CBSE Board
Class XII Mathematics
Sample Paper 3 - Solution

Part A

1. Correct option: D

Explanation:-

Determinant A denoted as $\left[a_{ij}\right]$ and determinanat B as $\left[b_{ij}\right]$

$\Rightarrow A + B = \left[a_{ij}\right] + \left[b_{ij}\right]$

$\Rightarrow A + B = \left[a_{ij} + b_{ij}\right]$

$\Rightarrow \det(A + B) = \det\left[a_{ij} + b_{ij}\right]$

$\det(A + B) = 0$ ... (Given)

$\Rightarrow \det\left[a_{ij} + b_{ij}\right] = 0$

$\Rightarrow a_{ij} + b_{ij} = 0$

$\Rightarrow A + B = 0$

2. Correct option: A

Explanation:-

$\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$

$\Rightarrow \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = 0$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$

$\Rightarrow \vec{a} \cdot \left(\vec{b} - \vec{c}\right) = 0$ and $\vec{a} \times \left(\vec{b} - \vec{c}\right) = 0$

Obviously $\vec{b} - \vec{c} = 0 \Rightarrow \vec{b} = \vec{c}$.

Also,

$\Rightarrow \left|\vec{a}\right|\left|\left(\vec{b} - \vec{c}\right)\right| \cos\theta = 0$ and $\left|\vec{a}\right|\left|\left(\vec{b} - \vec{c}\right)\right| \sin\theta = 0$

If $\theta = \frac{\pi}{2} \Rightarrow \sin\theta = 1 \Rightarrow \vec{b} - \vec{c} = 0 \Rightarrow \vec{b} = \vec{c}$

3. Correct option: A

Explanation:-

Total balls in 1st box = 3 white + 1 black = 4

Total balls in 2nd box = 2 white + 2 black = 4

Total balls in 3rd box = 1 white + 3 black = 4

Probability of 2 white and 1 black balls = P(WWB) + P(WBW) + P(BWW)

$$= \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} + \frac{3}{4} \times \frac{2}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4}$$

$$= \frac{18 + 6 + 2}{64} = \frac{13}{32}$$

4. Correct option: B

Explanation:-

On substituting (2, 1) in $3x + y \le 6$, we get

$7 \le 6$, which is not true

Hence, (2, 1) does not lie in the half plane $3x + y \le 6$.

5. Correct option: C

Explanation:-

$$\tan^{-1}\left\{\frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 + x^2} + \sqrt{1 - x^2}}\right\} = \alpha$$

$$\frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 + x^2} + \sqrt{1 - x^2}} = \tan \alpha$$

$$\frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 + x^2} + \sqrt{1 - x^2}} \times \frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}} = \tan \alpha$$

$$\frac{1 - \sqrt{1 - x^4}}{x^2} = \tan \alpha$$

$$1 - \sqrt{1 - x^4} = x^2 \tan \alpha$$

$$\left(1 - x^2 \tan \alpha\right)^2 = 1 - x^4$$

$$x^4 - 2x^2 \tan \alpha + x^4 \tan^2 \alpha = 0$$

$$x^2\left(x^2 - 2 \tan \alpha + x^2 \tan^2 \alpha\right) = 0$$

$$x^2 = \frac{2 \tan \alpha}{1 + \tan^2 \alpha}$$

$$x^2 = \frac{2 \tan \alpha}{\sec^2 \alpha}$$

$$x^2 = 2 \tan \alpha \cos^2 \alpha$$

$$x^2 = 2 \sin \alpha \cos \alpha = \sin 2\alpha$$

6. **Correct option: D**

Explanation:-

$$\lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$\lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3} = \lim_{h \to 0} \frac{|3+h-3|\cos(3+h) - 0}{h}$$

$$\lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3} = \lim_{h \to 0} \frac{h\cos(3+h)}{h} = \cos 3$$

cos3 is not differentiable.

Function is differentiable on $R - \{3\}$.

7. **Correct option: C**

Explanation:-

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$M_{11} = d \Rightarrow A_{11} = d$

$M_{12} = c \Rightarrow A_{12} = -c$

$M_{21} = b \Rightarrow A_{21} = -b$

$M_{22} = a \Rightarrow A_{22} = a$

$$\Rightarrow Adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

8. **Correct option: A**

Explanation:-

$y = \sin x$

$$\text{slope of tangent} = \frac{dy}{dx} = \cos x$$

$$\Rightarrow \text{slope of normal} = -\frac{1}{\frac{dy}{dx}} = -\sec x$$

$$\Rightarrow \text{slope of normal at } (0,0) = -\frac{1}{\frac{dy}{dx}} = -1$$

Equation of normal is,

$y - 0 = -1(x - 0)$

$\Rightarrow y = -x$

$\Rightarrow x + y = 0$

9. **Correct option: B**

Explanation:-

$P\left(\hat{i} - \hat{j} + 2k\right), Q\left(2\hat{i} - k\right)$ and $R\left(2\hat{j} + k\right)$

$\overrightarrow{PQ} = \hat{i} + \hat{j} - 3k$

$\overrightarrow{PR} = -\hat{i} + 3\hat{j} - k$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$$

$\overrightarrow{PQ} \times \overrightarrow{PR} = 8\hat{i} + 4\hat{j} + 4k$

$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = 4\sqrt{6}$

$$\text{Unit vector} = \frac{8\hat{i} + 4\hat{j} + 4k}{4\sqrt{6}} = \frac{1}{\sqrt{6}}\left(2\hat{i} + \hat{j} + k\right)$$

