8

Frequency Response Analysis

Frequency response analysis implies varying ω from zero to ∞ and observing corresponding variation in magnitude and phase angle of the response.

Frequency Response Analysis for Second Order System

Transfer Function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Resonant Frequency

It is defined as, the frequency at which the magnitude has maximum value.

$$\omega_r = \omega_n\sqrt{1-2\xi^2}$$

Resonant Peak

It is maximum value of magnitude occuring at resonant frequency ω_r.

$$|M_r| = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

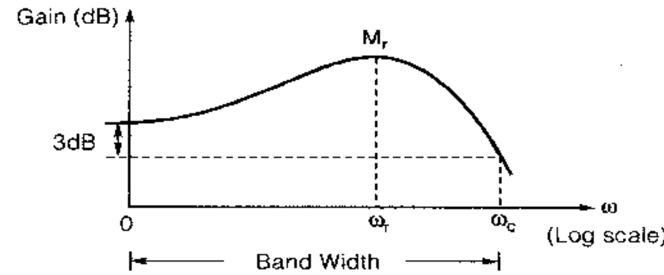
Phase Angle at Resonant Frequency

$$\phi_r = -\tan^{-1}\left[\frac{\sqrt{(1-2\xi^2)}}{\xi}\right]$$

Remember:

- As ζ approaches zero, ω_r approaches ω_n and M_r approaches to infinity.
- For $0 < \zeta \le 1/\sqrt{2}$, the resonant frequency always has a value less than ω_n and the resonant peak has a value greater than 1.

Cut-off Frequency and Bandwidth



- Cut-off frequency

$$\omega_c = \omega_n\left[1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right]^{1/2}$$

- Bandwidth

$$B.W. = \omega_c$$

Stability from Frequency Response Plot

- Phase Margin (P.M.)

$$P.M. = 180^\circ + \angle G(j\omega)H(j\omega)$$

....at gain cross over frequency

- Gain Margin (G.M.)

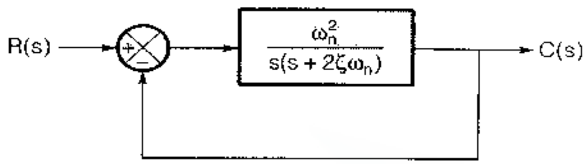
$$G.M. = \frac{1}{|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}} = \frac{1}{X}$$ ... at phase crossover frequency

$$G.M.\ (dB) = 20\log\left(\frac{1}{X}\right)$$

Remember:

- **For stable systems**
  (i) Gain cross over frequency < phase cross-over frequency
  (ii) G.M. and P.M. both are positive
- **For unstable systems**
  (i) Gain cross over frequency > phase cross-over frequency
  (ii) G.M. and P.M. both are negative
- **For marginally stable systems**
  (i) Gain cross over frequency = Phase cross-over frequency
  (ii) G.M. and P.M. both are zero

G.M. and P.M. for Second Order System



- Gain cross-over frequency

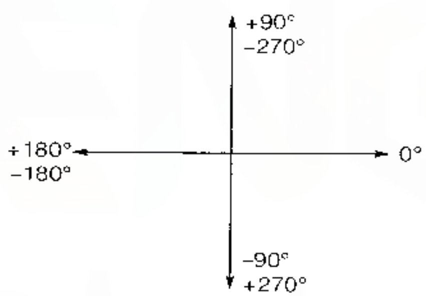
$$\omega_{gc} = \omega_n\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}$$

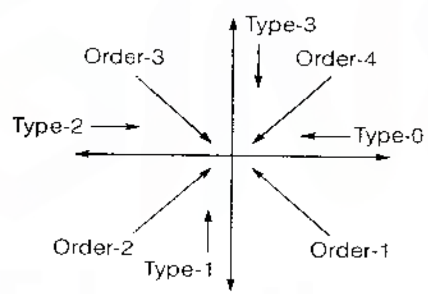
- Phase margin

$$\text{P.M.} = \tan^{-1}\left[\frac{2\zeta}{\sqrt{-2\zeta^2 + 4\zeta^4 + 1}}\right] \approx 100\zeta$$

Polar Plot

It is a plot of absolute values of magnitude and phase angle in degree of open loop transfer function G(jω) H(jω) versus ω drawn on polar coordinates.

