CHAPTER XX.

LIMITING VALUES AND VANISHING FRACTIONS.

262. If a be a constant finite quantity, the fraction $\frac{a}{x}$ can be made as small as we please by sufficiently increasing x; that is, we can make $\frac{a}{x}$ approximate to zero as nearly as we please by taking x large enough; this is usually abbreviated by saying, "when x is infinite the limit of $\frac{a}{x}$ is zero."

Again, the fraction $\frac{a}{x}$ increases as x decreases, and by making x as small as we please we can make $\frac{a}{x}$ as large as we please; thus when x is zero $\frac{a}{x}$ has no finite limit; this is usually expressed by saying, "when x is zero the limit of $\frac{a}{x}$ is infinite."

263. When we say that a quantity *increases without limit* or *is infinite*, we mean that we can suppose the quantity to become greater than any quantity we can name.

Similarly when we say that a quantity *decreases without limit*, we mean that we can suppose the quantity to become smaller than any quantity we can name.

The symbol ∞ is used to denote the value of any quantity which is indefinitely increased, and the symbol 0 is used to denote the value of any quantity which is indefinitely diminished.

264. The two statements of Art. 262 may now be written symbolically as follows :

$$\text{if } x \text{ is } \infty, \text{ then } \frac{a}{x} \text{ is } 0;$$

$$\text{if } x \text{ is } 0, \text{ then } \frac{a}{x} \text{ is } \infty.$$

But in making use of such concise modes of expression, it must be remembered that they are only convenient abbreviations of fuller verbal statements.

265. The student will have had no difficulty in understanding the use of the word *limit*, wherever we have already employed it; but as a clear conception of the ideas conveyed by the words *limit* and *limiting value* is necessary in the higher branches of Mathematics we proceed to explain more precisely their use and meaning.

266. DEFINITION. If $y = f(x)$, and if when x approaches a value a, the function $f(x)$ can be made to differ by as little as we please from a fixed quantity b, then b is called the **limit** of y when $x = a$.

For instance, if S denote the sum of n terms of the series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$; then $S = 2 - \frac{1}{2^{n-1}}$. [Art. 56.]

Here S is a function of n, and $\frac{1}{2^{n-1}}$ can be made as small as we please by increasing n; that is, the limit of S is 2 when n is infinite.

267. We shall often have occasion to deal with expressions consisting of a series of terms arranged according to powers of some common letter, such as

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$$

where the coefficients a_0, a_1, a_2, a_3, ... are finite quantities independent of x, and the number of terms may be limited or unlimited.

It will therefore be convenient to discuss some propositions connected with the limiting values of such expressions under certain conditions.

268. *The limit of the series*

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \dots\dots$$

when x *is indefinitely diminished is* a_0.

Suppose that the series consists of an *infinite* number of terms.

Let b be the greatest of the coefficients $a_1, a_2, a_3, \dots$; and let us denote the given series by $a_0 + S$; then

$$S < bx + bx^2 + bx^3 + \dots;$$

and if $x < 1$, we have $$S < \frac{bx}{1-x}.$$

Thus when x is indefinitely diminished, S can be made as small as we please; hence the limit of the given series is a_0.

If the series consists of a *finite* number of terms, S is less than in the case we have considered, hence *a fortiori* the proposition is true.

269. *In the series*

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \dots,$$

by taking x *small enough we may make any term as large as we please compared with the sum of all that follow it; and by taking* x *large enough we may make any term as large as we please compared with the sum of all that precede it.*

The ratio of the term a_nx^n to the sum of all that follow it is

$$\frac{a_nx^n}{a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \dots}, \text{ or } \frac{a_n}{a_{n+1}x + a_{n+2}x^2 + \dots}.$$

When x is indefinitely small the denominator can be made as small as we please; that is, the fraction can be made as large as we please.

Again, the ratio of the term a_nx^n to the sum of all that precede it is

$$\frac{a_nx^n}{a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots}, \text{ or } \frac{a_n}{a_{n-1}y + a_{n-2}y^2 + \dots};$$

where $y = \dfrac{1}{x}$.

When x is indefinitely large, y is indefinitely small; hence, as in the previous case, the fraction can be made as large as we please.

270. The following particular form of the foregoing proposition is very useful.

In the expression

$$a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x + a_0,$$

consisting of a finite number of terms in *descending* powers of x, by taking x small enough the last term a_0 can be made as large as we please compared with the sum of all the terms that precede it, and by taking x large enough the first term $a_n x^n$ can be made as large as we please compared with the sum of all that follow it.

Example 1. By taking n large enough we can make the first term of $n^4 - 5n^3 - 7n + 9$ as large as we please compared with the sum of all the other terms; that is, we may take the first term n^4 as the equivalent of the whole expression, with an error as small as we please provided n be taken large enough.

Example 2. Find the limit of $\dfrac{3x^3 - 2x^2 - 4}{5x^3 - 4x + 8}$ when (1) x is infinite; (2) x is zero.

