Ratio and Proportion

Exercise -6.1

Solution 1:

1. Ratio of 63 to 36 is $\frac{63}{36} = \frac{9 \times 7}{9 \times 4} = \frac{7}{4}$

2. Ratio of 44 to 99 is $\frac{44}{99} = \frac{11 \times 4}{11 \times 9} = \frac{4}{9}$

3. Ratio of 161 to 115 is $\frac{161}{115} = \frac{23 \times 7}{23 \times 5} = \frac{7}{5}$

4. Ratio of 135 to 405 is $\frac{135}{405} = \frac{135 \times 1}{135 \times 3} = \frac{1}{3}$

Solution 2:

1. 55 cm, 2 m

   2 m = 2 × 100 = 200 cm

   Ratio of 55 cm to 200 cm $= \frac{55}{200} = \frac{5 \times 11}{5 \times 40} = \frac{11}{40}$

   $\therefore$ Ratio of 55 cm to 2 m = 11 : 40

2. 3.5 Kg, 6500 gm

   3.5 Kg = 3.5 × 1000 = 3500 gm

   Ratio of 3500 gm to 6500 gm $= \frac{3500}{6500} = \frac{500 \times 7}{500 \times 13} = \frac{7}{13}$

   $\therefore$ Ratio of 3.5 Kg to 6500 gm = 7 : 13

3. 3 min 54 sec, 2 min 6 sec

   3 min 54 sec = 3 × 60 + 54 = 180 + 54 = 234 sec

   2 min 6 sec = 2 × 60 + 6 = 120 + 6 = 126 sec

   Ratio of 234 sec to 126 sec $= \frac{234}{126} = \frac{18 \times 13}{18 \times 7} = \frac{13}{7}$

   $\therefore$ Ratio of 3 min 54 sec to 2 min 6 sec = 13 : 7

4. Rs. 11, Rs. 15 and paise 40

   Rs. 11 = 11 × 100 = 1100 paise

   Rs. 15 and paise 40 = 15 × 100 + 40 = 1500 + 40 = 1540 paise

   Ratio of 1100 paise to 1540 paise $= \frac{1100}{1540} = \frac{220 \times 5}{220 \times 7} = \frac{5}{7}$

   $\therefore$ Ratio of Rs. 11 to Rs. 15 and paise 40 = 5 : 7

Solution 3:

1. 117 cm : 52 cm

$= \frac{117}{52}$

$= \frac{13 \times 9}{13 \times 4}$

$= \frac{9}{4}$

$= 9 : 4$

2. 8 years 4 months = 8 x 12 + 4 = 96 + 4 = 100 months

11 years 8 months = 11 x 12 + 8 = 132 + 8 = 140 months

11 years 8 months : 8 years 4 months

$= \frac{140}{100}$

$= \frac{20 \times 7}{20 \times 5}$

$= \frac{7}{5}$

$= 7 : 5$

3. 2.40 m : 1.44 m

$= \frac{2.40}{1.44}$

$= \frac{0.48 \times 5}{0.48 \times 3}$

$= \frac{5}{3}$

$= 5 : 3$

Solution 4:

1. 108 : 100

$= \frac{108}{100}$

$= \frac{4 \times 27}{4 \times 25}$

$= \frac{27}{25}$

2. 25 : 100

$= \frac{25}{100}$

$= \frac{25 \times 1}{25 \times 4}$

$= \frac{1}{4}$

$= 1 : 4$

3. 60 %

$= \frac{60}{100}$

$= \frac{20 \times 3}{20 \times 5}$

$= \frac{3}{5}$

$= 3 : 5$

4. 225 %

$= \frac{225}{100}$

$= \frac{25 \times 9}{25 \times 4}$

$= \frac{9}{4}$

$= 9 : 4$

Solution 5:

1. $\frac{14}{20}$

$= \frac{14}{20}$

$= \frac{14 \times 5}{20 \times 5}$

$= \frac{70}{100}$

$= 70\ \%$

2. 3 : 50

$= \frac{3}{50}$

$= \frac{3 \times 2}{50 \times 2}$

$= \frac{6}{100}$

$= 6\ \%$

3. $\frac{9}{25}$

$= \frac{9 \times 4}{25 \times 4}$

$= \frac{36}{100}$

$= 36\ \%$

4. $\frac{5}{16}$

$= \frac{5 \times \frac{100}{16}}{16 \times \frac{100}{16}}$

$= \frac{\frac{500}{16}}{100}$

$= \frac{500}{16}\ \%$

$= 31.25\ \%$

Solution 6:

1. For ratios $\frac{9}{7}, \frac{3}{2}$

We have $9 \times 2 = 18$ and $3 \times 7 = 21$

But, $9 \times 2 < 3 \times 7$

$\therefore \frac{9}{7} < \frac{3}{2}$

2. For ratios $\frac{5}{4}, \frac{\sqrt{3}}{\sqrt{2}}$

We have $5 \times \sqrt{2} = \sqrt{25} \times \sqrt{2} = \sqrt{50}$ and

$4 \times \sqrt{3} = \sqrt{16} \times \sqrt{3} = \sqrt{48}$

But, $\sqrt{50} > \sqrt{48}$

$\therefore \frac{5}{4} > \frac{\sqrt{3}}{\sqrt{2}}$

3. For ratios $\frac{2.5}{7}, \frac{3}{8}$

We have $2.5 \times 8 = 20$ and $7 \times 3 = 21$

But, $20 < 21$

$\therefore \frac{2.5}{7} < \frac{3}{8}$

4. $\frac{3\sqrt{3}}{2\sqrt{2}}, \frac{2\sqrt{2}}{3\sqrt{3}}$

We have $3\sqrt{3} \times 3\sqrt{3} = (3 \times 3) \times \left(\sqrt{3} \times \sqrt{3}\right) = 9 \times 3 = 27$ and

$2\sqrt{2} \times 2\sqrt{2} = (2 \times 2) \times \left(\sqrt{2} \times \sqrt{2}\right) = 4 \times 2 = 8$

But, $27 > 8$

$\therefore \frac{3\sqrt{3}}{2\sqrt{2}} > \frac{2\sqrt{2}}{3\sqrt{3}}$

Solution 7:

Let the common multiple be x.

