Vector calculas

Dot product

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

Cross Product

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta$$

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

∇

$\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$\nabla \rightarrow$ Scalar $\phi(x,y,z) \longrightarrow \nabla\phi$ Vector

Grad ϕ

$\nabla \rightarrow$ Vector $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\rightarrow \nabla . \vec{F}$ Scalar

div $\vec{F}$

$\rightarrow \nabla \times \vec{F}$ Vector

Curl $\vec{F}$

$$\phi(x, y, z) = c$$

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial t} = 0$$

$$\left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \cdot \left(\hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}\right) = 0$$

$$\underbrace{\nabla \phi}_{\text{Normal}} \cdot \underbrace{\frac{d\vec{r}}{dt}}_{\text{tengent (Rate of change of position)}} = 0$$

* The gradient of a scalar point function $\phi(x, y, c) = c$ is given by

$$\text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$\text{Grad}\,\phi = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

which is a vector point function

* The unit normal vector to the surface $\phi(x, y, z)$ is given by $\hat{n} = \dfrac{\nabla \phi}{|\nabla \phi|}$

* If θ is the angle between surface $F(x,y,z) = \phi c$ & $\phi(x,y,z) = c$ at a point P

then $$\cos\theta = \frac{\nabla f_{at P} \cdot \nabla \phi_{at P}}{|\nabla f_{at P}|\,|\nabla f_{at P}|}$$

* if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ & $r = \sqrt{x^2+y^2+z^2}$

then (i) $$\nabla(f(r)) = \frac{f'(r)}{r}\vec{r}$$

(ii) $$\nabla^2(f(r)) = f''(r) + \frac{2}{r}f'(r)$$

* The directional derivative of surface $\phi(x,y,z)$ at point P in the direction of a vector $\vec{a}$ is given by $\nabla\phi_{at P} \cdot \frac{\vec{a}}{|\vec{a}|}$

* The maximum directional derivative to the surface $\phi(x,y,z)$ at point P is given by
$= |\nabla\phi_{at P}|$

$\Rightarrow$ $$D.D. = |\nabla\phi| \cdot \frac{\vec{a}}{|\vec{a}|} = |\nabla\phi|\,|1|\cos 0$$

$$D.D. = |\nabla\phi_{at P}|$$

* The Divergence of a Vector point function $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is given by

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial y}$$ which is a scalar point function.

$\nabla\cdot\vec{F} = 0 \Leftrightarrow F$ is a solenoidel vector

* $\boxed{\text{div}\left(\text{curl}\,\vec{F}\right) = 0}$ $\Rightarrow$ $\nabla\cdot\left(\nabla\times\vec{F}\right) = 0$

* The curl of a vector point function $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is given by

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\text{Curl}\,\vec{F} = i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$\nabla\times\vec{F} = 0 \Leftrightarrow F$ is a irrotation vector

* $\boxed{\text{Curl}\left(\text{Grad}\,\phi\right) = 0}$ $\quad \nabla\times\left(\nabla\phi\right) = 0$

Ques $\nabla(r^n)$ — ?

$$\nabla(f(r)) = \frac{f'(r)}{r}\vec{r}$$

$$f(r) = r^n \quad , \quad f'(r) = n r^{n-1}$$

$$\nabla(r^n) = \frac{n r^{n-1}}{r}\vec{r}$$

$$\nabla(r^n) = n r^{n-2}\vec{r}$$

Ques $\nabla(\ln r)$ — ?

$$f(r) = \ln r \quad , \quad f'(r) = \frac{1}{r}$$

$$\nabla(\ln r) = \frac{1/r}{r}\vec{r}$$

$$\nabla(\ln r) = \frac{1}{r^2}\vec{r}$$

Ques. $\text{div}\left(r^2 \nabla(\ln r)\right) = ?$

$$\nabla\cdot\left(r^2\left(\frac{1}{r^2}\vec{r}\right)\right)$$

$$= \nabla\cdot(\vec{r}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z}$$

$$= 1 + 1 + 1 = 3$$

Que $\nabla \times \vec{r}$ _ ?

$$\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$$= i(0-0) - j(0-0) + \hat{k}(0-0)$$

$$\nabla \times \vec{r} = 0$$

Ques. $\nabla^2\left(\frac{1}{r}\right)$ - ?

