Limits and Derivatives

Limit of a Function Using Intuitive Approach

- For a function $f(x)$, if for x closes to a implies that $f(x)$ closes to l, then l is called the **limit** of function $f(x)$ at a.

- l is the limit of function $f(x)$ is written as $\lim_{x \to a} f(x) = l$ [read as "limit of $f(x)$ is l, when x tends to a" or "for $x \to a$ (x tends to a), $f(x) \to l$ ($f(x)$ tends to l)]

- If $f(x) = x^3 - 2$, then for x very close to 3, $f(x)$ will be very close to 25. This can be written as $\lim_{x \to 3} (x^3 - 2) = 25$. So, limiting value of $x^3 - 2$ at x closes to 3 is 25.

Solved Examples

Example 1:
For $f(x) = x(a - 3x)$, find the value of a at which the limits of function $f(x)$ when x tends to 4 and when it tends to 5 are the same?

Solution:

It is given that

$f(x) = x(a - 3x)$

$\Rightarrow f(x) = ax - 3x^2$

The limit of function $f(x)$ when x tends to 4 is calculated as follows:

x	3.9	3.95	3.99	3.999	4.001	4.01	4.05	4.1
f(x)	$3.9a -$ 45.63	$3.95a -$ 46.8075	$3.99a -$ 47.7603	$3.999a -$ 47.976003	$4.001a -$ 48.024003	$4.01a -$ 48.2403	$3.05a -$ 49.2075	$4.1a -$ 50.43

$$\therefore \lim_{x \to 4} f(x) = \lim_{x \to 4}(ax - 3x^2) = 4a - 48$$

The limit of function $f(x)$ when x tends to 5 is calculated as follows:

x	4.9	4.95	4.99	4.999	5.001	5.01	5.05	5.1
f(x)	$4.9a -$ 72.03	$4.95a -$ 73.5075	$4.99a -$ 74.7003	$4.999a -$ 74.970003	$5.001a -$ 75.030003	$5.01a -$ 75.3003	$5.05a -$ 76.5075	$5.1a -$ 78.03

$$\therefore \lim_{x \to 5} f(x) = \lim_{x \to 5}(ax - 3x^2) = 5a - 75$$

We have to find the particular value of *a* at which the limits of function *f*(*x*) when *x* tends to 4 and when it tends to 5 are equal.

$$\therefore \lim_{x \to 4} f(x) = \lim_{x \to 5} f(x)$$
$$\Rightarrow 4a - 48 = 5a - 75$$
$$\Rightarrow a = 27$$

Thus, the limiting values of $f(x) = x(a - 3x)$ when *x* tends to 4 and 5 are equal for $a = 27$.

Example 2:

Show that the limit value of $g(y) = [2y - 5]$ does not exist when *y* tends to 2.

Solution:

The given function is

$g(y) = [2y - 5]$.

Clearly, *g*(*y*) is a greatest integer function

Hence, $g(y) = \begin{cases} a - 1, \text{ for } a - 1 < g(y) < a \\ a, \text{ for } a \le g(y) < a + 1 \end{cases}$

Where, *a* is an integer

The limit of *g*(*y*) when *y* tends to 2 is calculated as follows:

y	1.9	1.95	1.99	1.999	2.001	2.01	2.05	2.1
***g*(*y*)**	−2	−2	−2	−2	−1	−1	−1	−1

We may observe that

Left hand limit of the function = $\lim_{y \to 2^-} g(y) = -2$ whereas the right hand limit = $\lim_{y \to 2^+} g(y) = -1$.

Since the left hand and the right hand limits of the function are not equal, the given function does not have a limiting value.

Example 3:

For what real and complex values of *b*, $\lim_{t \to b} v(t) \neq v(b)$, where $v(t) = \frac{(t^4 - 16)(t^2 - 16)}{(t^3 - 1)(2t^2 - t - 28)}$?

Solution:

We know that if a function *v*(*t*) is defined at *t* = *b*, then $\lim_{t \to b} v(t) = v(b)$, else not.

Since $\lim_{t \to b} v(t) \neq v(b)$, we need to find the value of *b*, i.e., *t*, where *v*(*t*) does not exist.

This is only possible, if

$$
\begin{aligned}
&(t^3 - 1)(2t^2 - t - 28) = 0 \\
&\Rightarrow (t-1)(t^2 + t + 1)(2t^2 - 8t + 7t - 28) = 0 \\
&\Rightarrow (t-1)(t^2 + t + 1)[2t(t-4) + 7(t-4)] = 0 \\
&\Rightarrow (t-1)(t^2 + t + 1)(t-4)(2t+7) = 0 \\
&\Rightarrow t = 1 \text{ or } 4 \text{ or } \frac{-7}{2} \text{ or } \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} \\
&\Rightarrow t = 1 \text{ or } 4 \text{ or } \frac{-7}{2} \text{ or } \frac{-1 \pm i\sqrt{3}}{2}
\end{aligned}
$$

So, for *b* = 1, 4, $\frac{-7}{2}$ as real values and $b = \frac{-1 \pm i\sqrt{3}}{2}$ as the complex values, $\lim_{t \to b} v(t) \neq v(b)$, where $v(t) = \frac{(t^4 - 16)(t^2 - 16)}{(t^3 - 1)(2t^2 - t - 28)}$.

Limit of a Polynomial and a Rational Function

Algebra of Limits

- If *f* and *g* are two functions such that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

- $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

  The limit of the sum of two functions is the sum of the limits of the functions.