The given numbers are 3x and 5x respectively.

Given that their sum is 360.

$\therefore\ 3x + 5x = 360$

$\therefore\ 8x = 360$

$\therefore\ x = \frac{360}{8}$

$\therefore\ x = 45$

$\therefore$ The first number $= 3x = 3 \times 45 = 135$

and the second number $= 5x = 5 \times 45 = 225$

The numbers are 135 and 225.

Solution 8:

Let the common multiple be x.

$\therefore$ The present ages of Shreya and Kavita are 3x and 4x years respectively.

Five years hence,

Shreya's age will be $(3x + 5)$ years and

Kavita's age will be $(4x + 5)$ years

From the given condition,

$\frac{3x + 5}{4x + 5} = \frac{4}{5}$

$\therefore\ 5(3x + 5) = 4(4x + 5)$

$\therefore\ 15x + 25 = 16x + 20$

$\therefore\ 16x - 15x = 25 - 20$

$\therefore\ x = 5$

Shreya's present age $= 3x = 3 \times 5 = 15$ years

Kavita's present age $= 4x = 4 \times 5 = 20$ years

Solution 9:

Let the common multiple be x.

$\therefore$ The length and breadth are 5x cm and 3x cm respectively.

The area of a rectangle = length x breadth

$\therefore 5x \times 3x = 29.4$

$\therefore 15x^2 = 29.4$

$\therefore x^2 = \frac{29.4}{15}$

$\therefore x^2 = 1.96$

$\therefore x = 1.4$

Length of the rectangle = 5x = 5 x 1.4 = 7 cm

Breadth of the rectangle = 3x = 3 x 1.4 = 4.2 cm

Solution 10:

Let the common multiple be x.
∴ The measures of the angles are 2x°, 3x°, 4x° and x° respectively.
By the angle sum property of a quadrilateral,
$2x° + 3x° + 4x° + x° = 360°$
∴ $10x° = 360°$
∴ $x° = 36°$
$m\angle D = x° = 36°$
$m\angle A = 2x° = 2 \times 36° = 72°$
$m\angle B = 3x° = 3 \times 36° = 108°$
$m\angle C = 4x° = 4 \times 36° = 144°$
Now, $m\angle A + m\angle B = 72° + 108° = 180°$ and
$m\angle C + m\angle D = 144° + 36° = 180°$
side AD ∥ side BC (By interior angle test for parallel lines.)
Again, $m\angle A + m\angle D = 72° + 36° \neq 180°$ and
$m\angle B + m\angle C = 108° + 144° \neq 180°$
∴ Side AB is not parallel to side CD.
∴ Only one pair of opposite sides is parallel,
∴ □ABCD is a trapezium.

Solution 11(i):

$$\frac{a}{b} - \frac{a+1}{b+1} = \frac{ab + a - ab - b}{b(b+1)}$$

$$= \frac{a-b}{b(b+1)}$$

Here we have $a < b$

$\therefore\ a - b < 0$

$b(b+1) > 0$

$\therefore \dfrac{a-b}{b(b+1)}$ is negative

$\therefore \dfrac{a}{b} < \dfrac{a+1}{b+1}$

Solution 11(ii):

$$\frac{3}{5} - \frac{3+c}{5+c} = \frac{15 + 3c - 15 - 5c}{5(5+c)}$$

$$= \frac{-2c}{5(5+c)}$$

Here we have $c > 0$,

$\therefore -2c < 0$ and $5(5+c) > 0$

$\therefore \dfrac{-2c}{5(5+c)}$ is neagtive

$\therefore \dfrac{3}{5} < \dfrac{3+c}{5+c}$

Solution 11(iii):

$$\frac{a-1}{b-1}-\frac{a+1}{b+1}$$

$$=\frac{(a-1)(b+1)-(a+1)(b-1)}{(b-1)(b+1)}$$

$$=\frac{ab+a-b-1-ab+a-b+1}{(b-1)(b+1)}$$

$$=\frac{2a-2b}{(b-1)(b+1)}$$

$$=\frac{2(a-b)}{(b-1)(b+1)}$$

Here we have $a > b$ and $b \neq \pm 1$

$$\therefore \frac{2(a-b)}{(b-1)(b+1)} \text{ is positive}$$

$$\therefore \frac{a-1}{b-1} > \frac{a+1}{b+1}$$

Exercise – 6.2

Solution 1:

i. $\frac{a}{b} = \frac{5}{8}$ ...(Given)

$\therefore \frac{b}{a} = \frac{8}{5}$ ...(Invertendo)

$\therefore b : a = 8 : 5$

ii. $\frac{a}{b} = \frac{5}{8}$ ...(Given)

$\therefore \frac{a+b}{b} = \frac{5+8}{8}$ ...(Componendo)

$\therefore \frac{a+b}{b} = \frac{13}{8}$

iii. $\frac{a}{b} = \frac{5}{8}$ ...(Given)

$\therefore \frac{a-b}{b} = \frac{5-8}{8}$ ...(Dividendo)

$\therefore \frac{a-b}{b} = \frac{-3}{8}$

iv. $\frac{a}{b} = \frac{5}{8}$ ...(Given)

$\therefore \frac{a+b}{a-b} = \frac{5+8}{5-8}$ ...(Componendo and Dividendo)