$$\nabla^2(f(r)) = f''(r) + \frac{2}{r} f'(r)$$

$$f(r) = \frac{1}{r}$$

$$f'(r) = \frac{-1}{r^2}, \quad f''(r) = \frac{2}{r^3}$$

$$\nabla^2\left(\frac{1}{r}\right) = \frac{2}{r^3} + \frac{2}{r}\left(-\frac{1}{r^2}\right)$$

$$\nabla^2\left(\frac{1}{r}\right) = 0$$

Q. 1
W.B.

$\phi : x^3 - y^3 + x^2 z$ at $(1, 1, -2)$

Unit normal vector = ?

$$= \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \left(i\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\cdot\ \phi$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial}{\partial y}\phi + \frac{\partial\phi}{\partial z}$$

$$= i\left(3x^2 + 2xz\right) + \hat{j}\left(-3y^2\right) + \hat{k}\left(x^2\right)$$

$$\nabla\phi_{\text{at } (1,1,-2)} = \hat{i}\left(3-4\right) + \hat{j}\left(-3\right) + \hat{k}\left(1\right)$$

$$= -\hat{i} - 3\hat{j} + \hat{k}$$

$$|\nabla\phi| = \sqrt{1+9+1} = 1$$

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{-i - 3\hat{j} + \hat{k}}{\sqrt{11}}$$

Q.2

$\vec{F} = r^n \vec{r} = r^n\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

$\nabla\cdot\vec{F} = 0 \quad = \quad \vec{F} = r^n x\,\hat{i} + r^n y\,\hat{j} + r^n z\,\hat{k}$

$$\frac{\partial}{\partial x}\left(r^n x\right) + \frac{\partial}{\partial y}\left(r^n y\right) + \frac{\partial}{\partial z}\left(r^n z\right) = 0$$

$$r^n + x\,n r^{n-1}\frac{\partial r}{\partial x} + r^n + y\,n r^{n-1}\frac{\partial r}{\partial y} + r^n + z\,n r^{n-1}\frac{\partial r}{\partial z} = 0$$

$$\Rightarrow 3r^n + n r^{n-1}\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right) = 0$$

$$\Rightarrow 3r^n + n r^{n-1}\cdot r = 0$$

$$\Rightarrow (3+n) r = 0$$

$$\Rightarrow n = -3$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^3}$ which is a degree 1 homogeneous function

$$x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z} = nr = r$$

Q.3 $\phi = x^2yz + 4xz^2$ $\qquad a = 2\hat{i} - \hat{j} - 2\hat{k}$

$$\nabla\phi = \cancel{(2xyz)}\hat{i}\,(2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$$\nabla\phi \text{ at } (1,-2,-1) = \left(2\times1\times(-2)(-1) + 4(-1)^2\right)\hat{i} + \left(1^2(-1)\right)\hat{j} + \left(1^2(-2) + 8(1)(-1)\right)\hat{k}$$

$$= 8\hat{i} - \hat{j} - 10\hat{k}$$

$$\text{So. D.D.} = \vec{\nabla}\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \frac{(2\hat{i} - \hat{j} - 2\hat{k})}{\sqrt{4+1+4}}$$

$$= \frac{16 + 1 + 20}{3} = \cancel{33}$$

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{37}{3}$$

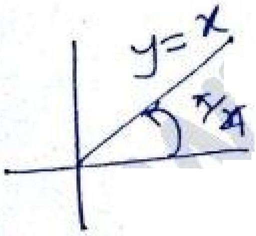
Q.4 $\phi : x^{2/3} + y^{2/3}$ at $(8, 8)$

$$\nabla\phi = \frac{2}{3}(x)^{-1/3}\hat{i} + \frac{2}{3}(y)^{-1/3}$$

$$\nabla\phi_{(8,8)} = \frac{1}{3}\hat{i} + \frac{1}{3}\hat{j} \qquad \vec{a} = \hat{i} + \hat{j} \quad \text{at } y = x$$

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = \left(\frac{1}{3}\hat{i} + \frac{1}{3}\hat{i}\right) \cdot \frac{(i + j)}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{3}$$



$$\vec{r} = r\cos\pi/4\,\hat{i} + r\sin\pi/4\,\hat{j}$$

$$\vec{r} = \frac{r}{\sqrt{2}}\hat{i} + \frac{r}{\sqrt{2}}\hat{j}$$

$$\frac{\vec{r}}{r} = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}$$

Ques1 $\vec{F} = \dfrac{(x, y, z)}{(x^2+y^2+z^2)^{3/2}}$, $(x,y,z) \in R^3$ then $\nabla \cdot \vec{F} = ?$

