# 2. Matrices

**EXERCISE 2.1**

**(1)Construct a matrix A $= [a_{ij}]_{3\times 2}$ whose element $a_{ij}$ is given by**

**(i)** $a_{ij} = \frac{(i-j)^2}{5-i}$

**Solution:**

**(i)** $A = [a_{ij}]_{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

Now, $a_{ij} = \frac{(i-j)^2}{5-i}$

$$\therefore \ a_{11} = \frac{(1-1)^2}{5-1} = \frac{0}{4} = 0$$

$$a_{12} = \frac{(1-2)^2}{5-1} = \frac{1}{4}$$

$$a_{21} = \frac{(2-1)^2}{5-2} = \frac{1}{3}$$

$$a_{22} = \frac{(2-2)^2}{5-2} = \frac{0}{3} = 0$$

$$a_{31} = \frac{(3-1)^2}{5-3} = \frac{4}{2} = 2$$

$$a_{32} = \frac{(3-2)^2}{5-3} = \frac{1}{2}$$

$$\therefore \ A = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \\ 2 & \frac{1}{2} \end{bmatrix}$$

**(ii)** $a_{ij} = i - 3j$

**Solution:**

$a_{ij} = i - 3j$

$\therefore a_{11} = 1 - 3(1) = 1 - 3 = - 2$

$a_{12} = 1 - 3(2) = 1 - 6 = - 5$

$a_{21} = 2 - 3(1 = 2 - 3 = - 1$

$a_{22} = 2 - 3(2) = 2 - 6 = - 4$

$a_{31} = 3 - 3(1) = 3 - 3 = 0,$

$a_{32} = 3 - 3(2) = 3 - 6 = - 3$

$$\therefore A = \begin{bmatrix} -2 & -5 \\ -1 & -4 \\ 0 & -3 \end{bmatrix}.$$

**(iii)** $a_{ij} = \frac{(i+j)^3}{5}$

**Solution:**

$a_{ij} = \frac{(i+j)^3}{5}$

$\therefore$ a11 =

$\frac{(1+1)^3}{5} = \frac{2^3}{5} = \frac{8}{5}, a_{12} = \frac{(1+2)^3}{5} =$

$a_{21} =$

$\frac{(2+1)^3}{5} = \frac{3^3}{5} = \frac{27}{5}, a_{22} = \frac{(2+2)^2}{5} =$

$a_{31} =$

$\frac{(3+1)^3}{5} = \frac{4^3}{5} = \frac{64}{5}, a_{32} = \frac{(3+2)^2}{5} =$

$\therefore A = \begin{bmatrix} \frac{8}{5} & \frac{27}{5} \\ \frac{27}{5} & \frac{64}{5} \\ \frac{64}{5} & \frac{125}{5} \end{bmatrix}$

$= \begin{bmatrix} 8 & 27 \\ 27 & 64 \\ 64 & 125 \end{bmatrix}.$

**(2) Classify each of the following matrices as a row, a column, a square, a diagonal, a scalar, a unit, an upper traingular, a lower triangular matrix.**

$(i)\begin{bmatrix} 3 & -2 & 4 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

**Solution:**

Since, all the elements below the diagonal are zero, it is an **upper tringular matrix.**

$(ii)\begin{bmatrix} 5 \\ 4 \\ -3 \end{bmatrix}$

**Solution:**

This matrix has only one column, it is a **column matrix.**

$(iii)\begin{bmatrix} 9 & \sqrt{2} & -3 \end{bmatrix}$

**Solution:**

This matrix has only one row, it is a **row matrix.**

$(iv)\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

**Solution:**

Since, diagonal elements are equal and non diagonal elements are zero, it is a **scalar matrix.**

$(v)\begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ -7 & 3 & 1 \end{bmatrix}$

**Solution:**

Since, all the elements above the diagonal are zero, it is a **lower triangular matrix.**

$(vi)\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

**Solution:**

Since, all the non-diagonal elements are zero, it is a **diagonal matrix.**

$(vii)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Solution:**

Since, diagonal elements are 1 and non-diagonal elements are 0, it is an **identity (or unit) matrix.**

**(3) Which of the following matrices are singular or non singular?**

$(i)\begin{bmatrix} a & b & c \\ p & q & r \\ 2a-p & 2b-q & 2c-r \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ 2a-p & 2b-q & 2c-r \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} a & b & c \\ p & q & r \\ 2a-p & 2b-q & 2c-r \end{vmatrix}$

By $R_3 + R_2$, we get,

$|A| = \begin{vmatrix} a & b & c \\ p & q & r \\ 2a & 2b & 2c \end{vmatrix}$

$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ a & b & c \end{vmatrix}$

$= 2 \times 0$ ... $[\because R_1 \equiv R_3]$

$= 0$

$\therefore$ A is a **singular** matrix.

$(ii)\begin{bmatrix} 5 & 0 & 5 \\ 1 & 99 & 100 \\ 6 & 99 & 105 \end{bmatrix}$

**Solution:**

Let $B = \begin{bmatrix} 5 & 0 & 5 \\ 1 & 99 & 100 \\ 6 & 99 & 105 \end{bmatrix}$

$\therefore |B| = \begin{vmatrix} 5 & 0 & 5 \\ 1 & 99 & 100 \\ 6 & 99 & 105 \end{vmatrix}$

By $R_3 - R_2$, we get

$|B| = \begin{vmatrix} 5 & 0 & 5 \\ 1 & 99 & 100 \\ 5 & 0 & 5 \end{vmatrix}$

$= 0$ ... $[\because R_1 \equiv R_3]$

$\therefore$ B is a **singular** matrix.

$(iii)\begin{bmatrix} 3 & 5 & 7 \\ -2 & 1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$

**Solution:**

Let $C = \begin{pmatrix} 3 & 5 & 7 \\ -2 & 1 & 4 \\ 3 & 2 & 5 \end{pmatrix}$

$\therefore\ |C| = \begin{vmatrix} 3 & 5 & 7 \\ -2 & 1 & 4 \\ 3 & 2 & 5 \end{vmatrix}$

$= 3(5-8) - 5(-10-12) + 7(-4-3)$

$= -9 + 110 - 49 = 52 \neq 0$

$\therefore$ C is a **non-singular** matrix.

$(iv)$ $\begin{bmatrix} 7 & 5 \\ -4 & 7 \end{bmatrix}$

**Solution:**

Let $D = \begin{pmatrix} 7 & 5 \\ -4 & 7 \end{pmatrix}$

$\therefore\ |D| = \begin{vmatrix} 7 & 5 \\ -4 & 7 \end{vmatrix}$

$= 49 - (-20) = 69 \neq 0$

$\therefore$ D is a **non-singular** matrix.

**(4) Find K if the following matrices are singular.**

$(i)$ $\begin{bmatrix} 7 & 3 \\ -2 & K \end{bmatrix}$

**Solution:**

Let $A = \begin{pmatrix} 7 & 3 \\ -2 & k \end{pmatrix}$

Since, A is a singular matrix, $|A| = 0$

$\therefore\ \begin{vmatrix} 7 & 3 \\ -2 & k \end{vmatrix} = 0$

$\therefore\ 7k - (-6) = 0$

$\therefore\ 7k = -6 \qquad \therefore\ k = -\frac{6}{7}$

$(ii)$ $\begin{bmatrix} 4 & 3 & 1 \\ 7 & K & 1 \\ 10 & 9 & 1 \end{bmatrix}$

**Solution:**

Let $B = \begin{pmatrix} 4 & 3 & 1 \\ 7 & k & 1 \\ 10 & 9 & 1 \end{pmatrix}$

Since, B is a singular matrix, $|B| = 0$

$\therefore\ \begin{vmatrix} 4 & 3 & 1 \\ 7 & k & 1 \\ 10 & 9 & 1 \end{vmatrix} = 0$

$\therefore\ 4(k-9) - 3(7-10) + 1(63-10k) = 0$

$\therefore\ 4k - 36 + 9 + 63 - 10k = 0$

$\therefore\ -6k + 36 = 0$

$\therefore\ 6k = 36 \qquad \therefore\ k = 6.$

$(iii)$ $\begin{bmatrix} K-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$

**Solution:**

Let $C = \begin{pmatrix} k-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{pmatrix}$

Since, C is a singular matrix, $|C| = 0$

$\therefore\ \begin{vmatrix} k-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{vmatrix} = 0$

$\therefore\ (k-1)(4+4) - 2(12-2) + 3(-6-1) = 0$

$\therefore\ 8k - 8 - 20 - 21 = 0$

$\therefore\ 8k = 49$

$\therefore\ k = \frac{49}{8}.$

## EXERCISE 2.2

(1) If $A = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 3 \\ -1 & 4 \\ -2 & 1 \end{bmatrix}$

**Show that (i) A+B=B+A (ii) (A+B) + C = A + (B+C)**

**Solution:**

(i) $A + B = \begin{pmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 2 & 2 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} 2-1 & -3+2 \\ 5+2 & -4+2 \\ -6+0 & 1+3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 7 & -2 \\ -6 & 4 \end{pmatrix} \quad \ldots (1)$

$B + A = \begin{pmatrix} -1 & 2 \\ 2 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{pmatrix}$

$= \begin{pmatrix} -1+2 & 2-3 \\ 2+5 & 2-4 \\ 0-6 & 3+1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 7 & -2 \\ -6 & 4 \end{pmatrix} \quad \ldots (2)$

From (1) and (2), we get

A + B = B + A.

**(ii)** $A+B=\begin{bmatrix}2 & -3\\5 & -4\\-6 & 1\end{bmatrix}+\begin{bmatrix}-1 & 2\\2 & 2\\0 & 3\end{bmatrix}$

$=\begin{bmatrix}2-1 & -3+2\\5+2 & -4+2\\-6+0 & 1+3\end{bmatrix}=\begin{bmatrix}1 & -1\\7 & -2\\-6 & 4\end{bmatrix}$

$\therefore\ (A+B)+C=\begin{bmatrix}1 & -1\\7 & -2\\-6 & 4\end{bmatrix}+\begin{bmatrix}4 & 3\\-1 & 4\\-2 & 1\end{bmatrix}$

$=\begin{bmatrix}1+4 & -1+3\\7-1 & -2+4\\-6-2 & 4+1\end{bmatrix}=\begin{bmatrix}5 & 2\\6 & 2\\-8 & 5\end{bmatrix}\quad\ldots(1)$

Also, $B+C=\begin{bmatrix}-1 & 2\\2 & 2\\0 & 3\end{bmatrix}+\begin{bmatrix}4 & 3\\-1 & 4\\-2 & 1\end{bmatrix}$

$=\begin{bmatrix}-1+4 & 2+3\\2-1 & 2+4\\0-2 & 3+1\end{bmatrix}=\begin{bmatrix}3 & 5\\1 & 6\\-2 & 4\end{bmatrix}$

$\therefore\ A+(B+C)=\begin{bmatrix}2 & -3\\5 & -4\\-6 & 1\end{bmatrix}+\begin{bmatrix}3 & 5\\1 & 6\\-2 & 4\end{bmatrix}$

$=\begin{bmatrix}2+3 & -3+5\\5+1 & -4+6\\-6-2 & 1+4\end{bmatrix}=\begin{bmatrix}5 & 2\\6 & 2\\-8 & 5\end{bmatrix}\quad\ldots(2)$

From (1) and (2), we get

(A + B) + C = A + (B + C).

