7

Waves on Transmission Lines

In the wave motion discussed so far four major points have emerged. They are

1. Individual particles in the medium oscillate about their equilibrium positions with simple harmonic motion but do not propagate through the medium.
2. Crests and troughs and all planes of equal phase are transmitted through the medium to give the wave motion.
3. The wave or phase velocity is governed by the product of the inertia of the medium and its capacity to store potential energy; that is, its elasticity.
4. The impedance of the medium to this wave motion is governed by the ratio of the inertia to the elasticity (see table on p. 546).

In this chapter we wish to investigate the wave propagation of voltages and currents and we shall see that the same physical features are predominant. Voltage and current waves are usually sent along a geometrical configuration of wires and cables known as transmission lines. The physical scale or order of magnitude of these lines can vary from that of an oscilloscope cable on a laboratory bench to the electric power distribution lines supported on pylons over hundreds of miles or the submarine telecommunication cables lying on an ocean bed.

Any transmission line can be simply represented by a pair of parallel wires into one end of which power is fed by an a.c. generator. Figure 7.1a shows such a line at the instant when the generator terminal A is positive with respect to terminal B, with current flowing out of the terminal A and into terminal B as the generator is doing work. A half cycle later the position is reversed and B is the positive terminal, the net result being that along each of the two wires there will be a distribution of charge as shown, reversing in sign at each half cycle due to the oscillatory simple harmonic motion of the charge carriers (Figure 7.1b). These carriers move a distance equal to a fraction of a wavelength on either side of their equilibrium positions. As the charge moves current flows, having a maximum value where the product of charge density and velocity is greatest.

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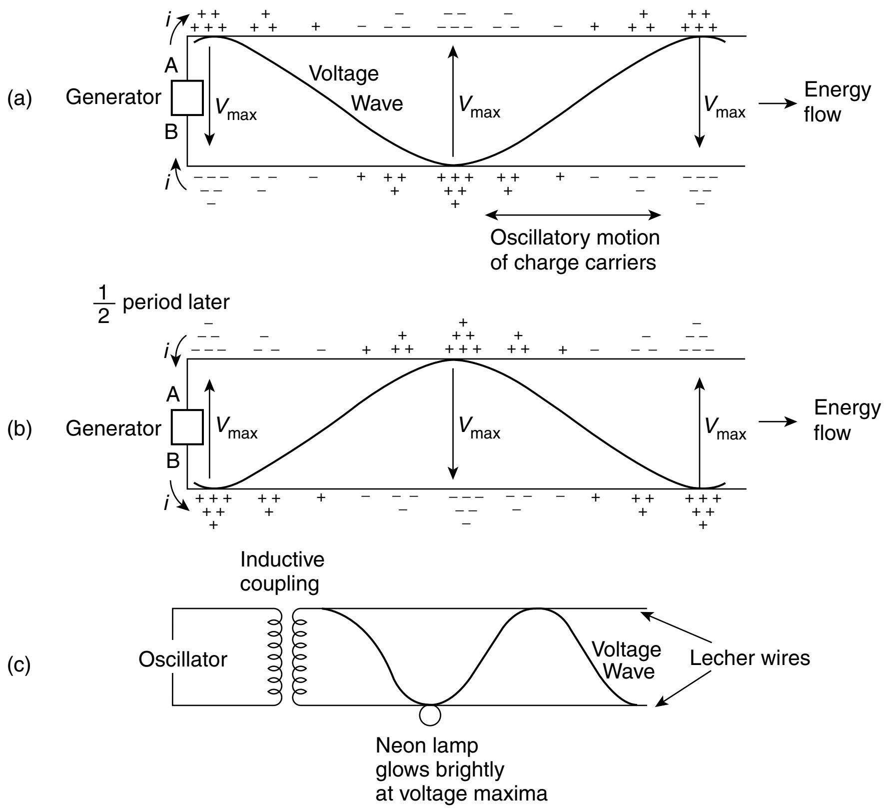


Figure 7.1 Power fed continuously by a generator into an infinitely long transmission line. Charge distribution and voltage waves for (a) generator terminal positive at A and (b) a half period later, generator terminal positive at B. Laboratory demonstration (c) of voltage maxima along a Lecher wire system. The neon lamp glows when held near a position of V_{max}

The existence along the cable of maximum and minimum current values varying simple harmonically in space and time describes a current wave along the cable. Associated with these currents there are voltage waves (Figure 7.1a), and if the voltage and current at the generator are always in phase then power is continuously fed into the transmission line and the waves will always be carrying energy away from the generator. In a laboratory the voltage and current waves may be shown on a Lecher Wire sysem (Figure 7.1c).

In deriving the wave equation for both voltage and current to obtain the velocity of wave propagation we shall concentrate our attention on a short element of the line having a length very much less than that of the waves. Over this element we may consider the variables to change linearly to the first order and we can use differentials.

The currents which flow will generate magnetic flux lines which thread the region between the cables, giving rise to a self inductance L_0 per unit length measured in henries per metre. Between the lines, which form a condenser, there is an electrical capacitance C_0

per unit length measured in farads per metre. In the absence of any resistance in the line these two parameters completely describe the line, which is known as *ideal* or *lossless*.

Ideal or Lossless Transmission Line

Figure 7.2 represents a short element of zero resistance of an ideal transmission line length $\mathrm{d}x \ll \lambda$ (the voltage or current wavelength). The self inductance of the element is $L_0\,\mathrm{d}x$ and its capacitance if $C_0\,\mathrm{d}x$ F.

If the rate of change of voltage per unit length at constant time is $\partial V/\partial x$, then the voltage difference between the ends of the element $\mathrm{d}x$ is $\partial V/\partial x\,\mathrm{d}x$, which equals the voltage drop from the self inductance $-(L_0\,\mathrm{d}x)\partial I/\partial t$.

Thus

$$\frac{\partial V}{\partial x}\,\mathrm{d}x = -(L_0\,\mathrm{d}x)\frac{\partial I}{\partial t}$$

or

$$\frac{\partial V}{\partial x} = -L_0\frac{\partial I}{\partial t} \tag{7.1}$$

If the rate of change of current per unit length at constant time is $\partial I/\partial x$ there is a loss of current along the length $\mathrm{d}x$ of $-\partial I/\partial x\,\mathrm{d}x$ because some current has charged the capacitance $C_0\,\mathrm{d}x$ of the line to a voltage V.

If the amount of charge is $q = (C_0\,\mathrm{d}x)V$,

$$\mathrm{d}I = \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial}{\partial t}(C_0\,\mathrm{d}x)V$$

so that

$$\frac{-\partial I}{\partial x}\,\mathrm{d}x = \frac{\partial}{\partial t}(C_0\,\mathrm{d}x)V$$

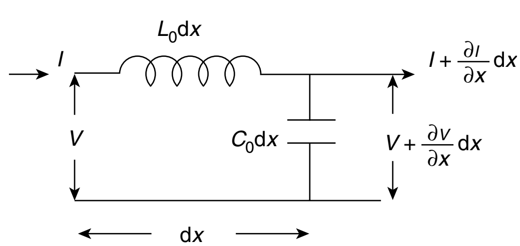


Figure 7.2 Representation of element of an ideal transmission line of inductance L_0 H per unit length and capacitance C_0 F per unit length. The element length $\ll \lambda$, the voltage and current wavelength

or

$$\frac{-\partial I}{\partial x} = C_0 \frac{\partial V}{\partial t} \tag{7.2}$$

Since $\partial^2/\partial x \partial t = \partial^2/\partial t\, \partial x$ it follows, by taking $\partial/\partial x$ of equation (7.1) and $\partial/\partial t$ of equation (7.2) that

$$\frac{\partial^2 V}{\partial x^2} = L_0 C_0 \frac{\partial^2 V}{\partial t^2} \tag{7.3}$$

a pure wave equation for the voltage with a velocity of propagation given by $v^2 = 1/L_0 C_0$.