$$F = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}} = \frac{\vec{r}}{(r^2)^{3/2}} = \frac{\vec{r}}{r^3}$$

$$F = r^{-3}\,\vec{r}$$

→ Solenoidal Vactor

$$\nabla \cdot \vec{F} = 0$$

Question if $|\vec{F}| = r^n$ & $\nabla \cdot \vec{F} = 0$ then $n = ?$

$$\vec{F} = r^n \frac{\vec{r}}{r} = r^{n-1}\,\vec{r}$$

$\vec{F} = r^{n-1}\vec{r}$, & $\nabla \cdot \vec{F} = 0$

previous problem $n-1 = -3$

$n = -2$

Q24 $\vec{F} = 3xy^2\,\hat{i} - 4yz\,\hat{j} + xz^2\,\hat{k}$

$$\nabla \cdot (\nabla \times \vec{F}) = 0$$

Q26

$$V = \int_3^{4.5}\int_{\pi/8}^{\pi/4}\int_3^{5} \rho^2 \sin\phi \, d\rho \, d\phi \, dz$$

$$= \int_3^{4.5}\int_{\pi/8}^{\pi/4} \left[\frac{\rho^3}{3}\right]_3^5 \sin\phi \, d\phi \, dz$$

$$= \frac{98}{3}(2)\int_3^{4.5}\int_{\pi/8}^{\pi/4} \sin\phi \, d\phi \, dz$$

$$= (2)\frac{98}{3}\int_3^{4.5} \left[-\cos\phi\right]_{\pi/8}^{\pi/4} dz.$$

$$= (2)\frac{98}{3}\left[-\cos\left(\frac{\pi}{4}\right) - \left(-\cos\frac{\pi}{8}\right)\right](4.5-3)$$

Vol =

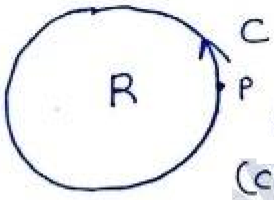
Green's Theorem :-

transformation b/w double integrals and line integral

If $F_1(x,y)$ & $F_2(x,y)$ are two differential Function of x & y defined on a regian R bounded by a simple closed curve c then.

$$\boxed{\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy}$$

workdone



$\vec{r} = x\hat{i} + y\hat{j}$ (Position

$d\vec{r} = dx\hat{i} + dy\hat{j}$ (ch

(change in position)

force $= \vec{F} = F_1\hat{i} + F_2\hat{j}$

$F.d\vec{r} = F_1 dx + F_2\,dy$

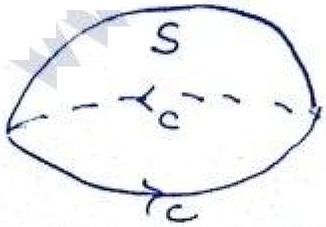
for whole Circle $\oint_C F_1 dx + F_2 dy$

Stokes Theorem :-

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is a differentible vector point function defined on an open surface S bounded by a simple closed curve c then

$$\boxed{\oint_C \vec{F}.d\vec{r} = \iint_R (\nabla \times \vec{F}).\hat{n}\,ds}$$

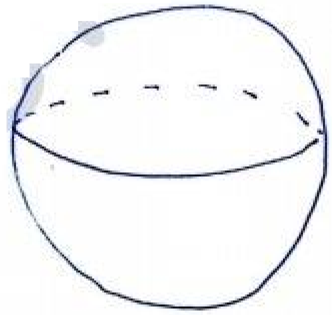
workdone



Gauss divergence theorem:-

If $\vec{F} = \vec{F_1}\hat{i} + F_2\hat{j} + F_3\hat{k}$ is a differentible vector point function defined on a closed surface S enclosing Volume then

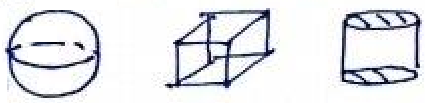
$$\boxed{\underbrace{\oiint_S \vec{F}\cdot\hat{n}\,ds}_{\text{flux}} = \iiint \nabla\cdot\vec{F}\,dv}$$



Remember

Surface (s)

- Closed surface → Gauss



- open surface → stokes

Simple Closed curve (c)

- Vector $F_1\hat{i}+F_2\hat{j}$ → Green's
- Vector $F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ → Stoke's

Q.5 $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^3}{c^2} = 1$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot F\,dv$$

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 3\iiint_V dv = 3V$$

where V is volume of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$V = \frac{4}{3}\pi abc$$

$$= 3\left(\tfrac{4}{3}\pi abc\right)$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 4\pi abc.$$