  For example, $\lim_{x \to 4} \left(x^{\frac{5}{2}} + x^{\frac{3}{2}} \right) = \lim_{x \to 4} x^{\frac{5}{2}} + \lim_{x \to 4} x^{\frac{3}{2}} = 4^{\frac{5}{2}} + 4^{\frac{3}{2}} = 32 + 8 = 40$

- $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

  The limit of the difference between two functions is the difference between the limits of the functions.

  For example, $\lim_{x \to 4} \left(x^{\frac{5}{2}} - x^{\frac{3}{2}} \right) = \lim_{x \to 4} x^{\frac{5}{2}} - \lim_{x \to 4} x^{\frac{3}{2}} = 4^{\frac{5}{2}} - 4^{\frac{3}{2}} = 32 - 8 = 24$

- $\lim_{x \to a} [f(x).g(x)] = \lim_{x \to a} f(x).\lim_{x \to a} g(x)$

  The limit of the product of two functions is the product of the limits of the functions.

  For example, $\lim_{x \to 4} \left(x^{\frac{5}{2}}.x^{\frac{3}{2}} \right) = \lim_{x \to 4} x^{\frac{5}{2}}.\lim_{x \to 4} x^{\frac{3}{2}} = 4^{\frac{5}{2}} \times 4^{\frac{3}{2}} = 32 \times 8 = 256$

- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, where $\lim_{x \to a} g(x) \neq 0$

  The limit of the quotient of the two functions is the quotient of the limits of the functions, where the denominator is not zero.

  For example, $\lim_{x \to 4} \frac{x^{\frac{5}{2}}}{x^{\frac{3}{2}}} = \frac{\lim_{x \to 4} x^{\frac{5}{2}}}{\lim_{x \to 4} x^{\frac{3}{2}}} = \frac{4^{\frac{5}{2}}}{4^{\frac{3}{2}}} = \frac{32}{8} = 4$

- $\lim_{x \to a} [k.f(x)] = k \lim_{x \to a} f(x)$, where *k* is a constant

  The limit of the product of a constant and a function is the product of the constant and the limit of that function.

For example, $\lim_{x \to 4} \left(\frac{9}{2} x^{\frac{5}{2}} \right) = \frac{9}{2} \lim_{x \to 4} x^{\frac{5}{2}} = \frac{9}{2} \times 4^{\frac{5}{2}} = \frac{9}{2} \times 32 = 144$

Limit of a Polynomial Function

- A function $p(x)$ is said to be a polynomial function if $p(x) = 0$ or $p(x) = \sum_{i=0}^{n} a_r x^r$, where $a_r \in$ R and $a_r \neq 0$ for some whole number r.
- The limit of a polynomial function $p(x)$ at $x = a$ is given by $\lim_{x \to a} p(x) = p(a)$

For example, the value of $\lim_{m \to n+3} \left(3m^3 - 9m^2n + 9mn^2 - 3n^3 - m + n - 80\right)$ can be calculated as follows:

$$
\begin{aligned}
&\lim_{m \to n+3} \left(3m^3 - 9m^2n + 9mn^2 - 3n^3 - m + n - 80\right) \\
&= \lim_{m-n \to 3} \left[3\left(m^3 - 3m^2n + 3mn^2 - n^3\right) - (m-n) - 80\right] \\
&= \lim_{m-n \to 3} \left[3(m-n)^3 - (m-n) - 80\right] \\
&= \left[3(3)^3 - (3) - 80\right] \\
&= 81 - 3 - 80 \\
&= -2
\end{aligned}
$$

Limit of a Rational Function

- A function $p(x)$ is said to be a rational function if $p(x) = \frac{q(x)}{r(x)}$, where $q(x)$ and $r(x)$ are polynomials such that $r(x) \neq 0$.
- The limit of a rational function $p(x)$ of the form $p(x) = \frac{q(x)}{r(x)}$ at $x = a$ is given by $\lim_{x \to a} p(x) = \frac{q(a)}{r(a)}$
- For example, to find the value of $\lim_{x \to 64} \frac{\sqrt{x} + 7}{\sqrt[3]{x} + 2}$, we may proceed as follows.

$$\lim_{x \to 64} \frac{\sqrt{x} + 7}{\sqrt[3]{x} + 2} = \frac{\sqrt{64} + 7}{\sqrt[3]{64} - 1} = \frac{8 + 7}{4 - 1} = \frac{15}{3} = 5$$

- For any positive integer *n*, $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

- To know its proof, let us look at the following video.

- For example, $\lim_{y \to 0} \frac{(y+5)^4 - 625}{y}$ can be calculated as follows.

$$\lim_{y \to 0} \frac{(y+5)^4 - 625}{y} = \lim_{y+5 \to 5} \frac{(y+5)^4 - 5^4}{(y+5) - 5} \qquad (y \to 0 \text{ shows that } y+5 \to 5)$$
$$= 4 \times 5^{4-1}$$
$$= 500$$

Solved Examples

Example 1:

Find the values of *a* and *b* if

$\lim_{n \to \infty} \frac{3a.(n+5)! - 2b.(n+4)!}{b.(n+5)! + a.(n+4)!} = -2$ and $\lim_{n \to \infty} \frac{(a+2b).(n+1)! - b.(n-1)!}{(2a-b+1).(n+1)! - a.(n-1)!} = \frac{-1}{2}$

Also, show that $\lim_{x \to 1} \frac{a+2b}{x} = \lim_{x \to \frac{-3}{2}} \frac{b-a}{x^2 - 1}$.