**(2)** $\boldsymbol{If\ A=\begin{bmatrix}1 & -2\\5 & 3\end{bmatrix}, B=\begin{bmatrix}1 & -3\\4 & -7\end{bmatrix}},$

**then find the matrix A - 2B + 6I, where I is the unit matrix of order 2.**

**Solution:**

$A-2B+6I=\begin{bmatrix}1 & -2\\5 & 3\end{bmatrix}-2\begin{bmatrix}1 & -3\\4 & -7\end{bmatrix}+6\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$=\begin{bmatrix}1 & -2\\5 & 3\end{bmatrix}-\begin{bmatrix}2 & -6\\8 & -14\end{bmatrix}+\begin{bmatrix}6 & 0\\0 & 6\end{bmatrix}$

$=\begin{bmatrix}1-2+6 & -2-(-6)+0\\5-8+0 & 3-(-14)+6\end{bmatrix}$

$=\begin{bmatrix}5 & 4\\-3 & 23\end{bmatrix}.$

**(3)** $\boldsymbol{If\ A=\begin{bmatrix}1 & 2 & -3\\-3 & 7 & -8\\0 & -6 & 1\end{bmatrix}, B=\begin{bmatrix}9 & -1 & 2\\-4 & 2 & 5\\4 & 0 & -3\end{bmatrix}}$

**then find the matrix C such that A + B + C is a zero matrix.**

**Solution:**

A + B + C = 0

$\therefore C=-A-B$

$=-\begin{bmatrix}1 & 2 & -3\\-3 & 7 & -8\\0 & -6 & 1\end{bmatrix}-\begin{bmatrix}9 & -1 & 2\\-4 & 2 & 5\\4 & 0 & -3\end{bmatrix}$

$=\begin{bmatrix}-1 & -2 & 3\\3 & -7 & 8\\0 & 6 & -1\end{bmatrix}-\begin{bmatrix}9 & -1 & 2\\-4 & 2 & 5\\4 & 0 & -3\end{bmatrix}$

$=\begin{bmatrix}-1-9 & -2-(-1) & 3-2\\3-(-4) & -7-2 & 8-5\\0-4 & 6-0 & -1-(-3)\end{bmatrix}$

$\therefore\ C=\begin{bmatrix}-10 & -1 & 1\\7 & -9 & 3\\-4 & 6 & 2\end{bmatrix}.$

**(4)** $\boldsymbol{If\ A=\begin{bmatrix}1 & -2\\3 & -5\\-6 & 0\end{bmatrix}, B=\begin{bmatrix}-1 & -2\\4 & 2\\1 & 5\end{bmatrix}}$

$\boldsymbol{and\ C=\begin{bmatrix}2 & 4\\-1 & -4\\3 & 6\end{bmatrix}}$

**find the matrix X such that 3A – 4B + 5X = C.**

**Solution:**

3A – 4B + 5X = C

$\therefore 5X=C-3A+4B$

$=\begin{bmatrix}2 & 4\\-1 & -4\\-3 & 6\end{bmatrix}-3\begin{bmatrix}1 & -2\\3 & -5\\-6 & 0\end{bmatrix}+4\begin{bmatrix}-1 & -2\\4 & 2\\1 & 5\end{bmatrix}$

$=\begin{bmatrix}2 & 4\\-1 & -4\\-3 & 6\end{bmatrix}-\begin{bmatrix}3 & -6\\9 & -15\\-18 & 0\end{bmatrix}+\begin{bmatrix}-4 & -8\\16 & 8\\4 & 20\end{bmatrix}$

$=\begin{bmatrix}2-3+(-4) & 4-(-6)-8\\-1-9+16 & -4-(-15)+8\\-3-(-18)+4 & 6-0+20\end{bmatrix}$

$=\begin{bmatrix}-5 & 2\\6 & 19\\19 & 26\end{bmatrix}$

$\therefore\ X=\frac{1}{5}\begin{bmatrix}-5 & 2\\6 & 19\\19 & 26\end{bmatrix}=\begin{bmatrix}-1 & \frac{2}{5}\\\frac{6}{5} & \frac{19}{5}\\\frac{19}{5} & \frac{26}{5}\end{bmatrix}$

**(5)** $\boldsymbol{If\ A=\begin{bmatrix}5 & 1 & -4\\3 & 2 & 0\end{bmatrix}, find\ (A^T)^T}.$

**Solution:**

$$= \begin{bmatrix} 5 & 1 & -4 \\ 3 & 2 & 0 \end{bmatrix}$$

$$\therefore\ A^T = \begin{bmatrix} 5 & 3 \\ 1 & 2 \\ -4 & 0 \end{bmatrix}$$

$$\therefore\ (A^T)^T = \begin{bmatrix} 5 & 1 & -4 \\ 3 & 2 & 0 \end{bmatrix} = A.$$

**(6) If $A = \begin{bmatrix} 7 & 3 & 1 \\ -2 & -4 & 1 \\ 5 & 9 & 1 \end{bmatrix}$, find $(A^T)^T$.**

**Solution:**

$$A = \begin{bmatrix} 7 & 3 & 1 \\ -2 & -4 & 1 \\ 5 & 9 & 1 \end{bmatrix}$$

$$\therefore\ A^T = \begin{bmatrix} 7 & -2 & 5 \\ 3 & -4 & 9 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\therefore\ (A^T)^T = \begin{bmatrix} 7 & 3 & 1 \\ -2 & -4 & 1 \\ 5 & 9 & 1 \end{bmatrix} = A.$$

**(7) Find a, b, c if $\begin{bmatrix} 1 & \frac{3}{5} & a \\ b & -5 & -7 \\ -4 & c & 0 \end{bmatrix}$ is a symmetric matrix.**

**Solution:**

$$Let\ A = \begin{bmatrix} 1 & \frac{3}{5} & a \\ b & -5 & -7 \\ -4 & c & 0 \end{bmatrix}$$

Since, A is a symmetric matrix, $a_{ij} = a_{ji}$ for all $i$ and $j$

$\therefore\ a_{13} = a_{31},\ a_{12} = a_{21}$ and $a_{23} = a_{32}$

$\therefore\ a = -4,\ \frac{3}{5} = b$ and $-7 = c$

$\therefore\ a = -4,\ b = \frac{3}{5}$ and $c = -7.$

*Alternative Method :*

$$\text{Let } A = \begin{bmatrix} 1 & \frac{3}{5} & a \\ b & -5 & -7 \\ -4 & c & 0 \end{bmatrix}$$

$$\text{Then } A^T = \begin{bmatrix} 1 & b & -4 \\ \frac{3}{5} & -5 & -7 \\ a & -7 & 0 \end{bmatrix}$$

Since, A is symmetric matrix, $A = A^T$

$$\therefore \begin{bmatrix} 1 & \frac{3}{5} & a \\ b & -5 & -7 \\ -4 & c & 0 \end{bmatrix} = \begin{bmatrix} 1 & b & -4 \\ \frac{3}{5} & -5 & -7 \\ a & -7 & 0 \end{bmatrix}$$

By equality of matrices

$a = -4,\ b = \frac{3}{5}$ and $c = -7.$

**(8) Find $x, y, z$ if $\begin{bmatrix} 0 & -5i & x \\ y & 0 & z \\ \frac{3}{2} & -\sqrt{2} & 0 \end{bmatrix}$ is a skew symmetric matrix**

**Solution:**

$$\text{Let } A = \begin{bmatrix} 0 & -5i & x \\ y & 0 & z \\ \frac{3}{2} & -\sqrt{2} & 0 \end{bmatrix}$$

Since, A is skew-symmetric matrix,

$a_{ij} = -a_{ji}$ for all $i$ and $j$.

$\therefore\ a_{13} = -a_{31},\ a_{12} = -a_{21}$ and $a_{23} = -a_{32}$

$\therefore\ x = -\frac{3}{2},\ -5i = -y$ and $z = -(-\sqrt{2})$

$\therefore\ x = -\frac{3}{2},\ y = 5i$ and $z = \sqrt{2}.$

**(9) For each of the following matrices, find its transpose and state whether it is symmetric, skew - symmetric or neither.**

**(i) $\begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$**

**Solution:**

Let $A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$

Since, $A = A^T$, A is a symmetric matrix.

$(ii) \begin{bmatrix} 2 & 5 & 1 \\ -5 & 4 & 6 \\ -1 & -6 & 3 \end{bmatrix}$

**Solution:**

Let $B = \begin{bmatrix} 2 & 5 & 1 \\ -5 & 4 & 6 \\ -1 & -6 & 3 \end{bmatrix}$

Then $B^T = \begin{bmatrix} 2 & -5 & -1 \\ 5 & 4 & -6 \\ 1 & 6 & 3 \end{bmatrix}$

$\therefore B \neq B^T$

Also,

$$-B^T = -\begin{bmatrix} 2 & -5 & -1 \\ 5 & 4 & -6 \\ 1 & 6 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 1 \\ -5 & -4 & 6 \\ -1 & -6 & -3 \end{bmatrix}$$

$\therefore B \neq -B^T$.

Hence, B is neither symmetric nor skew-symmetric matrix.

$(iii) \begin{bmatrix} 0 & 1+2i & i-2 \\ -1-2i & 0 & -7 \\ 2-i & 7 & 0 \end{bmatrix}$

**Solution:**

Let $C = \begin{bmatrix} 0 & 1+2i & i-2 \\ -1-2i & 0 & -7 \\ 2-i & 7 & 0 \end{bmatrix}$

Then $C^T = \begin{bmatrix} 0 & -1-2i & 2-i \\ 1+2i & 0 & 7 \\ i-2 & -7 & 0 \end{bmatrix}$

$\therefore -C^T = \begin{bmatrix} 0 & 1+2i & i-2 \\ -1-2i & 0 & -7 \\ 2-i & 7 & 0 \end{bmatrix}$

$\therefore C = -C^T$

Hence, C is skew-symmetric matrix.

**(10) Construct the matrix A = $[a_{ij}]_{3\times3}$ where $a_{ij} = i - j$. State whether A is symmetric or skew symmetric.**

**Solution:**

$$A = [a_{ij}]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now, $a_{ij} = i - j$ for all $i$ and $j$

$\therefore a_{11} = 1 - 1 = 0,\ a_{12} = 1 - 2 = -1$

$a_{13} = 1 - 3 = -2,\ a_{21} = 2 - 1 = 1$

$a_{22} = 2 - 2 = 0,\ a_{23} = 2 - 3 = -1$

$a_{31} = 3 - 1 = 2,\ a_{32} = 3 - 2 = 1,\ a_{33} = 3 - 3 = 0$

$\therefore A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

Since, $a_{ij} = i - j = -(j - i) = -a_{ji}$ for all $i$ and $j$,

A is skew-symmetric matrix.

**(11) Solve the following equations for X and Y, if 3X - Y =**

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ ***and*** $X - 3Y = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$

**Solution:**

$3X - Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ .. (1)

$X - 3Y = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$ ... (2)

Multiplying (1) by 3, we get

$9X - 3y = 3\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ ... (3)

Subtracting (2) from (3), we get

$$8X = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-0 & -3-(-1) \\ -3-0 & 3-(-1) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix}$$

$$\therefore X = \frac{1}{8}\begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} \\ -\frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

Substituting the value of X in (1), we get

$$3\begin{bmatrix} \frac{3}{8} & -\frac{1}{4} \\ -\frac{3}{8} & \frac{1}{2} \end{bmatrix} - Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\therefore\ Y = \begin{bmatrix} \frac{9}{8} & -\frac{3}{4} \\ -\frac{9}{8} & \frac{3}{2} \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{8}-1 & -\frac{3}{4}-(-1) \\ -\frac{9}{8}-(-1) & \frac{3}{2}-1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} \\ -\frac{1}{8} & \frac{1}{2} \end{bmatrix}$$

Hence, $X = \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} \\ -\frac{3}{8} & \frac{1}{2} \end{bmatrix}$ and $Y = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} \\ -\frac{1}{8} & \frac{1}{2} \end{bmatrix}$.