Q6 $x^2+y^2+z^2 = 1$

$$\iint_S \left(x\sin y,\ \cos^2 x,\ 2z - z\sin y\right)\cdot\underbrace{(x,y,z)}_{\hat{n}}\,ds = ?$$

$$F = x\sin y\,\hat{i} + \cos^2 x\,\hat{j} + (2z - z\sin y)\hat{k}$$

$$\iiint_V \nabla\cdot F\,dv$$

$$\nabla\cdot F = \sin y + 0 + 2 - \sin y$$

$$\nabla\cdot F = 2$$

$$= \iiint_V 2\,dv = 2V$$

$$= 2\times\tfrac{4}{3}\pi(1)^3$$

$$= 8\pi/3$$

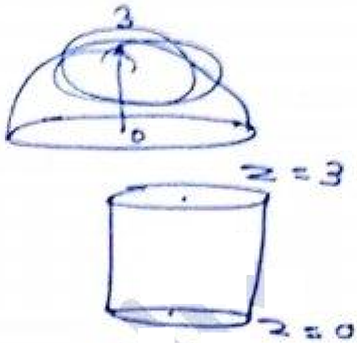
Q.1 $\int_S (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\cdot\hat{n}\, ds$

$x^2+y^2 = 4,\ z = 0\ \&\ z = 3$

Ans $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$

$\oiint_S \hat{F}\cdot\hat{n}\, ds = \iiint \nabla\cdot\vec{F}\, dv$

$\nabla\cdot\vec{F} = (4-4y+2z)$



$= \iiint (4-4y+2z)\, dv$

$= \iiint (4-4y+2z)\, dx\,dy\,dz$

$x^2+y^2=4$
$x = -2$ to 2
$y = -\sqrt{x^2-4}$ to $\sqrt{x^2-4}$

$= \iiint 4\, dx\,dy\,dz - 4\int_0^3\int_0^2\int_0^2 y\, dx\,dy\,dz + 2\int_0^3\int_0^2\int_0^2 dx\,dy\,dz$

$= 4V - 4\,(2-0)\,\frac{1}{2}(4-0)\,(3-0) + 2(2-0)\,(2-0)\,(3-0)$

$= 4V - 24 + 24 = \oint$ Next page

$= 4(\pi(2)^2 \times 3)$

$= 48\pi$

$*\quad x^2+y^2+z^2=1$

$$\nabla\phi = \hat{i}(2x)+\hat{j}(2y)+\hat{k}(2z)$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2+4z^2} = \sqrt{4} = 2$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2}$$

$$\hat{n} = x\hat{i}+y\hat{j}+z\hat{k}$$

Q.8 $\quad \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot F\,d\vec{V}$

$$\iint_S [(2x^2+3x)-y^2+5z^2)ds$$

$$\iint_S \{(2x+3)\hat{i}-y\hat{j}+5z\hat{k}\}\cdot(x\hat{i}+y\hat{j}+z\hat{k})\,ds$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2x+3)-\frac{\partial}{\partial y}y+\frac{\partial}{\partial z}(5z)$$

$$\nabla\cdot F = 2-1+5 = 6$$

$$\underline{So} = \iiint_V \nabla\cdot F\,d\vec{V} = 6\times\tfrac{4}{3}\pi(1)^3 = 8\pi$$

Q.7 Continue

$$\iiint (4 - 4y + 2z)\, dz\, dy\, dz$$

$$= \int_{x=-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{3} (4 - 4y + 2z)\, dz\; dy\; dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4z - 4yz + z^2\right)\Big|_{0}^{3}\; dy\, dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\, dy\, dx$$

$$= \underbrace{\int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\, dy\, dx}_{\text{Area of Circle}} - \underbrace{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 12y\, dy\, dx}_{\text{odd function}}$$

$$= 21\pi\left(\pi(2)^2\right)$$

$$= 84\pi \quad \text{Ans}$$

Q.10

$$\int_C (2x^2+y^2)\,dx + e^y\,dy$$

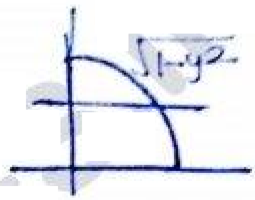
$$y=0, \quad x=0, \quad x^2+y^2=1$$

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right) dx\,dy$$

$$F_1 = (2x^2+y^2) \qquad F_2 = e^y$$

$$\frac{dF_1}{dy} = 2y \qquad \frac{dF_2}{dx} = 0$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} (0-2y)\,dx\,dy$$