(1) In the numerator and denominator we may disregard all terms but the first; hence the limit is $\dfrac{3x^3}{5x^3}$, or $\dfrac{3}{5}$.

(2) When x is indefinitely small the limit is $\dfrac{-4}{8}$, or $-\dfrac{1}{2}$.

Example 3. Find the limit of $\sqrt[x]{\dfrac{1+x}{1-x}}$ when x is zero.

Let P denote the value of the given expression; by taking logarithms we have

$$\log P = \frac{1}{x} \{\log(1+x) - \log(1-x)\}$$

$$= 2\left(1 + \frac{x^2}{3} + \frac{x^4}{5} + \ldots\right). \qquad \text{[Art. 226.]}$$

Hence the limit of $\log P$ is 2, and therefore the value of the limit required is e^2.

VANISHING FRACTIONS.

271. Suppose it is required to find the limit of

$$\frac{x^2 + ax - 2a^2}{x^2 - a^2}$$

when $x = a$.

If we put $x = a + h$, then h will approach the value zero as x approaches the value a.

Substituting $a + h$ for x,

$$\frac{x^2 + ax - 2a^2}{x^2 - a^2} = \frac{3ah + h^2}{2ah + h^2} = \frac{3a + h}{2a + h};$$

and when h is indefinitely small the limit of this expression is $\frac{3}{2}$.

There is however another way of regarding the question; for

$$\frac{x^2 + ax - 2a^2}{x^2 - a^2} = \frac{(x - a)(x + 2a)}{(x - a)(x + a)} = \frac{x + 2a}{x + a},$$

and if we *now* put $x = a$ the value of the expression is $\frac{3}{2}$, as before.

If in the given expression $\frac{x^2 + ax - 2a^2}{x^2 - a^2}$ we put $x = a$ *before* simplification it will be found that it assumes the form $\frac{0}{0}$, the value of which is indeterminate; also we see that it has this form in consequence of the factor $x - a$ appearing in both numerator and denominator. Now we cannot divide by a *zero factor*, but as long as x is not absolutely equal to a the factor $x - a$ may be removed, and we then find that the nearer x approaches to the value a, the nearer does the value of the fraction approximate to $\frac{3}{2}$, or in accordance with the definition of Art. 266,

$$\text{when } x = a, \text{ the limit of } \frac{x^2 + ax - 2a^2}{x^2 - a^2} \text{ is } \frac{3}{2}.$$

272. If $f(x)$ and $\phi(x)$ are two functions of x, each of which becomes equal to zero for some particular value a of x, the fraction $\dfrac{f(a)}{\phi(a)}$ takes the form $\dfrac{0}{0}$, and is called a **Vanishing Fraction.**

Example 1. If $x=3$, find the limit of

$$\frac{x^3-5x^2+7x-3}{x^3-x^2-5x-3}.$$

When $x=3$, the expression reduces to the indeterminate form $\frac{0}{0}$; but by removing the factor $x-3$ from numerator and denominator, the fraction becomes $\dfrac{x^2-2x+1}{x^2+2x+1}$. When $x=3$ this reduces to $\frac{1}{4}$, which is therefore the required limit.

Example 2. The fraction $\dfrac{\sqrt{3x-a}-\sqrt{x+a}}{x-a}$ becomes $\frac{0}{0}$ when $x=a$.

To find its limit, multiply numerator and denominator by the surd conjugate to $\sqrt{3x-a}-\sqrt{x+a}$; the fraction then becomes

$$\frac{(3x-a)-(x+a)}{(x-a)(\sqrt{3x-a}+\sqrt{x+a})}, \text{ or } \frac{2}{\sqrt{3x-a}+\sqrt{x+a}};$$

whence by putting $x=a$ we find that the limit is $\dfrac{1}{\sqrt{2a}}$.

Example 3. The fraction $\dfrac{1-\sqrt[3]{x}}{1-\sqrt[5]{x}}$ becomes $\frac{0}{0}$ when $x=1$.

To find its limit, put $x=1+h$ and expand by the Binomial Theorem. Thus the fraction

$$=\frac{1-(1+h)^{\frac{1}{3}}}{1-(1+h)^{\frac{1}{5}}}=\frac{1-\left(1+\frac{1}{3}h-\frac{1}{9}h^2+\dots\right)}{1-\left(1+\frac{1}{5}h-\frac{2}{25}h^2+\dots\right)}$$

$$=\frac{-\frac{1}{3}+\frac{1}{9}h-\dots\dots}{-\frac{1}{5}+\frac{2}{25}h-\dots\dots}.$$

Now $h=0$ when $x=1$; hence the required limit is $\frac{5}{3}$.

273. Sometimes the roots of an equation assume an indeterminate form in consequence of some relation subsisting between the coefficients of the equation.

For example, if $$ax+b=cx+d,$$
$$(a-c)x=d-b,$$
$$x=\frac{d-b}{a-c}.$$

But if $c=a$, then x becomes $\frac{d-b}{0}$, or ∞; that is, the root of a simple equation is indefinitely great if the coefficient of x is indefinitely small.