**(12) Find matrices A and B, if 2A – B $= \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix}$ *and* $A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$**

**Solution:**

Given equations are

$$2A - B = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} \quad \ldots(i)$$

$$\text{and } A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix} \quad \ldots(ii)$$

By (i) − (ii) × 2, we get

$$3B = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 4 & 16 \\ -4 & 2 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} 6-6 & -6-4 & 0-16 \\ -4+4 & 2-2 & 1+14 \end{bmatrix}$$

$$\therefore\ 3B = \begin{bmatrix} 0 & -10 & -16 \\ 0 & 0 & 15 \end{bmatrix}$$

$$\therefore\ B = \frac{1}{3}\begin{bmatrix} 0 & -10 & -16 \\ 0 & 0 & 15 \end{bmatrix}$$

$$\therefore\ B = \begin{bmatrix} 0 & \frac{-10}{3} & \frac{-16}{3} \\ 0 & 0 & 5 \end{bmatrix}$$

By (i) × 2 − (ii), we get

$$3A = 2\begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -12 & 0 \\ -8 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 12-3 & -12-2 & 0-8 \\ -8+2 & 4-1 & 2+7 \end{bmatrix}$$

$$\therefore\ 3A = \begin{bmatrix} 9 & -14 & -8 \\ -6 & 3 & 9 \end{bmatrix}$$

$$\therefore\ A = \frac{1}{3}\begin{bmatrix} 9 & 14 & -8 \\ -6 & 3 & 9 \end{bmatrix}$$

$$\therefore\ A = \begin{bmatrix} 3 & \frac{-14}{3} & \frac{-8}{3} \\ -2 & 1 & 3 \end{bmatrix}.$$

**(13) *Find x and y, if***

$$\begin{bmatrix} 2x+y & -1 & 1 \\ 3 & 4y & 4 \end{bmatrix} + \begin{bmatrix} -1 & 6 & 4 \\ 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 5 \\ 6 & 18 & 7 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 2x+y & -1 & 1 \\ 3 & 4y & 4 \end{bmatrix} + \begin{bmatrix} -1 & 6 & 4 \\ 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 5 \\ 6 & 18 & 7 \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} 2x+y-1 & -1+6 & 1+4 \\ 3+3 & 4y+0 & 4+3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 5 \\ 6 & 18 & 7 \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} 2x+y-1 & 5 & 5 \\ 6 & 4y & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 5 \\ 6 & 18 & 7 \end{bmatrix}$$

By equality of matrices, we get

$2x + y - 1 = 3$ … (1)

and $4y = 18$ … (2)

From (2), $y = \frac{9}{2}$

Substituting $y = \frac{9}{2}$ in (1), we get

$$2x + \frac{9}{2} - 1 = 3$$

$$\therefore\ 2x = 3 - \frac{7}{2} = -\frac{1}{2}$$

$$\therefore\ x = -\frac{1}{4}$$

Hence, $x = -\frac{1}{4}$ and $y = \frac{9}{2}$.

**(14) If $\begin{bmatrix} 2a+b & 3a-b \\ c+2d & 2c-d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, find a, b, c and d.**

**Solution:**

$$\begin{bmatrix} 2a+b & 3a-b \\ c+2d & 2c-d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$$

By equality of matrices,

$2a + b = 2$ … (1)

$3a - b = 3$ … (2)

$c + 2d = 4$ … (3)

$2c - d = -1$ … (4)

Adding (1) and (2), we get

$5a = 5 \quad \therefore\ a = 1$

Substituting $a = 1$ in (1), we get

$2(1) + b = 2 \quad \therefore\ b = 0$

Multiplying equation (4) by 2, we get

$4c - 2d = -2$ … (5)

Adding (3) and (5), we get

$5c = 2 \quad \therefore\ c = \frac{2}{5}$

**(15)**

**There are two book shops owned by Suresh and Ganesh. Their sales (in ₹) for books in three subjects – Physics, Chemistry and Mathematics for two months, July and August 2017 are given by two matrices A and B :**

**July sales (in ₹), Physics, Chemistry, Mathematics**

$$A = \begin{bmatrix} 5600 & 6750 & 8500 \\ 6650 & 7055 & 8905 \end{bmatrix} \begin{matrix} \text{First Row Suresh,} \\ \text{Second Row Ganesh} \end{matrix}$$

**August Sales (in ₹), Physics, Chemistry, Mathematics**

$$B = \begin{bmatrix} 6650 & 7055 & 8905 \\ 7000 & 7500 & 10200 \end{bmatrix} \begin{matrix} \text{First Row Suresh,} \\ \text{Second Row Ganesh} \end{matrix}$$

**(i) Find the increase in sales in ₹ from July to August 2017.**

**(ii) If both book shops get 10% profit in the month of August 2017, find the profit for each book seller in each subject in that month.**

**Solution:**

The sales for the July and August 2017 for Suresh and Ganesh are given by the matrices A and B as :

**July Sales (in ₹)**

$$A = \begin{matrix} \text{Physics} & \text{Chemistry} & \text{Mathematics} & \\ \end{matrix}$$
$$A = \begin{bmatrix} 5600 & 6750 & 8500 \\ 6650 & 7055 & 8905 \end{bmatrix} \begin{matrix} \textbf{Suresh} \\ \textbf{Ganesh} \end{matrix}$$

**August Sales (in ₹)**

(Physics, Chemistry, Mathematics)

$$B = \begin{bmatrix} 6650 & 7055 & 8905 \\ 7000 & 7500 & 10200 \end{bmatrix} \begin{matrix} \textbf{Suresh} \\ \textbf{Ganesh} \end{matrix}$$

(i) The increase in sales (in ₹) from July to August 2017 is obtained by subtracting the matrix A from B.

$$\text{Now, } B - A = \begin{bmatrix} 6650 & 7055 & 8905 \\ 7000 & 7500 & 10200 \end{bmatrix} - \begin{bmatrix} 5600 & 6750 & 8500 \\ 6650 & 7055 & 8905 \end{bmatrix}$$

$$= \begin{bmatrix} 6650-5600 & 7055-6750 & 8905-8500 \\ 7000-6650 & 7500-7055 & 10200-8905 \end{bmatrix}$$

(Physics, Chemistry, Mathematics)

$$= \begin{bmatrix} 1050 & 305 & 405 \\ 350 & 445 & 1295 \end{bmatrix} \begin{matrix} \textbf{Suresh} \\ \textbf{Ganesh} \end{matrix}$$

Hence, the increase in sales (in ₹) from July to August 2017 for :

Suresh book shop : ₹ 1050 in Physics, ₹ 305 in Chemistry and ₹ 405 in Mathematics.

Ganesh book shop : ₹ 350 in Physics, ₹ 445 in Chemistry and ₹ 1295 in Mathematics.

**(ii)** Both the book shops get 10% profit in August 2017, the profit for each book seller in each subject in August 2017 is obtained by the scalar multiplication of matrix B by 10%, i.e. $\frac{10}{100} = \frac{1}{10}$.

$$\text{Now, } \frac{1}{10}B = \frac{1}{10}\begin{bmatrix} 6650 & 7055 & 8905 \\ 7000 & 7500 & 10200 \end{bmatrix}$$

(Physics, Chemistry, Mathematics)

$$= \begin{bmatrix} 665 & 705.5 & 890.5 \\ 700 & 750 & 1020 \end{bmatrix} \begin{matrix} \textbf{Suresh} \\ \textbf{Ganesh} \end{matrix}$$

Hence, the profit for Suresh book shop are ₹ 665 in Physics, ₹ 705.50 in Chemistry and ₹ 890.50 in Mathematics and for Ganesh book shop are ₹ 700 in Physics, ₹ 750 in Chemistry and ₹ 1020 in Mathematics.

## EXERCISE 2.3

**1) Evaluate**

$$\textbf{i)} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} [2 \quad -4 \quad 3] \quad \textbf{ii)} [2 \quad -1 \quad 3] \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

**Solution:**

$$\text{i)} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} [2 \quad -4 \quad 3] = \begin{bmatrix} 6 & -12 & 9 \\ 4 & -8 & 6 \\ 2 & -4 & 3 \end{bmatrix}.$$

$$\text{ii)} [2 \quad -1 \quad 3] \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = [8 - 3 + 3] = [8].$$

$$\textbf{2) If } A \begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & -3 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

**State whether AB = BA? Justify your answer.**

**Solution:**

$$AB = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+3+1 & -1+0+2 & -4+2+1 \\ 4+9+0 & 2+0+0 & 8+6+0 \\ 2-9+1 & 1-0+2 & 4-6+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ 13 & 2 & 14 \\ -6 & 3 & -1 \end{bmatrix} \quad \ldots (1)$$

$$BA = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+2+4 & 2+3-12 & 2+0+4 \\ -3+0+2 & 3+0-6 & 3+0+2 \\ -1+4+1 & 1+6-3 & 1+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -7 & 6 \\ -1 & -3 & 5 \\ 4 & 4 & 2 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), $AB \neq BA$.

**3) Show that AB = BA where,**

$$A = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$$

**Solution:**

$$AB = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+6-3 & -6+6-0 & 2-3+1 \\ -1+4-3 & -3+4-0 & 1-2+1 \\ -6+18-12 & -18+18-0 & 6-9+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots (1)$$

$$B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -2-3+6 & 3+6-9 & -1-3+4 \\ -4-2+6 & 6+4-9 & -2-2+4 \\ -6-0+6 & 9+0-9 & -3-0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), AB = BA.

**4) Verify A(BC) = (AB)C,**

$$\text{if } A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$$

$$and\ C = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 0 & -2 \end{pmatrix}$$

**Solution:**

$$BC = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 2 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6-4 & 4-0 & -2+4 \\ -3+2 & -2+0 & 1-2 \\ 0+6 & 0+0 & 0-6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ -1 & -2 & -1 \\ 6 & 0 & -6 \end{bmatrix}$$

$$\therefore A(BC) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ -1 & -2 & -1 \\ 6 & 0 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 2-0+6 & 4-0+0 & 2-0-6 \\ 4-3+0 & 8-6+0 & 4-3-0 \\ 0-4+30 & 0-8+0 & 0-4-30 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 & -4 \\ 1 & 2 & 1 \\ 26 & -8 & -34 \end{bmatrix} \quad \ldots (1)$$

$$\text{Also, } AB = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-0+0 & -2+0+3 \\ 4-3+0 & -4+3+0 \\ 0-4+0 & 0+4+15 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ -4 & 19 \end{bmatrix}$$

$$\therefore (AB)C = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ -4 & 19 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 2 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6+2 & 4+0 & -2-2 \\ 3-2 & 2-0 & -1+2 \\ -12+38 & -8+0 & 4-38 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 & -4 \\ 1 & 2 & 1 \\ 26 & -8 & -34 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), A(BC) = (AB)C.