Solution:

We have $\lim_{n \to \infty} \frac{3a.(n+5)! - 2b.(n+4)!}{b.(n+5)! + a.(n+4)!} = -2$

$$\Rightarrow \lim_{n\to\infty}\frac{[3a.(n+5)-2b](n+4)!}{[b.(n+5)+a](n+4)!}=-2$$

$$\Rightarrow \lim_{n\to\infty}\frac{3an+15a-2b}{bn+5b+a}=-2$$

$$\Rightarrow \lim_{n\to\infty}\frac{n\left(3a+\frac{15a-2b}{n}\right)}{n\left(b+\frac{5b+a}{n}\right)}=-2$$

$$\Rightarrow \frac{\lim_{n\to\infty}\left(3a+\frac{15a-2b}{n}\right)}{\lim_{n\to\infty}\left(b+\frac{5b+a}{n}\right)}=-2$$

$$\Rightarrow \frac{3a+0}{b+0}=-2$$

$$\Rightarrow 3a=-2b$$

$$\Rightarrow a=\frac{-2b}{3} \qquad \ldots (1)$$

We also have $\lim_{n\to\infty}\frac{(a+2b).(n+1)!-b.(n-1)!}{(2a-b+1).(n+1)!-a.(n-1)!}=\frac{-1}{2}$

$$\Rightarrow \lim_{n\to\infty} \frac{\left[(a+2b).n(n+1)-b\right](n-1)!}{\left[(2a-b+1).n(n+1)!-a\right](n-1)!} = \frac{-1}{2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{(a+2b)n^2+(a+2b)n-b}{(2a-b+1)n^2+(2a-b+1)n-a} = \frac{-1}{2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{n^2\left[(a+2b)+\frac{(a+2b)}{n}-\frac{b}{n^2}\right]}{n^2\left[(2a-b+1)+\frac{(2a-b+1)}{n}-\frac{a}{n^2}\right]} = \frac{-1}{2}$$

$$\Rightarrow \frac{\lim_{n\to\infty}\left[(a+2b)+\frac{(a+2b)}{n}-\frac{b}{n^2}\right]}{\lim_{n\to\infty}\left[(2a-b+1)+\frac{(2a-b+1)}{n}-\frac{a}{n^2}\right]} = \frac{-1}{2}$$

$$\Rightarrow \frac{a+2b}{2a-b+1} = \frac{-1}{2}$$

$$\Rightarrow \frac{\frac{-2b}{3}+2b}{2\times\frac{-2b}{3}-b+1} = \frac{-1}{2} \qquad \left[\text{Using equation (1)}\right]$$

$$\Rightarrow \frac{4b}{-7b+3} = \frac{-1}{2}$$

$$\Rightarrow 8b = 7b-3$$

$$\Rightarrow b = -3$$

Substituting the value of *b* in equation (1), we obtain

a = 2

Hence, *a* = 2 and *b* = −3

Now,

$$\lim_{x\to 1} \frac{a+2b}{x} = \frac{2+2(-3)}{1} = -4$$

and

$$\lim_{x \to \frac{-3}{2}} \frac{b-a}{x^2-1} = \frac{(-3)-2}{\left(\frac{-3}{2}\right)^2 - 1} = \frac{-5}{\frac{5}{4}} = -4$$

$$\Rightarrow \lim_{x \to 1} \frac{a+2b}{x} = \lim_{x \to \frac{-3}{2}} \frac{b-a}{x^2-1}.$$

Example 2:

Find the value of *n*, such that $\lim_{a \to b-3} \frac{(a-b)^{2n} - 9^n}{(a-b)^{3n} + 27^n} = -\frac{2}{729}$, where *n* is an odd number.

Solution:

$$\lim_{a \to b-3} \frac{(a-b)^{2n} - 9^n}{(a-b)^{3n} + 27^n} = -\frac{2}{729}$$

$$\Rightarrow \lim_{a-b \to -3} \frac{(a-b)^{2n} - 3^{2n}}{(a-b)^{3n} + 3^{3n}} = -\frac{2}{729} \qquad (a \to b-3 \Rightarrow a-b \to -3)$$

$$\Rightarrow \lim_{a-b \to -3} \frac{(a-b)^{2n} - (-3)^{2n}}{(a-b)^{3n} - (-3)^{3n}} = -\frac{2}{729} \qquad (\text{Since } n \text{ is an odd number, } \left(-3\right)^{2n} = 3^{2n} \text{ and } (-3)^{3n} = -3^{3n})$$

$$\Rightarrow \frac{\lim_{a-b \to -3} \frac{(a-b)^{2n} - (-3)^{2n}}{(a-b) - (-3)}}{\lim_{a-b \to -3} \frac{(a-b)^{3n} - (-3)^{3n}}{(a-b) - (-3)}} = -\frac{2}{729}$$

$$\Rightarrow \frac{2n(-3)^{2n-1}}{3n(-3)^{3n-1}} = -\frac{2}{729}$$

$$\Rightarrow \frac{2}{3(-3)^n} = -\frac{2}{729}$$

$$\Rightarrow (-3)^n = \frac{-729}{3}$$

$$\Rightarrow (-3)^n = -243 = (-3)^5$$

$$\Rightarrow n = 5$$

Example 3:

