Chapter 3

UNIPLANAR MOTION WHERE THE ACCELERATIONS PARALLEL TO FIXED AXES ARE GIVEN

Art 43

Ex. 1. $x=\cos 2\pi t$, and $y=\cos(\pi t+\epsilon)=-\sin \pi t$, since $y=0$ when $t=0$.

$$\therefore\ y^2=-\tfrac{1}{2}(x-1), \text{ which is a parabola.}$$

Ex. 2. (1) $x=a\cos 3nt$ and $y=b\cos 2nt$.

$$\therefore\ \frac{2x^2}{a^2}=1+\cos 6nt=1+\cos 2nt\,[4\cos^2 2nt-3]=1+\frac{y}{b}\left(\frac{4y^2}{b^2}-3\right)$$

$$=\left(1+\frac{y}{b}\right)\left(\frac{2y}{b}-1\right)^2.$$

(2) $x=a\cos 3nt$, and $y=b\cos\left(2nt+\frac{\pi}{2}\right)=-b\sin 2nt$.

$$\therefore\ \frac{2x^2}{a^2}-1=\cos 6nt=\cos 2nt\,(1-4\sin^2 2nt)=\sqrt{1-\frac{y^2}{b^2}}\left(1-\frac{4y^2}{b^2}\right).$$

End of Art 47

EXAMPLES ON CHAPTER 3

1. $r^2\omega=h=V.p=\sqrt{\mu}\,p\,.\,CD=\sqrt{\mu}\,br,$

since
$$\frac{p^2}{r^2}=\frac{b^2}{r'^2}=\frac{b^4}{CD^2}. \qquad \therefore\ \omega=\frac{b\sqrt{\mu}}{r}.$$

2. $V^2=\mu\,.\,CD^2=\mu\,(a^2+b^2-CP^2).$

$$\therefore\ v^2=\mu\,(a^2+b^2-a^2e^2-b^2)=\mu\left(2b^2-\frac{b^4}{a^2}\right),$$

$$v_1^2=\mu b^2, \text{ and } v_2^2=\mu a^2.$$

$$\therefore\ v^2v_2^2=\mu^2(2a^2b^2-b^4)=v_1^2(2v_2^2-v_1^2).$$

3. $\dot{x}=u+\omega y$, and $\dot{y}=v+\omega' x$.

$$\therefore\ dy\,(u+\omega y)=dx\,(v+\omega' x).$$

$$\therefore\ \omega y^2-\omega' x^2+2uy-2vx=\text{const.} \quad \therefore \text{ etc.}$$

4. $\ddot{x}=P\cos\omega t$, and $\ddot{y}=P\sin\omega t$.

$$\therefore\ x=-\frac{P}{\omega^2}\cos\omega t+At+B \text{ and } y=-\frac{P}{\omega^2}\sin\omega t+Ct+D.$$

If initially the particle is at the origin and moving with velocities u and v along the axes, then

$$x=\frac{P}{\omega^2}(1-\cos\omega t)+ut, \text{ and } y=\left(\frac{P}{\omega}+v\right)t-\frac{P}{\omega^2}\sin\omega t.$$

5. If the given vertical plane is at a distance b from the starting point, and OP, at any time t, meets it at a height ξ, then

$$\xi = b\frac{y}{x} = b\,\frac{vt-\frac{1}{2}gt^2}{ut} = \frac{b}{u}\left(v-\tfrac{1}{2}gt\right). \qquad \therefore\ \frac{d\xi}{dt} = -\frac{1}{2}\frac{bg}{u}.\ \text{Hence, etc.}$$

6. $\dfrac{d^2x}{dy^2} \div \sqrt{1+\left(\dfrac{dx}{dy}\right)^2} = \dfrac{\lambda}{y}$;

$$\therefore\ \log\left[\frac{dx}{dy}+\sqrt{1+\left(\frac{dx}{dy}\right)^2}\right] = \lambda\log y - \lambda\log a,\ \left(\text{since } \frac{dx}{dy}=0 \text{ initially}\right).$$

$$\therefore\ 2\frac{dx}{dy} = \left(\frac{y}{a}\right)^{\lambda} - \left(\frac{y}{a}\right)^{-\lambda}.$$

$$\therefore\ 2x = \frac{a}{1+\lambda}\left(\frac{y}{a}\right)^{\lambda+1} - \frac{a}{1-\lambda}\left(\frac{y}{a}\right)^{1-\lambda} - \left[\frac{a}{1+\lambda}-\frac{a}{1-\lambda}\right], \text{ etc.}$$

7. The acceleration of A is zero; hence the acceleration of B perpendicular to AB is $l\ddot{\theta}$, and this is zero since there are no forces in this direction, so that $\dot{\theta}$ is constant.

Hence A moves uniformly in a straight line, B is at a constant distance from A, and BA turns round A with constant angular velocity. The path AB is therefore a trochoid.

8. The minimum path is given by $\cos\theta = -\dfrac{V}{v}$ or $-\dfrac{v}{V}$.

If $V > v$, $\cos\theta = -\dfrac{v}{V}$, and the time $= \dfrac{aV}{v\sqrt{V^2-v^2}}$.

If $V < v$, $\cos\theta = -\dfrac{V}{v}$, and the time $= \dfrac{a}{\sqrt{v^2-V^2}}$.