10. **Correct option: D**

Explanation:-

$$I = \int_0^{\pi} \frac{1}{1 + \sin x}\, dx$$

$$I = \int_0^{\pi} \frac{1}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x}\, dx$$

$$I = \int_0^{\pi} \frac{1 - \sin x}{1 - \sin^2 x}\, dx$$

$$I = \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x}\, dx$$

$$I = \int_0^{\pi} \left(\sec^2 x - \tan x \sec x\right) dx$$

$$I = \left[\tan x - \sec x\right]_0^{\pi}$$

$$I = -\left(-1 - 1\right)$$

$$I = 2$$

11. Correct option: C

Explanation:-

$$I = \int \frac{1}{1 + e^{x}} dx$$

$$I = \int \frac{\frac{1}{e^{x}}}{\frac{1 + e^{x}}{e^{x}}} dx$$

$$I = \int \frac{e^{-x}}{1 + e^{-x}} dx$$

Put $1 + e^{-x} = t \Rightarrow -e^{-x}dx = dt$

$e^{-x}dx = -dt$

$$I = \int \frac{-dt}{t}$$

$I = -\log|t| + c$

$I = -\log\left|1 + e^{-x}\right| + c$

12. Correct option: D

Explanation:-

a * b = 3a + 4b – 2

2 * 7 = 3 × 2 + 4 × 7 – 2 = 32

13. Correct option: C

Explanation:-

If A = {a, b, c} and B = {1, 2, 3}

The function f: A → B is given by f = {(a, 2), (b, 3), (c, 1)}

Every element of set A is mapped to a unique element of set B, i.e. each element in set B has a unique pre image in B.

So, f is a one–one function

Range of f = {1, 2, 3} = B

So, f is an onto function

Thus, f is a bijective function

14. Correct option: A

Explanation:-

$$\frac{d\left(e^{x^2 + \tan x}\right)}{dx} = e^{x^2 + \tan x}\frac{d}{dx}(x^2 + \tan x) = e^{x^2 + \tan x}\left(2x + \sec^2 x\right)$$

15. Correct option: B

Explanation:-

Given curves are: $y^2 = x$ ... (1) and $x^2 = y$ ... (2)

From (1), $\frac{dy}{dx} = \frac{1}{2y}$

Slope of tangent to $y^2 = x$ at (1, 1) $= \frac{1}{2 \times 1} = \frac{1}{2}$

From (2), $2x = \frac{dy}{dx}$

Slope of tangent to $x^2 = y$ at (1, 1) = 2 × 1 = 2

Let angle between both curves be θ, then

$$\tan\theta = \left|\frac{2 - \frac{1}{2}}{1 + 2 \times \frac{1}{2}}\right| = \left|\frac{\frac{3}{2}}{2}\right| = \frac{3}{4}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

16. Correct option: C

Explanation:-

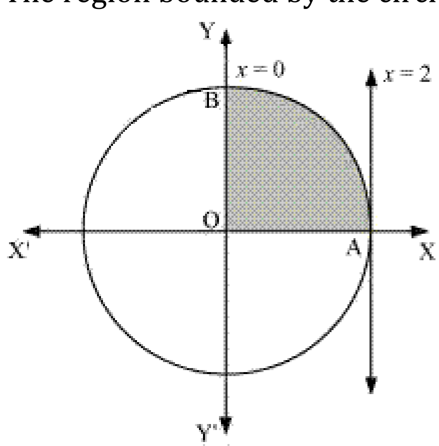
Equation of line through (−2, 1, 3) and parallel to $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$ is

$$\frac{x+2}{3} = \frac{y-1}{5} = \frac{z-3}{6}$$

17. Correct option: D

Explanation:-

The region bounded by the circle $x^2 + y^2 = 4$ in the 1st quadrant is:



Therefore, the required area is

$$A = \int_0^2 y\,dx = \int_0^2 \sqrt{4 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2\left(\frac{\pi}{2}\right)$$

$$= \pi \text{ units}$$

18. Correct option: B

Explanation:-

Given differential equation parallel to $\frac{dy}{dx} = \left(1 + x^2\right)\left(1 + y^2\right)$

$$\Rightarrow \frac{dy}{1 + y^2} = \left(1 + x^2\right)dx$$

Integrating both the sides, we get

$$\int \frac{dy}{1 + y^2} = \int \left(1 + x^2\right)dx$$

$$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + C$$

This is the required solution of the given differential equation.

19. Correct option: A

Explanation:-

$$\cot^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right) = \cot^{-1}(-1) = \frac{3\pi}{4}$$

20. Correct option: D

Explanation:-

Let l, m and n be the direction cosines of a line.

l = cos 90° = 0, m = cos 60° $= \frac{1}{2}$ and n = cos 30° $= \frac{\sqrt{3}}{2}$

Hence, the direction cosines of the line are $0, \frac{1}{2} \text{ and } \frac{\sqrt{3}}{2}$.

Part B

21. Let $y = \frac{x^3 \sqrt{5+x}}{(7-3x)^5 \sqrt[3]{8+5x}}$

Taking log on both the sides, we get

$$\log y = 3\log x + \frac{1}{2}\log(5+x) - 5\log(7-3x) - \frac{1}{3}\log(8+5x)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = 3.\frac{1}{x} + \frac{1}{2}.\frac{1}{5+x}.1 - 5.\frac{1}{7-3x}(-3) - \frac{1}{3}.\frac{1}{8+5x}.5$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{3}{x} + \frac{1}{2(5+x)} + \frac{15}{7-3x} - \frac{5}{3(8+5x)}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{3}{x} + \frac{1}{2(5+x)} + \frac{15}{7-3x} - \frac{5}{3(8+5x)}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3 \sqrt{5+x}}{(7-3x)^5 \sqrt[3]{8+5x}}\left[\frac{3}{x} + \frac{1}{2(5+x)} + \frac{15}{7-3x} - \frac{5}{3(8+5x)}\right]$$

OR

$y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow \frac{dy}{dx} = a\left[-\sin(\log x).\frac{1}{x}\right] + b.\left[\cos(\log x).\frac{1}{x}\right]$$

$$\Rightarrow x\frac{dy}{dx} = xy' = a\left[-\sin(\log x)\right] + b.\left[\cos(\log x)\right]$$