$\therefore \frac{a+b}{a-b} = \frac{13}{-3}$

$\therefore \frac{a+b}{a-b} = -\frac{13}{3}$

Solution 2(i):

$$\frac{p}{q} = \frac{6}{5} \qquad \ldots\text{(Given)}$$

$$\therefore \frac{p}{q} \times \frac{3}{4} = \frac{6}{5} \times \frac{3}{4}$$

$$\therefore \frac{3p}{4q} = \frac{9}{10}$$

$$\frac{3p + 4q}{3p - 4q} = \frac{9 + 10}{9 - 10} \qquad \ldots\text{(Componendo – Dividendo)}$$

$$\therefore \frac{3p + 4q}{3p - 4q} = \frac{19}{-1}$$

$$\therefore \frac{3p + 4q}{3p - 4q} = -\frac{19}{1}$$

Solution 2(ii):

$$\frac{p}{q} = \frac{6}{5} \qquad \ldots\text{(Given)}$$

$$\therefore \frac{p^2}{q^2} = \frac{36}{25} \qquad \ldots\text{(Squaring both the sides)}$$

$$\therefore \frac{p^2}{q^2} \times \frac{1}{2} = \frac{36}{25} \times \frac{1}{2}$$

$$\therefore \frac{p^2}{2q^2} = \frac{36}{50}$$

$$\therefore \frac{p^2}{2q^2} = \frac{18}{25}$$

$$\therefore \frac{p^2 + 2q^2}{2q^2} = \frac{18 + 25}{25} \qquad \ldots\text{(Componendo)}$$

$$\therefore \frac{p^2 + 2q^2}{2q^2} = \frac{43}{25}$$

Solution 2(iii):

$$\frac{p}{q} = \frac{6}{5} \qquad \ldots\text{(Given)}$$

$$\therefore 5p = 6q$$

$$\therefore 5p - 6q = 0$$

$$\text{Now, } \frac{5p - 6q}{3p + 4q} = \frac{0}{3p + 4q} = 0$$

$$\therefore \frac{5p - 6q}{3p + 4q} = 0$$

Solution 2(iv):

$$\frac{p}{q} = \frac{6}{5}$$

$$\therefore \frac{p^3}{q^3} = \frac{216}{125} \qquad \ldots\text{(Cubing both the sides)}$$

$$\frac{p^3 - q^3}{q^3} = \frac{216 - 125}{125} \qquad \ldots\text{(Dividendo)}$$

$$\therefore \frac{p^3 - q^3}{q^3} = \frac{91}{125}$$

$$\therefore \frac{q^3}{p^3 - q^3} = \frac{125}{91} \qquad \ldots\text{(Invertendo)}$$

Solution 3:

$$\frac{7a^2 + 2b^2}{7a^2 - 2b^2} = \frac{113}{13}$$

$$\frac{7a^2 + 2b^2 + 7a^2 - 2b^2}{7a^2 + 2b^2 - (7a^2 - 2b^2)} = \frac{113 + 13}{113 - 13} \qquad \ldots\text{(Compodendo – Dividendo)}$$

$$\therefore \frac{14a^2}{7a^2 + 2b^2 - 7a^2 + 2b^2} = \frac{126}{100}$$

$$\therefore \frac{14a^2}{4b^2} = \frac{126}{100}$$

$$\therefore \frac{a^2}{b^2} = \frac{126}{100} \times \frac{4}{14}$$

$$\therefore \frac{a^2}{b^2} = \frac{9}{25}$$

$$\therefore \frac{a}{b} = \pm\frac{3}{5} \qquad \ldots\text{(Taking square root)}$$

$$\therefore \frac{a}{b} = \frac{3}{5} \text{ or } \frac{a}{b} = -\frac{3}{5}$$

Solution 4(i):

$$\frac{x^2 - 5x + 12}{5x - 12} = \frac{x^2 - x - 4}{x + 4}$$

$$\therefore \frac{x^2 - 5x + 12 + 5x - 12}{5x - 12} = \frac{x^2 - x - 4 + x + 4}{x + 4} \quad \text{...(Componendo)}$$

$$\therefore \frac{x^2}{5x - 12} = \frac{x^2}{x + 4}$$

Now, for $x = 0$, the equation is satisfied

$\therefore$ $x = 0$ is one of the solutions,

When $x \neq 0$, $x^2 \neq 0$

$$\therefore \frac{1}{5x - 12} = \frac{1}{x + 4}$$

$\therefore 5x - 12 = x + 4$

$\therefore 5x - x = 4 + 12$

$\therefore 4x = 16$

$$\therefore x = \frac{16}{4}$$

$\therefore x = 4$

$\therefore$ $x = 0$ or $x = 4$ is the solution.

Solution 4(ii):

$$\frac{x^2 + 8x - 3}{8x - 3} = \frac{x^2 + 4x + 4}{4x + 4}$$

$$\therefore \frac{x^2 + 8x - 3 - (8x - 3)}{8x - 3} = \frac{x^2 + 4x + 4 - (4x + 4)}{4x + 4} \quad \text{...(Dividendo)}$$

$$\therefore \frac{x^2 + 8x - 3 - 8x + 3}{8x - 3} = \frac{x^2 + 4x + 4 - 4x - 4}{4x + 4}$$

$$\therefore \frac{x^2}{8x - 3} = \frac{x^2}{4x + 4}$$

$\therefore x^2 = 0$ or $8x - 3 = 4x + 4$

$\therefore x = 0$ or

$8x - 4x = 4 + 3$

$\therefore 4x = 7$

$$\therefore x = \frac{7}{4}$$

$\therefore$ $x = 0$ and $x = \frac{7}{4}$ is the solution.

Solution 4(iii):

$$\frac{10x^2 + 15x + 63}{5x^2 - 25x + 12} = \frac{2x + 3}{x - 5}$$

$$\therefore \frac{5x(2x + 3) + 63}{5x(x - 5) + 12} = \frac{2x + 3}{x - 5}$$

Substituting a for $2x + 3$ and b for $x - 5$,

$$\therefore \frac{5xa + 63}{5xb + 12} = \frac{a}{b}$$

$\therefore b(5xa + 63) = a(5xb + 12)$

$\therefore 5xab + 63b = 5xab + 12a$

$\therefore 63b = 12a$

$\therefore 21b = 4a$

Resubstituting values of a and b,

$21(x - 5) = 4(2x + 3)$

$\therefore 21x - 105 = 8x + 12$

$\therefore 21x - 8x = 12 + 105$

$\therefore 13x = 117$

$$\therefore x = \frac{117}{13}$$

$\therefore x = 9$

$\therefore x = 9$ is the solution.