The river is crossed in minimum time when the boat points straight across the river, and this time $= \dfrac{a}{v}$.

9. Let (h, k) be the starting point and u, v the components of the initial velocity. Then

$$\ddot{x}=0 \text{ and } \ddot{y} = -\frac{\mu}{y^2}.$$

Hence $\qquad \dot{x}=u$, and $\dot{y}^2 = 2\mu\left(\dfrac{1}{y}-\dfrac{1}{A}\right)$, where $\dfrac{2\mu}{A} = \dfrac{2\mu}{k} - v^2$.

$$\therefore\ \frac{x\sqrt{2\mu}}{u} = \int\sqrt{\frac{Ay}{A-y}}\,dy.$$

If A is positive, *i.e.* if $v^2 < \dfrac{2\mu}{k}$, put $y = A\cos^2\theta$, and then

$$\frac{x\sqrt{2\mu}}{u} = -A^{\frac{3}{2}}\int 2\cos^2\theta\,d\theta = -A^{\frac{3}{2}}\left(\theta+\tfrac{1}{2}\sin 2\theta\right)+B.$$

If A is negative, *i.e.* if $v^2 > \dfrac{2\mu}{k}$, put $y = A\cosh^2\theta$, and then

$$\frac{x\sqrt{2\mu}}{u} = A\sqrt{-A}\int(1+\cosh 2\theta)\,d\theta = A\sqrt{-A}\left(\theta+\tfrac{1}{2}\sinh 2\theta\right)+B.$$

In each case B is found from the fact that $y=k$ when $x=h$.

If $h=0$, $k=2a$, $u=\sqrt{\dfrac{\mu}{a}}$, and $v=0$, then $A=2a$. In this case $\dfrac{dy}{dx}$ is negative, and we have

$$y = 2a\cos^2\theta, \text{ and } x = 2a\left(\theta+\tfrac{1}{2}\sin 2\theta\right),$$

i.e. $\qquad 2a-y = a(1-\cos 2\theta)$ and $x = a(2\theta+\sin 2\theta)$,

which are the equations of a cycloid.

10. $\ddot{x}=0$, and $\ddot{y}=\dfrac{-\mu}{(2a+y)^2}=\dfrac{-4a^2g}{(2a+y)^2}=-g\left(1-\dfrac{y}{a}\right)$, the squares of $\dfrac{y}{a}$ being neglected.

Hence $\dot{x}=u$ and $\dot{y}^2=g(k-y)\left[2-\dfrac{k+y}{a}\right]$, since $\dot{y}$ is zero when $y=k$. Hence

$$\sqrt{2g}\,\frac{x-h}{u}=\int\frac{dy}{\sqrt{(k-y)\left(1-\dfrac{k+y}{2a}\right)}}=\int\frac{dy}{\sqrt{k-y}}\left(1+\frac{k+y}{4a}\right)$$

$$=-2\left(1+\frac{k}{2a}\right)\sqrt{k-y}+\frac{1}{6a}(k-y)^{\frac{3}{2}}=-2\sqrt{k-y}\left[1+\frac{5k+y}{12a}\right].$$

$$\therefore\ (h-x)^2=\frac{2u^2}{g}(k-y)\left(1+\frac{5k+y}{6a}\right).$$

11. $\ddot{x}=0$, and $\ddot{y}=\mu y$.

$$\therefore\ \dot{x}=\text{const.}=u,\text{ and }\dot{y}^2=\mu y^2+\text{const.}=\mu y^2+v^2.$$

$$\therefore\ \left(\frac{dy}{dx}\right)^2=\frac{v^2+\mu y^2}{u^2}.$$

$$\therefore\ \frac{x}{u}=\int\frac{dy}{\sqrt{v^2+\mu y^2}}=\frac{1}{\sqrt{\mu}}\cosh^{-1}\frac{y\sqrt{\mu}}{v}+\text{const.}$$

$$\therefore\ \frac{y\sqrt{\mu}}{v}=\cosh\left(\frac{x}{u}+C\right)=\tfrac{1}{2}\left[e^{C}e^{\frac{x}{u}}+e^{-C}e^{-\frac{x}{u}}\right],$$

i.e. $\qquad y=A\alpha^x+B\alpha^{-x}.$

If the acceleration is attractive, then

$$\frac{x}{u}=\int\frac{dy}{\sqrt{v^2-\mu y^2}}=\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{y\sqrt{\mu}}{v}+\text{const.},$$

i.e. is of the form $y=A\cos(ax+B)$.

12. $\ddot{x}=0$, and $\ddot{y}=\mu y$.

$$\therefore\ x=At+B,\text{ and }y=Ce^{\sqrt{\mu}t}+De^{-\sqrt{\mu}t},$$

where $0=B$, $\sqrt{\mu g^2}=A$, $g=C+D$ and $0=C-D$, the point of projection being $(0, g)$.

$$\therefore\ y=\frac{g}{2}\left(e^{\frac{x}{g}}+e^{-\frac{x}{g}}\right),\text{ a catenary.}$$

13. Let the rectangular hyperbola be $x^2-y^2=a^2$. ....(1)

Now $\ddot{x}=\mu x$, and $\ddot{y}=\mu y$.

$$\therefore\ x=A\cosh\sqrt{\mu}t+B\sinh\sqrt{\mu}t,\text{ and }y=C\cosh\sqrt{\mu}t+D\sinh\sqrt{\mu}t.$$

Now, when $t=0$, $x=a$, $\dot{x}=0$, and $y=0$.

$$\therefore\ A=a,\ B=0,\text{ and }C=0.$$

$$\therefore\ x=a\cosh\sqrt{\mu}t,\text{ and }y=D\sinh\sqrt{\mu}t.$$

Substituting in (1), we have $D=a$.

$$\therefore\ \tan\theta=\frac{y}{x}=\tanh\sqrt{\mu}t.$$

Chapter 3

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38. Let the coordinates of a particle referred to axes Ox and Oy be x and y at time t, and let its accelerations parallel to the axes at this instant be X and Y.