Evaluate $\lim_{x \to 0} \frac{\sqrt{4 + x^3} - \sqrt{4 + x}}{\sqrt{9 + x^7} - \sqrt{9 + x}}$

Solution:

$$\lim_{x\to 0}\frac{\sqrt{4+x^3}-\sqrt{4+x}}{\sqrt{9+x^7}-\sqrt{9+x}}=\frac{0}{0}\text{ form}$$

Hence,

$$\lim_{x\to 0}\frac{\sqrt{4+x^3}-\sqrt{4+x}}{\sqrt{9+x^7}-\sqrt{9+x}}$$

$$=\lim_{x\to 0}\left(\left(\sqrt{4+x^3}-\sqrt{4+x}\right)\times\frac{1}{\sqrt{9+x^7}-\sqrt{9+x}}\right)$$

$$=\lim_{x\to 0}\left(\frac{\left(\sqrt{4+x^3}-\sqrt{4+x}\right)\left(\sqrt{4+x^3}+\sqrt{4+x}\right)}{\sqrt{4+x^3}+\sqrt{4+x}}\times\frac{\sqrt{9+x^7}+\sqrt{9+x}}{\left(\sqrt{9+x^7}-\sqrt{9+x}\right)\left(\sqrt{9+x^7}+\sqrt{9+x}\right)}\right)$$

$$=\lim_{x\to 0}\left(\frac{(4+x^3)-(4+x)}{\sqrt{4+x^3}+\sqrt{4+x}}\times\frac{\sqrt{9+x^7}+\sqrt{9+x}}{(9+x^7)-(9+x)}\right)$$

$$=\lim_{x\to 0}\left(\frac{x(x^2-1)}{\sqrt{4+x^3}+\sqrt{4+x}}\times\frac{\sqrt{9+x^7}+\sqrt{9+x}}{x(x^6-1)}\right)$$

$$=\lim_{x\to 0}\left(\frac{x^2-1}{x^6-1}\times\frac{\sqrt{9+x^7}+\sqrt{9+x}}{\sqrt{4+x^3}+\sqrt{4+x}}\right)$$

$$=\lim_{x\to 0}\frac{x^2-1}{x^6-1}\times\lim_{x\to 0}\frac{\sqrt{9+x^7}+\sqrt{9+x}}{\sqrt{4+x^3}+\sqrt{4+x}}$$

$$=\frac{-1}{-1}\times\frac{3+3}{2+2}$$

$$=\frac{3}{2}$$

Limits of Trigonometric Functions

- Let *f* and *g* be two real-valued functions with the same domain, such that $f(x) \le g(x)$ for all x in the domain of definition. For some a, if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x)$ ≤ $\lim_{x\to a} g(x)$.
- For example, we know that $x^2 \le x^3$, for $x \in \mathbf{R}$ and $x \ge 1$. So, for any $a \in \mathbf{R}$ and $a \ge 1$, $\lim_{x\to a} x^2 \le \lim_{x\to a} x^3$.
- Two important limits are

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$
- To know the proofs of the above two formulae, let us look at the following video.

Solved Examples

Example 1:

Evaluate $\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{3}\sin\left(\frac{\pi}{2} - x\right) + \sin(\pi + x)}{3\pi\left(\frac{\pi}{3} - x\right)}$

Solution

$$\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{3}\sin\left(\frac{\pi}{2} - x\right) + \sin(\pi + x)}{3\pi\left(\frac{\pi}{3} - x\right)} = \lim_{\frac{\pi}{3} - x \to 0} \frac{\sqrt{3}\cos x - \sin x}{3\pi\left(\frac{\pi}{3} - x\right)}$$

$$= \frac{1}{3\pi} \cdot \lim_{\frac{\pi}{3} - x \to 0} \frac{2 \cdot \left[\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right]}{\frac{\pi}{3} - x}$$

$$= \frac{2}{3\pi} \cdot \lim_{\frac{\pi}{3} - x \to 0} \frac{\left[\sin\frac{\pi}{3}\cos x - \cos\frac{\pi}{3}\sin x\right]}{\frac{\pi}{3} - x}$$

$$= \frac{2}{3\pi} \cdot \lim_{\frac{\pi}{3} - x \to 0} \frac{\sin\left(\frac{\pi}{3} - x\right)}{\frac{\pi}{3} - x}$$

$$= \frac{2}{3\pi} \times 1$$

$$= \frac{2}{3\pi}$$

Example 2:

If $\lim_{x\to 0}\dfrac{\cos 4x-\sin\left(\dfrac{\pi}{2}+5x\right)}{x^2}=\dfrac{3a+b}{2}$ and $\lim_{x\to 0}\dfrac{\sin\left(\dfrac{\pi}{4}+5x\right)-\sin\left(\dfrac{\pi}{4}+3x\right)}{\cdot x}=\sqrt{4b-5a}$, then find the value of $\sqrt{5a+2b}$.