Differentiating,

$$xy'' + y' = a\left[-\cos(\log x).\frac{1}{x}\right] + b.\left[-\sin(\log x).\frac{1}{x}\right]$$

$$\Rightarrow x^2y'' + xy' = a\left[-\cos(\log x)\right] + b.\left[-\sin(\log x)\right] = -y$$

$$\Rightarrow x^2y'' + xy' + y = 0$$

22. Given differential equation is $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{\frac{x}{y} + \frac{y}{x}}\right) ....(i)$$

Let $v = \frac{y}{x}$

$$x\frac{dv}{dx} + v = \frac{dy}{dx}$$

From (i), we have

$$x\frac{dv}{dx} + v = \left(\frac{1}{\frac{1}{v} + v}\right)$$

$$\Rightarrow \left(-\frac{1}{v^3}-\frac{1}{v}\right)dv=\frac{1}{x}dx$$

Integrating on both the sides we have

$$\frac{1}{2v^2}-\log v=\log x+C$$

$$\Rightarrow \frac{x^2}{2y^2}=\log\left(\frac{y}{x}\times x\right)+C.........(ii)$$

Put $x=0, y=1$

$$0=\log(1)+C$$

$$C=0$$

From eq (ii) we have

$$\frac{x^2}{2y^2}=\log(y)$$

23. Let $I=\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

$$=\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\left[1-(\sin x-\cos x)^2\right]}dx$$

Put $\sin x-\cos x=t$

$$\Rightarrow (\cos x+\sin x)dx=dt$$

For $x=\pi/4, t=0$ and

For $x=0, t=-1$

$$\therefore\ I=\int_{-1}^{0}\frac{1}{9+16\left[1-(t)^2\right]}dt$$

$$=\int_{-1}^{0}\frac{1}{25-16(t)^2}dt$$

$$=\int_{-1}^{0}\frac{1}{5^2-(4t)^2}dt$$

$$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$$

$$=\frac{1}{10}\left[\log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}$$

$$=\frac{1}{10}\log\left|\frac{1}{\frac{1}{9}}\right|$$

$= \frac{1}{10}\log|9|$

OR

$$\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)\,dx$$

$$\int_0^{\frac{\pi}{2}} \left(\log\left(\frac{\sin^2 x}{\sin 2x}\right)\right)dx$$

$$\int_0^{\frac{\pi}{2}} \left(\log\left(\frac{\sin^2 x}{2\sin x\cos x}\right)\right)dx$$

$$\int_0^{\frac{\pi}{2}} \left(\log\left(\frac{\tan x}{2}\right)\right)dx \quad \dots (i)$$

We get,

$$I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\tan\left(\frac{\pi}{2} - x\right)}{2}\right)dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\cot x}{2}\right)dx \quad \dots (ii)$$

Additing (i) & (ii)

$$2I = \int_0^{\frac{\pi}{2}} \left[\log\left(\frac{\tan x}{2}\right) + \log\left(\frac{\cot x}{2}\right)\right]dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log\left[\left(\frac{\tan x}{2}\right)\left(\frac{\cot x}{2}\right)\right]dx$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\left(\frac{1}{4}\right)dx$$

$$\Rightarrow I = \frac{1}{2}\log\left(\frac{1}{4}\right)\times\left(\frac{\pi}{2}\right)$$

$$\Rightarrow I = \frac{1}{2}\log\left(\frac{1}{4}\right)^{\frac{1}{2}}\times\left(\frac{\pi}{2}\right)$$

$$\Rightarrow I = \frac{1}{2}\log\left(\frac{1}{2}\right)\times\left(\frac{\pi}{2}\right)$$

$$\Rightarrow I = \frac{\pi}{2}\log\frac{1}{2}$$

24. (i) $\sum_{i=0}^{7} P(X_i) = 1$

$\Rightarrow \left[0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k\right] = 1$

$\Rightarrow 10k^2 + 9k = 1 \Rightarrow 10k^2 + 9k - 1 = 0 \Rightarrow 10k^2 + 10k - k - 1 = 0$

$\Rightarrow 10k(k+1) - (k+1) = 0$

$\Rightarrow (k+1)(10k-1) = 0$

$\Rightarrow k = -1, k = \frac{1}{10}$

k being a probability cannot be negative

$\Rightarrow k = \frac{1}{10}$

(ii) $P(X < 3) = P(0) + P(1) + P(2) = 0 + k + 2k = 3k = \frac{3}{10}$

(iii) $P(X > 5) = P(6) + P(7) = 2k^2 + 7k^2 + k = 2\left(\frac{1}{10}\right)^2 + 7\left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right) = \frac{19}{100}$

(iv) $P(1 \le X < 3) = P(1) + P(2) = k + 2k = 3k = \frac{3}{10}$

25. Given: $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$

Let x = sin y $\Rightarrow y = \sin^{-1} x$

$\Rightarrow \sin^{-1}(1 - \sin y) - 2\sin^{-1}(\sin y) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}(1 - \sin y) - 2y = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}(1 - \sin y) = \frac{\pi}{2} + 2y$

$\Rightarrow 1 - \sin y = \sin\left(\frac{\pi}{2} + 2y\right)$

$\Rightarrow 1 - \sin y = \cos 2y \ \ldots \left(\text{As } \sin\left(\frac{\pi}{2} + x\right) = \cos x\right)$

Since $\cos 2y = 1 - 2\sin^2 y$, we have

$1 - \sin y = 1 - 2\sin^2 y$

$2\sin^2 y - \sin y = 0$

$\sin y(2\sin y - 1) = 0$

$\sin y = 0$ or $2\sin y = 1$

$\sin y = 0$ or $\sin y = \frac{1}{2}$

i.e. x = 0 or $x = \frac{1}{2}$

But $x = \frac{1}{2}$ does not satisfy the given equation.

Hence, x = 0.