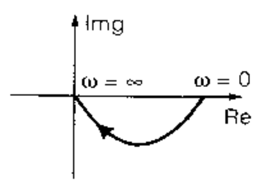




General Shapes of Polar Plot

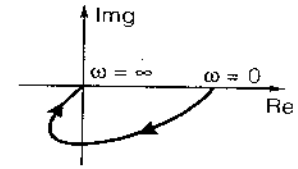
1. Type-0/Order-1

$$G(s) = \frac{1}{1 + Ts}$$



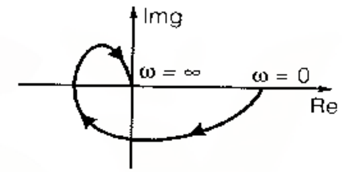
2. Type-0/Order-2

$$G(s) = \frac{1}{(1 + T_1 s)(1 + T_2 s)}$$



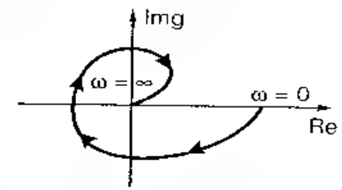
3. Type-0/Order-3

$$G(s) = \frac{1}{(1 + T_1 s)(1 + T_2 s)(1 + T_3 s)}$$



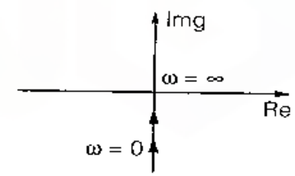
4. Type-0/Order-4

$$G(s) = \frac{1}{(1 + T_1 s)(1 + T_2 s)(1 + T_3 s)(1 + T_4 s)}$$



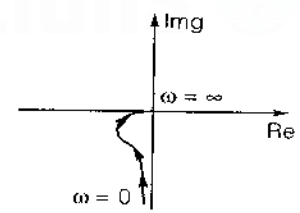
5. Type-1/Order-1

$$G(s) = \frac{1}{s}$$



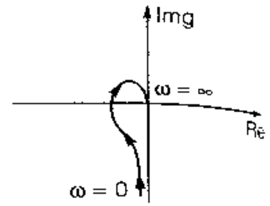
6. Type-1/Order-2

$$G(s) = \frac{1}{s(1 + Ts)}$$



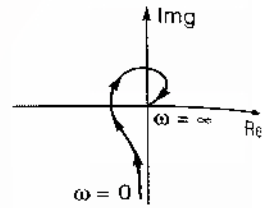
7. Type-1/Order-3

$$G(s) = \frac{1}{s(1+T_1s)(1+T_2s)}$$



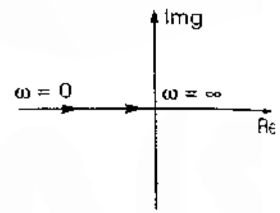
8. Type-1/Order-4

$$G(s) = \frac{1}{s(1+T_1s)(1+T_2s)(1+T_3s)}$$



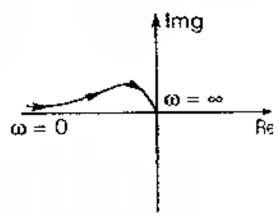
9. Type-2/Order-2

$$G(s) = \frac{1}{s^2}$$



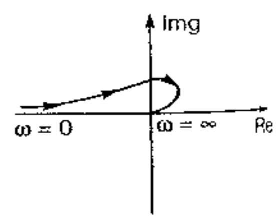
10. Type-2/Order-3

$$G(s) = \frac{1}{s^2(1+Ts)}$$



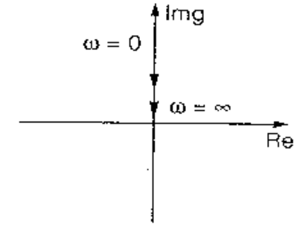
11. Type-2/Order-4

$$G(s) = \frac{1}{s^2(1+T_1s)(1+T_2s)}$$



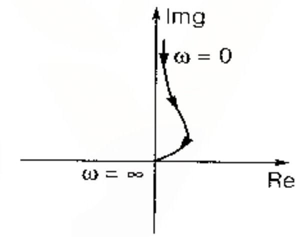
12. Type-3/Order-3

$$G(s) = \frac{1}{s^3}$$



13. Type-3/Order-4

$$G(s) = \frac{1}{s^3(1+Ts)}$$



Note:

- For standard transfer function of type-2 and type-3 systems, the polar plot intersects negative real axis as many times as there are zeros in open loop transfer function.