Similarly $\partial/\partial t$ of (7.1) and $\partial/\partial x$ of (7.2) gives

$$\frac{\partial^2 I}{\partial x^2} = L_0 C_0 \frac{\partial^2 I}{\partial t^2} \tag{7.4}$$

showing that the current waves propagate with the same velocity $v^2 = 1/L_0 C_0$. We must remember here, in checking dimensions, that L_0 and C_0 are defined per unit length.

So far then, the oscillatory motion of the charge carriers (our particles in a medium) has led to the propagation of voltage and current waves with a velocity governed by the product of the magnetic inertia or inductance of the medium and its capactiy to store potential energy.

Coaxial Cables

Many transmission lines are made in the form of coaxial cables, e.g. a cylinder of dielectric material such as polythene having one conductor along its axis and the other surrounding its outer surface. This configuration has an inductance per unit length of

$$L_0 = \frac{\mu}{2\pi} \log_e \frac{r_2}{r_1}\ \mathrm{H}$$

where r_1 and r_2 are the radii of the inner and outer conductors respectively and μ is the magnetic permeability of the dielectric (henries per metre). Its capacitance per unit length

$$C_0 = \frac{2\pi\varepsilon}{\log_e r_2/r_1}\ \mathrm{F}$$

where ε is the permittivity of the dielectric (farads per metre) so that $v^2 = 1/L_0 C_0 = 1/\mu\varepsilon$.

The velocity of the voltage and current waves along such a cable is wholly determined by the properties of the dielectric medium. We shall see in the next chapter on electromagnetic waves that μ and ε represent the inertial and elastic properties of any medium in which such waves are propagating; the velocity of these waves will be given by $v^2 = 1/\mu\varepsilon$. In free space these parameters have the values

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$$
$$\varepsilon_0 = (36\pi \times 10^9)^{-1}\ \mathrm{F\,m^{-1}}$$

and v^2 becomes $c^2 = (\mu_0\varepsilon_0)^{-1}$ where c is the velocity of light, equal to 3×10^8 m s^{-1}.

As we shall see in the next section the ratio of the voltage to the current in the waves travelling along the cable is

$$\frac{V}{I} = Z_0 = \sqrt{\frac{L_0}{C_0}}$$

where Z_0 defines the impedance seen by the waves moving down an infinitely long cable. It is called the Characteristic Impedance.

We write $\varepsilon = \varepsilon_r \, \varepsilon_0$ where ε_r is the relative permittivity (dielectric constant) of a material and $\mu = \mu_r \mu_0$, where μ_r is the relative permeability. Polythene, which commonly fills the space between r_1 and r_2, has $\varepsilon_r \approx 10$ and $\mu_r \approx 1$.

Hence

$$Z_0 = \sqrt{\frac{L_0}{C_0}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}} \log_e \frac{r_2}{r_1} = \frac{1}{2\pi} \frac{1}{\sqrt{\varepsilon_r}} \log_e \frac{r_2}{r_1} \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

where

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.6 \ \Omega$$

Typically, the ratio r_2/r_1 varies between 2 and 10^2 and for a laboratory cable using polythene $Z_0 \approx 50-75 \ \Omega$ with a signal speed $\approx c/3$ where c is the speed of light.

Coaxial cables can be made to a very high degree of precision and the time for an electrical signal to travel a given length can be accurately calculated because the velocity is known.

Such a cable can be used as a 'delay line' in order to separate the arrival of signals at a given point by very small intervals of time.

Characteristic Impedance of a Transmission Line

The solutions to equations (7.3) and (7.4) are, of course,

$$V_+ = V_{0+} \sin \frac{2\pi}{\lambda}(vt - x)$$

and

$$I_+ = I_{0+} \sin \frac{2\pi}{\lambda}(vt - x)$$

where V_0 and I_0 are the maximum values and where the subscript + refers to a wave moving in the positive x-direction. Equation (7.1), $\partial V/\partial x = -L_0 \, \partial I/\partial t$, therefore gives $-V'_+ = -vL_0 I'_+$, where the superscript refers to differentiation with respect to the bracket $(vt - x)$.

Integration of this equation gives

$$V_+ = vL_0 I_+$$

where the constant of integration has no significance because we are considering only oscillatory values of voltage and current whilst the constant will change merely the d.c. level.

The ratio

$$\frac{V_+}{I_+} = vL_0 = \sqrt{\frac{L_0}{C_0}}\ \Omega$$

and the value of $\sqrt{L_0/C_0}$, written as Z_0, is a constant for a transmission line of given properties and is called the *characteristic impedance*. Note that it is a pure resistance (no dimensions of length are involved) and it is the impedance seen by the wave system propagating along an infinitely long line, just as an acoustic wave experiences a specific acoustic impedance ρc. The physical correspondence between ρc and $L_0 v = \sqrt{L_0/C_0} = Z_0$ is immediately evident.

The value of Z_0 for the coaxial cable considered earlier can be shown to be

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\log_e \frac{r_2}{r_1}$$

Electromagnetic waves in free space experience an impedance $Z_0 = \sqrt{\mu_0/\varepsilon_0} = 376.6\ \Omega$.

So far we have considered waves travelling only in the x-direction. Waves which travel in the negative x-direction will be represented (from solving the wave equation) by

$$V_- = V_{0-} \sin \frac{2\pi}{\lambda}(vt + x)$$

and

$$I_- = I_{0-} \sin \frac{2\pi}{\lambda}(vt + x)$$

where the negative subscript denotes the negative x-direction of propagation.

Equation (7.1) then yields the results that

$$\frac{V_-}{I_-} = -vL_0 = -Z_0$$

so that, in common with the specific acoustic impedance, a negative sign is introduced into the ratio when the waves are travelling in the negative x-direction.

When waves are travelling in both directions along the transmission line the total voltage and current at any point will be given by

$$V = V_+ + V_-$$

and

$$I = I_+ + I_-$$

When a transmission line has waves only in the positive direction the voltage and current waves are always in phase, energy is propagated and power is being fed into the line by the generator at all times. This situation is destroyed when waves travel in both directions;

waves in the negative x-direction are produced by reflection at a boundary when a line is terminated or mismatched; we shall now consider such reflections.

(Problems 7.1, 7.2)

Reflections from the End of a Transmission Line

Suppose that a transmission line of characteristic impedance Z_0 has a finite length and that the end opposite that of the generator is terminated by a load of impedance Z_L as shown in Figure 7.3.

A wave travelling to the right $(V_+,\ I_+)$ may be reflected to produce a wave $(V_-,\ I_-)$

The boundary conditions at Z_L must be $V_+ + V_- = V_L$, where V_L is the voltage across the load and $I_+ + I_- = I_L$. In addition $V_+/I_+ = Z_0$, $V_-/I_- = -Z_0$ and $V_L/I_L = Z_L$. It is easily shown that these equations yield

$$\frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

(the voltage amplitude reflection coefficient),

$$\frac{I_-}{I_+} = \frac{Z_0 - Z_L}{Z_L + Z_0}$$

(the current amplitude reflection coefficient),

$$\frac{V_L}{V_+} = \frac{2Z_L}{Z_L + Z_0}$$

and

$$\frac{I_L}{I_+} = \frac{2Z_0}{Z_L + Z_0}$$

in complete correspondence with the reflection and transmission coefficients we have met so far. (See Summary on p. 546.)

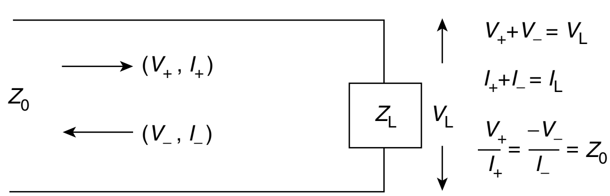


Figure 7.3 Transmission line terminated by impedance Z_L to produce reflected waves unless $Z_L = Z_0$, the characteristic impedance

We see that if the line is terminated by a load $Z_L = Z_0$, its characteristic impedance, the line is matched, all the energy propagating down the line is absorbed and there is no reflected wave. When $Z_L = Z_0$, therefore, the wave in the positive direction continues to behave as though the transmission line were infinitely long.

