix} = 8 \neq 0$

So, rank of $P + Q$ is 2.

167. Sol.

$$f(\lambda) = \lambda^3 - 4\lambda^2 + a\lambda + 30 = 0$$

Now 2 is one of roots of this equation

So, $2^3 - 4 \times 2^2 + a \times 2 + 30 = 0$

$\Rightarrow 8 - 16 + 2a + 30 = 0$

$\Rightarrow a = -11$

So, the equation is $\lambda^3 - 4\lambda^2 - 11\lambda + 30 = 0$

Now, by polynomials division we get

$$\frac{\lambda^3 - 4\lambda^2 - 11\lambda + 30}{\lambda - 2} = \lambda^2 - 2\lambda - 15$$

roots of $\lambda^2 - 2\lambda - 15 = 0$ are

$$\lambda = \frac{2 \pm \sqrt{4+60}}{2} = \frac{2 \pm 8}{2} = 5 \text{ and} -3$$

So the eigen values are 2, 5 and –3, the maximum absolute eigen value is 5.

168. (c)

The equation $Ax = b$ becomes according to what is given

$$[a_1, a_2, ...a_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_1 + a_2 + a_3 + \dots a_n$$

where a_i are column vectors in R^n but since we have C_i (not all zero) such that,

$$\sum C_j a_j = 0$$

it means the n column vectors are not linearly independent and hence

rank (A) < n

So we have infinitely many solutions one of which will be J_n, where J_n denotes a n-dim vector of all 1.

169. (a)

$$[A] = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

$$[A - \lambda I] = \begin{bmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$(3 - \lambda)(1 - \lambda) - 8 = 0$$

$$3 - 4\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

170. (a)

$$AB^T = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 38 & 28 \\ 32 & 56 \end{bmatrix}$$

171. (c)

Given that P is inverse of Q.

$P = Q^{-1}$ $P = Q^{-1}$

$PQ = Q^{-1}Q$ $QP = QQ^{-1}$

$PQ = I$ $QP = I$

$\therefore$ $PQ = QP = I$

172. (a)

$$A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$$

Ch. equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & -1 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$5 - 5\lambda - \lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$\lambda = 3, 3$$

Algebraic multiplicity of eigen value 3 is 2. It has only one independent eigen vector exists.

173. (d)

$x + y + z = 4$ ...(1)

$x - y + z = 0$ ...(2)

$2x + y + z = 5$ ...(3)

Adding (1) and (2) & (2) and (3) gives

$2x + 2z = 4$ and $3x + 2z = 5$ which gives $x = 1$, $z = 1$ and $y = 2$

Alt: Option (b) can be eliminated since they do not satisfy 1st condition. Only (d) satisfies 3rd equation.

174. (a)

If A is square matrix of order 100 has 15 distinct eigen values. The of minimal polynomial is a divisor of the characteristic polynomial. Every root of minimal polynomial is also a root of characteristic equation.

175. (c)

$$|Q| = \frac{3}{7}\left(-\frac{9}{49} - \frac{12}{49}\right) - \frac{2}{7}\left(\frac{18}{49} - \frac{4}{49}\right) + \frac{6}{7}\left(\frac{-36}{49} - \frac{6}{49}\right)$$

$= -1$

$$\text{Adj. } Q = \begin{bmatrix} -\frac{21}{49} & \frac{42}{49} & -\frac{14}{49} \\ -\frac{14}{49} & -\frac{21}{49} & -\frac{42}{49} \\ -\frac{42}{49} & -\frac{14}{49} & \frac{21}{42} \end{bmatrix}$$

$$\therefore \quad Q^{-1} = \frac{Adj Q}{|Q|} = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$$

Or $\because Q$ is orthogonal

$\therefore \quad Q^{-1} = Q^T$

176. (c)

Option (a): $|A| = 6 - 5 = 1$

Option (b): $|A| = 9 - 4 = 5$

Option (c): $|A| = 12 - 12 = 0$

Option (d): $|A| = 8 - 18 = -10$

Hence matrix (c) is singular.