OR

Given: $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x \ (x>0)$

$\Rightarrow 2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}x$

$\Rightarrow \tan\left[2\tan^{-1}\left(\frac{1-x}{1+x}\right)\right] = \tan\left[\tan^{-1}x\right]$

$$\Rightarrow \frac{2\tan\left[\tan^{-1}\left(\frac{1-x}{1+x}\right)\right]}{1-\left(\tan\left[\tan^{-1}\left(\frac{1-x}{1+x}\right)\right]\right)^2} = x$$

$$\Rightarrow \frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2} = x$$

$$\Rightarrow \frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2} = x$$

$$\Rightarrow \frac{\left(1-x^2\right)}{2x} = x$$

$\Rightarrow 3x^2 = 1$

$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$

26. The two planes are $\vec{r}\cdot\ \hat{i}+\hat{j}+\hat{k}\ -1=0$ and $\vec{r}\cdot\ 2\hat{i}+3\hat{j}-\hat{k}\ =-4$

$\Rightarrow x+y+z-1=0$ and $2x+3y-z+4=0$

The required plane passes through the line common to two intersecting planes

$\Rightarrow x+y+z-1+k\ \ 2x+3y-z+4\ \ =0$

$\Rightarrow x\ \ 1+2k\ \ +y\ \ 1+3k\ \ +z\ \ 1-k\ \ +\ \ -1+4k\ \ =0 \ldots\ \ 1$

The required plane is parallel to x-axis whose direction ratios are 1, 0 and 0.

$\therefore 1\cdot\ \ 1+2k\ \ +0\cdot\ \ 1+3k\ \ +0\cdot\ \ 1-k\ \ =0$

$\Rightarrow\ \ 1+2k\ \ =0 \Rightarrow\ k=-\frac{1}{2}$

Substituting in (1), we get

$$x\left[1+2\left(-\frac{1}{2}\right)\right]+y\left[1+3\left(-\frac{1}{2}\right)\right]+z\left[1-\left(-\frac{1}{2}\right)\right]+\left[-1+4\left(-\frac{1}{2}\right)\right]=0$$

$$\Rightarrow x(0)+y\left(-\frac{1}{2}\right)+z\left(\frac{3}{2}\right)-3=0$$

$$\Rightarrow -y+3z-6=0$$

$$\Rightarrow y-3z+6=0$$

27. Equation of the plane passing through the point (2, 5, – 8) is:

$a(x-2)+b(y-5)+c(z+8)=0$

If the plane is perpendicular to the plane $2x-3y+4z+1=0$, then

$2a-3b+4c=0$

If the plane is perpendicular to the plane $4x+y-2z+6=0$, then

$4a+b-2c=0$

On solving equations (2) and (3), we get

$$\frac{a}{6-4}=\frac{b}{16+4}=\frac{c}{2+12}$$

$$\Rightarrow \frac{a}{2}=\frac{b}{20}=\frac{c}{14} \Rightarrow \frac{a}{1}=\frac{b}{10}=\frac{c}{7}$$

On substituting the proportional values of a, b and c in (1), we get

$(x-2)+10(y-5)+7(z+8)=0$

$x+10y+7z+4=0$

28. Given: $f(x)=\frac{x}{1+|x|}, x\in R; -1<x<1$

$$\Rightarrow f(x)=\begin{cases}\frac{x}{1+x}, & x\ge 0\\ \frac{x}{1-x}, & x<0\end{cases}$$

To show f is one-one i.e. $f(x_1)=f(x_2)\Rightarrow f(x_1)=f(x_2)$

Case I: Let x_1 be positive and x_2 be negative so $(x_1)\neq(x_2)$

Then $f(x_1)=\frac{x_1}{1+x_1}$ and $f(x_2)=\frac{x_2}{1-x_2}$

So $f(x_1)\neq f(x_2)$ and hence $(x_1)\neq(x_2) \Rightarrow f(x_1)\neq f(x_2)$

CaseII :Let both the numbers x_1 and x_2 be positive.

$$f(x_1)=f(x_2)\Rightarrow \frac{x_1}{1+x_1}=\frac{x_2}{1+x_2}\Rightarrow x_1+x_1x_2=x_2+x_1x_2\Rightarrow x_1=x_2$$

CaseIII :Let both the numbers x_1 and x_2 be negative.

$$f(x_1)=f(x_2)\Rightarrow \frac{x_1}{1-x_1}=\frac{x_2}{1-x_2}\Rightarrow x_1-x_1x_2=x_2-x_1x_2\Rightarrow x_1=x_2$$

Hence, f is one –one

For onto,

Case I: If x is non-negative then

$$f(x) = y = \frac{x}{1+x} \Rightarrow y + xy = x \Rightarrow x = \frac{y}{1-y}$$

$$\text{Now}, f\left(\frac{y}{1-y}\right) = \frac{\frac{y}{1-y}}{1+\frac{y}{1-y}} = y$$

Case II: If x is negative then

$$f(x) = y = \frac{x}{1-x} \Rightarrow y - xy = x \Rightarrow x = \frac{y}{1+y}$$

$$\text{Now}, f\left(\frac{y}{1+y}\right) = \frac{\frac{y}{1+y}}{1-\frac{y}{1+y}} = y$$

Therefore, for each x in A, there exist y in R such that f(y) = x.

Thus, f is an onto function.

Hence, f is a bijective function.

29.

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$$

$$\Rightarrow (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$$\Rightarrow (1 + xyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$$R_2 \rightarrow R_2 - R_1; R_3 \rightarrow R_3 - R_1$$

$$\Rightarrow (1 + xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y - x & y^2 - x^2 \\ 0 & z - x & z^2 - x^2 \end{vmatrix} = 0$$

$$\Rightarrow (1+xyz)(y-x)(z-x)\begin{vmatrix}1 & x & x^2\\0 & 1 & y+x\\0 & 1 & z+x\end{vmatrix}=0$$