**5) Verify that A(B+C) = AB + AC, if A =**

$$\text{if } A = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} and\ C = \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$$

**Solution:**

$$B + C \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+4 & 1+1 \\ 3+2 & -2-1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}$$

$$\therefore A(B+C) = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 12-10 & 8+6 \\ 6+15 & 4-9 \end{bmatrix} = \begin{bmatrix} 2 & 14 \\ 21 & -5 \end{bmatrix} \quad \ldots (1)$$

$$\text{Also, } AB = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4-6 & 4+4 \\ -2+9 & 2-6 \end{bmatrix} = \begin{bmatrix} -10 & 8 \\ 7 & -4 \end{bmatrix}$$

$$AC = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 16-4 & 4+2 \\ 8+6 & 2-3 \end{bmatrix} = \begin{bmatrix} 12 & 6 \\ 14 & -1 \end{bmatrix}$$

$$\therefore AB + AC = \begin{bmatrix} -10 & 8 \\ 7 & -4 \end{bmatrix} + \begin{bmatrix} 12 & 6 \\ 14 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -10+12 & 8+6 \\ 7+14 & -4-1 \end{bmatrix} = \begin{bmatrix} 2 & 14 \\ 21 & -5 \end{bmatrix} \ldots (2)$$

From (1) and (2), A(B + C) = AB + AC.

$$\textbf{6) } If\ A = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$$

**Show that matrix AB is non singular.**

**Solution:**

$$AB = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3+2 & 8+0-4 \\ -1-2+0 & -2+0-0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -3 & -2 \end{bmatrix}$$

$$\therefore \ |AB| = \begin{vmatrix} 3 & 4 \\ -3 & -2 \end{vmatrix}$$

$$= -6-(-12) = 6 \neq 0$$

Hence, AB is a non-singular matrix.

**7) If $A + I = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 4 & 2 \\ 0 & 7 & -3 \end{bmatrix}$,**

**Find the product (A + I) (A – I).**

**Solution:**

$$AB = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3+2 & 8+0-4 \\ -1-2+0 & -2+0-0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -3 & -2 \end{bmatrix}$$

$$\therefore \ |AB| = \begin{vmatrix} 3 & 4 \\ -3 & -2 \end{vmatrix}$$

$$= -6-(-12) = 6 \neq 0$$

Hence, AB is a non-singular matrix.

**8) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$,**

**show that $A^2$ - 4A is a scalar matrix.**

**Solution:**

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\therefore \ A^2 - 4A = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

which is a scalar matrix.

**9) If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$,**

**find k so that $A^2$ - 8A - kI= O, where I is a 2 × 2 unit and O is null matrix of order 2.**

**Solution:**

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1-0 & 0+0 \\ -1-7 & 0+49 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$$

$$\therefore \ A^2 - 8A - kI = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} - 8 \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} - k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} - \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$= \begin{bmatrix} 1-8-k & 0-0-0 \\ -8+8-0 & 49-56-k \end{bmatrix}$$

$$= \begin{bmatrix} -k-7 & 0 \\ 0 & -k-7 \end{bmatrix}$$

But, $A^2 - 8A - kI = 0$

$$\therefore \begin{bmatrix} -k-7 & 0 \\ 0 & -k-7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equality of matrices,

$-k-7=0 \quad \therefore \ k = -7.$

**10) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$,**

**prove that $A^2$ - 5A + 71 = 0, where I is 2 × 2 unit matrix.**

**Solution:**

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore A^2 - 5A + 7I = 0.$

**11) If $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}, B = \begin{pmatrix} 2 & a \\ -1 & b \end{pmatrix}$**

**And if $(A + B)^2 = A^2 + B^2$, find value of a and b.**

**Solution:**

$$(A + B)^2 = A^2 + B^2$$

$\therefore (A + B)(A + B) = A^2 + B^2$

$\therefore A^2 + AB + BA + B^2 = A^2 + B^2$

$\therefore AB + BA = 0$

$\therefore AB = -BA$

$$\therefore \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & a \\ -1 & b \end{bmatrix} = -\begin{bmatrix} 2 & a \\ -1 & b \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2-2 & a+2b \\ -2+2 & -a-2b \end{bmatrix} = -\begin{bmatrix} 2-a & 4-2a \\ -1-b & -2-2b \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & a+2b \\ 0 & -a-2b \end{bmatrix} = \begin{bmatrix} a-2 & 2a-4 \\ 1+b & 2+2b \end{bmatrix}$$

By the equality of matrices, we get

$0 = a - 2$ ... (1)

$0 = 1 + b$ ... (2)

$a + 2b = 2a - 4$ ... (3)

$-a - 2b = 2 + 2b$ ... (4)

From equations (1) and (2), we get

$a = 2$ and $b = -1$

The values of $a$ and $b$ satisfy equations (3) and (4) also.

Hence, $a = 2$ and $b = -1$.

**12) Find k, If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$**

**and $A^2 = kA - 2I$.**

**Solution:**

$$A^2 = A \cdot A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

$$kA - 2I = k\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3k & -2k \\ 4k & -2k \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$$

But, $A^2 = kA - 2I$

$$\therefore \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$$

By equality of matrices,

$1 = 3k - 2$ ... (1)

$-2 = -2k$ ... (2)

$4 = 4k$ ... (3)

$-4 = -2k - 2$ ... (4)

From (2), $k = 1$.

$k = 1$ also satisfies equation (1), (3) and (4).

Hence, $k = 1$.

**13) Find x and y, If**

$$\left\{4\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 4 \\ 2 & 1 & 1 \end{bmatrix}\right\} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

**Solution:**

$$\left\{4\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 4 \\ 2 & 1 & 1 \end{bmatrix}\right\} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \left\{\begin{bmatrix} 8 & -4 & 12 \\ 4 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 4 \\ 2 & 1 & 1 \end{bmatrix}\right\} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 5 & -1 & 8 \\ 2 & -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 10+1+8 \\ 4+1+7 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 19 \\ 12 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

By equality of matrices,

$x = 19$ and $y = 12$.

**14) Find x, y, z if**

$$\left\{3\begin{bmatrix}2 & 0\\0 & 2\\2 & 2\end{bmatrix} - 4\begin{bmatrix}1 & 1\\-1 & 2\\3 & 1\end{bmatrix}\right\}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}x-3\\y-1\\2z\end{bmatrix}$$

**Solution:**

$$\left\{3\begin{bmatrix}2 & 0\\0 & 2\\2 & 2\end{bmatrix} - 4\begin{bmatrix}1 & 1\\-1 & 2\\3 & 1\end{bmatrix}\right\}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}x-3\\y-1\\2z\end{bmatrix}$$

$$\therefore \left\{\begin{bmatrix}6 & 0\\0 & 6\\6 & 6\end{bmatrix} - \begin{bmatrix}4 & 4\\-4 & 8\\12 & 4\end{bmatrix}\right\}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}x-3\\y-1\\2z\end{bmatrix}$$

$$\therefore \begin{bmatrix}2 & -4\\4 & -2\\-6 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}x-3\\y-1\\2z\end{bmatrix}$$

$$\therefore \begin{bmatrix}2-8\\4-4\\-6+4\end{bmatrix} = \begin{bmatrix}x-3\\y-1\\2z\end{bmatrix}$$

$$\therefore \begin{bmatrix}-6\\0\\-2\end{bmatrix} = \begin{bmatrix}x-3\\y-1\\2z\end{bmatrix}$$

By equality of matrices,

$-6 = x-3$, $0 = y-1$ and $-2 = 2z$

$\therefore$ $x = -3$, $y = 1$ and $z = -1$.

**15) Jay and Ram are two friends. Jay wants to buy 4 pens and 8 notebooks, Ram wants to buy 5 pens and 12 notebooks. The price of One pen and one notebook was Rs. 6 and Rs.10 respectively. Using matrix multiplication, find the amount each one of them requires for buying the pens and notebooks.**

**Solution:**

**Solution :** The given data can be written in matrix form as :

**Number of Pens and Notebooks**

| | Pens | Notebooks | |
|---|---|---|---|
| A = | 4 | 8 | Jay |
| | 5 | 12 | Ram |

Price in ₹

$B = \begin{bmatrix}6\\10\end{bmatrix}$ **Pen** / **Notebook**

For finding the amount each one of them requires to buy the pens and notebook, we require the multiplication of the two matrices A and B.

$$\text{Consider } AB = \begin{bmatrix}4 & 8\\5 & 12\end{bmatrix}\begin{bmatrix}6\\10\end{bmatrix}$$

$$= \begin{bmatrix}24+80\\30+120\end{bmatrix} = \begin{bmatrix}104\\150\end{bmatrix}$$

Hence, Jay requires ₹ 104 and Ram requires ₹ 150 to buy the pens and notebooks.

## EXERCISE 2.4

**(1) Find $A^T$, if**

$(i)\ A = \begin{bmatrix}1 & 3\\-4 & 5\end{bmatrix}$ $(ii)\ A\begin{bmatrix}2 & -6 & 1\\-4 & 0 & 5\end{bmatrix}$

**Solution:**

**(i)** $A = \begin{bmatrix}1 & 3\\-4 & 5\end{bmatrix}$

$$\therefore A^T = \begin{bmatrix}1 & -4\\3 & 5\end{bmatrix}.$$

**(ii)** $A = \begin{bmatrix}2 & -6 & 1\\-4 & 0 & 5\end{bmatrix}$

$$\therefore A^T = \begin{bmatrix}2 & -4\\-6 & 0\\1 & 5\end{bmatrix}.$$

**(2) If A =$[a_{ij}]_{3\times3}$ where $a_{ij}$ = 2(i - j). Find A and $A^T$ State whether A and $A^T$ both are symmetric or skew symmetric matrices?**

**Solution:**

$$A = [a_{ij}]_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Given : $a_{ij} = 2(i-j)$

$\therefore\ a_{11} = 2(1-1) = 0,\ a_{12} = 2(1-2) = -2,$

$a_{13} = 2(1-3) = -4,\ a_{21} = 2(2-1) = 2,$

$a_{22} = 2(2-2) = 0,\ a_{23} = 2(2-3) = -2,$

$a_{31} = 2(3-1) = 4,\ a_{32} = 2(3-2) = 2,$

$a_{33} = 2(3-3) = 0$

$$\therefore\ A = \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$$

$$\therefore\ A^T = \begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & 2 \\ -4 & -2 & 0 \end{bmatrix}$$

$$\therefore\ -A^T = -\begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & 2 \\ -4 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$$

$\therefore\ A = -A^T$ and $A^T = -A$

Hence, A and $A^T$ are both skew-symmetric matrices.

**(3) If** $A = \begin{bmatrix} 5 & -3 \\ 4 & -3 \\ -2 & 1 \end{bmatrix}$, ***Prove that*** $(A^T)^T = A$.

**Solution :** $A = \begin{bmatrix} 5 & -3 \\ 4 & -3 \\ -2 & 1 \end{bmatrix}$

$$\therefore\ A^T = \begin{bmatrix} 5 & 4 & -2 \\ -3 & -3 & 1 \end{bmatrix}$$

$$\therefore\ (A^T)^T = \begin{bmatrix} 5 & -3 \\ 4 & -3 \\ -2 & 1 \end{bmatrix} = A.$$

**(4) If** $A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$, **Prove that** $A^T = A$.

**Solution:**

Solution : $A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$ ... (1)

$\therefore\ A^T = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$ ... (2)

From (1) and (2), $A^T = A$.

**(5) If** $A = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & -1 \\ -3 & 3 \end{bmatrix}$,

$C = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ -2 & 3 \end{bmatrix}$ ***then show that***

***(i)*** $(A+B)^T = A^T + B^T$

***(ii)*** $(A-C)^T = A^T - C^T$

**Solution:**

**(i)** $A + B = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 4 & -1 \\ -3 & 3 \end{bmatrix}$

$$\begin{bmatrix} 2+2 & -3+1 \\ 5+4 & -4-1 \\ -6-3 & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 9 & -5 \\ -9 & 4 \end{bmatrix}$$

$\therefore\ (A+B)^T = \begin{bmatrix} 4 & 9 & -9 \\ -2 & -5 & 4 \end{bmatrix}$ ... (1)

$A^T = \begin{bmatrix} 2 & 5 & -6 \\ -3 & -4 & 1 \end{bmatrix}$, $B^T = \begin{bmatrix} 2 & 4 & -3 \\ 1 & -1 & 3 \end{bmatrix}$

$\therefore\ A^T + B^T = \begin{bmatrix} 2 & 5 & -6 \\ -3 & -4 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 & -3 \\ 1 & -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2+2 & 5+4 & -6-3 \\ -3+1 & -4-1 & 1+3 \end{bmatrix}$

$= \begin{bmatrix} 4 & 9 & -9 \\ -2 & -5 & 4 \end{bmatrix}$ ... (2)

From (1) and (2),

$(A+B)^T = A^T + B^T$.