Solution 4(iv):

$$\frac{(5x + 2)^2 + (5x - 2)^2}{(5x + 2)^2 - (5x - 2)^2} = \frac{13}{12}$$

$$\therefore \frac{(5x + 2)^2 + (5x - 2)^2 + (5x + 2)^2 - (5x - 2)^2}{(5x + 2)^2 + (5x - 2)^2 - (5x + 2)^2 + (5x - 2)^2} = \frac{13 + 12}{13 - 12}$$

...(Componendo-Dividendo)

$$\therefore \frac{2(5x + 2)^2}{2(5x - 2)^2} = \frac{25}{1}$$

$$\therefore \frac{(5x + 2)^2}{(5x - 2)^2} = \frac{25}{1}$$

$$\therefore \frac{5x + 2}{5x - 2} = \pm 5 \qquad \text{...(Taking square root)}$$

$$\therefore \frac{5x + 2}{5x - 2} = 5 \quad \text{or} \quad \frac{5x + 2}{5x - 2} = -5$$

If $\dfrac{5x + 2}{5x - 2} = 5$

$5x + 2 = 5(5x - 2)$

$\therefore 5x + 2 = 25x - 10$

$\therefore 25x - 5x = 10 + 2$

$\therefore 20x = 12$

$\therefore x = \frac{12}{20}$

$\therefore x = \frac{3}{5}$

If $\frac{5x + 2}{5x - 2} = -5$

$5x + 2 = -5(5x - 2)$

$\therefore 5x + 2 = -25x + 10$

$\therefore 5x + 25x = 10 - 2$

$\therefore 30x = 8$

$\therefore x = \frac{8}{30}$

$\therefore x = \frac{4}{15}$

$\therefore x = \frac{3}{5}$ and $x = \frac{4}{15}$ is the solution.

Solution 4(v):

$$\frac{(3x-4)^3-(x+1)^3}{(3x-4)^3+(x+1)^3}=\frac{61}{189}$$

$$\therefore \frac{(3x-4)^3-(x+1)^3+(3x-4)^3+(x+1)^3}{(3x-4)^3-(x+1)^3-(3x-4)^3-(x+1)^3}=\frac{61+189}{61-189}$$

...(Componendo-Dividendo)

$$\therefore \frac{2(3x-4)^3}{-2(x+1)^3}=\frac{250}{-128}$$

$$\therefore \frac{(3x-4)^3}{-(x+1)^3}=\frac{125}{-64}$$

$$\therefore \frac{(3x-4)^3}{(x+1)^3}=\frac{125}{64}$$

$$\therefore \frac{3x-4}{x+1}=\frac{5}{4} \qquad \text{...(Taking cube root)}$$

$\therefore 4(3x-4)=5(x+1)$

$\therefore 12x-16=5x+5$

$\therefore 12x-5x=5+16$

$\therefore 7x=21$

$\therefore x=\frac{21}{7}$

$\therefore x=3$

$\therefore x=3$ is the sdution.

Exercise – 6.3

Solution 1:

Let $\frac{x}{y+z-x} = \frac{y}{z+x-y} = \frac{z}{x+y-z} = k$

By Theorem on Equal Ratios we get,

$$k = \frac{x+y+z}{y+z-x+z+x-y+x+y-z}$$

$$= \frac{x+y+z}{x+y+z}$$

$$= 1$$

$$\therefore \frac{x}{y+z-x} = \frac{y}{z+x-y} = \frac{z}{x+y-z} = 1$$

Solution 2:

Let $\frac{y+z}{a} = \frac{z+x}{b} = \frac{x+y}{c} = k$

By Theorem on Equal Ratios we get

$$k = \frac{z+x+x+y-y-z}{b+c-a}$$

$$= \frac{2x}{b+c-a} \qquad \ldots(i)$$

Also,

$$k = \frac{x+y+y+z-z-x}{c+a-b}$$

$$= \frac{2y}{c+a-b} \qquad \ldots(ii)$$

Also,

$$k = \frac{y+z+z+x-x-y}{a+b-c}$$

$$= \frac{2z}{a+b-c} \qquad \ldots(iii)$$

From (i), (ii) and (iii),

$$\frac{2x}{b+c-a} = \frac{2y}{c+a-b} = \frac{2z}{a+b-c}$$

$$\frac{x}{b+c-a} = \frac{y}{c+a-b} = \frac{z}{a+b-c} \qquad \ldots\left(\text{Multiplying each ratio by } \frac{1}{2}\right)$$

Solution 3:

$$\frac{a}{y+z-x} = \frac{b}{z+x-y} = \frac{c}{x+y-z}$$

$$\frac{y+z-x}{a} = \frac{z+x-y}{b} = \frac{x+y-z}{c} \quad \ldots(\text{Invertendo})$$

Let $\frac{y+z-x}{a} = \frac{z+x-y}{b} = \frac{x+y-z}{c} = k$

By Theorem on Equal Ratios,

$$k = \frac{z+x-y+x+y-z}{b+c} = \frac{2x}{b+c} \quad \ldots(i)$$

Again,

$$k = \frac{x+y-z+y+z-x}{c+a} = \frac{2y}{c+a} \quad \ldots(ii)$$

Also,

$$k = \frac{y+z-x+z+x-y}{a+b} = \frac{2z}{a+b} \quad \ldots(iii)$$

From (i), (ii) and (iii),

$$\frac{x}{b+c} = \frac{y}{c+a} = \frac{z}{a+b} \quad \ldots\left(\text{Multiplying each ratio by } \frac{1}{2}\right)$$

Solution 4:

$$\frac{z-y}{z-x} \times \frac{(-3)}{(-3)} = \frac{-3z+3y}{-3z+3x} \quad \ldots\left(\frac{a}{b} = \frac{ak}{bk}\right)$$

Now, $\frac{2x-3y}{3z+y} = \frac{-3z+3y}{-3z+3x} = \frac{x+3z}{2y-3x}$,

Let $\frac{2x-3y}{3z+y} = \frac{-3z+3y}{-3z+3x} = \frac{x+3z}{2y-3x} = k$

By Theorem on Equal Ratios,

$$k = \frac{2x-3y-3z+3y+x+3z}{3z+y-3z+3x+2y-3x}$$

$$= \frac{3x}{3y}$$

$$= \frac{x}{y}$$

Solution 5:

Let $\frac{ax + by}{x + y} = \frac{bx + az}{x + z} = \frac{ay + bz}{y + z} = k$

By Theorem on Equal Ratios,

$$k = \frac{ax + by + bx + az + ay + bz}{x + y + x + z + y + z}$$

$$= \frac{x(a + b) + y(a + b) + z(a + b)}{2(x + y + z)}$$

$$= \frac{(a + b)(x + y + z)}{2(x + y + z)}$$

$$= \frac{(a + b)}{2}$$

$\therefore$ Each ratio $= \frac{(a + b)}{2}$

Solution 6:

Let $\frac{x}{x + 2y + z} = \frac{y}{y + 2z + x} = \frac{z}{z + 2x + y} = k$

By Theorem on Equal Ratios,

$$k = \frac{x + y + z}{x + 2y + z + y + 2z + x + z + 2x + y}$$

$$= \frac{x + y + z}{4x + 4y + 4z}$$

$$= \frac{x + y + z}{4(x + y + z)}$$

$$= \frac{1}{4} \qquad \ldots(\because x + y + z \neq 0)$$

Solution 7:

Let $\frac{a}{x+y} = \frac{b}{y+z} = \frac{c}{z-x} = k$

$\therefore \frac{b}{y+z} = k \qquad \ldots(i)$

Also,

$\frac{a}{x+y} = \frac{c}{z-x}$

By Theorem on Equal Ratios,

$k = \frac{a+c}{x+y+z-x}$

$k = \frac{a+c}{y+z} \qquad \ldots(ii)$

$\therefore \frac{b}{y+z} = \frac{a+c}{y+z} \qquad \ldots(\text{from}(i) \text{ and } (ii))$

$\therefore b = a + c$

Solution 8:

$a(y+z) = b(z+x) = c(x+y)$

$\therefore \frac{y+z}{bc} = \frac{z+x}{ca} = \frac{x+y}{ab} \qquad \ldots(\text{dividing by abc})$

Let $\frac{y+z}{bc} = \frac{z+x}{ca} = \frac{x+y}{ab} = k$

By theorem on equal ratios,

$k = \frac{(x+y)-(z+x)}{ab-ca}$

$= \frac{x+y-z-x}{ab-ca}$

$= \frac{y-z}{a(b-c)} \qquad \ldots(i)$

Also,

$k = \frac{(y+z)-(x+y)}{bc-ab}$

$= \frac{y+z-x-y}{bc-ab}$

$= \frac{z-x}{b(c-a)} \qquad \ldots(ii)$

Also,

$$k = \frac{(z + x) - (y + z)}{ca - bc}$$

$$= \frac{z + x - y - z}{ca - bc}$$

$$= \frac{x - y}{c(a - b)} \qquad \ldots(iii)$$

From (i), (ii) and (iii),

$$\frac{y - z}{a(b - c)} = \frac{z - x}{b(c - a)} = \frac{x - y}{c(a - b)}$$

Solution 9:

$$\frac{12x^2 + 18x + 12}{18x^2 + 12x + 58} = \frac{2x + 3}{3x + 2}$$

$$\therefore \frac{12x^2 + 18x + 12}{18x^2 + 12x + 58} = \frac{6x(2x + 3)}{6x(3x + 2)}$$

Let $\frac{12x^2 + 18x + 12}{18x^2 + 12x + 58} = \frac{6x(2x + 3)}{6x(3x + 2)} = k$

By Theorem on Equal Ratios,

$$k = \frac{12x^2 + 18x + 12 - 6x(2x + 3)}{18x^2 + 12x + 58 - 6x(3x + 2)}$$

$$= \frac{12x^2 + 18x + 12 - 12x^2 - 18x}{18x^2 + 12x + 58 - 18x^2 - 12x}$$

$$= \frac{12}{58}$$

$$= \frac{6}{29}$$

$$\therefore \frac{2x + 3}{3x + 2} = \frac{6}{29}$$

$$\therefore 29(2x + 3) = 6(3x + 2)$$

$$\therefore 58x + 87 = 18x + 12$$

$$\therefore 58x - 18x = 12 - 87$$

$$\therefore 40x = -75$$

$$\therefore x = \frac{-75}{40}$$

$$\therefore x = \frac{-15}{8}$$

$\therefore x = \frac{-15}{8}$ is the solution

Solution 10:

$$\frac{16y^2 - 20y + 9}{8y^2 + 12y + 21} = \frac{4y - 5}{2y + 3}$$

$$\therefore \frac{16y^2 - 20y + 9}{8y^2 + 12y + 21} = \frac{4y(4y - 5)}{4y(2y + 3)}$$

Let $\frac{16y^2 - 20y + 9}{8y^2 + 12y + 21} = \frac{4y(4y - 5)}{4y(2y + 3)} = k$

By Theorem on Equal Ratios,

$$k = \frac{16y^2 - 20y + 9 - 4y(4y - 5)}{8y^2 + 12y + 21 - 4y(2y + 3)}$$

$$= \frac{16y^2 - 20y + 9 - 16y^2 + 20y}{8y^2 + 12y + 21 - 8y^2 - 12y}$$

$$= \frac{9}{21}$$

$$= \frac{3}{7}$$

$$\therefore \frac{4y - 5}{2y + 3} = \frac{3}{7}$$

$\therefore 7(4y - 5) = 3(2y + 3)$

$\therefore 28y - 35 = 6y + 9$

$\therefore 28y - 6y = 35 + 9$

$\therefore 22y = 44$

$$\therefore y = \frac{44}{22}$$

$\therefore y = 2$

$\therefore y = 2$ is the solution

Exercise – 6.4

Solution 1(i)(1):

Let the fourth proportional be x

$\therefore \frac{14}{21} = \frac{4}{x}$

$\therefore 14 \times x = 4 \times 21$

$\therefore x = \frac{4 \times 21}{14}$

$\therefore x = \frac{4 \times 3}{2}$

$\therefore x = 6$

$\therefore$ 6 is the fourth proportional to 14, 21, 4.