The equations of motion are then

$$\frac{d^2x}{dt^2} = X \qquad \text{...(1),}$$

and $$\frac{d^2y}{dt^2} = Y \qquad \text{...(2).}$$

Integrating each of these equations twice, we have two equations containing four arbitrary constants. These latter are determined from the initial conditions, *viz.* the initial values x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

From the two resulting equations we then eliminate t, and obtain a relation between x and y which is the equation to the path.

39. *Parabolic motion under gravity, supposed constant, the resistance of the air being neglected.*

Let the axis of y be drawn vertically upward, and the axis of x horizontal. Then the horizontal acceleration is zero, and the vertical acceleration is $-g$.

Hence the equations of motion are

$$\frac{d^2x}{dt^2} = 0, \text{ and } \frac{d^2y}{dt^2} = -g \qquad \text{...(1).}$$

Integrating with respect to t, we have

$$\frac{dx}{dt} = A, \text{ and } \frac{dy}{dt} = -gt + C \qquad ...(2).$$

Integrating a second time,

$$x = At + B, \text{ and } y = -g\frac{t^2}{2} + Ct + D \qquad ...(3).$$

If the particle be projected from the origin with a velocity μ at an angle α with the horizon, then when $t = 0$ we have $x = y = 0$, $\frac{dx}{dt} = u\sin\alpha$.

Hence from (2) and (3) we have initially $u\cos\alpha = A$, $u\sin\alpha = C$, $0 = B$ and $0 = D$.

$$\therefore \ \text{(3) gives} \qquad x = u\cos\alpha t, \quad \text{and} \quad y = u\sin\alpha t - \frac{1}{2}gt^2.$$

Eliminating t, we have

$$y = x\tan\alpha - \frac{g}{2}\frac{x^2}{u^2\cos^2\alpha}$$

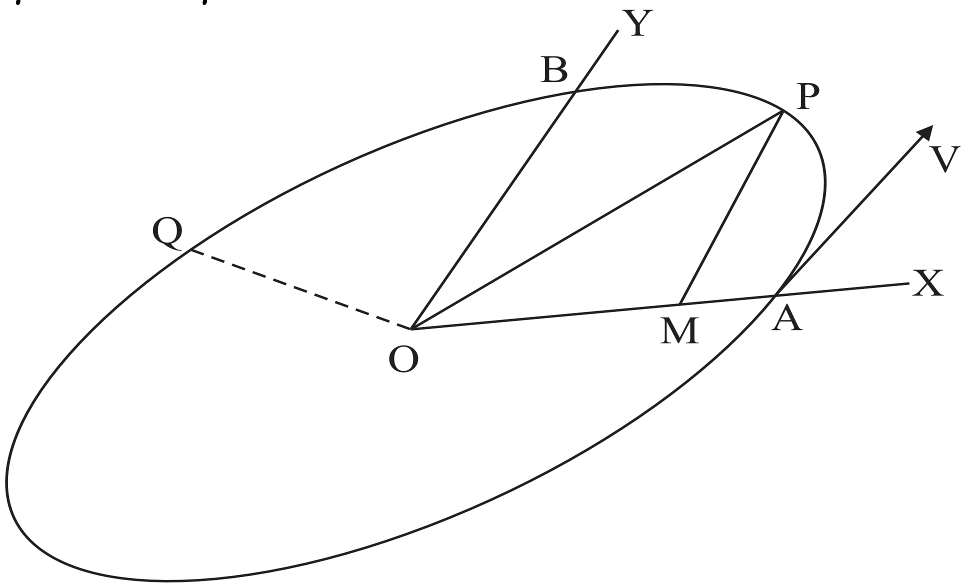
which is the equation to a parabola.

40. *A particle describes a path with an acceleration which is always directed towards a fixed point and varies directly as the distance from it; to find the path.*

Let O be the centre of acceleration and A the point of projection. Take OA as the axis of x and OY parallel to the direction of the initial velocity, V, of projection.

Let P be any point on the path, and let MP be the ordinate of P.

The acceleration, $\mu.PO$, along PO is equivalent, by the triangle of accelerations, to accelerations along PM and MO equal respectively to $\mu.PM$ and $\mu.MO$.