**(ii)** $A - C = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ -2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2-1 & -3-2 \\ 5-(-1) & -4-4 \\ -6-(-2) & 1-3 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 6 & -8 \\ -4 & -2 \end{bmatrix}$$

$\therefore\ (A-C)^T = \begin{bmatrix} 1 & 6 & -4 \\ -5 & -8 & -2 \end{bmatrix}$ ... (1)

$A^T = \begin{bmatrix} 2 & 5 & -6 \\ -3 & -4 & 1 \end{bmatrix}$, $C^T = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 4 & 3 \end{bmatrix}$

$\therefore\ A^T - C^T = \begin{bmatrix} 2 & 5 & -6 \\ -3 & -4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & -2 \\ 2 & 4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2-1 & 5-(-1) & -6-(-2) \\ -3-2 & -4-4 & 1-3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 6 & -4 \\ -5 & -8 & -2 \end{bmatrix}$ ... (2)

From (1) and (2),

$(A-C)^T = A^T - C^T$.

**(6) If $A = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 4 & -1 \end{bmatrix}$, then find $C^T$, such that 3A – 2B + C = I, where I is the unit matrix of order 2.**

**Solution:**

$$3A - 2B + C = I$$

$$\therefore\ C = I - 3A + 2B$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix} + 2\begin{bmatrix} -1 & 3 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 12 \\ -6 & 9 \end{bmatrix} + \begin{bmatrix} -2 & 6 \\ 8 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-15+(-2) & 0-12+6 \\ 0-(-6)+8 & 1-9-2 \end{bmatrix}$$

$$\therefore\ C = \begin{bmatrix} -16 & -6 \\ 14 & -10 \end{bmatrix}$$

$$\therefore\ C^T = \begin{bmatrix} -16 & 14 \\ -6 & -10 \end{bmatrix}.$$

**(7) If $A = \begin{bmatrix} 7 & 3 & 0 \\ 0 & 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 1 & -4 \end{bmatrix}$ then find (i) $A^T + 4B^T$ (ii) $5A^T - 5B^T$**

**Solution:**

$$A = \begin{bmatrix} 7 & 3 & 0 \\ 0 & 4 & -2 \end{bmatrix},\ B = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 1 & -4 \end{bmatrix}$$

$$\therefore\ A^T = \begin{bmatrix} 7 & 0 \\ 3 & 4 \\ 0 & -2 \end{bmatrix},\ B = \begin{bmatrix} 0 & 2 \\ -2 & 1 \\ 3 & -4 \end{bmatrix}$$

$$\text{(i)}\ A^T + 4B^T = \begin{bmatrix} 7 & 0 \\ 3 & 4 \\ 0 & -2 \end{bmatrix} + 4\begin{bmatrix} 0 & 2 \\ -2 & 1 \\ 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 \\ 3 & 4 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ -8 & 4 \\ 12 & -16 \end{bmatrix}$$

$$= \begin{bmatrix} 7+0 & 0+8 \\ 3-8 & 4+4 \\ 0+12 & -2-16 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ -5 & 8 \\ 12 & -18 \end{bmatrix}.$$

$$\text{(ii)}\ 5A^T - 5B^T = 5\begin{bmatrix} 7 & 0 \\ 3 & 4 \\ 0 & -2 \end{bmatrix} - 5\begin{bmatrix} 0 & 2 \\ -2 & 1 \\ 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 35 & 0 \\ 15 & 20 \\ 0 & -10 \end{bmatrix} - \begin{bmatrix} 0 & 10 \\ -10 & 5 \\ 15 & -20 \end{bmatrix}$$

$$= \begin{bmatrix} 35-0 & 0-10 \\ 15-(-10) & 20-5 \\ 0-15 & -10-(-20) \end{bmatrix}$$

$$= \begin{bmatrix} 35 & -10 \\ 25 & 15 \\ -15 & 10 \end{bmatrix}.$$

**(8) If $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & -4 \\ 3 & 5 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -1 & 0 \end{bmatrix}$, verify that $(A + 2B + 3C)^T = A^T + 2B^T + 3C^T$**

**Solution:**

$$A + 2B + 3C$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} + 2\begin{bmatrix} 2 & 1 & -4 \\ 3 & 5 & -2 \end{bmatrix} + 3\begin{bmatrix} 0 & 2 & 3 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 2 & -8 \\ 6 & 10 & -4 \end{bmatrix} + \begin{bmatrix} 0 & 6 & 9 \\ -3 & -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 0+2+6 & 1-8+9 \\ 3+6-3 & 1+10-3 & 2-4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 & 2 \\ 6 & 8 & -2 \end{bmatrix}$$

$$\therefore\ (A + 2B + 3C)^T = \begin{bmatrix} 5 & 6 \\ 8 & 8 \\ 2 & -2 \end{bmatrix} \qquad \ldots (1)$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix},\ B^T = \begin{bmatrix} 2 & 3 \\ 1 & 5 \\ -4 & -2 \end{bmatrix},\ C^T = \begin{bmatrix} 0 & -1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

$$\therefore\ A^T + 2B^T + 3C^T$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} + 2\begin{bmatrix} 2 & 3 \\ 1 & 5 \\ -4 & -2 \end{bmatrix} + 3\begin{bmatrix} 0 & -1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 2 & 10 \\ -8 & -4 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 6 & -3 \\ 9 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 3+6-3 \\ 0+2+6 & 1+10-3 \\ 1-8+9 & 2-4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 8 & 8 \\ 2 & -2 \end{bmatrix} \qquad \ldots (2)$$

From (1) and (2),

$(A + 2B + 3C)^T = A^T + 2B^T + 3C^T$

**(9) If $A = \begin{bmatrix} -1 & 2 & 1 \\ -3 & 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}$,**

**Prove that $(A + B^T)^T = A^T + B$**

**Solution:**

$A = \begin{bmatrix} -1 & 2 & 1 \\ -3 & 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}$

$\therefore A^T = \begin{bmatrix} -1 & -3 \\ 2 & 2 \\ 1 & -3 \end{bmatrix}, B^T = \begin{bmatrix} 2 & -3 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

$\therefore A + B^T = \begin{bmatrix} -1 & 2 & 1 \\ -3 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & -3 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} -1+2 & 2-3 & 1-1 \\ -3+1 & 2+2 & -3+3 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & 0 \end{bmatrix}$

$\therefore (A + B^T)^T = \begin{bmatrix} 1 & -2 \\ -1 & 4 \\ 0 & 0 \end{bmatrix}$ ... (1)

$A^T + B = \begin{bmatrix} -1 & -3 \\ 2 & 2 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} -1+2 & -3+1 \\ 2-3 & 2+2 \\ 1-1 & -3+3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 4 \\ 0 & 0 \end{bmatrix}$ ... (2)

From (1) and (2),

$(A + B^T)^T = A^T + B.$

**(10) Prove that $A + A^T$ is a symmetric and $A - A^T$ is a skew symmetric matrix, where**

**(i) $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -2 & -3 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 5 & 2 & -4 \\ 3 & -7 & 2 \\ 4 & -5 & -3 \end{bmatrix}$**

**Solution:**

**(i)** $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -2 & -3 & 2 \end{bmatrix}$ $\therefore A^T = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix}$

$\therefore A + A^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -2 & -3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -2 \\ 2 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1+1 & 2+3 & 4-2 \\ 3+2 & 2+2 & 1-3 \\ -2+4 & -3+1 & 2+2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 5 & 2 \\ 5 & 4 & -2 \\ 2 & -2 & 4 \end{bmatrix}$

This is a symmetric matrix (by definition).

Also, $A - A^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -2 & -3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ 2 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1-1 & 2-3 & 4-(-2) \\ 3-2 & 2-2 & 1-(-3) \\ -2-4 & -3-1 & 2-2 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 & 6 \\ 1 & 0 & 4 \\ -6 & -4 & 0 \end{bmatrix}$

This is a skew-symmetric matrix (by definition).

(ii) $A = \begin{bmatrix} 5 & 2 & -4 \\ 3 & -7 & 2 \\ 4 & -5 & -3 \end{bmatrix}$

$\therefore A^T = \begin{bmatrix} 5 & 3 & 4 \\ 2 & -7 & -5 \\ -4 & 2 & -3 \end{bmatrix}$

$\therefore A + A^T =$

$\begin{bmatrix} 5 & 2 & -4 \\ 3 & -7 & 2 \\ 4 & -5 & -3 \end{bmatrix} + \begin{bmatrix} 5 & 3 & 4 \\ 2 & -7 & -5 \\ -4 & 2 & -3 \end{bmatrix}$

$= \begin{bmatrix} 5+5 & 2+3 & -4+4 \\ 3+2 & -7-7 & 2-5 \\ 4-4 & -5+2 & -3-3 \end{bmatrix}$

$\therefore A + A^T = \begin{bmatrix} 10 & 5 & 0 \\ 5 & -14 & -3 \\ 0 & -3 & -6 \end{bmatrix}$

$\therefore (A + A^T)^T = \begin{bmatrix} 10 & 5 & 0 \\ 5 & -14 & -3 \\ 0 & -3 & -6 \end{bmatrix}$

$\therefore (A + A^T)^T = A + A^T$, i.e., $A + A^T = (A + A^T)^T$

$\therefore A + A^T$ is symmetric matrix.

Also, $A - A^T =$

$\begin{bmatrix} 5 & 2 & -4 \\ 3 & -7 & 2 \\ 4 & -5 & -3 \end{bmatrix} - \begin{bmatrix} 5 & 3 & 4 \\ 2 & -7 & -5 \\ -4 & 2 & -3 \end{bmatrix}$

$= \begin{bmatrix} 5-5 & 2-3 & -4-4 \\ 3-2 & -7+7 & 2+5 \\ 4+4 & -5-2 & -3+3 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 & -8 \\ 1 & 0 & 7 \\ 8 & -7 & 0 \end{bmatrix}$

$\therefore (A - A^T)^T = \begin{bmatrix} 0 & 1 & 8 \\ -1 & 0 & -7 \\ -8 & 7 & 0 \end{bmatrix}$

$= -\begin{bmatrix} 0 & -1 & -8 \\ 1 & 0 & 7 \\ 8 & -7 & 0 \end{bmatrix}$

$\therefore (A - A^T)^T = -(A - A^T)$,

i.e., $A - A^T = (A - A^T)^T$

$\therefore A - A^T$ is a skew-symmetric matrix.

**(11) Express each of the following matrix as the sum of a symmetric and a skew symmetric matrix.**

$(i) \begin{bmatrix} \mathbf{4} & \mathbf{-2} \\ \mathbf{3} & \mathbf{-5} \end{bmatrix}$ $(ii) \begin{bmatrix} \mathbf{3} & \mathbf{3} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{-2} & \mathbf{1} \\ \mathbf{-4} & \mathbf{-5} & \mathbf{2} \end{bmatrix}$

**Solution:**

(i) Let $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}$

$\therefore A + A^T = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}$

$= \begin{bmatrix} 4+4 & -2+3 \\ 3-2 & -5-5 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix}$

This is a symmetric matrix.