Solution:

$$\lim_{x\to 0}\frac{\cos 4x-\sin\left(\frac{\pi}{2}+5x\right)}{x^2}=\frac{3a+b}{2}$$

$$\Rightarrow \frac{3a+b}{2}=\lim_{x\to 0}\frac{\cos 4x-\cos 5x}{x^2}$$

$$=\lim_{x\to 0}\frac{2\sin\left(\frac{5x+4x}{2}\right)\sin\left(\frac{5x-4x}{2}\right)}{x^2}$$

$$=2\lim_{x\to 0}\frac{\sin\frac{9x}{2}\cdot\sin\frac{x}{2}}{x^2}$$

$$=2\lim_{x\to 0}\frac{\sin\frac{9x}{2}}{x}\times\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}$$

$$=2\times\frac{9}{2}\lim_{\frac{9x}{2}\to 0}\frac{\sin\frac{9x}{2}}{\frac{9x}{2}}\times\frac{1}{2}\cdot\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

$$=9\times 1\times\frac{1}{2}\times 1$$

$$=\frac{9}{2}$$

$\Rightarrow 3a + b = 9$

$\Rightarrow b = 9 - 3a$ …(1)

It is also given that

$$\lim_{x\to 0}\frac{\sin\left(\frac{\pi}{4}+5x\right)-\sin\left(\frac{\pi}{4}+3x\right)}{x}=\sqrt{4b-5a}$$

$$\Rightarrow \sqrt{4b-5a}=\lim_{x\to 0}\frac{2\sin\left[\frac{\left(\frac{\pi}{4}+5x\right)-\left(\frac{\pi}{4}+3x\right)}{2}\right]\cdot\cos\left[\frac{\left(\frac{\pi}{4}+5x\right)+\left(\frac{\pi}{4}+3x\right)}{2}\right]}{x}$$

$$=2\lim_{x\to 0}\frac{\sin x\cdot\cos\left(\frac{\pi}{4}+4x\right)}{x}$$

$$=2\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\cos\left(\frac{\pi}{4}+4x\right)$$

$$=2\times 1\times\frac{1}{\sqrt{2}}$$

$$=\sqrt{2}$$

$\Rightarrow 4b - 5a = 2$

From (1), we have

$4(9 - 3a) - 5a = 2$

$36 - 17a = 2$

$17a = 34$

$a = 2$

Substituting $a = 2$ in equation (1), we obtain $b = 3$

Now, $\sqrt{5a+2b}=\sqrt{5\times 2+2\times 3}=\sqrt{16}=4$

Derivative of a Function

- Suppose *f* is a real-valued function and *a* is a point in the domain of definition. If the limit $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists, then it is called the derivative of *f* at *a*. The derivative of *f* at *a* is denoted by $f'(a)$.

- $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- Suppose *f* is a real-valued function. The derivative of *f* {denoted by $f'(x)$ or $\frac{d}{dx}[f(x)]$ } is defined by

  $$\frac{d}{dx}[f(x)] = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  This definition of derivative is called the **first principle** of derivative.

- For example, the derivative of $y = (ax-b)^{10}$ is calculated as follows.

  We have $y = f(x) = (ax-b)^{10}$; using the first principle of derivative, we obtain

  $$\begin{aligned}\frac{dy}{dx} = f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[a(x+h)-b]^{10} - (ax-b)^{10}}{h} \\ &= \lim_{h \to 0} \frac{\left[a(x+h)-b-(ax-b)\right] \cdot \sum_{r=0}^{9} [a(x+h)-b]^{9-r} (ax-b)^r}{h} \\ &= \lim_{h \to 0} \frac{ah}{h} \cdot \lim_{h \to 0} \sum_{r=0}^{9} [a(x+h)-b]^{9-r} ax-b)^r \\ &= a \sum_{r=0}^{9} (ax-b)^{9-r} \cdot (ax-b)^r \\ &= a[(ax-b)^{9-0} \cdot (ax-b)^0 + (ax-b)^{9-1} \cdot (ax-b)^1 + \ldots + (ax-b)^{9-9} \cdot (ax-b)^9] \\ &= 10a(ax-b)^9\end{aligned}$$

Solved Examples

Example 1:

Find the derivative of $f(x) = \text{cosec}^2\, 2x + \tan^2 4x$. Also, find $f'(x)$ at $x = \frac{\pi}{6}$.

Solution:

The derivative of $f(x) = \text{cosec}^2\, 2x + \tan^2 4x$ is calculated as follows.

$$f'(x)=\lim_{h\to 0}\frac{\operatorname{cosec}^2 2(x+h)+\tan^2 4(x+h)-\left[\operatorname{cosec}^2 2(x)+\tan^2 4(x)\right]}{h}$$

$$=\lim_{h\to 0}\frac{\left[\operatorname{cosec}^2(2x+2h)-\operatorname{cosec}^2 2x\right]+\left[\tan^2(4x+4h)-\tan^2(4x)\right]}{h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{1}{\sin^2(2x+2h)}-\frac{1}{\sin^2 2x}\right)+\left(\frac{\sin^2(4x+4h)}{\cos^2(4x+4h)}-\frac{\sin^2 4x}{\cos^2 4x}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{\sin^2 2x-\sin^2(2x+2h)}{\sin^2 2x\ \sin^2(2x+2h)}\right)+\left(\frac{\sin^2(4x+4h)\cos^2 4x-\cos^2(4x+4h)\sin^2 4x}{\cos^2 4x\ \cos^2(4x+4h)}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\left[\sin 2x-\sin(2x+2h)\right]\left[\sin 2x+\sin(2x+2h)\right]}{h\ \sin^2 2x\ \sin^2(2x+2h)}$$

$$+\lim_{h\to 0}\frac{\left[\sin(4x+4h)\cos 4x-\cos(4x+4h)\sin 4x\right]\left[\sin(4x+4h)\cos 4x+\cos(4x+4h)\sin 4x\right]}{h\ \cos^2 4x\ \cos^2(4x+4h)}$$