Applying $R_3 \to R_3 - R_2$

$$\Rightarrow (1+xyz)(y-x)(z-x)\begin{vmatrix}1 & x & x^2\\0 & 1 & x+y\\0 & 0 & z-y\end{vmatrix}=0$$

$$\Rightarrow (1+xyz)(x-y)(y-z)(z-x)=0$$

Given, x , y , z are different

$$\Rightarrow 1+xyz=0 \Rightarrow xyz=-1$$

30. Let $I=\int \frac{x^2}{x^4+x^2-2}dx$

$$I=\int \frac{x^2}{(x^2-1)(x^2+2)}dx$$

$$=\int \frac{x^2}{(x-1)(x+1)(x^2+2)}dx$$

Using partial fraction,

$$\frac{x^2}{(x-1)(x+1)(x^2+2)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+2}$$

$$\frac{x^2}{(x-1)(x+1)(x^2+2)}=\frac{A(x+1)(x^2+2)+B(x^2+2)(x-1)+(Cx+D)(x-1)(x+1)}{(x-1)(x+1)(x^2+2)}$$

Equating the coefficients from both the numerators we get,

$A+B+C=0$ ....(1)

$A-B+D=1$ ....(2)

$2A+2B-C=0$ ....(3)

$2A-2B-D=0$ ....(4)

Solving the above equations we get,

$$A=\frac{1}{6}, B=-\frac{1}{6}, C=0, D=\frac{2}{3}$$

Our Integral becomes,

$$\int \frac{x^2}{(x-1)(x+1)(x^2+2)}dx=\int \left(\frac{1}{6(x-1)}-\frac{1}{6(x+1)}+\frac{2}{3(x^2+2)}\right)dx$$

$$= \frac{1}{6}\log\ x-1\ -\frac{1}{6}\log\ x+1\ +\frac{2}{3}\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$$

$$= \frac{1}{6}\left[\log\ x-1\ -\log\ x+1\ +2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]+C$$

31. Given function is $f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & x\neq 1\\ 4k, & x=1\end{cases}$

A function f(x) is continuous at a point x = a if $\lim_{x\to a} f(x)=f(a)$

Now,

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x^2-1}{x-1}$$

$$=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}$$

$$=\lim_{x\to 1}(x+1)=2$$

As it is given that f(x) is continuous at x = 1, we have

$$\lim_{x\to 1} f(x)=f(1)$$

$$\Rightarrow\ 2=4k$$

$$\Rightarrow\ k=\frac{1}{2}$$

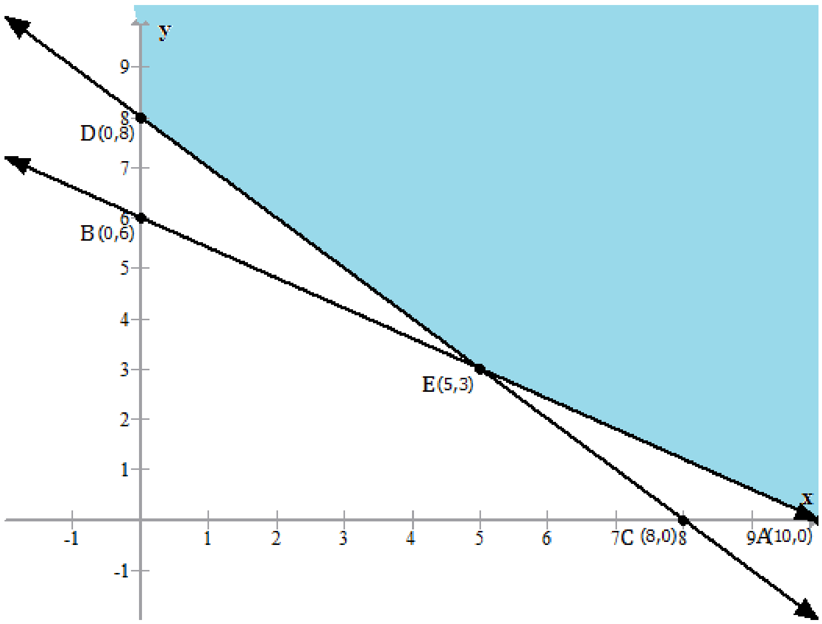
Part C

32. Let the two carpenters work for x days and y days respectively.
Our problem is to minimize the objective function.
C = 150x + 200y
Subject to the constraints

$6x + 10y \geq 60 \Leftrightarrow 3x + 5y \geq 30$

$4x + 4y \geq 32 \Leftrightarrow x + y \geq 8$

And
$x \geq 0,\ y \geq 0$
Feasible region is shaded.



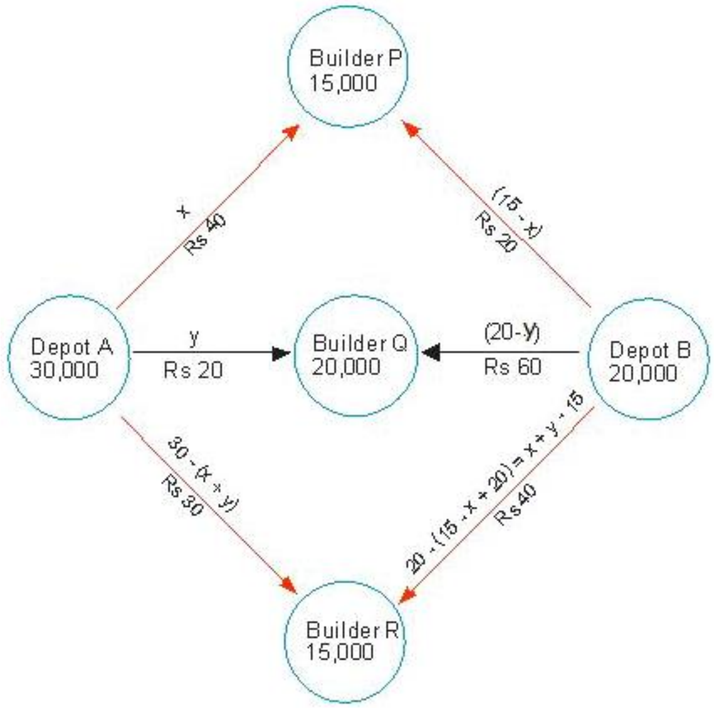
This region is unbounded.

Corner points	Objective function values C = 150x + 200y
A(10, 0)	1500
E(5, 3)	1350
D(0, 8)	1600

The labour cost is the least, when carpenter A works for 5 days and carpenter B works for 3 days.