Solution 1(i)(2):

Let the fourth proportion be x

$\therefore \frac{a^2}{ab} = \frac{b^2}{x}$

$\therefore a^2 \times x = b^2 \times ab$

$\therefore x = \frac{b^2 \times ab}{a^2}$

$\therefore x = \frac{b^3}{a}$

$\frac{b^3}{a}$ is the fourth proportion to a^2, ab, b^2.

Solution 1(i)(3):

Let the fourth proportion be x

$\therefore \frac{5}{\sqrt{75}} = \frac{\sqrt{48}}{x}$

$\therefore x = \frac{\sqrt{48} \times \sqrt{75}}{5}$

$\therefore x = 4\sqrt{3} \times \sqrt{3}$

$\therefore x = 4 \times 3$

$\therefore x = 12$

12 is the fourth proportion to $5, \sqrt{75}, \sqrt{48}$.

Solution 1(ii)(1):

Let the mean proportional be x.

$\therefore x^2 = 8 \times 18$

$\therefore x^2 = 144$

$\therefore x = 12$

$\therefore$ The mean proportional is 12.

Solution 1(ii)(2):

Let the mean proportional be x.

$\therefore x^2 = 8 \times 32$

$\therefore x^2 = 256$

$\therefore x = 16$

$\therefore$ The mean proportional is 16.

Solution 1(ii)(3):

Let the mean proportional be x.

$\therefore x^2 = ak \times ak^3$

$\therefore x^2 = a^2k^4$

$\therefore x = ak^2$

$\therefore$ The mean proportional is ak^2.

Solution 1(iii):

4.8, 6.0, x and 8.5 are in proportion

$\therefore \frac{4.8}{6.0} = \frac{x}{8.5}$

$\therefore x = \frac{4.8 \times 8.5}{6.0}$

$\therefore x = 0.8 \times 8.5$

$\therefore x = 6.8$

Solution 1(iv):

$6a^3b$, $12a^2b^2$, k and $48ab^3$

$\therefore \frac{6a^3b}{12a^2b^2} = \frac{k}{48ab^3}$

$\therefore k = \frac{6a^3b \times 48ab^3}{12a^2b^2}$

$\therefore k = 6ab \times 4ab$

$\therefore k = 24a^2b^2$

Solution 2:

$\frac{p+q}{p-q}$, x and $p^2 - q^2$ are in continued proportion.

$\therefore \frac{\frac{p+q}{p-q}}{x} = \frac{x}{p^2 - q^2}$

$\therefore x^2 = \frac{p+q}{p-q} \times (p^2 - q^2)$

$\therefore x^2 = \frac{p+q}{p-q} \times (p-q)(p+q)$

$\therefore x^2 = (p+q) \times (p+q)$

$\therefore x^2 = (p+q)^2$

$\therefore x = (p+q)$

Solution 3:

x + 1, x + 5, x + 7 and x + 19 are in proportion.

$\therefore \frac{x+1}{x+5} = \frac{x+7}{x+19}$

$\therefore (x+1)(x+19) = (x+7)(x+5)$

$\therefore x^2 + 20x + 19 = x^2 + 12x + 35$

$\therefore 20x - 12x = 35 - 19$

$\therefore 8x = 16$

$\therefore x = \frac{16}{8}$

$\therefore x = 2$

Solution 4:

Let x be the number to be added to each of the given numbers.

Then the numbers $(1 + x)$, $(9 + x)$, $(13 + x)$ and $(45 + x)$ are in proportion.

$\therefore \frac{(1 + x)}{(9 + x)} = \frac{(13 + x)}{(45 + x)}$

$\therefore (1 + x)(45 + x) = (13 + x)(9 + x)$

$\therefore 45 + 46x + x^2 = 117 + 22x + x^2$

$\therefore 46x - 22x = 117 - 45$

$\therefore 24x = 72$

$\therefore x = \frac{72}{24}$

$\therefore x = 3$

Thus, 3 should be added to each of the given numbers

to make them in proportion.

Solution 5:

Let x be the number to be subtracted from each of the given numbers.

Then the numbers $(13 - x)$, $(25 - x)$ and $(55 - x)$ are in continued proportion.

$\therefore (25 - x)^2 = (13 - x)(55 - x)$

$\therefore 625 - 50x + x^2 = 715 - 68x + x^2$

$\therefore -50x + 68x = 715 - 625$

$\therefore 18x = 90$

$\therefore x = \frac{90}{18}$

$\therefore x = 5$

5 should be subtracted from each of the given numbers.

Solution 6:

$(a + b + c)(a - b + c) = a^2 + b^2 + c^2$

$\therefore [(a + c) + b][(a + c) - b] = a^2 + b^2 + c^2$

$\therefore (a + c)^2 - b^2 = a^2 + b^2 + c^2$

$\therefore a^2 + 2ac + c^2 = a^2 + 2b^2 + c^2$

$\therefore 2ac = 2b^2$

$\therefore b^2 = ac$

Thus, a, b, and c are in continued proportion.

Solution 7:

$(x - 4)$ is the geometric mean of $(x - 5)$ and $(x - 2)$

$\therefore (x - 4)^2 = (x - 5)(x - 2)$

$\therefore x^2 - 8x + 16 = x^2 - 7x + 10$

$\therefore -8x + 7x = 10 - 16$

$\therefore -x = -6$

$\therefore x = 6$

Solution 8:

a, b, and c are in continued proportion.

$\therefore ac = b^2 \qquad \ldots(i)$

$$\begin{aligned} \text{L.H.S.} &= (ab + bc + ac)^2 \\ &= (ab + bc + b^2)^2 && \ldots[\text{from}(i)] \\ &= [b(a + c + b)]^2 \\ &= b^2(a + c + b)^2 \\ &= ac(a + b + c)^2 && \ldots[\text{from}(i)] \\ &= \text{R.H.S.} \end{aligned}$$

$\therefore (ab + bc + ac)^2 = ac(a + b + c)^2$

Solution 9:

$$\frac{(ab + c^2)}{(a - b + c)} = \frac{(b + c)}{1} \qquad \ldots(\text{given})$$

$\therefore ab + c^2 = (b + c)(a - b + c)$

$\therefore ab + c^2 = b(a - b + c) + c(a - b + c)$

$\therefore ab + c^2 = ab - b^2 + bc + ac - bc + c^2$

$\therefore 0 = -b^2 + ac$

$\therefore b^2 = ac$

$\therefore$ b is the geometric mean between a and c.