177. (d)

$$A = \begin{bmatrix} 2 & -4 \\ 4 & -2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -4 \\ 4 & -2-\lambda \end{vmatrix} = 0$$

$$-4 - 2\lambda + 2\lambda + \lambda^2 + 16 = 0$$

$$\lambda^2 + 12 = 0$$

$$\lambda = \pm 2\sqrt{3}i \quad \text{(Complex eigen values)}$$

(1) $\lambda = 2\sqrt{3}i$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-2\sqrt{3}i & -4 \\ 4 & -2-2\sqrt{3}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2 - 2\sqrt{3}i x_1 = 4x_2$$

$$\frac{x_1}{4} = \frac{x_2}{2-2\sqrt{3}i}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2-2\sqrt{3}i \end{bmatrix}$$

(2) $\lambda = -2\sqrt{3}i$

Consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2+2\sqrt{3}i & -4 \\ 4 & -2+2\sqrt{3}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2 + 2\sqrt{3}i x_1 = 4x_2$$

$$\frac{x_1}{4} = \frac{x_2}{2+2\sqrt{3}i}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2+2\sqrt{3}i \end{bmatrix}$$

$\therefore$ Complex Eigen values and complex Eigen vectors.

178. (b)

$$A = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{pmatrix}$$

$$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 4R_1$$

$$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & -3 & 3 & 9 \end{pmatrix}$$

$$R_3 \to R_3 - \frac{3}{2}R_2$$

$$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Number of non zero rows = 2

rank of A = 2

179. (b)

$$\begin{bmatrix} -4 & 1 & -1 \\ -1 & -1 & -1 \\ 7 & -3 & 1 \end{bmatrix}$$

$$R_1 \longleftrightarrow R_2 \quad \begin{bmatrix} -1 & -1 & -1 \\ -4 & 1 & -1 \\ 7 & -3 & 1 \end{bmatrix}$$

$R_2 - 4R_1,\ R_3 + 7R_1$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 5 & 3 \\ 0 & -10 & -6 \end{bmatrix}$$

$$R_3 + 2R_2 \quad \begin{bmatrix} -1 & -1 & -1 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of non zero rows = 2

rank = 2

180. Sol.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = 4$$

$$|A^{-1}| = \frac{1}{|A|} = \frac{1}{4}$$

181. (c)

Eigen vectors corresponding to distinct eigen values are linearly independent.

Hence, "S2 implies S1".

182. Sol.

$AX = 0$ has infinitely many solutions

So, $|A| = 0$

$$\begin{vmatrix} k & 2k \\ k^2 - k & k^2 \end{vmatrix} = 0$$

$$k^3 - 2k^3 + 2k^2 = 0$$

$$k^2(2 - k) = 0$$

$$k = 0, 2$$

$\Rightarrow$ "two" distinct values of k

183. Sol.

A is 2×2 matrix

$$Tr\, A = 4$$

$$\lambda_1 + \lambda_2 = 4$$

$$tr(A^2) = 5$$

$$\lambda_1^2 + \lambda_2^2 = 5$$

$$(\lambda_1 + \lambda^2)^2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_1 \lambda_2$$

$$16 = 5 + 2\lambda_1 \lambda_2$$

$$2\lambda_1 \lambda_2 = 11$$

$$\lambda_1 \lambda_2 = \frac{11}{2}$$

$$|A| = \frac{11}{2} = 5.5$$

184. Sol.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$(1 - \lambda)\,((2 - \lambda)\,(-2 - \lambda)) - 1(0 - 0) = 0$$

$$\lambda = 1, 2, -2$$

Eigen values of	A are	$1, 2, -2$
Eigen values of	A^3 are	$1, 8, -8$
	A^2 are	$1, 4, 4$
	$4A$ are	$4, 8, -8$
	$5I$ are	$5, 5, 5$
$A^3 - A^2 - 4A + 5I$ are		$1, 1, 1$

$$\therefore \quad |B| = (1)(1)(1) = 1$$

185. (a)

$$N^2 = 0$$

Let Eigen values of N are $\lambda_1, \lambda_2, \lambda_3$;