Short Circuited Transmission Line ($Z_L = 0$)

If the ends of the transmission line are short circuited (Figure 7.4), $Z_L = 0$, and we have

$$V_L = V_+ + V_- = 0$$

so that $V_+ = -V_-$, and there is total reflection with a phase change of π, But this is the condition, as we saw in an earlier chapter, for the existence of standing waves; we shall see that such waves exist on the transmission line.

At any position x on the line we may express the two voltage waves by

$$V_+ = Z_0 I_+ = V_{0+}\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}$$

and

$$V_- = -Z_0 I_- = V_{0-}\,\mathrm{e}^{\mathrm{i}(\omega t + kx)}$$

where, with total reflection and π phase change, $V_{0+} = -V_{0-}$. The total voltage at x is

$$V_x = (V_+ + V_-) = V_{0+}(\mathrm{e}^{-\mathrm{i}kx} - \mathrm{e}^{\mathrm{i}kx})\,\mathrm{e}^{\mathrm{i}\omega t} = (-\mathrm{i})2V_{0+}\sin kx\,\mathrm{e}^{\mathrm{i}\omega t}$$

and the total current at x is

$$I_x = (I_+ + I_-) = \frac{V_{0+}}{Z_0}(\mathrm{e}^{-\mathrm{i}kx} + \mathrm{e}^{\mathrm{i}kx})\,\mathrm{e}^{\mathrm{i}\omega t} = \frac{2V_{0+}}{Z_0}\cos kx\,\mathrm{e}^{\mathrm{i}\omega t}$$

We see then that at any point x along the line the voltage V_x varies as sin kx and the current I_x varies as cos kx, so that voltage and current are 90° out of phase in space. In addition the $-\mathrm{i}$ factor in the voltage expression shows that the voltage lags the current 90° in time, so that if we take the voltage to vary with cos ωt from the $\mathrm{e}^{\mathrm{i}\omega t}$ term, then the current

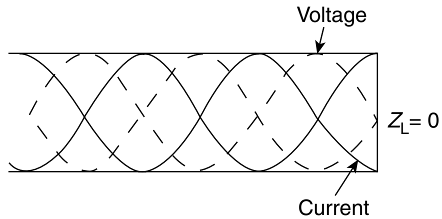


Figure 7.4 Short circuited transmission line of length $(2n + 1)\lambda/4$ produces a standing wave with a current maximum and zero voltage at end of line

will vary with $-\sin \omega t$. If we take the time variation of voltage to be as $\sin \omega t$ the current will change with $\cos \omega t$.

Voltage and current at all points are 90° out of phase in space and time, and the power factor $\cos \phi = \cos 90° = 0$, so that no power is consumed. A standing wave system exists with equal energy propagated in each direction and the total energy propagation equal to zero. Nodes of voltage and current are spaced along the transmission line as shown in Figure 7.4, with I always a maximum where $V = 0$ and vice versa.

If the current I varies with $\cos \omega t$ it will be at a maximum when $V = 0$; when V is a maximum the current is zero. The energy of the system is therefore completely exchanged each quarter cycle between the magnetic inertial energy $\frac{1}{2}L_0 I^2$ and the electric potential energy $\frac{1}{2}C_0 V^2$.

(Problems 7.3, 7.4, 7.5, 7.6, 7.7, 7.8, 7.9, 7.10, 7.11)

The Transmission Line as a Filter

The transmission line is a continuous network of impedances in series and parallel combination. The unit section is shown in Figure 7.5(a) and the continuous network in Figure 7.5(b).

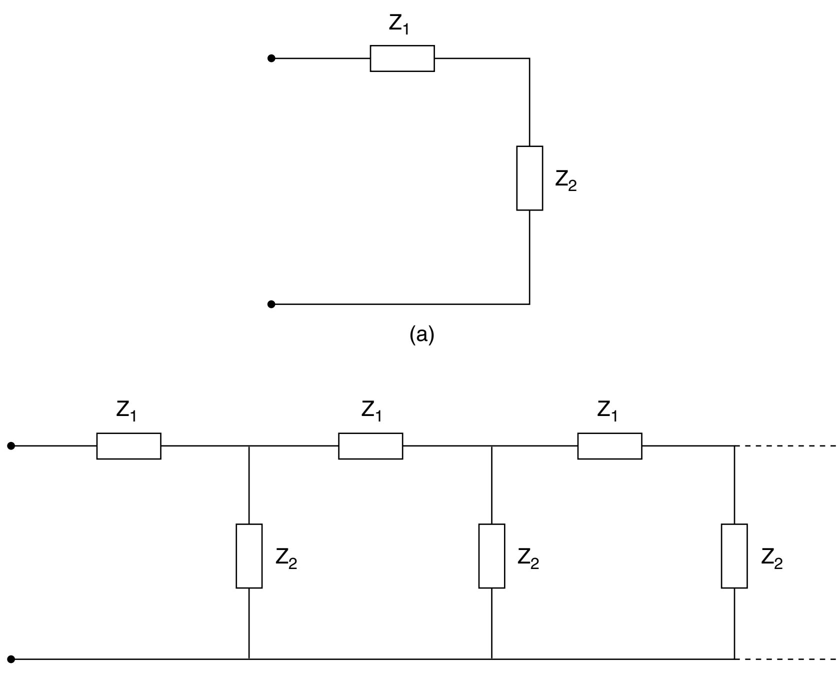


Figure 7.5 (a) The elementary unit of a transmission line. (b) A transmission line formed by a series of such units

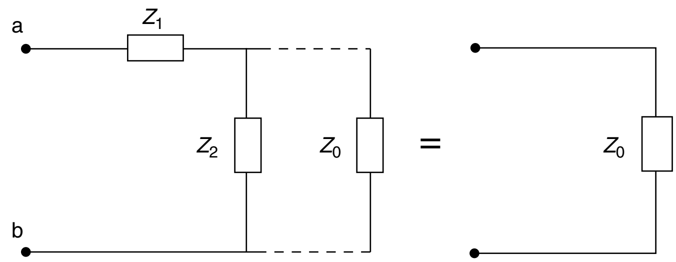


Figure 7.6 A infinite series of elemenetary units presents a characteristic impedance Z_0 to a wave travelling down the transmission line. Adding an extra unit at the input terminal leaves Z_0 unchanged

If we add an infinite series of such sections a wave travelling down the line will meet its characteristic impedance Z_0. Figure 7.6 shows that, adding an extra section to the beginning of the line does not change Z_0. The impedance in Figure 7.6 is

$$Z = Z_1 + \left(\frac{1}{Z_2} + \frac{1}{Z_0}\right)^{-1}$$

or

$$Z = Z_1 + \frac{Z_2 Z_0}{Z_2 + Z_0} = Z_0$$

so the characteristic impedance is

$$Z_0 = \frac{Z_1}{2} + \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}$$

Note that $Z_1/2$ is half the value of the first impedance in the line so if we measure the impedance from a point half way along this impedance we have

$$Z_0 = \left(\frac{Z_1^2}{4} + Z_1 Z_2\right)^{1/2}$$

We shall, however, use the larger value of Z_0 in what follows.