Nyquist Stability Criteria

Open loop transfer function

$$G(s)H(s) = \frac{K(s \pm Z_1)}{s(s \pm P_1)}$$

$$1 + G(s)\,H(s) = 0$$

$$1 + \frac{K(s \pm Z_1)}{s(s \pm P_1)} = 0$$

$$\frac{s(s \pm P_1) + K(s \pm Z_1)}{s(s \pm P_1)} = 0 \qquad \text{...(i)}$$

Nyquist stability criteria states that the number of encirclements about the critical point (−1 + j0) should be equal to poles of open loop transfer function G(s) H(s), which are in the right half of s-plane.

i.e. $\boxed{N = P - Z}$

For stability (i) $Z = 0$

and (ii) $N = P$

where, P = Number of open loop poles in RHS of s-plane

Z = Number of closed loop poles in RHS of s-plane

N = Number of encirclements about of the point (−1 + j0) by G(s) H(s) plot. The positive direction of encirclements being anti-clockwise.

Bode Plot

The variation of magnitude of sinusoidal transfer function expressed in decibel and the corresponding phase angle in degrees being plotted w.r.t. frequency on a logarithmic scale in rectangular axis. The plot thus obtained is known as Bode plot.

Procedure to Draw Bode Plot

- s is replaced by jω to convert into frequency domain.
- Find magnitude in terms of ω and write in terms of dB.

Magnitude in dB $\boxed{M_{dB} = 20\log|G(j\omega)H(j\omega)|}$

- Find the phase angle ∠ϕ

$$\boxed{\angle\phi = \tan^{-1}\left(\frac{\text{imaginary part}}{\text{real part}}\right)}$$

- With required approximation by varying the frequency minimum to maximum, draw the magnitude and phase plot.

❑ If $G(s)H(s) = K$

then $\boxed{|M_{dB}| = 20\log K}$

System gain K shift the magnitude plot either in upward or downward by "20 logK".

❑ Slope

$$\boxed{\text{Slope} = \frac{d|M|_{dB}}{d\log\omega}}$$

Note:

- The magnitude plot should be started at the frequency of 0.1 with opposite sign of the slope and it should be passes through 0 dB line and intersect at ω = 1, when K = 1.

❑ n poles at origin gives

Slope = −20n dB/dec

∠ϕ = −90n°

❑ n zeros at origin gives

Slope = +20n dB/dec

∠ϕ = +90n°

Remember:

- The initial slope of the magnitude plot is given by the poles or zero located at origin.

❑ Corner frequency

The frequency at which slope changes from one level to another level. Corner frequencies are nothing but a finite poles and finite zeroes location in the form of magnitude.

❑ If $G(s)H(s) = \frac{1}{(1+sT)^n}$

Frequency	Slope	Phase (ϕ)
Below corner frequency	0	0
Above corner frequency	−20n dB/dec	−90n°

❑ If $G(s)H(s) = (1 + sT)^n$

Frequency	Slope	Phase (ϕ)
Below corner frequency	0	0
Above corner frequency	20n dB/dec	90n°

Error

Maximum error between the exact and asymptotic plot occurs at corner frequency

Frequency	Error in magnitude plot	Error in phase plot
$\frac{0.1}{T}$	0 dB	–5.6°
$\frac{0.5}{T}$	0.96 dB	–26.56°
$\frac{1}{T}$	3 dB	–45°
$\frac{10}{T}$	0 dB	–5.6°

Remember:

- The error is nothing but difference between actual value and asymptotic value.
- Error is maximum at corner frequency.
- Error is even function w.r.t. corner frequency.
- 6 dB/oct = 20 dB/dec.

Classification of system

1. Minimum Phase System

A system in which all finite poles and finite zeros are located in left half of s-plane then it is called as minimum phase system.

2. Non Minimum Phase System

A system in which one or more zeros lies in the right half of s-plane. remaining all the poles and zeros lies on the left half of s-plane.

3. All Pass System

A system in which zeros lies on the right half of s-plane, poles lies on the left half of s-plane and which are symmetrical about imaginary axis. For all pass system magnitude should be 1 and phase angle is –180°.

Note:

- Non minimum system = Minimum phase system * all pass system
- $\angle\phi_{MNPS} = \angle\phi_{MPS} + \angle\phi_{ALP}$

Constant Magnitude Loci (M-circles)

- Radius of M-circles

$$\text{Radius} = \frac{M}{1-M^2}$$

- Centre of M-circles

$$\text{Centre} = \left(\frac{M^2}{1-M^2}, 0\right)$$

Constant Phase Angles Loci (N-circles)

- Radius of N-circles

$$\text{Radius} = \sqrt{\frac{1}{4} + \frac{1}{4N^2}}$$

- Centre of N-circles

$$\text{Centre} = \left(-\frac{1}{2}, \frac{1}{2N}\right)$$

■■■■