OR

Let the depot A transport x thousand bricks to builder P and y thousand bricks to builder Q and 30 - (x + y) thousand bricks to builder R. Let the depot B transport (15 – x) thousand bricks to builder P and (20 – y) thousand bricks to builder Q and (x + y) - 15 thousand bricks to builder R



Then, the LPP can be stated mathematically as follows:

Minimize Z = 30x – 30y + 1800

Subject to

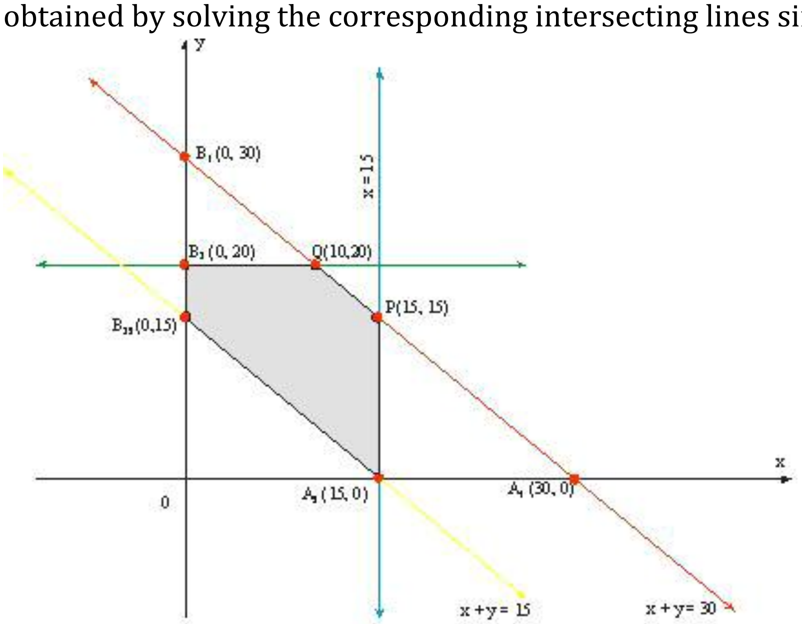
$x + y \leq 30$

$x + y \leq 15$

$x \leq 20$

$y \leq 15$ and $x \geq 0$, $y \geq 0$

To solve this LPP graphically, we first convert inequations into equations and then draw the corresponding lines. The feasible region of the LPP is shaded in the figure given below. The co-ordinates of the corner points of the feasible region A_2 PQ B_3 B_2 are A_2 (15, 0), P (15, 15), Q (10, 20), B_3 (0, 20) and B_2 (0, 15). These points have been obtained by solving the corresponding intersecting lines simultaneously.

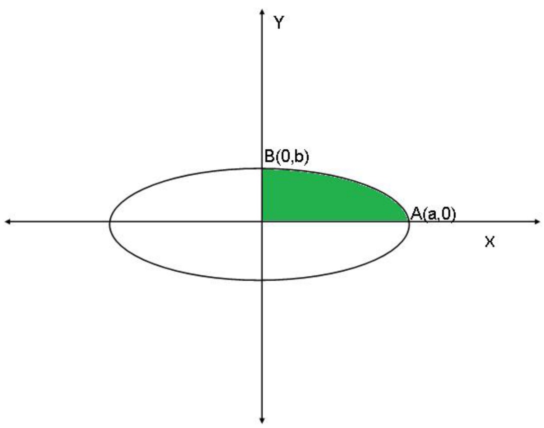


The value of the objective function at the corner points of the feasible region are given in the following table

Point (x, y) X = 30x – 30y + 1800	Value of the objective function
A_2 (15, 0) P (15, 15) Q (10, 20) B_3 (0, 20) B_2 (0, 15)	Z = 30 × 15 – 30 × 0 + 1800 = 2250 Z = 30 × 15 – 30 × 15 + 1800 = 1800 Z = 30 × 10 – 30 × 20 + 1800 = 1500 Z = 30 × 0 – 30 × 20 + 1800 = 1200 Z = 30 × 0 – 30 × 15 + 1800 = 1350

Clearly, Z is minimum at x = 0, y = 20 and the minimum value Z is 1200.
Thus, the manufacturer should supply 0, 20 and 10 thousand bricks to builders P, Q and R from depot A and 15, 0 and 5 thousand bricks to builders P, Q and R from dept B respectively. In this case the minimum transportation cost will be Rs. 1200.

33. (i) Between the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the x-axis between x = 0 to x = a is



$$\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$$

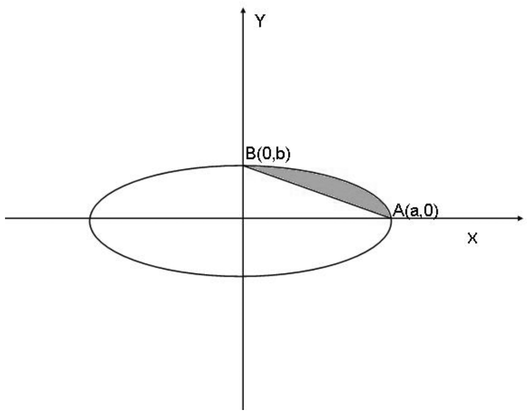
$$=\frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\frac{b}{2a}\left[\left(0+a^2\sin^{-1}1\right)-\left(0+a^2\sin^{-1}0\right)\right]$$

$$=\frac{b}{2a}\left[a^2\times\frac{\pi}{2}\right]$$

$$=\frac{1}{4}\pi ab$$

(ii) Area of triangle AOB is in the first quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where OA = a and OB = a



The required area is same as the area enclosed between the chord AB and arc AB of the ellipse

= Area of ellipse in 1st quadrant – Area of triangle AOB

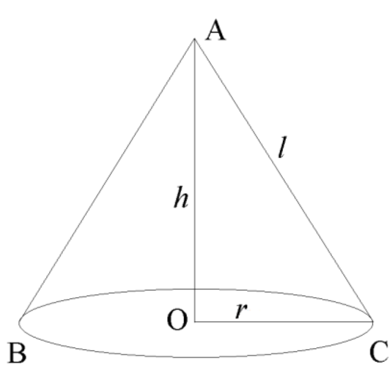
$$= \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx - \frac{1}{2}ab$$

$$= \frac{1}{4}\pi ab - \frac{1}{2}ab$$

$$= \frac{(\pi-2)ab}{4}$$

(iii) Ratio = $\dfrac{\frac{1}{4}\pi ab}{\frac{(\pi-2)}{4}ab} = \dfrac{\pi}{\pi-2}$

34.