In Figure 7.7 we now consider the currents and voltages at the far end of the transmission line. Any V_n since it is across Z_0 is given by $V_n = I_n Z_0$

Moreover

$$V_n - V_{n+1} = I_n Z_1 = V_n \frac{Z_1}{Z_0}$$

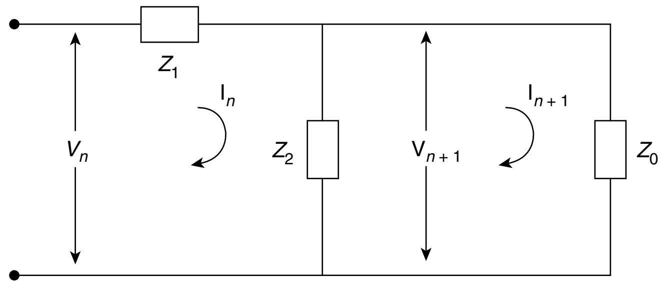


Figure 7.7 The propagation constant $\alpha = V_{n+1}/V_n = Z_0 - 1/Z_0$ for all sections of the transmission line

So

$$\frac{V_{n+1}}{V_n} = 1 - \frac{Z_1}{Z_0} = \frac{Z_0 - Z_1}{Z_0}$$

a result which is the same for all sections of the line.

We define a propagation factor

$$\alpha = \frac{V_{n+1}}{V_n} = \frac{Z_0 - Z_1}{Z_0}$$

which, with

$$Z_0 = \frac{Z_1}{2} + \left(\frac{Z_1^2}{4} + Z_1 Z_2\right)^{1/2}$$

gives

$$\alpha = \frac{\left(\sqrt{Z}_0 - \dfrac{Z_1}{2}\right)}{\left(\sqrt{Z}_0 + \dfrac{Z_1}{2}\right)}$$
$$= 1 + \frac{Z_1}{2Z_2} - \left[\left(1 + \frac{Z_1}{2Z_2}\right)^2 - 1\right]^{1/2}$$

In all practical cases Z_1/Z_2 is real since

1. there is either negligible resistance so that Z_1 and Z_2 are imaginary or
2. the impedances are purely resistive.

So, given (1) or (2) we see that if

(a) $\left(1+\dfrac{Z_1}{2Z_2}\right)^2 = \left[1+\dfrac{Z_1}{Z_2}\left(1+\dfrac{Z_1}{4Z_2}\right)\right] \geq 1 \qquad \text{then } \alpha \text{ is real, and}$

(b) $\left(1+\dfrac{Z_1}{2Z_2}\right)^2 < 1 \qquad \text{then } \alpha \text{ is complex.}$

For α real we have $Z_1/4Z_2 \geq 0 \quad$ or ≤ -1.

If $Z_1/4Z_2 \geq 0$, then $0 < \alpha < 1$, the currents in successive sections decrease progressively and since α is real and positive there is no phase change from one section to another.

If $Z_1/4Z_2 \leq -1$, then $\alpha \leq 0$, and there is again a progressive decrease in current amplitudes along the network but here α is negative and there is a π phase change for each successive section.

When α is complex we have

$$-1 < \frac{Z_1}{4Z_2} < 0$$

and

$$\alpha = 1 + \frac{Z_1}{2Z_2} - i\left[1 - \left(1 + \frac{Z_1}{2Z_2}\right)^2\right]^{1/2}$$

Note that $|\alpha| = 1$ so we can write

$$\alpha = \cos\beta - i\ \sin\ \beta = e^{-i\beta}$$

where

$$\cos\beta = 1 + \frac{Z_1}{2Z_2}$$

The current amplitude remains constant along the transmission line but the phase is retarded by β with each section. If Z_1 and Z_2 are purely resistive α is fixed and the attenuation is constant for all voltage inputs.

If Z_1 is an inductance with Z_2 a capacitance (or vice versa) the division between α real and α complex occurs at certain frequencies governed by their relative magnitudes.

If $Z_1 = \mathrm{i}\omega L$ and $Z_2 = 1/\mathrm{i}\omega C$ for an input voltage $V = V_0 \mathrm{e}^{i\omega t}$ then $|\alpha| = 1$ when $0 \leq \omega^2 LC \leq 4$.

So the line behaves as a low pass filter with a cut-off frequency $\omega_c = 2/\sqrt{LC}$ Above this frequency there is a progressive decrease in amplitude with a phase change of π in each section, Figure 7.8a.

If the positions of Z_1 and Z_2 are now interchanged so that $Z_1 = 1/\mathrm{i}\omega C$ is now a capacitance and Z_2 is now an inductance with $Z_2 = \mathrm{i}\omega L$ the transmisson line becomes a

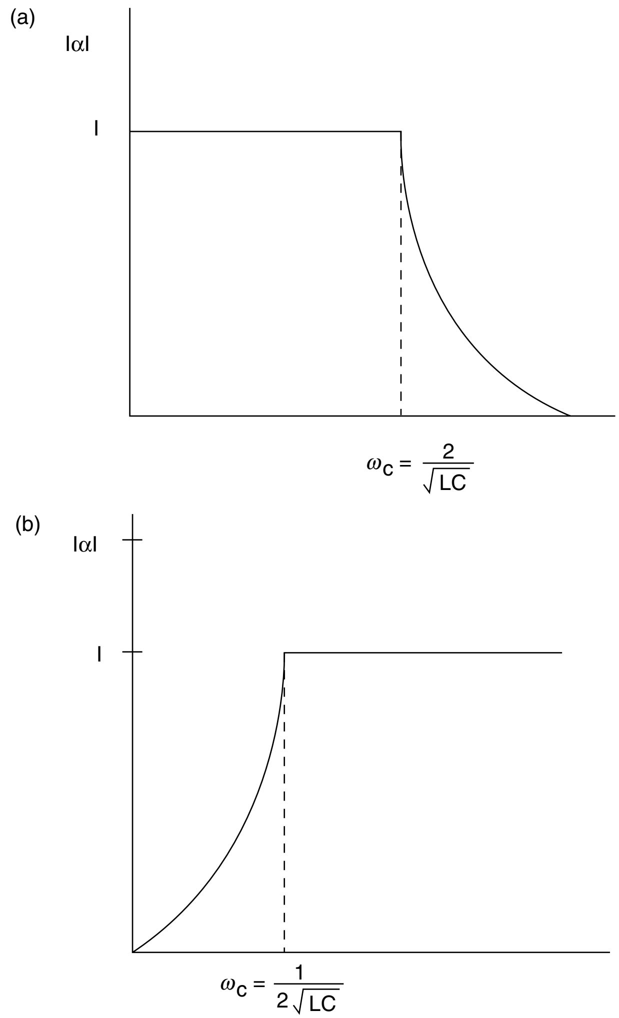


Figure 7.8 (a) When $Z_1 = i\omega L$ and $Z_2 = (i\omega L)^{-1}$ the transmission line acts as a low-pass filter. (b) Reversing the positions of Z_1 and Z_2 changes the transmission line into a high-pass filter

high pass filter with zero attenuation for $0 \leq 1/\omega^2 LC \leq 4$ that is for all frequencies above $\omega_C = (1/2\sqrt{LC})$ Figure 7.8b.