Hence the equations of motion are

$$\frac{d^2x}{dt^2} = -\mu x \qquad ...(1),$$

and $$\frac{d^2y}{dt^2} = -\mu y \qquad ...(2).$$

The solutions of these equations are, as in Art. 22,

$$x = A\cos[\sqrt{\mu}t + B] \qquad ...(3),$$

and $$y = C\cos[\sqrt{\mu}t + D] \qquad ...(4).$$

The initial conditions are that when $t = 0$, then

$$x = OA = a, \quad \frac{dx}{dt} = 0, \quad y = 0, \quad \text{and} \quad \frac{dy}{dt} = V.$$

Hence, from (3), $a = A\cos B$ and $0 = -a\sin B$.
These give $B = 0$ and $A = a$.
Also, from (4), similarly we have $0 = C\cos D$, and $V = -C\sqrt{\mu}\sin D$.

$$\therefore\ D = \frac{\pi}{2}, \text{ and } C = -\frac{V}{\sqrt{\mu}}.$$

$$\therefore \quad \text{(3) and (4) give} \quad x = a\cos\sqrt{\mu}t \qquad ...(5),$$

$$\text{and} \quad y = -\frac{V}{\sqrt{\mu}}\cos\left[\sqrt{\mu}t + \frac{\pi}{2}\right] = \frac{V}{\sqrt{\mu}}\sin(\sqrt{\mu}t) \qquad ...(6).$$

$$\therefore\ \frac{x^2}{a^2} + \frac{y^2}{\frac{V^2}{\mu}} = 1.$$

The locus of P is therefore an ellipse, referred to OX and OY as a pair of conjugate diameters.

Also, if the ellipse meet OY in B, then $OB = \dfrac{V}{\sqrt{\mu}}$ i.e. $V = \sqrt{\mu}\times$ semi-diameter conjugate to OA.

Since any point on the path may be taken as the point of projection, this result will be always true, so that at any point the velocity $=$ $\sqrt{\mu}\times$ semi-conjugate diameter.

[This may be independently derived from (5) and (6). For (Velocity at $P)^2$

$$= \dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\omega$$

$$= a^2\mu\sin^2(\sqrt{\mu}t) + V^2\cos^2(\sqrt{\mu}t) - 2aV\sqrt{\mu}\sin(\sqrt{\mu}t)\cos(\sqrt{\mu}t)\cos\omega$$

$$= \mu\left[a^2 + \frac{V}{\mu} - a^2\cos^2(\sqrt{\mu}t) - \frac{V^2}{\mu}\sin^2\sqrt{\mu}t \right.$$
$$\left. - \frac{2aV}{\sqrt{\mu}}\sin(\sqrt{\mu}t)\cos(\sqrt{\mu}t)\cos\omega\right]$$

$$= \mu\left[a^2 + \frac{V^2}{\mu} - x^2 - y^2 - 2xy\cos\omega\right] = \mu\left(a^2 + \frac{V^2}{\mu} - OP^2\right)$$

$= \mu\times$ Square of semi-diameter conjugate to OP]

From equations (5) and (6) it is clear that the values of x and y are the same at time $t+\dfrac{2\pi}{\sqrt{\mu}}$ as they are at time t.

Hence the time of describing the ellipse is $\dfrac{2\pi}{\sqrt{\mu}}$.

41. If a particle possess two simple harmonic motions, in perpendicular directions and of the same period, it is easily seen that its path is an ellipse.

If we measure the time from the time when the x-vibration has its maximum value, we have

$$x = a\cos nt \qquad ...(1),$$

$$\text{and}\ \ y = b\cos(nt+\varepsilon) \qquad ...(2),$$

where a, b are constants.

(2) gives $\dfrac{y}{b} = \cos nt\cos\varepsilon - \sin nt\sin\varepsilon = \dfrac{x}{a}\cos\varepsilon - \sin\varepsilon\sqrt{1-\dfrac{x^2}{a^2}}$.

$$\therefore\quad \left(\frac{y}{b}-\frac{x}{a}\cos\varepsilon\right)^2 = \sin^2\varepsilon\left(1-\frac{x^2}{a^2}\right)$$

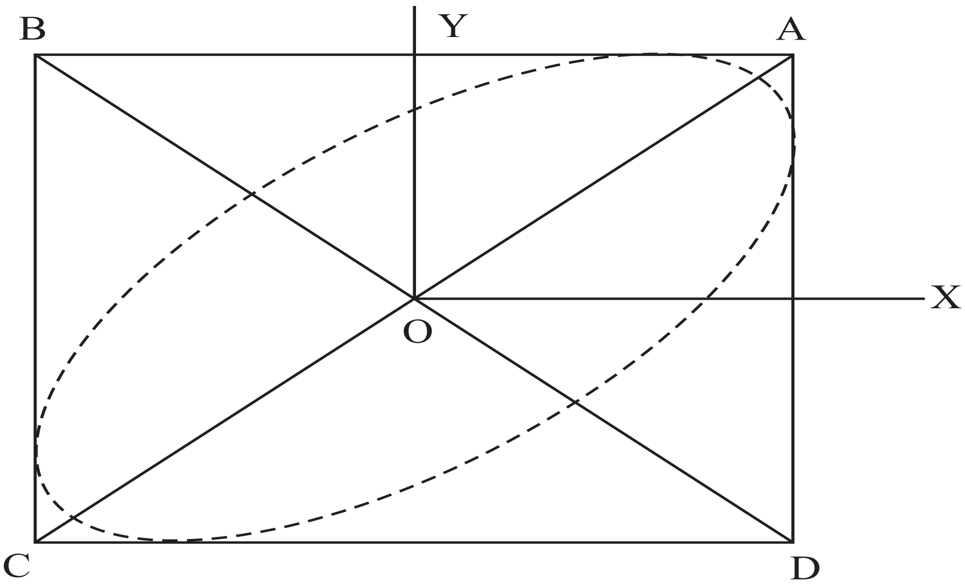
i.e., $\dfrac{x^2}{a^2}-\dfrac{2xy}{ab}\cos\varepsilon+\dfrac{y^2}{b^2}=\sin^2\varepsilon$ ...(3).