Here, Volume 'V' of the cone is $V = \frac{1}{3}\pi r^2 h \Rightarrow r^2 = \frac{3V}{\pi h} \ldots (1)$

Surface area $S = \pi r l = \pi\sqrt{h^2+r^2} \ldots (2)$

Where h = height of the cone

r = radius of the cone

l = Slant height of the cone

$$S^2 = \pi^2 r^2 \left(h^2 + r^2\right) \ldots \text{From } (2)$$

Let, $S_1 = S^2$

Substituting the value of r^2 from equation (1), we have,

$$S_1 = \frac{3\pi V}{h}\left(h^2 + \frac{3V}{\pi h}\right) = 3\pi V h + \frac{9V^2}{h^2}$$

Differentiating S_1 with respect to h, we get

$$\frac{dS_1}{dh} = 3\pi V - \frac{18V^2}{h^3} \ldots (iii)$$

Take $\frac{dS_1}{dh} = 0$ for maxima or minima

$$3\pi V - \frac{18V^2}{h^3} = 0$$

$$\Rightarrow h^3 = \frac{6V}{\pi}$$

Differentiating (iii) w.r.t h, we get

$$\frac{d^2S_1}{dh^2} = \frac{54V^2}{h^4}$$

$$\frac{d^2S_1}{dh^2} > 0 \text{ at } h^3 = \frac{6V}{\pi}$$

Therefore curved surface area is minimum at $\frac{\pi h^3}{6} = V$

Thus, $\frac{\pi h^3}{6} = \frac{1}{3}\pi r^2 h \Rightarrow h^2 = 2r^2 \Rightarrow h = \sqrt{2}r$

Hence for least curved surface the altitude is $\sqrt{2}$ times the radius.

OR

$$f(x) = (x - 2)^4 (x+1)^3$$

$$f'(x) = 3(x - 2)^4 (x+1)^2 + 4(x+1)^3 (x - 2)^3$$

$$= (x - 2)^3 (x+1)^2 \left[3(x - 2) + 4(x+1)\right]$$

$$= (x - 2)^3 (x+1)^2 \left[3x - 6 + 4x + 4\right]$$

$$= (x - 2)^3 (x+1)^2 \left[7x - 2\right]$$

$$f'(x) = 0 \Rightarrow (x - 2)^3 (x+1)^2 \left[7x - 2\right] = 0 \Rightarrow x = -1, \frac{2}{7}, 2$$

Let us examine the behavior of f'(x) slightly to the left and right of each of these three values of x.

(i) x = –1

When $x < -1$, $f'(x) > 0$

When $x > -1$ and $x < \frac{2}{7}$, $f'(x) > 0$

Therefore, $x = -1$ is neither a point of local maxima not minima

(ii) $x = \frac{2}{7}$

When $x < \frac{2}{7}$ and $x > 1$, $f'(x) > 0$

When $x > \frac{2}{7}$ and $x < 2$, $f'(x) < 0$

$\Rightarrow x < \frac{2}{7}$ is a point of local maxima

$$f\left(\frac{2}{7}\right) = \left(\frac{2}{7} - 2\right)^4 \left(\frac{2}{7} + 1\right)^3 = \left(-\frac{12}{7}\right)^4 \left(\frac{9}{7}\right)^3 = \frac{2^8 \times 3^{10}}{7^7}$$

Thus, the local maximum value is $\frac{2^8 \times 3^{10}}{7^7}$.

(iii) $x = 2$

When $x < 2$, $f'(x) < 0$

When $x > 2$, $f'(x) > 0$

Therefore, $x = 2$ is a point of local minima

$f(2) = (2 - 2)^4(2 + 1)^3 = 0$

Thus, the local minimum value is 0.

35. Let E_1, E_2, E_3 and A be the events defined as follows:

Let E_1 be the event that the student knows the answer.

Let E_2 be the event that the student guesses the answer.

Let E_3 be the event that the student copies the answer.

Let A be the event that the answer is correct.

$P(E_1) = \frac{1}{2}; P(E_2) = \frac{1}{4}; P(E_3) = \frac{1}{4};$

Probability that he answers correctly given that he knew the answer is 1.

That is, $P(A | E_1) = 1$

If E_2 has already occurred, then the student guesses.

Since there are four choices out of which only one is correct, therefore, the probability that he answers correctly given that he has made a guess is $\frac{1}{4}$.

That is $P(A | E_2) = \frac{1}{4}$.

It is given that, $P(A | E_3) = \frac{3}{4}$

By Bayes' Theorem, we have,

Required Probability

$= P(E_1 \mid A)$

$$= \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)P(A|E_3)}$$

$$= \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4}}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{16} + \frac{3}{16}}$$

$$= \frac{\frac{1}{2}}{\frac{8}{16} + \frac{1}{16} + \frac{3}{16}}$$

$$= \frac{\frac{1}{2}}{\frac{12}{16}}$$

$$= \frac{1}{2} \times \frac{16}{12}$$

$$= \frac{1}{2} \times \frac{4}{3}$$

$$= \frac{2}{3}$$

Thus, the probability that student knows the answer, given that he answered it correctly is $\frac{2}{3}$.

Arjun is dishonest, as he copies from the other student(s).

Copying once may become habit forming as he may continue resort to dishonest means in the coming years.

OR

Let E_1 be the event that a red ball is transferred from bag A to bag B

Let E_2 be the event that a black ball is transferred from bag A to bag B

$\therefore E_1$ and E_2 are mutually exclusive and exhaustive.