$$=\lim_{h\to 0}\frac{2\cos(2x+h)\sin(-h)\times 2\sin(2x+h)\cos(-h)}{h\ \sin^2 2x\ \sin^2(2x+2h)}+\lim_{h\to 0}\frac{\sin(4x+4h-4x)\sin(4x+4h+4x)}{h\ \cos^2 4x\ \cos^2(4x+4h)}$$

$$=-4\lim_{h\to 0}\frac{\sin h}{h}\times\lim_{h\to 0}\frac{\cos(2x+h)\times\sin(2x+h)\cos(h)}{\sin^2 2x\ \sin^2(2x+2h)}+4\lim_{4h\to 0}\frac{\sin(4h)}{4h}\times\lim_{h\to 0}\frac{\sin(4x+4h+4x)}{\cos^2 4x\ \cos^2(4x+4h)}$$

$$=-4\times 1\times\frac{\cos 2x}{\sin^3 2x}+4\times 1\times\frac{\sin 8x}{\cos^4 4x}$$

$$=-4\cot 2x\ \operatorname{cosec}^2 2x+\frac{8\sin 4x\ \cos 4x}{\cos^4 4x}$$

$$=-4\cot 2x\ \operatorname{cosec}^2 2x+8\tan 4x\ \sec^2 4x$$

At $x=\frac{\pi}{6}$, $f'\left(\frac{\pi}{6}\right)$ is given by

$$f'\left(\frac{\pi}{6}\right)=-4\cot\left(\frac{\pi}{3}\right)\operatorname{cosec}^2\left(\frac{\pi}{3}\right)+8\tan\left(\frac{2\pi}{3}\right)\sec^2\left(\frac{2\pi}{3}\right)$$

$$=-4\times\frac{1}{\sqrt{3}}\times\left(\frac{2}{\sqrt{3}}\right)^2+8(-\sqrt{3})\times(-2)^2$$

$$=\frac{-16}{3\sqrt{3}}-32\sqrt{3}$$

$$=\frac{-304}{3\sqrt{3}}$$

Example 2:

If $y = (ax^2 + x + b)^2$, then find the values of a and b, such that $\frac{dy}{dx} = 4x^2(4x+3) + 2(13x+3)$.

Solution:

It is given that $y = (ax^2 + x + b)^2$

$$\Rightarrow \frac{dy}{dx} = \lim_{h \to 0} \frac{\left[a(x+h)^2 + (x+h) + b\right]^2 - \left[ax^2 + x + b\right]^2}{h}$$
$$= \lim_{h \to 0} \frac{\left[a(x+h)^2 + (x+h) + b - (ax^2 + x + b)\right]\left[a(x+h)^2 + (x+h) + b + (ax^2 + x + b)\right]}{h}$$
$$= \lim_{h \to 0} \frac{\left[a(2xh + h^2) + h\right]\left[a(x+h)^2 + (x+h) + b + (ax^2 + x + b)\right]}{h}$$
$$= \lim_{h \to 0} \frac{h\left[a(2x+h)+1\right]}{h} \times \lim_{h \to 0}\left[a(x+h)^2 + (x+h) + b + (ax^2 + x + b)\right]$$
$$= (2ax+1) \times 2(ax^2 + x + b)$$
$$= 4a^2x^3 + 6ax^2 + (4ab+2)x + 2b$$
$$\Rightarrow 4x^2(4x+3) + 2(13x+3) = 4a^2x^3 + 6ax^2 + (4ab+2)x + 2b$$
$$\Rightarrow 4a^2x^3 + 6ax^2 + (4ab+2)x + 2b = 16x^3 + 12x^2 + 26x + 6$$

Comparing the coefficients of x^3, x^2, x, and the constant terms of the above expression, we obtain

$$4a^2 = 16,\ 6a = 12,\ 4ab + 2 = 26 \text{ and } 2b = 6$$
$$\Rightarrow a = \pm 2,\ a = 2,\ b = 3 \text{ and } b = 3$$
$$\Rightarrow a = 2 \text{ and } b = 3$$

Example 3:

What is the derivative of y with respect to x, if $y = \sqrt{\frac{ax+b}{cx-d}}$?

Solution:

It is given that $y = \sqrt{\frac{ax+b}{cx-d}}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\frac{ax+b}{cx-d}}\right)$$

$$= \lim_{h\to 0} \frac{\sqrt{\frac{a(x+h)+b}{c(x+h)-d}} - \sqrt{\frac{ax+b}{cx-d}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{[a(x+h)+b](cx-d)} - \sqrt{[c(x+h)-d][ax+b]}}{h\sqrt{[c(x+h)-d][cx-d]}}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{[a(x+h)+b](cx-d)} - \sqrt{[c(x+h)-d][ax+b]}\right)\times\left(\sqrt{[a(x+h)+b](cx-d)} + \sqrt{[c(x+h)-d][ax+b]}\right)}{h\left(\sqrt{[c(x+h)-d][cx-d]}\right)\left(\sqrt{[a(x+h)+b](cx-d)} + \sqrt{[c(x+h)-d][ax+b]}\right)}$$

$$= \lim_{h\to 0} \frac{[a(x+h)+b](cx-d) - [c(x+h)-d][ax+b]}{h\left(\sqrt{[c(x+h)-d][cx-d]}\right)\left(\sqrt{[a(x+h)+b](cx-d)} + \sqrt{[c(x+h)-d][ax+b]}\right)}$$