This always represents an ellipse whose principal axes do not, in general, coincide with the axes of coordinates, but which is always inscribed in the rectangle $x=\pm a$, $y=\pm b$.

The figure drawn is an ellipse where ε is equal to about $\dfrac{\pi}{3}$.

If $\varepsilon = 0$, equation (3) gives $\dfrac{x}{a}-\dfrac{y}{b}=0$, *i.e.* the straight line AC.

If $\varepsilon=\pi$, it gives $\dfrac{x}{a}+\dfrac{y}{b}=0$, *i.e.* the straight line BD.



In the particular case when $\varepsilon = \frac{\pi}{2}$, *i.e.* when the phase of the y-vibration at zero time is one-quarter of the periodic time, equation (3) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

i.e. the path is an ellipse whose principal axes are in the direction of the axes of x and y and equal to the amplitudes of the component vibrations in these directions.

If in addition $a = b$, *i.e.* if the amplitudes of the component vibrations are the same, the path is a circle.

42. If the period of the y-vibration is one-half that of the x-vibration, the equations are

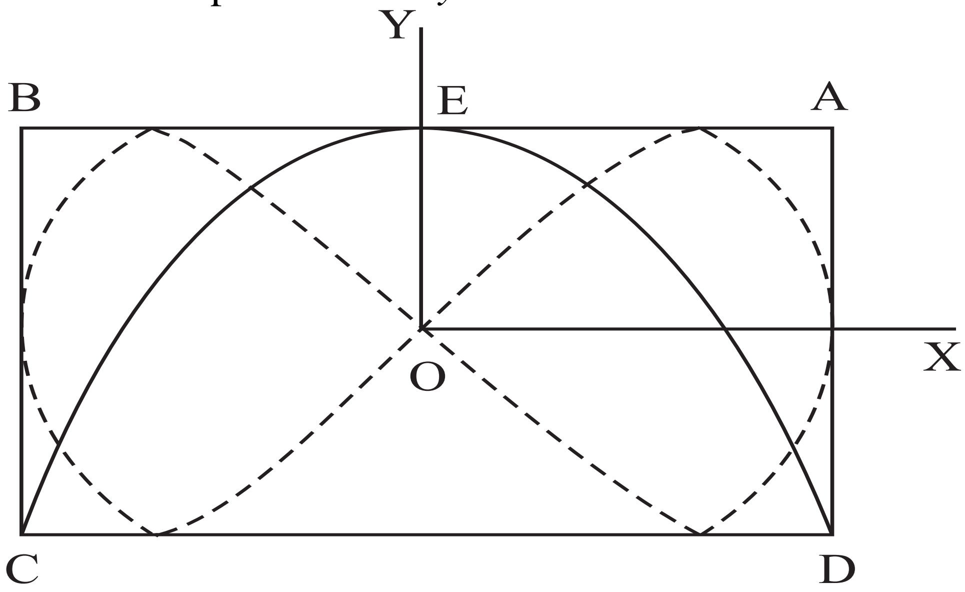
$$x = a\cos nt \quad \text{and} \quad y = b\cos(2nt + \varepsilon).$$

Hence, by eliminating t, we have as the equation to the path

$$\frac{y}{b} = \cos\varepsilon.\left[\frac{2x^2}{a^2} - 1\right] - \sin\varepsilon.\frac{2x}{a}\sqrt{1 - \frac{x^2}{a^2}}.$$

On rationalization, this equation becomes one of the fourth degree.

The dotted curve in the figure is the path when $\varepsilon = -\frac{\pi}{2}$, *i.e.* when the phase of the y-vibration at zero time is negative and equal to one-quarter of the period of the y-vibration.



When $\varepsilon = \pi$, *i.e.*, when the phase of the y-vibration at zero time is one-half of the y-period, the path becomes $x^2 = -\frac{a^2}{2b}(y-b)$, *i.e.* the parabola CED.

When $\varepsilon = 0$ the path is similarly the parabola

$$x^2 = \frac{a^2}{2b}(y+b).$$

For any other value of ε the path is more complicated.

Curves, such as the preceding, obtained by compounding simple harmonic motions in two directions are known as **Lissajous' figures**. For other examples with different ratios of the periods, and for different values of the zero phases, the student may refer to any standard book on Physics.

These curves may be drawn automatically by means of a pendulum, or they may be constructed geometrically.

43. Ex. 1. A point moves in a plane so that its projection on the axis of x performs a harmonic vibration of period one second with an amplitude of one foot; also its projection on the perpendicular axis of y performs a harmonic vibration of period two seconds with an amplitude of one foot. It being given that the origin is the centre of the vibrations, and that the point (1, 0) is on the path, find its equation and draw it.

Ex. 2. A point moves in a path produced by the combination of two simple harmonic vibrations in two perpendicular directions, the periods of the components being as 2:3; find the paths described (1) if the two vibrations have zero phase at the same instant, and (2) if the vibration of greater period be of phase one-quarter of its period when the other vibration is of zero phase. Trace the paths, and find their equations.