(Problem 7.12)

Effect of Resistance in a Transmission Line

The discussion so far has concentrated on a transmission line having only inductance and capacitance, i.e. wattless components which consume no power. In practice, of course, no

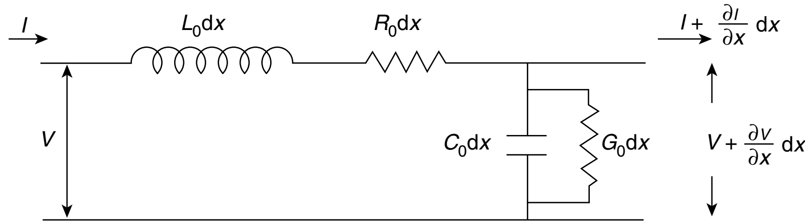


Figure 7.9 Real transmission line element includes a series resistance $R_0\,\Omega$ per unit length and a shunt conductance G_0 S per unit length

such line exists: there is always some resistance in the wires which will be responsible for energy losses. We shall take this resistance into account by supposing that the transmission line has a series resistance $R_0\Omega$ per unit length and a short circuiting or shunting resistance between the wires, which we express as a shunt conductance (inverse of resistance) written as G_0, where G_0 has the dimensions of siemens per metre. Our model of the short element of length dx of the transmission line now appears in Figure 7.9, with a resistance $R_0\,\mathrm{d}x$ in series with $L_0\,\mathrm{d}x$ and the conductance $G_0\,\mathrm{d}x$ shunting the capacitance $C_0\,\mathrm{d}x$. Current will now leak across the transmission line because the dielectric is not perfect. We have seen that the time-dependence of the voltage and current variations along a transmission line may be written

$$V = V_0\,\mathrm{e}^{\mathrm{i}\omega t} \quad \text{and} \quad I = I_0\,\mathrm{e}^{\mathrm{i}\omega t}$$

so that

$$L_0\frac{\partial I}{\partial t} = \mathrm{i}\omega L_0 I \quad \text{and} \quad C_0\frac{\partial V}{\partial t} = \mathrm{i}\omega C_0 V$$

The voltage and current changes across the line element length dx are now given by

$$\frac{\partial V}{\partial x} = -L_0\frac{\partial I}{\partial t} - R_0 I = -(R_0 + \mathrm{i}\omega L_0)I \qquad (7.1\mathrm{a})$$

$$\frac{\partial I}{\partial x} = -C_0\frac{\partial V}{\partial t} - G_0 V = -(G_0 + \mathrm{i}\omega C_0)V \qquad (7.2\mathrm{a})$$

since $(G_0\,\mathrm{d}x)V$ is the current shunted across the condenser. Inserting $\partial/\partial x$ of equation (7.1a) into equation (7.2a) gives

$$\frac{\partial^2 V}{\partial x^2} = -(R_0 + \mathrm{i}\omega L_0)\frac{\partial I}{\partial x} = (R_0 + \mathrm{i}\omega L_0)(G_0 + \mathrm{i}\omega C_0)V = \gamma^2 V$$

where $\gamma^2 = (R_0 + \mathrm{i}\omega L_0)(G_0 + \mathrm{i}\omega C_0)$, so that γ is a complex quantity which may be written

$$\gamma = \alpha + \mathrm{i}k$$

Inserting $\partial/\partial x$ of equation (7.2a) into equation (7.1a) gives

$$\frac{\partial^2 I}{\partial x^2} = -(G_0 + \mathrm{i}\omega C_0)\frac{\partial V}{\partial x} = (R_0 + \mathrm{i}\omega L_0)(G_0 + \mathrm{i}\omega C_0)I = \gamma^2 I$$

an equation similar to that for V.

The equation

$$\frac{\partial^2 V}{\partial x^2} - \gamma^2 V = 0 \tag{7.5}$$

has solutions for the x-dependence of V of the form

$$V = A\,\mathrm{e}^{-\gamma x} \quad \text{or} \quad V = B\,\mathrm{e}^{+\gamma x}$$

where A and B are constants.

We know already that the time-dependence of V is of the form $\mathrm{e}^{\mathrm{i}\omega t}$, so that the complete solution for V may be written

$$V = (A\,\mathrm{e}^{-\gamma x} + B\,\mathrm{e}^{\gamma x})\,\mathrm{e}^{\mathrm{i}\omega t}$$

or, since $\gamma = \alpha + \mathrm{i}k$,

$$\begin{aligned} V &= (A\,\mathrm{e}^{-\alpha x}\,\mathrm{e}^{-\mathrm{i}kx} + B\,\mathrm{e}^{\alpha x}\,\mathrm{e}^{+\mathrm{i}kx})\,\mathrm{e}^{\mathrm{i}\omega t} \\ &= A\,\mathrm{e}^{-\alpha x}\,\mathrm{e}^{\mathrm{i}(\omega t - kx)} + B\,\mathrm{e}^{\alpha x}\,\mathrm{e}^{\mathrm{i}(\omega t + kx)} \end{aligned}$$

The behaviour of V is shown in Figure 7.10—a wave travelling to the right with an amplitude decaying exponentially with distance because of the term $\mathrm{e}^{-\alpha x}$ and a wave travelling to the left with an amplitude decaying exponentially with distance because of the term $\mathrm{e}^{\alpha x}$.

In the expression $\gamma = \alpha + \mathrm{i}k$, γ is called the propagation constant, α is called the attenuation or absorption coefficient and k is the wave number.

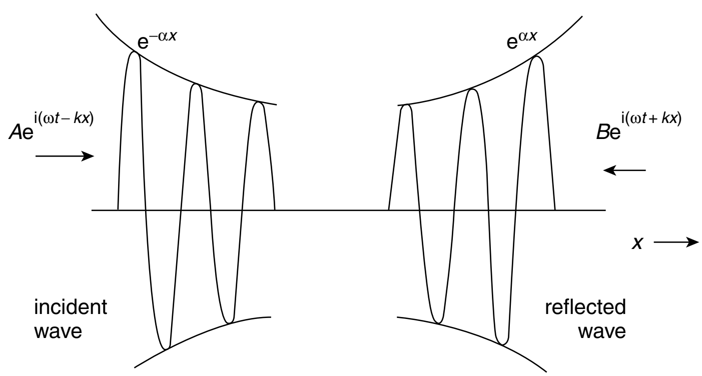


Figure 7.10 Voltage and current waves in both directions along a transmission line with resistance. The effect of the dissipation term is shown by the exponentially decaying wave in each direction

The behaviour of the current wave I is exactly similar and since power is the product VI, the power loss with distance varies as $(\mathrm{e}^{-\alpha x})^2$; that is, as $\mathrm{e}^{-2\alpha x}$.

We would expect this behaviour from our discussion of damped simple harmonic oscillations. When the transmission line properties are purely inductive (inertial) and capacitative (elastic), a pure wave equation with a sine or cosine solution will follow. The introduction of a resistive or loss element produces an exponential decay with distance along the transmission line in exactly the same way as an oscillator is damped with time.

Such a loss mechanism, resistive, viscous, frictional or diffusive, will always result in energy loss from the propagating wave. These are all examples of random collision processes which operate in only one direction in the sense that they are thermodynamically irreversible. At the end of this chapter we shall discuss their effects in more detail.

Characteristic Impedance of a Transmission Line with Resistance

In a lossless line we saw that the ratio $V_+/I_+ = Z_0 = \sqrt{L_0/C_0} = Z_0\,\Omega$, a purely resistive term. In what way does the introduction of the resistance into the line affect the characteristic impedance?

The solution to the equation $\partial^2 I/\partial x^2 = \gamma^2 I$ may be written (for the x-dependence of I) as

$$I = (A'\,\mathrm{e}^{-\gamma x} + B'\,\mathrm{e}^{\gamma x})$$

so that equation (7.2a)

$$\frac{\partial I}{\partial x} = -(G_0 + \mathrm{i}\omega C_0)V$$

gives

$$-\gamma(A'\,\mathrm{e}^{-\gamma x} - B'\,\mathrm{e}^{\gamma x}) = -(G_0 + \mathrm{i}\omega C_0)V$$

or

$$\frac{\sqrt{(R_0 + \mathrm{i}\omega L_0)(G_0 + \mathrm{i}\omega C_0)}}{G_0 + \mathrm{i}\omega C_0}(A'\,\mathrm{e}^{-\gamma x} - B'\,\mathrm{e}^{\gamma x}) = V = V_+ + V_-$$

But, except for the $\mathrm{e}^{\mathrm{i}\omega t}$ term,

$$A'\,\mathrm{e}^{-\gamma x} = I_+$$

the current wave in the positive x-direction, so that

$$\sqrt{\frac{R_0 + \mathrm{i}\omega L_0}{G_0 + \mathrm{i}\omega C_0}}I_+ = V_+$$

or

$$\frac{V_+}{I_+} = \sqrt{\frac{R_0 + \mathrm{i}\omega L_0}{G_0 + \mathrm{i}\omega C_0}} = Z_0'$$

for a transmission line with resistance. Similarly $B' \mathrm{e}^{\gamma x} = I_-$ and

$$\frac{V_-}{I_-} = -\sqrt{\frac{R_0 + \mathrm{i}\omega L_0}{G_0 + \mathrm{i}\omega C_0}} = -Z_0'$$

The presence of the resistance term in the complex characteristic impedance means that power will be lost through Joule dissipation and that energy will be absorbed from the wave system.

