# 1. Arithmetic Progressions

- **Sequence:** A sequence is an arrangement of numbers in definite order according to some rule.
  Also, we define a sequence as a function whose domain is the set of natural numbers or some subset of the type {1, 2, 3… *k*}.
    - A sequence containing finite number of terms is called a finite sequence.
    - sequence containing infinite number of terms is called an infinite sequence.

- A general sequence can be written as

  $a_1, a_2, a_3 \dots a_{n-1}, a_n, \dots$

  Here, $a_1, a_2 \dots$ etc. are called the terms of the sequence and $a_n$ is called the general term or n$^{th}$ of the sequence.

- **Fibonacci sequence:** An arrangement of numbers such as 1, 2, 4, 6, 10 … has no visible pattern.
  However, the sequence is generated by the recurrence relation given by
  $a_1 = 1, a_2 = 2, a_3 = 4$
  $a_n = a_{n-2} + a_{n-1}, n > 3$
  This sequence is called the Fibonacci sequence.

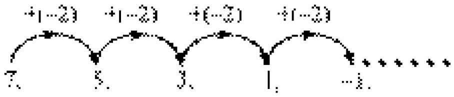
- Let $a_1, a_2, \dots a_n, \dots$ be a given sequence. Accordingly, the sum of this sequence is given by the expression
  $a_1 + a_2 + \dots + a_n + \dots$
  This is called the series associated with the given sequence.
  The series is finite or infinite according as the given sequence.
  A series is usually represented in a compact form using sigma notation (∑).
  This means the series $a_1 + a_2 + \dots + a_{n-1} + a_n \dots$ can be written as $\sum_{k=1}^{n} a_k$.

- **The Concept of Arithmetic Progression**
    - An arithmetic progression is a list of numbers in which the difference between any two consecutive terms is equal.
    - In an AP, each term, except the first term, is obtained by adding a fixed number called common difference to the preceding term.
    - The common difference of an AP can be positive, negative or zero.

  **Example 1:**

1. 

is an AP whose first term and common difference are 3 and 3 respectively.

2. 

is an AP whose first term and common difference are 7 and –2 respectively.

3. 

is an AP whose first term and common difference are –7 and 0 respectively.

- The general form of an AP can be written as $a$, $a + d$, $a + 2d$, $a + 3d$ …, where $a$ is the first term and $d$ is the common difference.
- A given list of numbers i.e., $a_1$, $a_2$, $a_3$ … forms an AP if $a_{k+1} - a_k$ is the same for all values of $k$.

**Example 2:**

Which of the following lists of numbers forms an AP? If it forms an AP, then write its next three terms.

**(a)** –4, 0, 4, 8, …

**(b)** 2, 4, 8, 16, …

**Solution:**

**(a)** –4, 0, 4, 8, …

$a_2 - a_1 = 0 - (-4) = 4$
$a_3 - a_2 = 4 - 0 = 4$
$a_4 - a_3 = 8 - 4 = 4$
$a_{n+1} - a_n = 4$; for all values of $n$

Therefore, the given list of numbers forms an AP with 4 being its common difference.
The next three terms of the AP are 8 + 4 = 12, 12 + 4 = 16, 16 + 4 = 20
Hence, AP: –4, 0, 4, 8, 12, 16, 20 …

**(b)** 2, 4, 8, 16, …

$a_2 - a_1 = 4 - 2 = 2$
$a_3 - a_2 = 8 - 4 = 4$
$a_3 - a_2 \neq a_2 - a_1$

Therefore, the given list of numbers does not form an AP.

- **The terminology related to arithmetic progression**
  - An arithmetic progression is a list of numbers in which each term is obtained by adding a fixed number to the preceding term except the first term.
  - The fixed number is called the common difference ($d$) of the A.P. The common difference can be either positive or negative or zero.
- **The general form of an A.P.**
  - $a$, $(a + d)$, $(a + 2d)$, $(a + 3d)$, ... , $[a + (n - 1)d]$, ... where *a is the* first term and *d is* common difference
- **Type of AP**
  - Finite AP: The APs have finite number of terms.
  - Infinite AP: The APs have not finite number of terms.