Solution 10:

$b - c = \frac{2b}{a}$ ...(given)

$\therefore a(b - c) = 2b$

$\therefore a(b - c) = (a - b) \times b$ ...(given $2 = a - b$)

$\therefore ab - ac = ab - b^2$

$\therefore b^2 = ac$

Solution 11:

Let a, b, c, d and e be the five numbers in continued proportion.

Second term is 6 and fourth term is 54.

$\therefore$ a, 6, c, 54 and e are in continued proprotion.

$\therefore c^2 = 6 \times 54$

$\therefore c^2 = 6 \times 6 \times 3 \times 3$

$\therefore c^2 = 6^2 \times 3^2$

$\therefore c = 6 \times 3$

$\therefore c = 18$

Further, a, 6 and c are in continued proportion.

$\therefore 6^2 = a \times c$

$\therefore 36 = a \times 18$

$\therefore a = \frac{36}{18}$

$\therefore a = 2$

Now, c, 54 and e are in continued proportion.

$\therefore 54^2 = c \times e$

$\therefore 54^2 = 18 \times e$

$\therefore e = \frac{54 \times 54}{18}$

$\therefore e = 3 \times 54$

$\therefore e = 162$

2, 6, 18, 54 and 162 are the required numbers.

Solution 12:

Let a, b, c, d, e be the five numbers in continued proportion.

Then $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e}$

The first term is 1 and the last term is 256.

$\therefore \frac{1}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{256}$

Now, $\frac{1}{b} = \frac{d}{256}$

$\therefore bd = 256 \quad \ldots(i)$

$\frac{b}{c} = \frac{c}{d}$

$\therefore bd = c^2 \quad \ldots(ii)$

From (i) and (ii),

$c^2 = 256$

$\therefore c = 16$

Now,

$\frac{1}{b} = \frac{b}{c}$

$\therefore b^2 = c$

$\therefore b^2 = 16$

$\therefore b = 4$

Also,

$\frac{c}{d} = \frac{d}{256}$

$\therefore d^2 = c \times 256$

$\therefore d^2 = 16 \times 256$

$\therefore d = 4 \times 16$

$\therefore d = 64$

$\therefore$ 1, 4, 16, 64 and 256 are the required numbers.

Exercise – 6.5

Solution 1:

Let $\frac{a}{b} = \frac{c}{d} = k$

$\therefore$ $a = bk$ and $c = dk$

$\text{L.H.S.} = \frac{a}{b} = \frac{bk}{b} = k$

$\text{R.H.S.} = \sqrt{\frac{a^2 + c^2}{b^2 + d^2}}$

$= \sqrt{\frac{b^2k^2 + d^2k^2}{b^2 + d^2}}$

$= \sqrt{\frac{k^2(b^2 + d^2)}{b^2 + d^2}}$

$= \sqrt{k^2}$

$= k$

$\therefore$ L.H.S. = R.H.S.

$\therefore \frac{a}{b} = \frac{\sqrt{a^2 + b^2}}{\sqrt{b^2 + d^2}}$.

Solution 2:

a, b, c, and d are in proportion.

$\therefore$ Let $\frac{a}{b} = \frac{c}{d} = k$

$\therefore$ $a = bk$ and $c = dk$

$\text{L.H.S.} = \frac{a^2}{b^2} = \frac{b^2k^2}{b^2} = k^2$

$\text{R.H.S.} = \frac{ac}{bd} = \frac{bk \times dk}{bd} = k^2$

$\therefore$ L.H.S. = R.H.S.

$\therefore \frac{a^2}{b^2} = \frac{ac}{bd}$

Solution 3:

a, b, and c are in continued proportion.

$\therefore$ Let $\frac{a}{b} = \frac{b}{c} = k$

$\therefore$ $a = bk$ and $b = ck$

$\therefore$ $a = ck.k$

$\therefore$ $a = ck^2$

$$\text{L.H.S.} = \frac{(a+b)^2}{(b+c)^2}$$

$$= \frac{(ck^2 + ck)^2}{(ck + c)^2}$$

$$= \frac{[ck(k+1)]^2}{[c(k+1)]^2}$$

$$= \frac{c^2k^2}{c^2}$$

$$= k^2$$

$$\text{R.H.S.} = \frac{a^2 + b^2}{b^2 + c^2}$$

$$= \frac{(ck^2)^2 + (ck)^2}{(ck^2) + c^2}$$

$$= \frac{c^2k^4 + c^2k^2}{c^2k^2 + c^2}$$

$$= \frac{c^2k^2(k^2+1)}{c^2(k^2+1)}$$

$$= k^2$$

$\therefore$ L.H.S. = R.H.S.

$$\therefore \frac{(a+b)^2}{(b+c)^2} = \frac{a^2 + b^2}{b^2 + c^2}$$

Solution 4:

$b^2 = ac$

$\therefore$ Let $\frac{a}{b} = \frac{b}{c} = k$

$\therefore$ $a = bk$ and $b = ck$

$\therefore$ $a = ck.k$

$\therefore$ $a = ck^2$

$$\text{L.H.S.} = \frac{a - c}{b + c}$$

$$= \frac{ck^2 - c}{ck + c}$$

$$= \frac{c(k^2 - 1)}{c(k + 1)}$$

$$= \frac{(k + 1)(k - 1)}{(k + 1)}$$

$$= (k - 1)$$

$$\text{R.H.S.} = \frac{a - b}{b}$$

$$= \frac{ck^2 - ck}{ck}$$

$$= \frac{ck(k - 1)}{ck}$$

$$= (k - 1)$$

$\therefore$ L.H.S. = R.H.S.

$$\therefore \frac{a - c}{b + c} = \frac{a - b}{b}.$$

Solution 5:

Let $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k.$

Then $c = dk$, $b = ck = (dk)k = dk^2$

$\therefore b = dk^2$

$a = bk = (dk^2)k = dk^3$

$$\text{L.H.S.} = \frac{a^3 + b^3 + c^3}{b^3 + c^3 + d^3}$$

$$= \frac{d^3k^9 + d^3k^6 + d^3k^3}{d^3k^6 + d^3k^3 + d^3}$$

$$= \frac{d^3k^3(k^6 + k^3 + 1)}{d^3(k^6 + k^3 + 1)}$$

$$= k^3$$

$$\text{R.H.S.} = \frac{a}{d}$$

$$= \frac{dk^3}{d}$$

$$= k^3$$

$\therefore$ L.H.S. = R.H.S.

$$\therefore \frac{a^3 + b^3 + c^3}{b^3 + c^3 + d^3} = \frac{a}{d}$$

Solution 6:

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$

Then $a = bk$, $c = dk$ and $e = fk$.

$$\left(\frac{7a^4 - 3c^4 + 5e^4}{7b^4 - 3d^4 + 5f^4}\right)^{\frac{1}{4}} = \left(\frac{7b^4k^4 - 3d^4k^4 + 5f^4k^4}{7b^4 - 3d^4 + 5f^4}\right)^{\frac{1}{4}}$$

$$= \left[\frac{k^4(7b^4 - 3d^4 + 5f^4)}{(7b^4 - 3d^4 + 5f)^4}\right]^{\frac{1}{4}}$$

$$= (k^4)^{\frac{1}{4}}$$

$$= k$$

$$\therefore \text{Each ratio} = \left(\frac{7a^4 - 3c^4 + 5e^4}{7b^4 - 3d^4 + 5f^4}\right)^{\frac{1}{4}}$$

Solution 7:

b is the geometric mean of a and c.

$\therefore\ b^2 = ac$

$$\text{L.H.S.} = a^2b^2c^2\left[\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right]$$

$$= a^2b^2c^2\left(\frac{b^3c^3 + c^3a^3 + a^3b^3}{a^3b^3c^3}\right)$$

$$= (ac)^2b^2\left(\frac{b^3c^3 + (ca)^3 + a^3b^3}{(ac)^3b^3}\right)$$

$$= (b^2)^2 \cdot b^2\left[\frac{b^3c^3 + (b^2)^3 + a^3b^3}{(b^2)^3 \cdot b^3}\right] \qquad \ldots[\text{from (i)}]$$

$$= b^4 \cdot b^2\left(\frac{b^3c^3 + b^6 + a^3b^3}{b^6 \cdot b^3}\right)$$

$$= b^6 \cdot b^3\left(\frac{c^3 + b^3 + a^3}{b^6 \cdot b^3}\right)$$

$$= a^3 + b^3 + c^3$$

$= \text{R.H.S.}$

$$\therefore\ a^2b^2c^2\left[\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right] = a^3 + b^3 + c^3$$

Solution 8:

$$\frac{a}{(a+b)(a+c)} = \frac{1}{a+b+c+d}$$

$\therefore\ a(a+b+c+d) = (a+b)(a+c)$

$\therefore\ a^2 + ab + ac + ad = a^2 + ac + ab + bc$

$\therefore\ ad = bc$

$\therefore\ \frac{a}{b} = \frac{c}{d}$

$\therefore$ a, b, c, and d are in proportion.

Solution 9:

Let $\frac{u}{v} = \frac{w}{x} = \frac{y}{z} = k$

So, $u = vk$, $w = xk$ and $y = zk$

$$\left[\frac{uw^2 + 7wy^2 + 3yu^2}{vx^2 + 7xz^2 + 3zv^2}\right]^{\frac{1}{3}} = \left[\frac{vk \times x^2k^2 + 7 \times xk \times z^2k^2 + 3 \times zk \times v^2k^2}{vx^2 + 7xz^2 + 3zv^2}\right]^{\frac{1}{3}}$$

$$= \left[\frac{vx^2k^3 + 7\,xz^2k^3 + 3\,zv^2k^3}{vx^2 + 7xz^2 + 3zv^2}\right]^{\frac{1}{3}}$$

$$= \left[\frac{k^3\left(vx^2 + 7xz^2 + 3\,zv^2\right)}{\left(vx^2 + 7xz^2 + 3zv^2\right)}\right]^{\frac{1}{3}}$$

$$= \left(k^3\right)^{\frac{1}{3}}$$

$$= k$$

$\therefore$ Each ratio $= \left[\frac{uw^2 + 7wy^2 + 3yu^2}{vx^2 + 7xz^2 + 3zv^2}\right]^{\frac{1}{3}}$

Solution 10:

Let $\frac{x}{b + c - a} = \frac{y}{c + a - b} = \frac{z}{a + b - c} = k$

Then $x = k(b + c - a)$,

$y = k(c + a - b)$ and

$z = k(a + b - c)$.

$$x(b - c) = k(b + c - a)(b - c)$$

$$= k\left(b^2 - c^2 - ab + ca\right) \qquad \ldots(i)$$

$$y(c - a) = k(c + a - b)(c - a)$$

$$= k\left(c^2 - a^2 - bc + ab\right) \qquad \ldots(ii)$$

$$z(a - b) = k(a + b - c)(a - b)$$

$$= k\left(a^2 - b^2 - ca + bc\right) \qquad \ldots(iii)$$

$\therefore$ from (i), (ii) and (iii),

$$x(b - c) + y(c - a) + z(a - b)$$

$$= k\left(b^2 - c^2 - ab + ca\right) + k\left(c^2 - a^2 - bc + ab\right) + k\left(a^2 - b^2 - ca + bc\right)$$

$$= k\left(b^2 - c^2 - ab + ca + c^2 - a^2 - bc + ab + a^2 - b^2 - ca + bc\right)$$

$$= k\left(b^2 - c^2 + c^2 - a^2 + a^2 - b^2 - ab + ca - bc + ab - ca + bc\right)$$

$$= k(0)$$

$$= 0$$

$\therefore x(b - c) + y(c - a) + z(a - b) = 0$