Also, $A - A^T = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}$

$= \begin{bmatrix} 4-4 & -2-3 \\ 3-(-2) & -5-(-5) \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$

This is a skew-symmetric matrix.

Now, $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$= \frac{1}{2}\begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 4 & \frac{1}{2} \\ \frac{1}{2} & -5 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}$.

(ii) Let $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

$\therefore A + A^T = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3+3 & 3-2 & -1-4 \\ -2+3 & -2-2 & 1-5 \\ -4-1 & -5+1 & 2+2 \end{bmatrix}$

$= \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$

This is a symmetric matrix.

Also, $A - A^T$

$= \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3-3 & 3-(-2) & -1-(-4) \\ -2-3 & -2-(-2) & 1-(-5) \\ -4-(-1) & -5-1 & 2-2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$

This is a skew-symmtric matrix.

Now, $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$\therefore A = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$.

**(12) If $A = \begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{-2} \\ \mathbf{4} & \mathbf{1} \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{0} & \mathbf{3} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$, verify that (i) $(AB)^T = B^TA^T$ (ii) $(BA)^T = A^TB^T$**

**Solution:**

$A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 & -4 \\ 2 & -1 & 1 \end{bmatrix}$

$\therefore A^T = \begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 1 \end{bmatrix}$, $B^T = \begin{bmatrix} 0 & 2 \\ 3 & -1 \\ -4 & 1 \end{bmatrix}$

(i) $AB = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & -4 \\ 2 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0-2 & 6+1 & -8-1 \\ 0-4 & 9+2 & -12-2 \\ 0+2 & 12-1 & -16+1 \end{bmatrix} = \begin{bmatrix} -2 & 7 & -9 \\ -4 & 11 & -14 \\ 2 & 11 & -15 \end{bmatrix}$$

$$\therefore (AB)^T = \begin{bmatrix} -2 & -4 & 2 \\ 7 & 11 & 11 \\ -9 & -14 & -15 \end{bmatrix} \quad \ldots (1)$$

$$B^TA^T = \begin{bmatrix} 0 & 2 \\ 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0-2 & 0-4 & 0+2 \\ 6+1 & 9+2 & 12-1 \\ -8-1 & -12-2 & -16+1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -4 & 2 \\ 7 & 11 & 11 \\ -9 & -14 & -15 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2),

$(AB)^T = B^TA^T$.

(ii) $BA = \begin{bmatrix} 0 & 3 & -4 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0+9-16 & 0-6-4 \\ 4-3+4 & -2+2+1 \end{bmatrix} = \begin{bmatrix} -7 & -10 \\ 5 & 1 \end{bmatrix}$$

$$\therefore (BA)^T = \begin{bmatrix} -7 & 5 \\ -10 & 1 \end{bmatrix} \quad \ldots (1)$$

$$A^TB^T = \begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & -1 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9-16 & 4-3+4 \\ 0-6-4 & -2+2+1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 5 \\ -10 & 1 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), $(BA)^T = A^TB^T$.

## EXERCISE 2.5

**1) Apply the given elementary transformation on each of the following matrices.**

***i)*** $\begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}, R_1 \leftrightarrow R_2$

**Solution:**

Let $A = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

By $R_1 \leftrightarrow R_2$, we get

$A \sim \begin{bmatrix} 2 & 2 \\ 3 & -4 \end{bmatrix}$

***ii)*** $\begin{bmatrix} 2 & 4 \\ 1 & -5 \end{bmatrix}, C_1 \leftrightarrow C_2$

**Solution:**

Let $B = \begin{bmatrix} 2 & 4 \\ 1 & -5 \end{bmatrix}$

By $C_1 \leftrightarrow C_2$, we get

$B \sim \begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix}$.

***iii)*** $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ $3R_2$ *and* $C_2 \to C_2 - 4C_1$

**Solution:**

Let $C = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

By $3R_2$, we get

$C \sim \begin{bmatrix} 3 & 1 & -1 \\ 3 & 9 & 3 \\ -1 & 1 & 3 \end{bmatrix}$

By $C_2 - 4C_1$ on C, we get

$C \sim \begin{bmatrix} 3 & -11 & -1 \\ 1 & -1 & 1 \\ -1 & 5 & 3 \end{bmatrix}$.

**2)** $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$ **into an upper triangular matrix by suitable row transformations.**

**Solution:**

**Solution :** Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{pmatrix}$

By $R_2 - 2R_1$ and $R_3 - 3R_1$, we get

$$A \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 5 & -2 \end{pmatrix}$$

By $\left(\frac{1}{3}\right) R_2$, we get

$$A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 5 & -2 \end{bmatrix}$$

By $R_3 - 5R_2$, we get

$$A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

This is an upper triangular matrix.

**3) Find the cofactor of the following matrices**

*i)* $\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}$

**Solution:**

Let $A = \begin{pmatrix} 1 & 2 \\ 5 & -8 \end{pmatrix}$

Here, $a_{11} = 1$, $M_{11} = -8$

$\therefore A_{11} = (-1)^{1+1} M_{11} = -8$

$a_{12} = 2$, $M_{12} = 5$

$\therefore A_{12} = (-1)^{1+2} M_{12} = -1(5) = -5$

$a_{21} = 5$, $M_{21} = 2$

$\therefore A_{21} = (-1)^{2+1} M_{21} = -1(2) = -2$

$a_{22} = -8$, $M_{22} = 1$

$\therefore A_{22} = (-1)^{2+2} M_{22} = 1.$

$$\therefore \text{cofactor matrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} -8 & -5 \\ -2 & 1 \end{pmatrix}.$$

*ii)* $\begin{bmatrix} 5 & 8 & 7 \\ -1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

**Solution:**

(ii) Let $A = \begin{pmatrix} 5 & 8 & 7 \\ -1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix}$

The cofactor of $a_{ij}$ is given by $A_{ij} = (-1)^{i+j} M_{ij}$

Now, $M_{11} = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -2 - 1 = -3$

$\therefore A_{11} = (-1)^{1+1} M_{11} = 1(-3) = -3$

$M_{12} = \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} = -1 - (-2) = 1$

$\therefore A_{12} = (-1)^{1+2} M_{12} = -1(1) = -1$

$M_{13} = \begin{vmatrix} -1 & -2 \\ -2 & 1 \end{vmatrix} = -1 - 4 = -5$

$\therefore A_{13} = (-1)^{1+3} M_{13} = 1(-5) = -5$

$M_{21} = \begin{vmatrix} 8 & 7 \\ 1 & 1 \end{vmatrix} = 8 - 7 = 1$

$\therefore A_{21} = (-1)^{2+1} M_{21} = -1(1) = -1$

$M_{22} = \begin{vmatrix} 5 & 7 \\ -2 & 1 \end{vmatrix} = 5 + 14 = 19$

$\therefore A_{22} = (-1)^{2+2} M_{22} = 1(19) = 19$

$M_{23} = \begin{vmatrix} 5 & 8 \\ -2 & 1 \end{vmatrix} = 5 - (-16) = 21$

$\therefore A_{23} = (-1)^{2+3} M_{23} = -1(21) = -21$

$M_{31} = \begin{vmatrix} 8 & 7 \\ -2 & 1 \end{vmatrix} = 8 - (-14) = 22$

$\therefore A_{31} = (-1)^{3+1} M_{31} = 1(22) = 22$

$M_{32} = \begin{vmatrix} 5 & 7 \\ -1 & 1 \end{vmatrix} = 5 - (-7) = 12$

$\therefore A_{32} = (-1)^{3+2} M_{32} = -1(12) = -12$

$M_{33} = \begin{vmatrix} 5 & 8 \\ -1 & -2 \end{vmatrix} = -10 - (-8) = -2$

$\therefore A_{33} = (-1)^{3+3} M_{33} = 1(-2) = -2$

$$\therefore \text{cofactor matrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} -3 & -1 & -5 \\ -1 & 19 & -21 \\ 22 & -12 & -2 \end{pmatrix}.$$

**4) Find the adjoint of the following matrices**

$i)\begin{bmatrix}2 & -3\\3 & 5\end{bmatrix}$

**Solution:**

$$\text{(i) } A=\begin{bmatrix}2 & -3\\3 & 5\end{bmatrix}$$

Here, $a_{11}=2, M_{11}=5$

$\therefore A_{11}=(-1)^{1+1}(5)=5$

$a_{12}=-3, M_{12}=3$

$\therefore A_{12}=(-1)^{1+2}(3)=-3$

$a_{21}=3, M_{21}=-3$

$\therefore A_{21}=(-1)^{2+1}(-3)=3$

$a_{22}=5, M_{22}=2$

$\therefore A_{22}=(-1)^{2+2}=2$

$\therefore$ the cofactor matrix $=\begin{bmatrix}A_{11} & A_{12}\\A_{21} & A_{22}\end{bmatrix}=\begin{bmatrix}5 & -3\\3 & 2\end{bmatrix}$

$\therefore$ adj $A=\begin{bmatrix}5 & 3\\-3 & 2\end{bmatrix}$.

$ii)\begin{bmatrix}1 & -1 & 2\\-2 & 3 & 5\\-2 & 0 & -1\end{bmatrix}$

**Solution:**

Let $A=\begin{bmatrix}1 & -1 & 2\\-2 & 3 & 5\\-2 & 0 & -1\end{bmatrix}$

The cofactor of $a_{ij}$ is given by $A_{ij}=(-1)^{i+j}M_{ij}$

Now, $M_{11}=\begin{vmatrix}3 & 5\\0 & -1\end{vmatrix}=-3-0=-3$

$\therefore A_{11}=(-1)^{1+1}(-3)=-3$

$M_{12}=\begin{vmatrix}-2 & 5\\-2 & -1\end{vmatrix}=2+10=12$

$\therefore A_{12}=(-1)^{1+2}(12)=-12$

$M_{13}=\begin{vmatrix}-2 & 3\\-2 & 0\end{vmatrix}=0+6=6$

$\therefore A_{13}=(-1)^{1+3}(6)=6$

$M_{21}=\begin{vmatrix}-1 & 2\\0 & -1\end{vmatrix}=1-0=1$

$\therefore A_{21}=(-1)^{2+1}(1)=-1$

$M_{22}=\begin{vmatrix}1 & 2\\-2 & -1\end{vmatrix}=-1+4=3$

$\therefore A_{22}=(-1)^{2+2}(3)=3$

$M_{23}=\begin{vmatrix}1 & -1\\-2 & 0\end{vmatrix}=0-2=-2$

$\therefore A_{23}=(-1)^{2+3}(-2)=2$

$M_{31}=\begin{vmatrix}-1 & 2\\3 & 5\end{vmatrix}=-5-6=-11$

$\therefore A_{31}=(-1)^{3+1}(-11)=-11$

$M_{32}=\begin{vmatrix}1 & 2\\-2 & 5\end{vmatrix}=5+4=9$

$\therefore A_{32}=(-1)^{3+2}(9)=-9$

$M_{33}=\begin{vmatrix}1 & -1\\-2 & 3\end{vmatrix}=3-2=1$

$\therefore A_{33}=(-1)^{3+3}(1)=1.$

$\therefore$ the cofactor matrix $=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{bmatrix}$

$=\begin{bmatrix}-3 & -12 & 6\\-1 & 3 & 2\\-11 & -9 & 1\end{bmatrix}$

$\therefore adj\,A=\begin{bmatrix}-3 & -1 & -11\\-12 & 3 & -9\\6 & 2 & 1\end{bmatrix}$

**5) Find the inverse of the following matrices by the adjoint method**

$i)\begin{bmatrix}3 & -1\\2 & -1\end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix} = -3-(-2) = -1 \neq 0$

$\therefore$ $A^{-1}$ exists.