We shall discuss this aspect in some detail in the next chapter on electromagnetic waves, but for the moment we shall examine absorption from a different (although equivalent) viewpoint.

(Problems 7.13, 7.14)

The Diffusion Equation and Energy Absorption in Waves

On p. 23 of Chapter 1 we discussed quite briefly the effect of random processes. We shall now look at this in more detail. The wave equation

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

is only one of a family of equations which have a double differential with respect to space on the left hand side.

In three dimensions the left hand side would be of the form

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

which, in vector language, is called the divergence of the gradient or div grad and is written $\nabla^2\phi$.

Five members of this family of equations may be written (in one dimension) as

1. Laplace's Equation

$$\frac{\partial^2 \phi}{\partial x^2} = 0 \quad (\text{for } \phi(x) \text{ only})$$

2. Poisson's Equation

$$\frac{\partial^2 \phi}{\partial x^2} = \text{constant} \quad (\text{for } \phi(x) \text{ only})$$

3. Helmholtz Equation

$$\frac{\partial^2 \phi}{\partial x^2} = \text{constant} \times \phi$$

4. Diffusion Equation

$$\frac{\partial^2\phi}{\partial x^2} = +\text{ve constant} \times \frac{\partial\phi}{\partial t}$$

5. Wave Equation

$$\frac{\partial^2\phi}{\partial x^2} = +\text{ve constant} \times \frac{\partial^2\phi}{\partial t^2}$$

Laplace's and Poisson's equations occur very often in electrostatic field theory and are used to find the values of the electric field and potential at any point. We have already met the Helmholtz equation in this chapter as equation (7.5), where the constant was positive (written γ^2) and we have seen its behaviour when the constant is negative, for it is then equivalent to the equation for standing waves (p. 124). The constant in the wave equation is of course $1/c^2$ where c is the wave velocity. Where the wave equation has an 'acceleration' or $\partial^2\phi/\partial t^2$ term on the right hand side, the diffusion equation has a 'velocity' or $\partial\phi/\partial t$ term.

All equations, however, have the same term $\partial^2\phi/\partial x^2$ on the left hand side, and we must ask: 'What is its physical significance?'

We know that the values of the scalar ϕ will depend upon the point in space at which it is measured. Suppose we choose some point at which ϕ has the value ϕ_0 and surround this point by a small cube of side l, over the volume of which ϕ may take other values. If the average value of ϕ over the small cube is written $\bar{\phi}$, then the difference between the average $\bar{\phi}$ and the value at the centre of the cube ϕ_0 is given by

$$\bar{\phi} - \phi_0 = \text{constant} \times \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)_0$$

This statement is proved in the appendix at the end of this chapter and is readily understood by those familiar with triple integration. The left hand side of any of these equations therefore measures the value

$$\bar{\phi} - \phi_0$$

In Laplace's equation the difference is zero, so that ϕ has a constant value over the volume considered. Poisson's equation tells us that the difference is constant and Helmholtz equation states that the value of ϕ at any point in the volume is proportional to this difference. The first two equations are 'steady state', i.e. they do not vary with time.

The Helmholtz equation states that if the constant is positive the behaviour of ϕ with space grows or decays exponentially, e.g. γ^2 is positive in equation (7.5), but if the constant is negative, ϕ will vary sinusoidally or cosinusoidally with space as the displacement varies with time in simple harmonic motion and the equation becomes the time independent wave equation for standing waves. This equation says nothing about the time behaviour of ϕ, which will depend only upon the function ϕ itself.

Both the diffusion and wave equations are time-derivative dependent. The diffusion equation states that the 'velocity' or change of ϕ with time at a point in the volume is proportional to the difference $\bar{\phi} - \phi_0$, whereas the wave equation states that the 'acceleration' $\partial^2\phi/\partial t^2$ depends on this difference.

The wave equation recalls the simple harmonic oscillator, where the difference from the centre ($\bar{x} = 0$) was a measure of the force or acceleration term; both the oscillator and the wave equation have time varying sine and cosine solutions with maximum velocity $\partial\phi/\partial t$ at the zero displacement from equilibrium; that is, where the difference $\bar{\phi} - \phi_0 = 0$.

The diffusion equation, however, describes a different kind of behaviour. It describes a non-equilibrium situation which is moving towards equilibrium at a rate governed by its distance from equilibrium, so that it reaches equilibrium in a time which is theoretically infinite. Readers will have already met this situation in Newton's Law of Cooling, where a hot body at temperatue T_0 stands in a room of lower temperature $\bar{T}$. The rate at which the body cools, i.e. the value of $\partial T/\partial t$, depends on $\bar{T} - T_0$; a cooling graph of this experiment is given in Figure 7.11. The greatest rate of cooling occurs when the temperature difference is greatest and the process slows down as the system approaches equilibrium. Here, of course, $\bar{T} - T_0$ and $\partial T/\partial t$ are both negative.

All non-equilibrium processes of this kind are unidirectional in the sense that they are thermodynamically irreversible. They involve the transport of mass in diffusion, the transport of momentum in friction or viscosity and the transport of energy in conductivity. All such processes involve the loss of useful energy and the generation of entropy.

They are all processes which are governed by random collisions, and we found in the first chapter, where we added vectors of constant length and random phase, that the average distance travelled by particles involved in these processes was proportional, not to the time, but to the square root of the time.

Rewriting the diffusion equation as

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{d}\frac{\partial\phi}{\partial t}$$

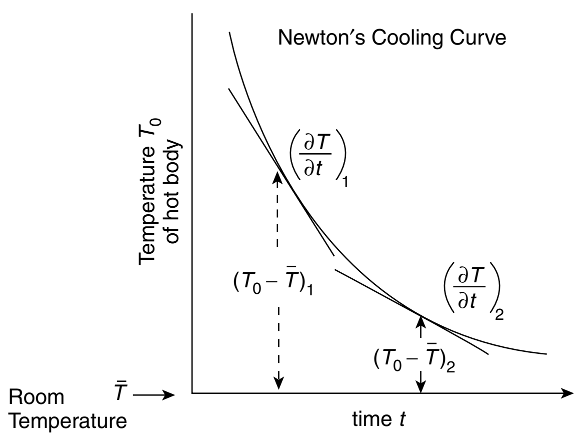


Figure 7.11 Newton's cooling curve shows that the rate of cooling of a hot body $\partial T/\partial t$ depends on the temperature difference between the body and its surrounding, this difference being directly measured by $\partial^2 T/\partial x^2$

we see that the dimensions of the constant d, called the diffusivity, are given by

$$\frac{\phi}{\text{length}^2} = \frac{1}{d}\frac{\phi}{\text{time}}$$

so that d has the dimensions of length^2/time. The interpretation of this as the square of a characteristic length varying with the square root of time has already been made in Chapter 1.

In a viscous process d is given by η/ρ, where η is the coefficient of viscosity and ρ is the density. In thermal conductivity $d = K/\rho C_p$, where K is the coefficient of thermal conductivity, ρ is the density and C_p is the specific heat at constant pressure.

A magnetic field which is non-uniformly distributed in a conductor has a diffusivity $d = (\mu\sigma)^{-1}$, where μ is the permeability and σ is the conductivity.