Eigen values of N^2 are $\lambda^2_1, \lambda^2_2, \lambda^2_3$

But $N^2 = 0$

$$\Rightarrow \quad \lambda_1^2 = 0,\ \lambda_2^2 = 0,\ \lambda_3^2 = 0$$

$\therefore$ Eigen values of N are 0, 0, 0

186. Sol.

$$(x - 2)^2 = 2x - 1$$

$$x^2 - 6x + 5 = 0$$

$$x = 1, 5$$

Smallest value of x is 1

187. (c)

Put $k = -3$ in options,

$$3x + 2ky = -2 \quad \Rightarrow 3x - 6y = -2$$

$$Kx + 6y = 2 \quad \Rightarrow 3x - 6y = -2$$

Both the equations are reducing in the same form, thus the system will have infinite number of solutions.

188. Sol.

$$u = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A = uv^T$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$(1 - \lambda)\,(2 - \lambda) - 2 = 0$$

$$\lambda^2 - 3\lambda = 0$$

$$\lambda(\lambda - 3) = 0$$

$$\lambda = 0$$

or, $\lambda = 3$

The largest eigen value is 3.

189. (d)

Only Eigen vector is $\begin{bmatrix}1\\4\end{bmatrix}$ multiples means that eigen value is repeated since if eigen values were distinct we will get one more independent eigen vector. So, II P has repeated eigen values is true.

I need not be true since $\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$ has repeated eigen values and yet it is invertible. III is true since if a 2 × 2 matrix has only one linearly independent eigen vector, surely it cannot be diagonalized.

190. (b)

Let $\begin{bmatrix}1\\a\end{bmatrix}$ be eigen vector for eigen value.

$\begin{bmatrix}1\\b\end{bmatrix}$ be eigen vector for eigen value.

$$Ax = \lambda x$$

$$\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}1\\a\end{bmatrix} = 1\begin{bmatrix}1\\a\end{bmatrix}$$

$$1 + 2a = 1$$
$$2a = a$$
$$a = 0$$

$$\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}1\\b\end{bmatrix} = 2\begin{bmatrix}1\\b\end{bmatrix}$$

$$1 + 2b = 2$$
$$2b = 2b$$
$$2b = 1$$
$$b = \frac{1}{2}$$
$$a + b = \frac{1}{2}$$

191. (c)

$$A = \begin{bmatrix}3 & -1 & -1\\-1 & 3 & -1\\-1 & -1 & 3\end{bmatrix}$$

Characteristic equation is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix}3-\lambda & -1 & -1\\-1 & 3-\lambda & -1\\-1 & -1 & 3-\lambda\end{vmatrix} = 0$$

$$(3-\lambda)((3-\lambda)^2 - 1) + 1(-3 + \lambda - 1) - 1(1 + 3 - \lambda)$$
$$(3-\lambda)(\lambda^2 - 6\lambda + 9 - 1) + -4 + \lambda - 4 + \lambda$$
$$(3-\lambda)(\lambda^2 - 6\lambda + 8) - 8 + 2\lambda = 0$$
$$3\lambda^2 - 18\lambda + 24 - \lambda^3 + 6\lambda^2 - 8\lambda - 8 + 2\lambda$$
$$(-\lambda^3 + 9\lambda^2 - 24\lambda + 16 = 0)$$
$$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$$
$$\lambda = 1, 4, 4$$

192. (b)

$$\text{Consider } (AB) = \begin{bmatrix}1 & 1 & 2 & 2\\2 & -1 & 3 & 2\\5 & -1 & a & b\end{bmatrix}$$

$$R_2 - 2R_1,\ R_3 - 5R_1$$

$$= 2\begin{bmatrix}1 & 1 & 2 & 2\\0 & -3 & -1 & -2\\0 & -6 & a-10 & b-10\end{bmatrix}$$

$$R_3 - 2R_2$$

$$= \begin{bmatrix}1 & 1 & 2 & 2\\0 & -3 & -1 & -2\\0 & 0 & a-8 & b-6\end{bmatrix}$$

$$a = 8$$
$$b = 6$$

∴ Infinite many solutions.