$$= \int_0^1 -2y\left\{\sqrt{1-y^2}\right\} dy$$

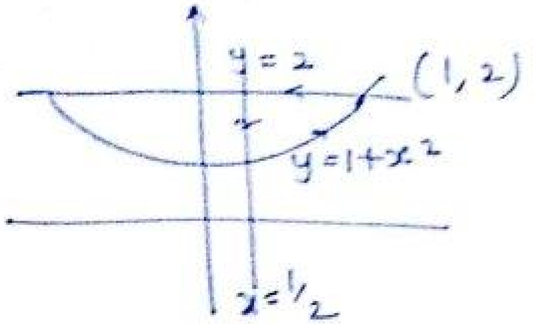
$$= -2\int_0^1 y\sqrt{1-y^2}\,dy$$

$$1-y^2 = t$$
$$-2y\,dy = dt$$
$$y\,dt = -\frac{dt}{2}$$

$$= -2\left(-\frac{1}{2}\right)\int_1^0 t\,dt$$

$$= 1\left[\frac{t^2}{2}\right]_1^0 = \frac{1}{2}\left[0-1\right] = -\frac{1}{2}$$

Q.9



Green

$$\int_c f_1 dx + \int f_2 dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$$

$$f_1 = \frac{e^y}{x} \qquad f_2 = e^y \ln x + x$$

$$\frac{df_1}{dy} = \frac{e^y}{x} \qquad \frac{df_2}{dx} = \frac{e^y}{x} + 1$$

$$= \iint \left(-\frac{e^y}{x} + \frac{e^y}{x} + 1\right) dx\,dy$$

$$= \int_{1/2}^{1} \left(\int_{1+x^2}^{2} dy\right) dx = \int_{1/2}^{1} \left(y\Big|_{1+x^2}^{2}\right.$$

$$= \int_{1/2}^{1} (2 - 1 - x^2)\, dx$$

$$= \int_{1/2}^{1} \left[x - \frac{x^3}{3}\right]_{1/2}^{1}$$

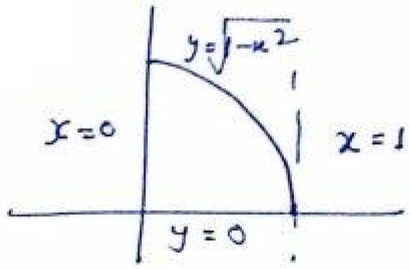
$$= \frac{2}{3} - \frac{11}{24} = \frac{5}{24}$$

Q.10

Green $\int_C F_1 dx + F_2 dy = \iint_R \left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right) dx\, dy$

$F_1 = 2x^2 + y^2 \qquad F_2 = e^y$

$\frac{dF_1}{dy} = 2y \qquad \frac{dF_2}{dx} = 0$



$= \iint (0-2y)\, dx\, dy$

$= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} -2y\, dy\right) dx$

$= \int_0^1 -2\left(\frac{y^2}{2}\right)\Big|_0^{\sqrt{1-x^2}} dx$

$= \int_0^1 -(\sqrt{1-x^2})^2\, dx = \int_0^1 (x^2-1) dx$

$= \frac{x^3}{3} - x \Big|_0^1$

$= \frac{1}{3} - 1 = -\frac{2}{3}$

Q.11

$$\int_{(1,1,0)}^{(2,3,2)} yz\,dx + xz\,dy + xy\,dz$$

$$\int_{(1,1,0)}^{(2,3,2)} d(xyz) = xyz\Big|_{1,1,0}^{2,3,2}$$

$$= (2\times 3\times 2) - (1\times 1\times 1)$$

$$= 12$$

Q.19

$$\vec{F} = 5x^2\,\hat{i} + (xz-y)\,\hat{j} + 3z\,\hat{k}$$

$(0,0,0) \rightarrow (1,1,1)$

$$d\vec{s} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$(0,0,0)$ $(1,1,1)$