- In an A.P., except the first term, all the terms can be obtained by adding the common difference to the previous term.
- In an A.P., except the last term, all the terms can be obtained by subtracting the common difference from its subsequent term.

**Example:**

Find the first four terms of an A.P. whose first term is 9 and the common difference is 6.

**Solution:**

$a = 9, d = 6$

$a_2 = a + d = 9 + 6 = 15$

$a_3 = a + 2d = 9 + 2 \times 6 = 9 + 12 = 21$

$a_4 = a + 3d = 9 + 3 \times 6 = 9 + 18 = 27$

The first four terms are 9, 15, 21, 27.

- **$n^{th}$ term of an AP**

  The $n^{th}$ term ($a_n$) of an AP with first term $a$ and common difference $d$ is given by $a_n = a + (n - 1) d$. Here, $a_n$ is called the general term of the AP.

- **$n^{th}$ term from the end of an AP**

  The $n^{th}$ term from the end of an AP with last term $l$ and common difference $d$ is given by $l - (n - 1) d$.

  **Example:**
  Find the $12^{th}$ term of the AP 5, 9, 13 …
  **Solution:**
  Here, $a = 5, d = 9 - 5 = 4, n = 12$

  $$\begin{aligned} a_{12} &= a + (n - 1) d \\ &= 5 + (12 - 1) 4 \\ &= 5 + 11 \times 4 \\ &= 5 + 44 \\ &= 49 \end{aligned}$$

- **Sum of $n$ terms of an AP**
  - The sum of the first $n$ terms of an AP is given by $S_n = \frac{n}{2}[2a + (n - 1)d]$ Sn=n22a+n-1d , where $a$ is the first term and $d$ is the common difference.

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  - If there are only $n$ terms in an AP, then $S_n = \frac{n}{2}[a + l]$ Sn=n2a+l, where $l = a_n$ is the last term.

**Example** :

Find the value of 2 + 10 + 18 + …. + 802.

**Solution**:

2, 10, 18… 802 is an AP where $a = 2$, $d = 8$, and $l = 802$.

Let there be $n$ terms in the series. Then,

$a_n = 802$

$\Rightarrow a + (n - 1)\,d = 802$

$\Rightarrow 2 + (n - 1)\,8 = 802$

$\Rightarrow 8(n - 1) = 800$

$\Rightarrow n - 1 = 100$

$\Rightarrow n = 101$

Thus, required sum = $\frac{n}{2}(a + l) = \frac{101}{2}(2 + 802) = 40602$ n2a+l = 10122+802 = 40602

- **Geometric Progression:** A sequence is said to be a geometric progression (G.P.) if the ratio of any term to its preceding term is the same throughout. This constant factor is called the common ratio and it is denoted by $r$.

- In standard form, the G.P. is written as $a$, $ar$, $ar^2$ ... where, $a$ is the first term and $r$ is the common ratio.

- **General Term of a G.P.:** The $n^{th}$ term (or general term) of a G.P. is given by $a_n = ar^{n-1}$

**Example:** Find the number of terms in G.P. 5, 20, 80 ... 5120.

**Solution:** Let the number of terms be $n$.

Here $a = 5$, $r = 4$ and $t_n = 5120$

$n^{th}$ term of G.P. = $ar^{n-1}$

$\therefore 5(4)^{n-1} = 5120$

$\Rightarrow 4^{n-1} = \frac{5120}{5} = 1024$

$\Rightarrow (2)^{2n-2} = (2)^{10}$

$\Rightarrow 2n - 2 = 10$

$\Rightarrow 2n = 12$

$\therefore n = 6$

- **Sum of n Term of a G.P.:** The sum of $n$ terms ($S_n$) of a G.P. is given by

$$S_n = \begin{cases} \frac{a(1-r^n)}{1-r}, & \text{if } r < 1 \quad \text{or} \quad \frac{a(r^n-1)}{r-1}, \text{ if } r > 1 \\ na, & \text{if } r = 1 \end{cases}$$

**Example:** Find the sum of the series 1 + 3 + 9 + 27 + ... to 10 terms.

**Solution:** The sequence 1, 3, 9, 27, ... is a G.P.

Here, $a = 1$, $r = 3$.