$P(E_1) = 3/7$; $P(E_2) = 4/7$

Let E be the event that a red ball is drawn from bag B

$$P(E|E_1) = \frac{4+1}{(4+1)+5} = \frac{5}{10} = \frac{1}{2}$$

$$P(E|E_2) = \frac{3+1}{(5+1)+4} = \frac{4}{10} = \frac{2}{5}$$

$\therefore$ Required probability $= P(E_2|E) = \dfrac{P(E|E_2)P(E_2)}{P(E|E_1)P(E_1) + P(E|E_2)P(E_2)}$

$$= \frac{\frac{4}{10} \times \frac{4}{7}}{\frac{1}{2} \times \frac{3}{7} + \frac{4}{10} \times \frac{4}{7}}$$

$$= \frac{\frac{16}{70}}{\frac{3}{14} + \frac{16}{70}}$$

$$= \frac{\frac{16}{70}}{\frac{31}{70}}$$

$$= \frac{16}{31}$$

$\therefore$ Required probability $= P(E_1|E) = \dfrac{P(E|E_1)P(E_1)}{P(E|E_1)P(E_1) + P(E|E_2)P(E_2)}$

$$= \frac{\frac{1}{2} \times \frac{3}{7}}{\frac{1}{2} \times \frac{3}{7} + \frac{4}{10} \times \frac{4}{7}}$$

$$= \frac{\frac{3}{14}}{\frac{3}{14} + \frac{16}{70}}$$

$$= \frac{\frac{3}{14}}{\frac{31}{70}}$$

$$= \frac{15}{31}$$

36. We have,

$$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan\left(\frac{x}{2}\right) \\ \tan\left(\frac{x}{2}\right) & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\tan\left(\frac{x}{2}\right) \\ \tan\left(\frac{x}{2}\right) & 1 \end{bmatrix}$$

$$I - A = \begin{bmatrix} 1 & \tan\left(\frac{x}{2}\right) \\ -\tan\left(\frac{x}{2}\right) & 1 \end{bmatrix}$$

$$\left(I - A\right)\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} = \begin{bmatrix} 1 & \tan\left(\frac{x}{2}\right) \\ -\tan\left(\frac{x}{2}\right) & 1 \end{bmatrix}\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \tan\left(\frac{x}{2}\right) \\ -\tan\left(\frac{x}{2}\right) & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)} & -\dfrac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)} \\ \dfrac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)} & \dfrac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)} \end{bmatrix}$$

for simplicity take $\tan\left(\frac{x}{2}\right) = t$

$$= \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1 - t}{1 + t^2} & -\dfrac{2t}{1 + t^2} \\ \dfrac{2t}{1 + t^2} & \dfrac{1 - t}{1 + t^2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1 + t^2}{1 + t^2} & -\dfrac{t\left(1 + t^2\right)}{1 + t^2} \\ \dfrac{t\left(1 + t^2\right)}{1 + t^2} & \dfrac{1 + t^2}{1 + t^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan\left(\frac{x}{2}\right) \\ \tan\left(\frac{x}{2}\right) & 1 \end{bmatrix} = I + A$$

37. Let the equation of required plane be $\ell x + my + nz + p = 0$ ... (1)

Plane passes through points (1, 2, 3) and (0, –1, 0)

∴ (1, 2, 3) and (0, –1, 0) satisfies the equation (1)

$\ell + 2m + 3n + p = 0$ ... (2)

$-m + p = 0$

$\Rightarrow p = m$ ... (3)

d.c.'s of line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{-3}$ are 2, 3, –3

d.c.'s normal to plane are ℓ, m, and n normal to the plane will be ⊥ to line

i.e., $2\ell + 3m - 3n = 0$ ... (4)

$\Rightarrow \ell = \frac{3}{2}(n - m)$

From (2) and (3) we have

$\ell + 3m + 3n = 0$

$\ell = -3(m + n)$ ... (5)

From (4) and (5)

$\frac{3}{2}(n - m) = -3(m + n)$

$n - m = -2m - 2n$

$\Rightarrow 3n = -m$ or $m = -3n$

$\ell = -3(-3n + n) = -3 \times -2n$

Using $\ell = 6n$, $m = -3n$ & $p = -3n$ in (1) we have required equation as

$6x - 3y + z - 3 = 0$

OR

Let the equation of plane be $ax + by + cz + d = 0$ .... (1)

Since the plane passes through the point A (0, 0, 0) and B(3, –1, 2), we have

$a \times 0 + b \times 0 + c \times 0 + d = 0$

$\Rightarrow d = 0$ .... (2)

Similarly for point B (3, –1, 2), $a \times 3 + b \times (-1) + c \times 2 + d = 0$

$3a - b + 2c = 0$ ... (Using d = 0) (3)

Given equation of the line is $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$

We can also write the above equation as $\frac{x-4}{1} = \frac{y-(-3)}{-4} = \frac{x-(-1)}{7}$

The required plane is parallel to the above line.

Therefore, $a \times 1 + b \times (-4) + c \times 7 = 0$

$\Rightarrow a - 4b + 7c = 0$ ... (4)

Cross multiplying equations (3) and (4), we obtain:

$$\frac{a}{(-1)\times 7-(-4)\times 2}=\frac{b}{2\times 1-3\times 7}=\frac{c}{3\times(-4)-1\times(-1)}$$

$$\Rightarrow \frac{a}{-7+8}=\frac{b}{2-21}=\frac{c}{-12+1}$$

$$\Rightarrow \frac{a}{1}=\frac{b}{-19}=\frac{c}{-11}=k$$

$$\Rightarrow a=k, b=-19k, c=-11k$$

Substituting the values of a, b and c in equation (1), we obtain the equation of plane as:

$$kx-19ky-11kz+d=0$$

$$\Rightarrow k\left(x-19y-11z\right)=0 \text{ ... From equation } (2)$$

$$\Rightarrow x-19y-11z=0$$

So, the equation of the required plane is x – 19y – 11z = 0