First we have to find the cofactor matrix

$= [A_{ij}]_{2 \times 2}$, where $A_{ij} = (-1)^{i+j} M_{ij}$

Now, $A_{11} = (-1)^{1+1} M_{11} = -1$

$A_{12} = (-1)^{1+2} M_{12} = -2$

$A_{21} = (-1)^{2+1} M_{21} = -(-1) = 1$

$A_{22} = (-1)^{2+2} M_{22} = 3$

$\therefore$ the cofactor matrix

$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}$

$\therefore$ adj $A = \begin{bmatrix} -1 & 1 \\ -2 & 3 \end{bmatrix}$

$\therefore$ $A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-1}\begin{bmatrix} -1 & 1 \\ -2 & 3 \end{bmatrix}$

$\therefore$ $A^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$.

*ii)* $\begin{bmatrix} 2 & -2 \\ 4 & 5 \end{bmatrix}$

**Solution:**

*Refer to the solution of Q. 5 (i).*

**Ans.** $\frac{1}{18}\begin{bmatrix} 5 & 2 \\ -4 & 2 \end{bmatrix}$.

*iii)* $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{vmatrix}$

$= 1(10-0) - 2(0-0) + 3(0-0)$

$= 10 \neq 0$

$\therefore$ $A^{-1}$ exist.

First we have to find the cofactor matrix

$= [A_{ij}]_{3 \times 3}$, where $A_{ij} = (-1)^{i+j} M_{ij}$

Now, $A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = 10 - 0 = 10$

$A_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} = -(0-0) = 0$

$A_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$

$A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -(10-0) = -10$

$A_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = 5 - 0 = 5$

$A_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = -(0-0) = 0$

$A_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 8 - 6 = 2$

$A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -(4-0) = -4$

$A_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

$\therefore$ the cofactor matrix

$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}$

$\therefore$ adj $A = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$

$\therefore$ $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

$\therefore$ $A^{-1} = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$.

**6) Find the inverse of the following matrices by the transformation method.**

*i)* $\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1-4 = -5 \neq 0$

$\therefore$ $A^{-1}$ exists.

**$A^{-1}$ by Row transformations :**

We write $AA^{-1} = I$

$\therefore \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By $R_2 - 2R_1$, we get

$\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

By $\left(-\frac{1}{5}\right) R_2$, we get

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$

By $R_1 - 2R_2$, we get

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ ... (1)

**$A^{-1}$ by Column transformations :**

We write $A^{-1}A = I$

$\therefore A^{-1}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By $C_2 - 2C_1$, we get

$A^{-1}\begin{bmatrix} 1 & 0 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

By $\left(-\frac{1}{5}\right) C_2$, we get

$A^{-1}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & -\frac{1}{5} \end{bmatrix}$

By $C_1 - 2C_2$, we get

$A^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ ... (2)

From (1) and (2),

$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, which is unique.

**Note :** If it is asked to find $A^{-1}$ by elementary transformation, we can use either row transformation or column transformation.

*ii)* $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

**Solution:**

Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix}$

$= 2(3-0) - 0(15-0) - 1(5-0)$

$= 6 - 0 - 5 = 1 \neq 0$

$\therefore A^{-1}$ exists.

We write $AA^{-1} = I$

$\therefore \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $3R_1$, we get

$\begin{bmatrix} 6 & 0 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_1 - R_2$, we get

$\begin{bmatrix} 1 & -1 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_2 - 5R_1$, we get

$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 6 & 15 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -15 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_2 - 5R_3$, we get

$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -15 & 6 & -5 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_1 + R_2$ and $R_3 - R_2$, we get

$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} -12 & 5 & -5 \\ -15 & 6 & -5 \\ 15 & -6 & 6 \end{bmatrix}$

By $\left(\frac{1}{3}\right) R_3$, we get

$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -12 & 5 & -5 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

By $R_1 + 3R_3$, we get

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

**Note**: $A^{-1}$ can also be obtained by using column transformations taking $A^{-1}A = I$.

**7) Find the inverse $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ by elementary column transformation method.**

**Solution:**

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$\therefore |A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix}$

$= 1(2-6) - 0 + 1(0-2)$

$= -4 - 2 = -6 \neq 0$

$\therefore A^{-1}$ exists.

We write $A^{-1}A = I$

$\therefore A^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $C_3 - C_1$, we get

$A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $\left(\frac{1}{2}\right) C_2$, we get

$A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $C_3 - 3C_2$, we get

$$A^{-1}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 1 & -3\end{pmatrix} = \begin{bmatrix}1 & 0 & -1\\0 & \frac{1}{2} & -\frac{3}{2}\\0 & 0 & 1\end{bmatrix}$$

By $\left(-\frac{1}{3}\right)C_3$, we get

$$A^{-1}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 1 & 1\end{pmatrix} = \begin{bmatrix}1 & 0 & \frac{1}{3}\\0 & \frac{1}{2} & \frac{1}{2}\\0 & 0 & -\frac{1}{3}\end{bmatrix}$$

**8) Find the inverse $\begin{bmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{bmatrix}$ of by the elementary row transformation.**

**Solution:**

$$\text{Let } A = \begin{pmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{pmatrix}$$

$$\text{Then } |A| = \begin{vmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{vmatrix}$$

$$= 1(7-20) - 2(7-10) + 3(4-2)$$

$$= -13 + 6 + 6 = -1 \neq 0$$

$\therefore$ $A^{-1}$ exists.

We write $AA^{-1} = I$

$$\therefore \begin{pmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{pmatrix} A^{-1} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

By $R_2 - R_1$ and $R_3 - 2R_1$, we get

$$\begin{pmatrix}1 & 2 & 3\\0 & -1 & 2\\0 & 0 & 1\end{pmatrix} A^{-1} = \begin{pmatrix}1 & 0 & 0\\-1 & 1 & 0\\-2 & 0 & 1\end{pmatrix}$$

By $(-1)R_2$, we get

$$\begin{pmatrix}1 & 2 & 3\\0 & 1 & -2\\0 & 0 & 1\end{pmatrix} A^{-1} = \begin{pmatrix}1 & 0 & 0\\1 & -1 & 0\\-2 & 0 & 1\end{pmatrix}$$

By $R_1 - 2R_2$, we get

$$\begin{pmatrix}1 & 0 & 7\\0 & 1 & -2\\0 & 0 & 1\end{pmatrix} A^{-1} = \begin{pmatrix}-1 & 2 & 0\\1 & -1 & 0\\-2 & 0 & 1\end{pmatrix}$$

By $R_1 - 7R_3$ and $R_3 + 2R_3$, we get,

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} A^{-1} = \begin{pmatrix}13 & 2 & -7\\-3 & -1 & 2\\-2 & 0 & 1\end{pmatrix}$$

$$\therefore \boldsymbol{A^{-1}} = \begin{bmatrix}\mathbf{13} & \mathbf{2} & \mathbf{-7}\\\mathbf{-3} & \mathbf{-1} & \mathbf{2}\\\mathbf{-2} & \mathbf{0} & \mathbf{1}\end{bmatrix}$$

**9) If A = $\begin{bmatrix}1 & 0 & 1\\0 & 2 & 3\\1 & 2 & 1\end{bmatrix}$ $and\ B = \begin{bmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{bmatrix}$ then find matrix X such that XA = B**

**Solution:**

$$XA = B$$

$$\therefore X\begin{pmatrix}1 & 0 & 1\\0 & 2 & 3\\1 & 2 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{pmatrix}$$

By $C_3 - C_1$, we get

$$X\begin{pmatrix}1 & 0 & 0\\0 & 2 & 3\\1 & 2 & 0\end{pmatrix} = \begin{pmatrix}1 & 2 & 2\\1 & 1 & 4\\2 & 4 & 5\end{pmatrix}$$

By $\left(\frac{1}{2}\right)C_2$, we get

$$X\begin{pmatrix}1 & 0 & 0\\0 & 1 & 3\\1 & 1 & 0\end{pmatrix} = \begin{bmatrix}1 & 1 & 2\\1 & \frac{1}{2} & 4\\2 & 2 & 5\end{bmatrix}$$

By $C_3 - 3C_2$, we get

$$X\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 1 & -3\end{pmatrix} = \begin{bmatrix}1 & 1 & -1\\1 & \frac{1}{2} & \frac{5}{2}\\2 & 2 & -1\end{bmatrix}$$

By $\left(-\frac{1}{3}\right)C_3$, we get

$$X\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix}=\begin{bmatrix}1&1&\frac{1}{3}\\1&\frac{1}{2}&-\frac{5}{6}\\2&2&\frac{1}{3}\end{bmatrix}$$

By $C_1-C_3$ and $C_2-C_3$, we get

$$X\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{2}{3}&\frac{2}{3}&\frac{1}{3}\\\frac{11}{6}&\frac{4}{3}&-\frac{5}{6}\\\frac{5}{3}&\frac{5}{3}&\frac{1}{3}\end{bmatrix}$$

$$\therefore X=\frac{1}{6}\begin{bmatrix}4&4&2\\11&8&-5\\10&10&2\end{bmatrix}.$$

**10) Find matrix X, if AX = B where A =** $\begin{bmatrix}1&2&3\\-1&1&2\\1&2&4\end{bmatrix}$ *and* $B=\begin{bmatrix}1\\2\\3\end{bmatrix}$

**Solution:**

$$AX=B$$

$$\therefore\begin{bmatrix}1&2&3\\-1&1&2\\1&2&4\end{bmatrix}X=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

By $R_2+R_1$ and $R_3-R_1$, we get

$$\begin{bmatrix}1&2&3\\0&3&5\\0&0&1\end{bmatrix}X=\begin{bmatrix}1\\3\\2\end{bmatrix}$$

By $\left(\frac{1}{3}\right)R_2$, we get

$$\begin{bmatrix}1&2&3\\0&1&\frac{5}{3}\\0&0&1\end{bmatrix}X=\begin{bmatrix}1\\1\\2\end{bmatrix}$$

By $R_1-2R_2$, we get

$$\begin{bmatrix}1&0&-\frac{1}{3}\\0&1&\frac{5}{3}\\0&0&1\end{bmatrix}X=\begin{bmatrix}-1\\1\\2\end{bmatrix}$$

By $R_1+\frac{1}{3}R_3$ and $R_2-\frac{5}{3}R_3$, we get

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}X=\begin{bmatrix}-\frac{1}{3}\\-\frac{7}{3}\\2\end{bmatrix}$$

$$\therefore X=\begin{bmatrix}-\frac{1}{3}\\-\frac{7}{3}\\2\end{bmatrix}.$$

**EXERCISE 2.6**

**1) Solve the following equations by method of inversion.**

*i)* $x+2y=2, 2x+3y=3$

**Solution:**

The given equations can be written in the matrix form as :

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

This is of the form $AX = B$, where

$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Let us find $A^{-1}$.

$$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$$

$\therefore$ $A^{-1}$ exists.

We write $AA^{-1} = I$

$$\therefore \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By $R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

By $(-1)R_2$, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

By $R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

Now, premultiply $AX = B$ by $A^{-1}$, we get

$$A^{-1}(AX) = A^{-1}B$$

$$\therefore (A^{-1}A)X = A^{-1}B$$

$$\therefore IX = A^{-1}B$$

$$\therefore X = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6+6 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

By equality of matrices,

$x = 0$, $y = 1$ is the required solution.