$$= \lim_{h\to 0} \frac{h[a(cx-d) - c(ax+b)]}{h\left(\sqrt{[c(x+h)-d][cx-d]}\right)\left(\sqrt{[a(x+h)+b](cx-d)} + \sqrt{[c(x+h)-d][ax+b]}\right)}$$

$$= \frac{a(cx-d) - c(ax+b)}{\left(\sqrt{[cx-d][cx-d]}\right)\left(\sqrt{(ax+b)(cx-d)} + \sqrt{(cx-d)(ax+b)}\right)}$$

$$= \frac{-(ad+bc)}{2(cx-d)\sqrt{(ax+b)(cx-d)}}$$

Derivatives of Trigonometric and Polynomial Functions

Derivatives of Trigonometric Functions and Standard Formulas

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(x^n) = nx^{n-1}$

For example, $\frac{d}{dx}(x^7) = 7x^{7-1} = 7x^6$

- $\frac{d}{dx}(C) = 0$, where C is a constant

Algebra of Derivatives

- If *f* and *g* are two functions such that their derivatives are defined in a common domain, then
- $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$
  This means that the derivative of the sum of two functions is the sum of the derivatives of the functions.
  For example, $\frac{d}{dx}\left(x^{\frac{5}{2}}+x^{\frac{3}{2}}\right) = \frac{d}{dx}\left(x^{\frac{5}{2}}\right)+\frac{d}{dx}\left(x^{\frac{3}{2}}\right) = \frac{5}{2}x^{\frac{5}{2}-1}+\frac{3}{2}x^{\frac{3}{2}-1} = \frac{5}{2}x^{\frac{3}{2}}+\frac{3}{2}x^{\frac{1}{2}}$
- $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x)-\frac{d}{dx}g(x)$

  This means that the derivative of the difference between two functions is the difference between the derivatives of the function.

  For example, $\frac{d}{dx}\left(\sin x - x^{\frac{1}{3}}\right) = \frac{d}{dx}(\sin x)-\frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \cos x-\frac{1}{3}x^{\frac{1}{3}-1} = \cos x-\frac{1}{3}x^{\frac{-2}{3}}$
- $\frac{d}{dx}[f(x).g(x)] = \frac{d}{dx}f(x).g(x)+f(x).\frac{d}{dx}g(x)$

  This is known as the **product** rule of derivative.
  For

  example,

  $\frac{d}{dx}(x^3\cos x) = \frac{d}{dx}(x^3).\cos x+(x^3).\frac{d}{dx}(\cos x) = 3x^2\cos x + x^3(-\sin x) = 3x^2\cos x - x^3\sin x$
- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}f(x).g(x)-f(x).\frac{d}{dx}g(x)}{[g(x)]^2}$, where $\frac{d}{dx}g(x) \neq 0$

  This is known as the **quotient rule** of derivative.

For example,

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\frac{d}{dx}(\sin x).\cos x - \sin x.\frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{\cos x.\cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

- $\frac{d}{dx}[k.f(x)] = k\frac{d}{dx}f(x)$, where *k* is a constant

This means that the derivative of the product of a constant and a function is the product of that constant and the derivative of that function.

For example,

$$\frac{d}{dx}(\sin 2x) = \frac{d}{dx}(2\sin x.\cos x)$$

$$= 2\frac{d}{dx}(\sin x.\cos x)$$

$$= 2\left(\frac{d}{dx}(\sin x).\cos x + \sin x.\frac{d}{dx}(\cos x)\right)$$

$$= 2[\cos x.\cos x + \sin x.(-\sin x)]$$

$$= 2(\cos^2 x - \sin^2 x)$$

$$= 2\cos 2x$$

Derivative of a Polynomial Function

- A function $p(x)$ is said to be a polynomial function if $p(x) = 0$ or $p(x)=\sum_{i=0}^{n} a_r x^r$, where $a_r \in$ R and $a_r \neq 0$ for some whole number r.
- The derivative of a polynomial function $p(x)=\sum_{i=0}^{n} a_r x^r$ is given by $\frac{d}{dx}\left[p(x)\right]=\sum_{i=1}^{n} r a_r x^{r-1}$

Solved Examples

Example 1:

If $y=\left(\sqrt{\frac{1+\cos 2x}{1-\cos 2x}}+\sqrt{\sec^2 x-1}\right)^{-1}+(1+x)^n$, then show that $\frac{dy}{dx}-n(1+x)^{n-1}=\cos 2x$.

Solution:

We have

$$\begin{aligned}
y&=\left(\sqrt{\frac{1+\cos 2x}{1-\cos 2x}}+\sqrt{\sec^2 x-1}\right)^{-1}+(1+x)^n\\
&=\left(\sqrt{\frac{\cos^2 x}{\sin^2 x}}+\sqrt{\tan^2 x}\right)^{-1}+\sum_{i=0}^{n} {}^nC_r x^r\\
&=(\cot x+\tan x)^{-1}+\left(1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3 \ldots+\frac{n(n-1)\ldots 2}{(n-1)!}x^{n-1}+\frac{n(n-1)\ldots 1}{n!}x^n\right)\\
&=\left(\frac{\cos x}{\sin x}+\frac{\sin x}{\cos x}\right)^{-1}+\left(1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3 \ldots+\frac{n(n-1)\ldots 2}{(n-1)!}x^{n-1}+\frac{n(n-1)\ldots 1}{n!}x^n\right)\\
&=\left(\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\right)^{-1}+\left(1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3 \ldots+\frac{n(n-1)\ldots 2}{(n-1)!}x^{n-1}+\frac{n(n-1)\ldots 1}{n!}x^n\right)\\
&=\sin x.\cos x+\left(1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3 \ldots+\frac{n(n-1)\ldots 2}{(n-1)!}x^{n-1}+\frac{n(n-1)\ldots 1}{n!}x^n\right)
\end{aligned}$$