Brownian motion is one of the best known examples of random collision processes. The distance x travelled in time t by a particle suffering multiple random collisions is given by Einstein's diffusivity relation

$$d = \frac{\overline{x}^2}{t} = \frac{2RT}{6\pi\eta N}$$

The gas law, $pV = RT$, gives RT as the energy of a mole of such particles at temperature T; a mole contains N particles, where N is Avogadro's number and $RT/N = kT$, the average energy of the individual particles, where k is Boltzmann's constant.

The process is governed, therefore, by the ratio of the energy of the particles to the coefficient of viscosity, which measures the frictional force. The higher the temperature, the greater is the energy, the less the effect of the frictional force and the greater the average distance travelled.

Wave Equation with Diffusion Effects

In natural systems we can rarely find pure waves which propagate free from the energy-loss mechanisms we have been discussing, but if these losses are not too serious we can describe the total propagation in space and time by a combination of the wave and diffusion equations.

If we try to solve the combined equation

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{d}\frac{\partial\phi}{\partial t}$$

we shall not obtain a pure sine or cosine solution.

Let us try the solution

$$\phi = \phi_m \mathrm{e}^{\mathrm{i}(\omega t - \gamma x)}$$

where ϕ_m is the maximum amplitude. This gives

$$\mathrm{i}^2\gamma^2 = \mathrm{i}^2\frac{\omega^2}{c^2} + \mathrm{i}\frac{\omega}{d}$$

or

$$\gamma^2 = \frac{\omega^2}{c^2} - \mathrm{i}\frac{\omega}{d}$$

giving a complex value for γ. But $\omega^2/c^2 = k^2$, where k is the wave number, and if we put $\gamma = k - \mathrm{i}\alpha$ we obtain

$$\gamma^2 = k^2 - 2\mathrm{i}k\alpha - \alpha^2 \approx k^2 - \mathrm{i}\,2\,k\alpha \quad \textit{if}\ \alpha \ll k$$

The solution for ϕ then becomes

$$\phi = \phi_m\,\mathrm{e}^{\mathrm{i}(\omega t - \gamma x)} = \phi_m\,\mathrm{e}^{-\alpha x}\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}$$

i.e. a sine or cosine oscillation of maximum amplitude ϕ_m which decays exponentially with distance. The physical significance of the condition $\alpha \ll k = 2\pi/\lambda$ is that many wavelengths λ are contained in the distance $1/\alpha$ before the amplitude decays to $\phi_m\,\mathrm{e}^{-1}$ at $x = 1/\alpha$. Diffusion mechanisms will cause attenuation or energy loss from the wave; the energy in a wave is proportional to the square of its amplitude and therefore decays as $\mathrm{e}^{-2\alpha x}$.

(Problems 7.15, 7.16, 7.17)

Appendix

Physical interpretation of

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial_2^2} \equiv \nabla^2\phi$$

At a certain point O of the scalar field, $\phi = \phi_0$. Constructing a cube around the point O having sides of length l gives for the average value over the cube volume

$$\bar{\phi}l^3 = \iiint_{-l/2}^{+l/2} \phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Expanding ϕ about the point O by a Taylor series gives

$$\begin{aligned}\phi = \phi_0 &+ \left(\frac{\partial\phi}{\partial x}\right)_0 x + \left(\frac{\partial\phi}{\partial y}\right)_0 y + \left(\frac{\partial\phi}{\partial z}\right)_0 z \\ &+ \frac{1}{2}\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 x^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)_0 y^2 + \left(\frac{\partial^2\phi}{\partial z^2}\right)_0 z^2\right] \\ &+ \left(\frac{\partial^2\phi}{\partial x\partial y}\right)_0 xy + \left(\frac{\partial^2\phi}{\partial y\partial z}\right)_0 yz + \left(\frac{\partial^2\phi}{\partial z\partial x}\right)_0 zx + \cdots\end{aligned}$$

Integrating from $-l/2$ to $+l/2$ removes all the functions of the form

$$\left(\frac{\partial\phi}{\partial x}\right)_0 x \quad \text{and} \quad \left(\frac{\partial^2\phi}{\partial x\partial y}\right)_0 xy$$

whose integrals are zero, leaving, since

$$\iiint_{-l/2}^{+l/2} x^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \frac{l^5}{12}$$

$$\bar{\phi} l^3 = \phi_0 l^3 + \frac{l^5}{24}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)_0$$

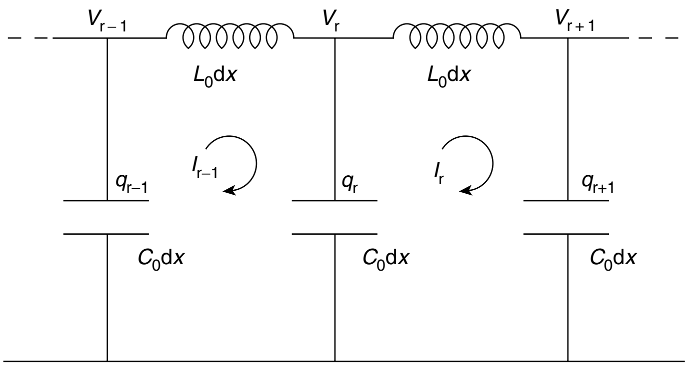
i.e.

$$\bar{\phi} - \phi_0 = \frac{l^2}{24}(\nabla^2 \phi)_0$$

where l is a constant.

Problem 7.1

The figure shows the mesh representation of a transmission line of inductance L_0 per unit length and capacitance C_0 per unit length. Use equations of the form



$$I_{r-1} - I_r = \frac{\mathrm{d}}{\mathrm{d}t} q_r = C_0 \, \mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}t} V_r$$

and

$$L_0 \, \mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}t} I_r = V_r - V_{r+1}$$

together with the method of the final section of Chapter 4 to show that the voltage and current wave equations are

$$\frac{\partial^2 V}{\partial x^2} = L_0 C_0 \frac{\partial^2 V}{\partial t^2}$$

and

$$\frac{\partial^2 I}{\partial x^2} = L_0 C_0 \frac{\partial^2 I}{\partial t^2}$$

Problem 7.2

Show that the characteristic impedance for a pair of Lecher wires of radius r and separation d in a medium of permeability μ and permittivity ε is given by

$$Z_0 = \frac{1}{\pi}\sqrt{\frac{\mu}{\varepsilon}}\log_e \frac{d}{r}$$

Problem 7.3

In a short-circuited lossless transmission line integrate the magnetic (inductive) energy $\frac{1}{2}L_0 I^2$ and the electric (potential) energy $\frac{1}{2}C_0 V^2$ over the last quarter wavelength (0 to $-\lambda/4$) to show that they are equal.

Problem 7.4

Show, in Problem 7.3, that the sum of the instantaneous values of the two energies over the last quarter wavelength is equal to the maximum value of either.

Problem 7.5

Show that the impedance of a real transmission line seen from a position x on the line is given by

$$Z_x = Z_0 \frac{A\,e^{-\gamma x} - B\,e^{+\gamma x}}{A\,e^{-\gamma x} + B\,e^{+\gamma x}}$$

where γ is the propagation constant and A and B are the current amplitudes at $x = 0$ of the waves travelling in the positive and negative x-directions respectively. If the line has a length l and is terminated by a load Z_L, show that

$$Z_L = Z_0 \frac{A\,e^{-\gamma l} - B\,e^{\gamma l}}{A\,e^{-\gamma l} + B\,e^{\gamma l}}$$

Problem 7.6

Show that the input impedance of the line of Problem 7.5; that is, the impedance of the line at $x = 0$, is given by

$$Z_i = Z_0\left(\frac{Z_0 \sinh\gamma l + Z_L \cosh\gamma l}{Z_0 \cosh\gamma l + Z_L \sinh\gamma l}\right)$$

$$\left(\textit{Note}: 2\cosh\gamma l = e^{\gamma l} + e^{-\gamma l} \atop 2\sinh\gamma l = e^{\gamma l} - e^{-\gamma l}\right)$$

Problem 7.7

If the transmission line of Problem 7.6 is short-circuited, show that its input impedance is given by

$$Z_{sc} = Z_0 \tanh\gamma l$$

and when it is open-circuited the input impedance is

$$Z_{0c} = Z_0 \coth\gamma l$$

By taking the product of these quantities, suggest a method for measuring the characteristic impedance of the line.