44. If in Art. 40 the acceleration be always from the fixed point and varying as the distance from it, we have similarly

$$x = a\cosh\sqrt{\mu}t, \quad \text{and} \quad y = \frac{V}{\sqrt{\mu}}\sinh\sqrt{\mu}t.$$

$$\therefore \qquad \frac{x^2}{a^2} - \frac{y^2}{\frac{V^2}{\mu}} = 1, \text{ so that the path is a hyperbola.}$$

45. *A particle describes a catenary under a force which acts parallel to its axis; find the law of the force and the velocity at any point of the path.*

Taking the directrix and axis of the catenary as the axes of x and y, we have as the equation to the catenary

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) \qquad ...(1).$$

Since there is no acceleration parallel to the directrix,

$$\therefore \quad \frac{d^2x}{dt^2} = 0.$$

$$\therefore \quad \frac{dx}{dt} = \text{ const. } = \mu \qquad ...(2).$$

Differentiating equation (1) twice, we have

$$\frac{dy}{dt} = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).\frac{dx}{dt} = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).u \qquad ...(3)$$

$$\text{and} \quad \frac{d^2y}{dt^2} = \frac{1}{2c}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right).\frac{dx}{dt}.u = \frac{u^2}{c^2}y$$

Also $(\text{velocity})^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$

$$= u^2 + \frac{u^2}{4}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)^2 = \frac{u^2}{4}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)^2 = \frac{u^2}{c^2}y^2$$

so that the velocity $= \frac{u}{c}y$.

Hence the velocity and acceleration at any point both vary as the distance from the directrix.

46. *A particle moves in one plane with an acceleration which is always towards and perpendicular to a fixed straight line in the plane and varies inversely as the cube of the distance from it; given the circumstances of projection, find the path.*

Take the fixed straight line as the axis of x.

Then the equations of motion are

$$\frac{d^2x}{dt^2} = 0 \qquad ...(1),$$

$$\text{and} \qquad \frac{d^2y}{dt^2} = -\frac{\mu}{y^3} \qquad ...(2).$$

$$\text{(1) Gives} \qquad x = At + B \qquad ...(3).$$

Multiplying (2) by $\frac{dy}{dt}$ and integrating, we have $\left(\frac{dy}{dt}\right)^2 = \frac{\mu}{y^2} + C.$

$$\therefore\ t = \int \frac{ydy}{\sqrt{\mu + Cy^2}} = \frac{1}{c}\sqrt{\mu + Cy^2} + D \qquad ...(4).$$

Let the particle be projected from a point on the axis of y distant b from the origin with component velocities u and v parallel to the axes. Then when $t = 0$, we have

$$x = 0, y = b, \frac{dx}{dt} = u, \text{ and } \frac{dy}{dt} = v.$$

$$\therefore\ A = u, B = 0, C = v^2 - \frac{\mu}{b^2}, \text{ and } D = -\frac{b^3 v}{b^2 v^2 - \mu}.$$

$\therefore$ (3) and (4) give $x = ut,$

and $\left(t + \frac{b^3 v}{b^2 v^2 - \mu}\right)^2 = \frac{\mu b^4}{(b^2 v^2 - \mu)^2} + \frac{y^3 b^2}{b^2 v^2 - \mu}.$

Eliminating t, we have as the equation to the path

$$\left(\frac{x}{u} - \frac{b^3 v}{\mu - b^2 v^2}\right)^2 + \frac{y^2 b^2}{\mu - b^2 v^2} = \frac{\mu b^4}{(\mu - b^2 v^2)^2}.$$

This is an ellipse or a hyperbola according as $\mu \lesseqgtr b^2 v^2$

If $\mu = b^2 v^2$, then $C = 0$ and equation (4) becomes

$$t = \int \frac{ydy}{\sqrt{\mu}} = \frac{y^2}{2\sqrt{\mu}} + D = \frac{y^2 - b^2}{\sqrt{\mu}}.$$

Hence the path in this case is $y^2 - b^2 = 2\sqrt{\mu}\frac{x}{u}$, *i.e.* a parabola.

The path is thus an ellipse, parabola, or hyperbola according as $v \lesseqqgtr \sqrt{\dfrac{\mu}{b^2}}$, *i.e.* according as the initial velocity perpendicular to the given line is less, equal to, or greater than the velocity that would be acquired in falling from infinity to the given point with the given acceleration.

$$\text{For the square of the latter } = -\int_{\infty}^{b} 2\frac{\mu}{y^3}dy = \left[\frac{\mu}{y^2}\right]_{\infty}^{b} = \frac{\mu}{b^2}.$$

COR. If the particle describe an ellipse and meets the axis of x it will not then complete the rest of the ellipse since the velocity parallel to the axis of x is always constant and in the same direction; it will proceed to describe a portion of another equal ellipse.

47. *If the velocities and accelerations at any instant of particles* $m_1, m_2, m_3, \ldots$ *parallel to any straight line fixed in space by* $v_1, v_2, v_3, \ldots$ *and* $f_1, f_2, f_3, \ldots$ *to find the velocity and acceleration of their centre of mass.*

If $x_1, x_2, x_3, \ldots$ be the distances of the particles at any instant measured along this fixed line from a fixed point, we have

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots}{m_1 + m_2 + \cdots}.$$

Differentiating with respect to t, we have

$$\bar{v} = \frac{d\bar{x}}{dt} = \frac{m_1v_1 + m_2v_2 + \cdots}{m_1 + m_2 + \cdots}. \qquad \ldots(1)$$

$$\text{and} \quad \bar{f} = \frac{d^2\bar{x}}{dt^2} = \frac{m_1f_1 + m_2f_2 + \cdots}{m_1 + m_2 + \cdots}. \qquad \ldots(2)$$

where $\bar{v}$ and $\bar{f}$ are the velocity and acceleration required.