Hence,

$$\frac{dy}{dx}=\frac{d}{dx}(\sin x.\cos x)+\frac{d}{dx}\left(1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3 \ldots+\frac{n(n-1)\ldots 2}{(n-1)!}x^{n-1}+\frac{n(n-1)\ldots 1}{n!}x^n\right)$$

Now,

$$\frac{d}{dx}(\sin x.\cos x)=\frac{d}{dx}(\sin x).\cos x+\sin x.\frac{d}{dx}(\cos x)$$
$$=\cos x.\cos x+\sin x(-\sin x)$$
$$=\cos^2 x-\sin^2 x$$
$$=\cos 2x$$

$$\frac{d}{dx}\left(1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3\ \dots+\frac{n(n-1)\dots 2}{(n-1)!}x^{n-1}+\frac{n(n-1)\dots 1}{n!}x^n\right)$$
$$=\frac{d}{dx}(1)+\frac{d}{dx}(nx)+\frac{d}{dx}\left(\frac{n(n-1)}{2!}x^2\right)+\frac{d}{dx}\left(\frac{n(n-1)(n-2)}{3!}x^3\right)\dots+\frac{d}{dx}\left(\frac{n(n-1)\dots 2}{(n-1)!}x^{n-1}\right)+\frac{d}{dx}\left(\frac{n(n-1)\dots 1}{n!}\right)x^n$$
$$=0+n\frac{d}{dx}(x)+\frac{n(n-1)}{2!}\frac{d}{dx}(x^2)+\frac{n(n-1)(n-2)}{3!}\frac{d}{dx}(x^3)+\dots+\frac{n(n-1)\dots 2}{(n-1)!}\frac{d}{dx}(x^{n-1})+\frac{n(n-1)\dots 1}{n!}\frac{d}{dx}(x^n)$$
$$=n+\frac{2n(n-1)}{2!}x+\frac{3n(n-1)(n-2)}{3!}x^2+\dots+\frac{(n-1).n(n-1)\dots 2}{(n-1)!}(x^{n-2})+\frac{n.n(n-1)\dots 1}{n!}(x^{n-1})$$
$$=n\left(1+(n-1)x+\frac{(n-1)(n-2)}{2!}x^2+\dots+\frac{(n-1)(n-2)\dots 2}{(n-2)!}x^{n-2}+\frac{(n-1)(n-2)\dots 1}{(n-1)!}x^{n-1}\right)$$
$$=n(1+x)^{n-1}$$

Hence,

$$\frac{dy}{dx}=\cos 2x+n(1+x)^n$$
$$\Rightarrow \cos 2x=\frac{dy}{dx}-n(1+x)^n$$

Example 2:

Find $\frac{dy}{dx}$ if $y=\frac{2x^7+3+\tan x}{x(\sin x-\cos x)}$.

Solution:

$$y=\frac{2x^7+3+\tan x}{x(\sin x-\cos x)}=\frac{2x^7+3+\tan x}{(x\sin x-x\cos x)}$$
$$\Rightarrow \frac{dy}{dx}=\frac{(2x^7+3+\tan x)'.(x\sin x-x\cos x)-(2x^7+3+\tan x).(x\sin x-x\cos x)'}{(x\sin x-x\cos x)^2} \quad \dots (1)$$
$$\because \left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$$

Now,

$$(2x^7+3+\tan x)' = \frac{d}{dx}(2x^7+3+\tan x)$$
$$= 2\frac{d}{dx}(x^7)+\frac{d}{dx}(3)+\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$
$$= 2\times 7x^6+0+\frac{\frac{d}{dx}(\sin x).\cos x-\sin x.\frac{d}{dx}(\cos x)}{\cos^2 x}$$
$$= 14x^6+\frac{\cos x.\cos x-\sin x.(-\sin x)}{\cos^2 x}$$
$$= 14x^6+\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$
$$= 14x^6+\sec^2 x$$

$$(x\sin x - x\cos x)' = \frac{d}{dx}(x\sin x - x\cos x)$$
$$= \frac{d}{dx}(x\sin x)-\frac{d}{dx}(x\cos x)$$
$$= \frac{d}{dx}(x).(\sin x)+x.\frac{d}{dx}(\sin x)-\left(\frac{d}{dx}(x).(\cos x)+x.\frac{d}{dx}(\cos x)\right) \qquad \left[(uv)'=u'v+uv'\right]$$
$$= \sin x + x\cos x - [\cos x + x(-\sin x)]$$
$$= (1+x)\sin x + (x-1)\cos x$$

On substituting all the values in equation (1), we obtain

$$\frac{dy}{dx} = \frac{(2x^7+3+\tan x)'.(x\sin x - x\cos x)-(2x^7+3+\tan x).(x\sin x - x\cos x)'}{(x\sin x - x\cos x)^2}$$
$$= \frac{(14x^6+\sec^2 x).(x\sin x - x\cos x)-(2x^7+3+\tan x).\left[(1+x)\sin x+(x-1)\cos x\right]}{x^2(\sin x-\cos x)^2}$$