274. The solution of the equations

$$ax+by+c=0, \qquad a'x+b'y+c'=0,$$

is $$x=\frac{bc'-b'c}{ab'-a'b}, \quad y=\frac{ca'-c'a}{ab'-a'b}.$$

If $ab'-a'b=0$, then x and y are both infinite. In this case $\frac{a'}{a}=\frac{b'}{b}=m$ suppose; by substituting for a', b', the second equation becomes $ax+by+\frac{c'}{m}=0$.

If $\frac{c'}{m}$ is not equal to c, the two equations $ax+by+c=0$ and $ax+by+\frac{c'}{m}=0$ differ only in their absolute terms, and being *inconsistent* cannot be satisfied by any finite values of x and y.

If $\frac{c'}{m}$ is equal to c, we have $\frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}$, and the two equations are now identical.

Here, since $bc'-b'c=0$ and $ca'-c'a=0$ the values of x and y each assume the form $\frac{0}{0}$, and the solution is *indeterminate*. In fact, in the present case we have really only *one* equation involving *two* unknowns, and such an equation may be satisfied by an unlimited number of values. [Art. 138.]

The reader who is acquainted with Analytical Geometry will have no difficulty in interpreting these results in connection with the geometry of the straight line.

275. We shall now discuss some peculiarities which may arise in the solution of a quadratic equation.

Let the equation be

$$ax^2 + bx + c = 0.$$

If $c = 0$, then

$$ax^2 + bx = 0;$$

whence $$x = 0, \text{ or } -\frac{b}{a};$$

that is, one of the roots is zero and the other is finite.

If $b = 0$, the roots are equal in magnitude and opposite in sign. [Art. 118.]

If $a = 0$, the equation reduces to $bx + c = 0$; and it appears that in this case the quadratic furnishes only one root, namely $-\frac{c}{b}$. But every quadratic equation has two roots, and in order to discuss the value of the other root we proceed as follows.

Write $\frac{1}{y}$ for x in the original equation and clear of fractions; thus

$$cy^2 + by + a = 0.$$

Now put $a = 0$, and we have

$$cy^2 + by = 0;$$

the solution of which is $y = 0$, or $-\frac{b}{c}$; that is, $x = \infty$, or $-\frac{c}{b}$.

Hence, *in any quadratic equation one root will become infinite if the coefficient of* x^2 *becomes zero.*

This is the form in which the result will be most frequently met with in other branches of higher Mathematics, but the student should notice that it is merely a convenient abbreviation of the following fuller statement:

In the equation $ax^2 + bx + c = 0$, if a is very small one root is very large, and as a is indefinitely diminished this root becomes indefinitely great. In this case the finite root approximates to $-\frac{c}{b}$ as its limit.

The cases in which more than one of the coefficients vanish may be discussed in a similar manner.

EXAMPLES. XX.

Find the limits of the following expressions,

(1) when $x=\infty$, (2) when $x=0$.

1. $\dfrac{(2x-3)(3-5x)}{7x^2-6x+4}$.

2. $\dfrac{(3x^2-1)^2}{x^4+9}$.

3. $\dfrac{(3+2x^3)(x-5)}{(4x^3-9)(1+x)}$.

4. $\dfrac{(x-3)(2-5x)(3x+1)}{(2x-1)^3}$.

5. $\dfrac{1-x^2}{2x^3-1} \div \dfrac{1-x}{2x^2}$.

6. $\dfrac{(3-x)(x+5)(2-7x)}{(7x-1)(x+1)^3}$.

Find the limits of

7. $\dfrac{x^3+1}{x^2-1}$, when $x=-1$.

8. $\dfrac{a^x-b^x}{x}$, when $x=0$.

9. $\dfrac{e^x-e^{-x}}{\log(1+x)}$, when $x=0$.

10. $\dfrac{e^{mx}-e^{ma}}{x-a}$, when $x=a$.

11. $\dfrac{\sqrt{x}-\sqrt{2a}+\sqrt{x-2a}}{\sqrt{x^2-4a^2}}$, when $x=2a$.

12. $\dfrac{\log(1+x^2+x^4)}{3x^2(1-2x)}$, when $x=0$.

13. $\dfrac{1-x+\log x}{1-\sqrt{2x-x^2}}$, when $x=1$.

14. $\dfrac{(a^2-x^2)^{\frac{1}{2}}+(a-x)^{\frac{3}{2}}}{(a^3-x^3)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}$, when $x=a$.

15. $\dfrac{\sqrt{a^2+ax+x^2}-\sqrt{a^2-ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$, when $x=0$.

16. $\left\{\left(\dfrac{n+1}{n}\right)^n-\dfrac{n+1}{n}\right\}^{-n}$, when $n=\infty$.

17. $n\log\dfrac{e}{\left(1+\dfrac{1}{n}\right)^{n-1}}$, when $n=\infty$.

18. $\sqrt[x]{\dfrac{a+x}{a-x}}$, when $x=0$.