Consider any two particles, m_1 and m_2, of the system and the mutual actions between them. These are, by Newton's Third Law, equal and opposite, and therefore their impulses resolved in any direction are equal and opposite. The changes in the momenta of the particles are thus, by Art. 15, equal and opposite, *i.e.* the sum of their momenta in any direction is thus unaltered by their mutual actions. Similarly for any other pair of particles of the system.

Hence the sum of the momenta of the system parallel to any line, and hence by (1) the momentum of the centre of mass, is unaltered by the mutual actions of the system.

If $P_1, P_2, \dots$ be the external forces acting on the particles $m_1, m_2, \dots$ parallel to the fixed line, we have

$$\begin{aligned} m_1 f_1 + m_2 f_2 \cdots &= (P_1 + P_2 + \cdots) + \text{ (The sum of the components} \\ &\qquad \text{of the internal actions on the particles)} \\ &= P_1 + P_2 + \cdots, \end{aligned}$$

since the internal actions are in equilibrium taken by themselves.

Hence equation (2) gives

$$(m_1 + m_2 + \dots)\bar{f} = P_1 + P_2 + \dots,$$

i.e. the motion of the centre of mass in any given direction is the same as if the whole of the particles of the system were collected at it, and all the external forces of the system applied at it parallel to the given direction.

Hence also *If the sum of the external forces acting on any given system of particles parallel to a given direction vanishes, the motion of the centre of gravity in that direction remains unaltered, and* *the total momentum of the system in that direction remains constant throughout the motion.*

This theorem is known as the Principle of the Conservation of Linear Momentum.

As an example, if a heavy chain be falling freely the motion of its centre of mass is the same as that of a freely falling particle.

EXAMPLES ON CHAPTER 3

1. A particle describes an ellipse with an acceleration directed towards the centre; show that its angular velocity about a focus is inversely proportional to its distance from that focus.
2. A particle is describing an ellipse under a force to the centre; if v, v_1 and v_2 are the velocities at the ends of the latus-rectum and major and minor axes respectively, prove that
$$v^2v_2^2 = v_1^2(2v_2^2 - v_1^2).$$
3. The velocities of a point parallel to the axes of x and y are $u + \omega y$ and $v + \omega' x$ respectively, where u, v, ω, and ω' are constants; show that its path is a conic section.
4. A particle moves in a plane under a constant force, the direction of which revolves with a uniform angular velocity; find equations to give the coordinates of the particle at any time t.
5. A small ball is projected into the air; show that it appears to an observer standing at the point of projection to fall past a given vertical plane with constant velocity.
6. A man starts from a point O and walks, or runs, with a constant velocity u along a straight road, taken as the axis of x. His dog starts at a distance a from O, his starting point being on the axis

of y which is perpendicular to Ox, and runs with constant velocity $\frac{u}{\lambda}$ in a direction which is always towards his master. show that the equation to his path is

$$2\left[x-\frac{a\lambda}{1-\lambda^2}\right]=y\left[\frac{1}{1+\lambda}\left(\frac{y}{a}\right)^{\lambda}-\frac{1}{1-\lambda}\left(\frac{a}{y}\right)^{\lambda}\right].$$

If $\lambda=1$, show that the path is the curve $2\left(x+\frac{a}{4}\right)=\frac{y^2}{2a}-a\log\frac{y}{a}$.
[The tangent at any point P of the path of the dog meets Ox at the point where the man then is, so that $\frac{dy}{dx}=-\frac{y}{ut-x}$.
Also $\frac{ds}{dt}=\frac{u}{\lambda}$. $\quad\therefore\quad -y\frac{dx}{dy}=ut-x=\lambda s-x$.

$$\therefore\quad -\frac{d}{dy}\left[y\frac{dx}{dy}\right]=\lambda\frac{ds}{dy}-\frac{dx}{dy},$$

$$\text{giving}\quad -y\frac{d^2x}{dy^2}=-\lambda\sqrt{1+\left(\frac{dx}{dy}\right)^2},\ \text{etc.}\ ]$$

7. A particle is fastened to one end, B, of a light thread and rests on a horizontal plane; the other end, A, of the thread is made to move on the plane with a given constant velocity in a given straight line; show that the path of the particle in space is a trochoid.
[Show that AB turns round A with a constant angular velocity.]
8. Two boats each move with a velocity v relative to the water and both cross a river of breadth a running with uniform velocity V. They start together, one boat crossing by the shortest path and the other in the shortest time. Show that the difference between the times of arrival is either

$$\frac{a}{v}\left\{\frac{V}{(V^2-v^2)^{1/2}}-1\right\}\ \text{or}\ \frac{a}{v}\left\{\frac{v}{(v^2-V^2)^{1/2}}-1\right\},$$

according as V or v is the greater.