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t \text{ (say)}$$

$x = t$, $y = t$, $z = t$

$dx = dt$, $dy = dt$ $dz = dt$

t varies from 0 to 1

work done $\int_C \vec{F} \cdot d\vec{r}$

$$= \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

$$= \int 5x^2\,dx + (xz - y)\,dy + 3z\,dz$$

$$= \int 5t^2\,dt + (t^2 - t)\,dt + 3t\,dt$$

$$= \int_0^1 (6t^2 + 2t)\,dt = \left(\frac{6}{3}t^3 + 2\frac{t^2}{2}\right)_0^1$$

$$= 6/3 + 2/2 = 2 + 1 = 3$$

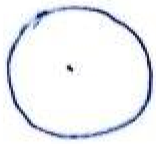
work done $= 3$

Q.13

$$\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$$

$x = \cos\theta$, $y = \sin\theta$, $z = 0$

$x^2 + y^2 = 1$ $\quad$ $\downarrow$ xy plane



$\hat{n} = \hat{k}$ (ACW)

$\hat{n} = -\hat{k}$ (CW)

By stokes

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,ds$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = i(0-1) - j(1-0) + \hat{k}(0-1)$$

$$\nabla \times \vec{F} = -\hat{i} - \hat{j} - \hat{k}$$

$$\int_C F\, d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$$

$$= \iint (-\hat{i} - \hat{j} - \hat{k}) \cdot \hat{k}\, ds$$

$$= \iint (-1)\, ds$$

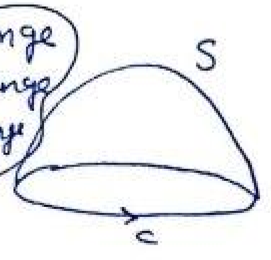
$= (-1)\, S$ where S is surface area of $x^2+y^2=1$

$$\int_C F \cdot d\vec{r} = -\pi \qquad S = \pi (1)^2 = \pi$$

Q.19 $\int_C \left((2x-y)\hat{i} - yz^2 \hat{j} - y^2 z \hat{k} \right) d\vec{r}$

Stock's

x change
y change
z change



$$\oint_C F \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$$

$$ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|} \text{ or } \frac{dy\,dz}{|\hat{n} \cdot \hat{i}|} \text{ or } \frac{dx\,dz}{|\hat{n} \cdot \hat{j}|}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & -yz^2 & -y^2 z \end{vmatrix}$$

$x^2+y^2=1$

$$\nabla \times \vec{F} = \hat{i}(-2yz + 2yz) - \hat{j}(0-0) + \hat{k}(0+1)$$

$$\nabla \times \vec{F} = \hat{k}$$

$$\int_C \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint \hat{k} \cdot (x\hat{i} + y\hat{j} + z\hat{k}) \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

$$= \iint z \frac{dx\,dy}{|(x\hat{i} + y\hat{j} + z\hat{k}) \cdot \hat{k}|}$$

$$= \iint_S dx\,dy = \text{Area of } x^2 + y^2 = 1$$

$$= \pi (1)^2 = \pi$$

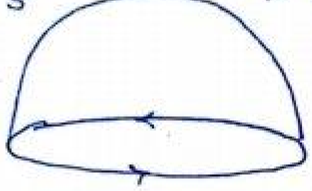
$\divideontimes$

$$ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}, \quad ds = \frac{dy\,dz}{|\hat{n} \cdot \hat{k}|}, \quad ds = \frac{dx\,dz}{|\hat{n} \cdot \hat{j}|}$$

Q.12 By strok's

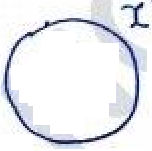
$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds = \int_C \vec{F} \cdot d\vec{r}$$

S $x^2+y^2+z^2=1$



C

C lies one one xy plane



$x^2+y^2=1 \qquad z=0$

$$\text{Along} = \int_C \vec{F} \cdot d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_C (2x-y)\,dx + \underbrace{(-yz^2)}_{\downarrow \, 0} dy - \underbrace{y^2 z}_{\downarrow \, 0} dz$$

along xy plane

$$= \int_C (2x-y)\,dx$$

in circle $x = \cos\theta \qquad y = \sin\theta$

$dx = -\sin\theta \, d\theta$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)(d\theta)$$

$$= \int_0^{2\pi} \left(-2\cos\theta\sin\theta + \sin^2\theta\right) d\theta$$

$$= \underbrace{\int_0^{2\pi} -2\sin 2\theta \, d\theta}_{0} + \int_0^{2\pi} \frac{1}{2} d\theta - \underbrace{\int_0^{2\pi} \frac{\cos 2\theta}{2} d\theta}_{0}$$

$$= \frac{1}{2}(\theta)_0^{2\pi} = \frac{1}{2}(2\pi - 0) = \pi \quad \underline{\text{Ans}}$$

* open surface line integral solve by stroke

$$\int_c \vec{F} \cdot d\vec{r}$$