Problem 7.8

Show that the input impedance of a short-circuited loss-free line of lenght l is given by

$$Z_i = i\sqrt{\frac{L_0}{C_0}}\tan\frac{2\pi l}{\lambda}$$

and by sketching the variation of the ratio $Z_i/\sqrt{L_0/C_0}$ with l, show that for l just greater than $(2n+1)\lambda/4$, Z_i is capacitative, and for l just greater than $n\lambda/2$ it is inductive. (This provides a positive or negative reactance to match another line.)

Problem 7.9

Show that a line of characteristic impedance Z_0 may be matched to a load Z_L by a loss-free quarter wavelength line of characteristic impedance Z_m if $Z_m^2 = Z_0 Z_L$.

(Hint—calculate the input impedance at the $Z_0 Z_m$ junction.)

Problem 7.10

Show that a short-circuited quarter wavelength loss-free line has an infinite impedance and that if it is bridged across another transmission line it will not affect the fundamental wavelength but will short-circuit any undesirable second harmonic.

Problem 7.11

Show that a loss-free line of characteristic impedance Z_0 and length $n\lambda/2$ may be used to couple two high frequency circuits without affecting other impedances.

Problem 7.12

A transmission line has $Z_1 = i\omega L$ and $Z_2 = (i\omega C)^{-1}$. If, for a range of frequencies ω, the phase shift per section β is very small show that $\beta = k$ the wave number and that the phase velocity is independent of the frequency.

Problem 7.13

In a transmission line with losses where $R_0/\omega L_0$ and $G_0/\omega C_0$ are both small quantities expand the expression for the propagation constant

$$\gamma = [(R_0 + \mathrm{i}\omega L_0)(G_0 + \mathrm{i}\omega C_0)]^{1/2}$$

to show that the attenuation constant

$$\alpha = \frac{R_0}{2}\sqrt{\frac{C_0}{L_0}} + \frac{G_0}{2}\sqrt{\frac{L_0}{C_0}}$$

and the wave number

$$k = \omega\sqrt{L_0 C_0} = \frac{\omega}{v}$$

Show that for $G_0 = 0$ the Q value of such a line is given by $k/2\alpha$.

Problem 7.14

Expand the expression for the characteristic impedance of the transmission line of Problem 7.13 in terms of the characteristic impedance of a lossless line to show that if

$$\frac{R_0}{L_0} = \frac{G_0}{C_0}$$

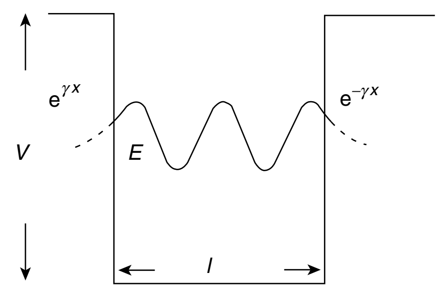
the impedance remains real because the phase effects introduced by the series and shunt losses are equal but opposite.

Problem 7.15

The wave description of an electron of total energy E in a potential well of depth V over the region $0 < x < l$ is given by Schrödinger's time independent wave equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0$$

where m is the electron mass and h is Planck's constant. (Note that $V = 0$ within the well.)



Show that for $E > V$ (inside the potential well) the solution for ψ is a standing wave solution but for $E < V$ (outside the region $0 < x < l$) the x dependence of ψ is $e^{\pm\gamma x}$, where

$$\gamma = \frac{2\pi}{h}\sqrt{2m(V - E)}$$

Problem 7.16

A localized magnetic field H in an electrically conducting medium of permeability μ and conductivity σ will diffuse through the medium in the x-direction at a rate given by

$$\frac{\partial H}{\partial t} = \frac{1}{\mu\sigma}\frac{\partial^2 H}{\partial x^2}$$

Show that the time of decay of the field is given approximately by $L^2\mu\sigma$, where L is the extent of the medium, and show that for a copper sphere of radius 1 m this time is less than 100s.

$$\mu \text{ (copper)} = 1 \cdot 26 \times 10^{-6}\ \mathrm{H\,m^{-1}}$$
$$\sigma \text{ (copper)} = 5 \cdot 8 \times 10^{7}\ \mathrm{S\,m^{-1}}$$

(If the earth's core were molten iron its field would freely decay in approximately 15×10^3 years. In the sun the local field would take 10^{10} years to decay. When σ is very high the local field will change only by being carried away by the movement of the medium—such a field is said to be 'frozen' into the medium—the field lines are stretched and exert a restoring force against the motion.)

Problem 7.17

A point x_0 at the centre of a large slab of material of thermal coductivity k, specific heat C and density ρ has an infinitely high temperature T at a time t_0. If the heat diffuses through the medium at a rate given by

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C}\frac{\partial^2 T}{\partial x^2} = d\frac{\partial^2 T}{\partial x^2}$$

show that the heat flow along the x-aixs is given by

$$f(\alpha, t) = \frac{r}{\sqrt{\pi}}\,\mathrm{e}^{-(r\alpha)^2},$$

where

$$\alpha = (x - x_0) \quad \text{and} \quad r = \frac{1}{2\sqrt{\mathrm{d}t}}$$

by inserting this solution in the differential equation. The solution is a Guassian function; its behaviour with x and t in this problem is shown in Fig. 10.12. At (x_0, t_0) the function is the Dirac delta function. The Guassian curves decay in height and widen with time as the heat spreads through the medium, the total heat, i.e. the area under the Gaussian curve, remaining constant.

Summary of Important Results

Lossless Transmission Line

Inductance per unit length $= L_0$ or μ

Capacitance per unit length $= C_0$ or ε

Wave Equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2}\,(\text{voltage})$$

$$\frac{\partial^2 I}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 I}{\partial t^2}\,(\text{current})$$

Phase Velocity

$$v^2 = \frac{1}{L_0 C_0} \quad \text{or} \quad \frac{1}{\mu\varepsilon}$$

Characteristic Impedance

$$Z_0 = \frac{V}{I} = \sqrt{\frac{L_0}{C_0}} \quad \text{or} \quad \sqrt{\frac{\mu}{\varepsilon}} \quad (\text{for right-going wave})$$

$(-Z_0$ for left-going wave$)$

Transmission Line with Losses

Resistane R_0 per unit length
Shunt conductance G_0 per unit length
Wave equation takes form

$$e^{i\omega t}\left(\frac{\partial^2 V}{\partial x^2} - \gamma^2 V\right) = 0 \quad \text{(same for } I\text{)}$$

where $\gamma = \alpha + ik$ is the propagation constant

$$\alpha = \text{attenuation coefficient}$$
$$k = \text{wave number}$$

giving

$$V = A\, e^{-\alpha x}\, e^{i(\omega t - kx)} + B\, e^{\alpha x}\, e^{i(\omega t + kx)}$$

Characteristic Impedance

$$Z_0' = \frac{V}{I} = \sqrt{\frac{R_0 + i\omega L_0}{G_0 + i\omega C_0}} \quad \text{(right-going wave)}$$
$$(-Z_0' \text{ for left-going wave})$$

Wave Attenuation

Energy absorption in a medium described by diffusion equation

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{d}\frac{\partial \phi}{\partial t}$$

Add to wave equation to account for attenuation giving

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \frac{1}{d}\frac{\partial \phi}{\partial t}$$

with exponentially decaying solution

$$\phi = \phi_m\, e^{-\alpha x}\, e^{i(\omega t - kx)}$$