[The angle that v makes with V being θ, the length of the path is

$$a.\frac{\sqrt{V^2+v^2+2Vv\cos\theta}}{v\sin\theta}$$

and the corresponding time is $\dfrac{a}{v\sin\theta}$. The condition for a minimum path gives

$$(v\cos\theta+V)(V\cos\theta+v)=0.]$$

9. A particle moves in one plane with an acceleration which is always perpendicular to a given line and is equal to $\mu\div$ (distance from the line)^2. Find its path for different velocities of projection.

   If it be projected from a point distant $2a$ from the given line with a velocity $\sqrt{\dfrac{\mu}{a}}$ parallel to the given line, show that its path is a cycloid.

10. If a particle travel with horizontal velocity u and rise to such a height that the variation in gravity must be taken account of as far as small quantities of the first order, show that the path is given by the equation

$$(h-x)^2=\frac{2u^2}{g}(k-y)\left(1+\frac{5k+y}{6a}\right),$$

   where $2a$ is the radius of the earth; the axes of x and y being horizontal and vertical, and h,k being the coordinates of the vertex of the path.

11. A particle moves in a plane with an acceleration which is parallel to the axis of y and varies as the distance from the axis of x; show that the equation to its path is of the form $y=Aa^x+Ba^{-x}$, when the acceleration is a repulsion.

If the acceleration is attractive, then the equation is of the form

$$y = A\cos[ax + B].$$

12. A particle moves under the action of a repulsive force perpendicular to a fixed plane and proportional to the distance from it. Find its path, and show that, if its initial velocity be parallel to the plane and equal to that which it would have acquired in moving from rest on the plane to the point of projection, the path is a catenary.
13. A particle describes a rectangular hyperbola, the acceleration being directed from the centre; show that the angle θ described about the centre in time t after leaving the vertex is given by the equation $\tan\theta = \tan h(\sqrt{\mu} t)$, where μ is acceleration at distance unity.
14. A particle moves freely in a semicircle under a force perpendicular to the bounding diameter; show that the force varies inversely as the cube of the ordinate to the diameter.
15. Show that a rectangular hyperbola can be described by a particle under a force parallel to an asymptote which varies inversely as the cube of its distance from the other asymptote.
16. A particle is moving under the influence of an attractive force $m\dfrac{\mu}{y^3}$ towards the axis of x. Show that, if it be projected from the point $(0, k)$ with component velocities U and V parallel to the axes of x and y, it will not strike the axis of x unless $\mu > V^2k^2$, and that in this case the distance of the point of impact from the origin is
$$\frac{Uk^2}{\mu^{1/2} - Vk}.$$
17. A plane has two smooth grooves at right angles cut in it, and two equal particles attracting one another according to the law of the inverse square are constrained to move one in each groove. Show

that the centre of mass of the two particles moves as if attracted to a centre of force placed at the intersection of the grooves and attracting as the inverse square of the distance.

ANSWERS WITH HINTS

Art. 43

Ex. 1 Parabola, $y^2 = -\frac{1}{2}(x-1)$

Ex. 2 (1) $\frac{2x^2}{a^2} = \left(1+\frac{y}{b}\right)\left(\frac{2y}{b}-1\right)^2$,

(2) $\frac{2x^2}{a^2} = 1+\sqrt{1-\frac{y^2}{b^2}}\left(1-\frac{4y^2}{b^2}\right)$.

Examples on Chapter 3 (end of Art. 47)

4. $x = \frac{P}{\omega^2}(1-\cos\omega t)+ut$,

$y = \left(\frac{P}{\omega}+v\right)t - \frac{P}{\omega^2}\sin\omega t$

9. Cycloid, $2a-y = a(1-\cos 2\theta)$, $x = a(2\theta+\sin 2\theta)$

12. Catenary, $y = \frac{g}{2}\left(e^{\frac{x}{g}}+e^{-\frac{x}{g}}\right)$

14. $x^2+y^2 = a^2$, $\dot{x} = C$ = const. then $\ddot{y} = -C^2\frac{a^2}{y^3}$

15. $xy = C$, $\dot{x} = V$ = const. then $\ddot{y} = \frac{2V^2}{C^2}y^3$

16. $\ddot{x} = 0$, and $\ddot{y} = -\frac{\mu}{y^3}$ $\therefore$ $\dot{x} = U$ and $\dot{y}^{\,2} = \frac{\mu}{y^2} - A$, where $A = \frac{\mu}{k^2} - V^2$, and A must be positive, i.e., $\mu > k^2V^2$ for otherwise $\dot{y}$, which is initially positive, will never be zero, and then the particle would not turn back towards the axis of x.

As long as $\dot{y}$ is positive, i.e., until $y = \sqrt{\dfrac{\mu}{A}}$, then $t = \displaystyle\int \frac{ydy}{\sqrt{\mu - Ay^2}} = \frac{1}{A}[kV - \sqrt{\mu - Ay^2}]$, since $y = k$ initially $\quad \therefore \; t_1 = \dfrac{kV}{A}$.

When $\dot{y}$ becomes negative, we have $t - t_1 = -\displaystyle\int \frac{ydy}{\sqrt{\mu - Ay^2}} = \frac{1}{A}\sqrt{\mu - Ay^2}$, so that, when $y = 0$, $t = t_1 + \dfrac{\sqrt{\mu}}{A} = \dfrac{kV + \sqrt{\mu}}{A} = \dfrac{k^2}{\sqrt{\mu} - kV}$, and $x = Ut =$ as given.