***ii)* $2x + y = 5, 3x + 5y = -3$**

**Solution:**

The given equations can be written in the matrix form as:

$$\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

By $2R_2$, we get,

$$\begin{bmatrix} 2 & 1 \\ 6 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

By $R_2 - 3R_1$, we get,

$$\begin{bmatrix} 2 & 1 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -21 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2x + y \\ 0 + 7y \end{bmatrix} = \begin{bmatrix} 5 \\ -21 \end{bmatrix}$$

By equality of matrices,

$2x + y = 5$ ........(1)

$7y = -21$.............(2)

From (2), $y = -3$

Substituting $y = -3$ in (1), we get,

$2x - 3 = 5$

$\therefore 2x = 8$

$\therefore x = 4$

Hence, $x = 4$, $y = -3$ is the required solution.

***iii)* $2x - y + z = 1, x + 2y + 3z = 8$ *and* $3x + y - 4z = 1$**

**Solution:**

The given equations can be written in the matrix form as :

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

This is of the form AX = B, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

Let us find $A^{-1}$.

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{vmatrix}$$

$= 2(-8-3) + 1(-4-9) + 1(1-6)$

$= -22 - 13 - 5 = -40 \neq 0$

$\therefore$ $A^{-1}$ exists.

We write $AA^{-1} = I$

$$\therefore \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_1 \leftrightarrow R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & -4 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -5 & -13 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

By $\left(-\frac{1}{5}\right) R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -13 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

By $R_1 - 2R_2$ and $R_3 + 5R_2$, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -8 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

By $\left(-\frac{1}{8}\right) R_3$, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{bmatrix}$$

By $R_1 - R_3$ and $R_2 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{11}{40} & \frac{3}{40} & \frac{1}{8} \\ -\frac{13}{40} & \frac{11}{40} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{40} \begin{bmatrix} 11 & 3 & 5 \\ -13 & 11 & 5 \\ 5 & 5 & -5 \end{bmatrix}$$

Now, premultiply AX = B by $A^{-1}$, we get

$A^{-1}(AX) = A^{-1}B$

$\therefore$ $(A^{-1}A)X = A^{-1}B$

$\therefore$ $IX = A^{-1}B$

$$\therefore X = \frac{1}{40} \begin{bmatrix} 11 & 3 & 5 \\ -13 & 11 & 5 \\ 5 & 5 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 11 + 24 + 5 \\ -13 + 88 + 5 \\ 5 + 40 - 5 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 40 \\ 80 \\ 40 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

By equality of matrices,

$x = 1, y = 2, z = 1$ is the required solution.

***iv)*** $\boldsymbol{x + y + z = 1, x - y + z = 2}$ ***and*** $\boldsymbol{x + y - z = 3}$

**Solution:**

The given equations can be written in the matrix form as:

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\1 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}$$

This is of the form AX = B, where

$$A=\begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\1 & 1 & -1\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}-1\\2\\3\end{bmatrix}$$

Let us find $A^{-1}$.

$$|A|=\begin{vmatrix}1 & 1 & 1\\0 & 1 & 1\\1 & 1 & -1\end{vmatrix}$$

$= 1(-1-1)-1(0-1)+1(0-1)$

$= -2+1-1$

$= -2 \neq 0$

$\therefore A^{-1}$ exists.

Consider $AA^{-1} = I$

$$\therefore \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\1 & 1 & -1\end{bmatrix}A-1=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

By $R_3 - R_1$, we get

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & -2\end{bmatrix}A-1=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$$

By $R_1 - R_2$, we get,

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & -2\end{bmatrix}A-1=\begin{bmatrix}1 & -1 & 0\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$$

By $\left(-\frac{1}{2}\right)R_3$, we get

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}A^{-1}=\begin{bmatrix}1 & -1 & 0\\0 & 1 & 0\\\frac{1}{2} & 0 & -\frac{1}{2}\end{bmatrix}$$

By $R_2 - R_3$, we get

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}A^{-1}=\begin{bmatrix}1 & -1 & 0\\-\frac{1}{2} & 1 & \frac{1}{2}\\\frac{1}{2} & 0 & -\frac{1}{2}\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}1 & -1 & 0\\-\frac{1}{2} & 1 & \frac{1}{2}\\\frac{1}{2} & 0 & -\frac{1}{2}\end{bmatrix}$$

Now, premultiply AX = B by A-1 , we get

A-1(AX) = A-1B

$\therefore$ (A-1A)X = A-1B

$\therefore$ IX = A-1B

$$\therefore X=\begin{bmatrix}-1-2+0\\\frac{1}{2}+2+\frac{3}{2}\\-\frac{1}{2}+0-\frac{3}{2}\end{bmatrix}$$

$$\therefore \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-3\\4\\-2\end{bmatrix}$$

$\therefore$ by equality of the matrices, x = -3, y = 4, z = - 2 is the required solution.

**2) Express the following equations in matrix form and solve them by method of reduction.**

**i) $x + 3y = 2, 3x + 5y = 4$**

**Solution:**

The given equations can be written in the matrix form as :

$$\begin{bmatrix}1 & 3\\3 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$$

By $R_2 - 3R_1$, we get

$$\begin{bmatrix}1 & 3\\0 & -4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\-2\end{bmatrix}$$

$$\therefore \begin{bmatrix}x+3y\\0-4y\end{bmatrix}=\begin{bmatrix}2\\-2\end{bmatrix}$$

By equality of matrices,

$x+3y=2$ ... (1)

$-4y=-2$ ... (2)

From (2), $y=\frac{1}{2}$

Substituting $y=\frac{1}{2}$ in (1), we get

$x+\frac{3}{2}=2$

$\therefore x=2-\frac{3}{2}=\frac{1}{2}$

Hence, $x=\frac{1}{2}$, $y=\frac{1}{2}$ is the required solution.

**ii) $3x - y = 1, 4x + y = 6$**

**Solution:**

The given equations can be written in the matrix form as :

$$\begin{bmatrix}3 & -1\\4 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\6\end{bmatrix}$$

By $4R_1$ and $3R_2$, we get

$$\begin{bmatrix}12 & -4\\12 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\18\end{bmatrix}$$

By $R_2 - R_1$, we get

$$\begin{bmatrix}12 & -4\\0 & 7\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\14\end{bmatrix}$$

$$\therefore \begin{bmatrix}12x-4y\\0+7y\end{bmatrix}=\begin{bmatrix}4\\14\end{bmatrix}$$

By equality of matrices,

$12x-4y=4$ ... (1)

$7y=14$ ... (2)

From (2), $y=2$

Substituting $y=2$ in (1), we get

$12x-8=4$

$\therefore 12x=12 \quad \therefore x=1$

Hence, $x=1$, $y=2$ is the required solution.

**iii)** $\mathbf{x + 2y + z = 8, 2x + 3y - z = 11 \text{ and } 3x - y - 2z = 5}$

**Solution:** The given equations can be written in the matrix form as:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 5 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & -7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -19 \end{bmatrix}$$

By $R_3 - 7R_2$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 16 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 16 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x + 2y + z \\ 0 - y - 3z \\ 0 + 0 + 16z \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 16 \end{bmatrix}$$

By equality of matrices,

$x + 2y + z = 8$ ... (1)

$-y - 3z = -5$ ... (2)

$16z = 16$ ... (3)

From (3), $z = 1$

Substituting $z = 1$ in (2), we get

$-y - 3 = -5, \therefore y = 2$

Substituting $y = 2, z = 1$ in (1), we get

$x + 4 + 1 = 8 \quad \therefore x = 3$

Hence, $x = 3, y = 2, z = 1$ is the required solution.

**iv)** $\mathbf{x + y + z = 1, 2x + 3y + 2z = 2 \text{ and } x + y + 2z = 4}$

**Solution:**

The given equations can be written in the matrix form as :

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x + y + z \\ 0 + y + 0 \\ 0 + 0 + z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

By equality of matrices,

$x + y + z = 1$ ... (1)

$y = 0$

$z = 3$

Substituting $y = 0, z = 3$ in (1), we get

$x + 0 + 3 = 1$

$\therefore x = -2$

Hence, x = -2, y = 0, z=3 is the required solution.

**3) The total cost of 3 T.V. and 2 V.C.R. is Rs. 35000. The shopkeeper wants profit of Rs. 1000 per T.V. and Rs. 500 per V.C.R. He sell 2 T.V. and 1 V.C.R. and he gets total revenue as Rs. 21500. Find the cost and selling price of T.V and V.C.R.**

**Solution:**

Let the cost of each T.V. be ₹ $x$ and each V.C.R. be ₹ $y$.

Then the total cost of 3 T.V. and 2 V.C.R. is ₹ $(3x + 2y)$ which is given to be ₹ 35000.

$\therefore 3x + 2y = 35000$

The shopkeeper wants profit of ₹ 1000 per T.V. and ₹ 500 per V.C.R.

The selling price of each T.V. is ₹ $(x + 1000)$ and of each V.C.R. is ₹ $(y + 500)$.

$\therefore$ selling price of 2 T.V. and 1 V.C.R is

₹ $[2(x + 1000) + (y + 500)]$ which is given to be ₹ 21500.

$\therefore 2(x + 1000) + (y + 500) = 21500$

$\therefore 2x + 2000 + y + 500 = 21500$

$\therefore 2x + y = 19000$

Hence, the system of linear equations is

$3x + 2y = 35000$

$2x + y = 19000$

The equations can be written in matrix form as :

$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 35000 \\ 19000 \end{bmatrix}$$

By $R_1 - 2R_2$, we get

$$\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3000 \\ 19000 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -x + 0 \\ 2x + y \end{bmatrix} = \begin{bmatrix} -3000 \\ 19000 \end{bmatrix}$$

By equality of matrices,

$-x = -3000$ ... (1)

$2x + y = 19000$ ... (2)

From (1), $x = 3000$

Substituting $x = 3000$ in (2), we get

$2(3000) + y = 19000$

$\therefore y = 19000 - 6000 = 13000$

Hence, the cost price of one T.V. is ₹ 3000 and of one V.C.R. is ₹13000 and the selling price of one T.V. is ₹ 4000 and of one V.C.R. is ₹ 13500.

**4) The sum of the cost of one Economic book, one Co-operation book and one account book is Rs. 420. The total cost of an Economic book, 2 Co-operation books and an Account book is Rs. 480. Also the total cost of an Economic book, 3 Co-operation book and 2 Account books is Rs. 600. Find the cost of each book.**

**Solution:**

Let the cost of 1 Economic book, 1 Cooperation book and 1 Account book be ₹ $x$, ₹ $y$ and ₹ $z$ respectively.

Then, from the given information

$x+y+z=420$

$x+2y+z=480$

$x+3y+2z=600$

These equations can be written in matrix form as :

$$\begin{bmatrix}1&1&1\\1&2&1\\1&3&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}420\\480\\600\end{bmatrix}$$

By $R_2-R_1$ and $R_3-R_1$, we get

$$\begin{bmatrix}1&1&1\\0&1&0\\0&2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}420\\60\\180\end{bmatrix}$$

$$\therefore \begin{bmatrix}x+y+z\\0+y+0\\0+2y+z\end{bmatrix}=\begin{bmatrix}420\\60\\180\end{bmatrix}$$

By equality of matrices,

$x+y+z=420$ ... (1)

$y=60$

$2y+z=180$ ... (2)

Substituting $y=60$ in (2), we get

$2(60)+z=180$

$\therefore z=180-120=60$

Substituting $y=60$, $z=60$ in (1), we get

$x+60+60=420$

$\therefore x=420-120=300$

Hence, the cost of each Economic book is ₹ 300, each Cooperation book is ₹ 60 and each Account book is ₹ 60.