Sum of $n$ terms of G.P. = $\frac{a(r^n-1)}{r-1}$ $[r > 1]$

$S_{10} = 1 + 3 + 9 + 27 + \ldots$ to 10 terms

$= \frac{1 \times [(3)^{10} - 1]}{(3-1)}$

$= \frac{59049-1}{2}$

$= \frac{59048}{2}$

$= 29524$

- Three consecutive terms can be taken as $\frac{a}{r}$, $a$, $ar$ ar, a, ar. Here, common ratio is $r$.
- Four consecutive terms can be taken as $\frac{a}{r^3}$, $\frac{a}{r}$, $ar$, $ar^3$ ar3, ar, ar, ar3. Here, common ratio is $r^2$r2.

- **Geometric Mean:** For any two positive numbers $a$ and $b$, we can insert a number G between them such that $a$, $G$, $b$ is $a$ G.P. Such a number i.e., G is called a geometric mean (G.M.) and is given by $G = \sqrt{ab}$

In general, if $G_1, G_2, \ldots, G_n$ be $n$ numbers between positive numbers $a$ and $b$ such that $a, G_1, G_2, \ldots, G_n, b$ is a G.P., then $G_1, G_2, \ldots, G_n$ are given by

$G_1 = ar$, $G_2 = ar^2, \ldots, G_n = ar^n$

Where, $r$ is calculated from the relation $b = ar^{n+1}$ , that is $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$.

**Example:** Insert three geometric means between 2 and 162.
**Solution:**
Let $G_1$, $G_2$, $G_3$ be 3 G.M.'s between 2 and 162.
Therefor 2, $G_1$, $G_2$, $G_3$, 162 are in G.P.
Let $r$ be the common ratio of G.P.
Here, $a = 2$, $b = 162$ and $n = 3$

$$r = \left(\frac{162}{2}\right)^{\frac{1}{3+1}} = (81)^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3$$

$G_1 = ar = 2 \times 3 = 6$
$G_2 = ar^2 = 2 \times (3)^2 = 2 \times 9 = 18$
$G_3 = ar^3 = 2 \times (3)^3 = 2 \times 27 = 54$

Thus, the required three geometric means between 2 and 162 are 6, 18, and 54.

- **Relation between A.M. and G.M.:** Let $A$ and $G$ be the respective A.M. and G.M. of two given positive real numbers $a$ and $b$. Accordingly, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$.

Then, we will always have the following relationship between the A.M. and G.M.: $A \geq G$

- **Sum of *n*-terms of some special series:**
  - Sum of first $n$ natural numbers
    $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
  - Sum of squares of the first $n$ natural numbers
    $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
  - Sum of cubes of the first $n$ natural numbers
    $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

**Example:** Find the sum of $n$ terms of the series whose $n^{\text{th}}$ term is $n(n + 1)(n - 2)$.

**Solution:** It is given that

$$\begin{aligned} a_n &= n(n+1)(n-2) \\ &= n(n^2 + n - 2n - 2) \\ &= n(n^2 - n - 2) \\ &= n^3 - n^2 - 2n \end{aligned}$$

Thus, the sum of $n$ terms is given by

$$S_n = \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k$$

$$= \left[\frac{n(n+1)}{2}\right]^2 - \frac{n(n+1)(2n+1)}{6} - \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} - \frac{2n+1}{3} - 2\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n(n+1)-2(2n+1)-12}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n-4n-2-12}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2-n-14}{6}\right]$$

$$= \frac{n(n+1)(3n^2-n-14)}{12}$$

$$= \frac{n(n+1)(3n^2-7n+6n-14)}{12}$$

$$= \frac{n(n+1)[n(3n-7)+2(3n-7)]}{12}$$

$$= \frac{n(n+1)(n+2)(3n-7)}{12}$$