CBSE Sample Paper-01 (Solved)
Mathematics
Class – XII

Time allowed: 3 hours | Maximum Marks: 100

General Instructions:

a) All questions are compulsory.
b) The question paper consists of 26 questions divided into three sections A, B and C. Section A comprises of 6 questions of one mark each, Section B comprises of 13 questions of four marks each and Section C comprises of 7 questions of six marks each.
c) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
d) Use of calculators is not permitted.

Section A

1. If $A^{-1}=\begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}, B^{-1}=\begin{bmatrix}3 & 0\\ -1 & 6\end{bmatrix}$ find $(AB)^{-1}$.

2. Find the angle between the vectors $\vec{a}=3i+4j$ *and* $\vec{b}=4i+3j$.

3. Without expanding prove that $\begin{vmatrix}1 & a & b+c\\ 1 & b & a+c\\ 1 & c & a+b\end{vmatrix}=0$.

4. Prove that greatest integer function is neither one-one nor onto.

5. If A,B are symmetric matrices of the same order, prove that AB-BA is skew symmetric.

6. An operation * defined on Z^+ is defined as a*b=a-b. Is * a binary operation? Justify.

Section B

7. If $F(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$, show that $F(x)F(y)=F(x+y)$

8. Prove that $\frac{d}{dx}\left(\cos^{-1}\sqrt{\frac{1+x}{2}}\right)=\frac{-1}{2\sqrt{1-x^2}}$.

9. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by f(x)=2x-3 and $g:\mathbb{R}\to\mathbb{R}$ be defined by $g(x)=\frac{x+3}{2}$. Show that $f\circ g=I_R=g\circ f$.

10. Find the point on the curve $y=x^3-11x+5$ at which tangent is y=x-11.

11. Prove that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$

12. Find the vector and Cartesian equation of the plane which passes through the point (5,2,-4) and perpendicular to the line with direction ratios 2,3,-1.

13. The probability of solving a specific problem independently by A and B are ½ and 1/3 respectively. If both try to solve the problem independently, find the probability that

(a) problem is solved (b) exactly one of them solves the problem.

14. Find the values of a and b such that the function defined by:

$$f(x)=\begin{cases}5, x\le 2\\ ax+b, 2<x<10\\ 21, x\ge 10\end{cases}$$ is continuous $\forall$ x.

15. The temperature T of a cooling object drops at a rate which is proportional to the difference T-S where S is the constant temperature of the surrounding medium. Thus, dT/dt=-c(T-S), where c>0. Solve the differential equation, given T(0)=40. Discuss two measures to prevent global warming.

16. If $\vec{a},\vec{b},\vec{c}$ are three vectors such that $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{b}\times\vec{c}=\vec{a}$; prove that $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular and $|\vec{b}|=1$ and $|\vec{c}|=|\vec{a}|$.

17. Integrate $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$.

18. Find the projection vector of 2i+3j-3k along 5j-k.

19. Check if the lines $\frac{x+3}{-3}=\frac{y-1}{1}=\frac{z-5}{5}$ and $\frac{x+1}{-1}=\frac{y-2}{2}=\frac{z-5}{5}$ are coplanar.

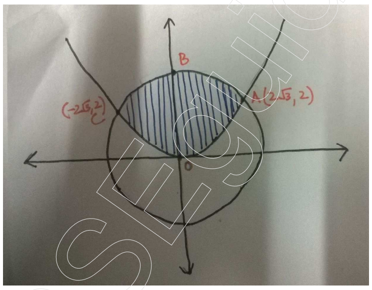
Section C

20. Prove that the semi vertical angle of the cone of maximum volume and given slant height is $\tan^{-1}(\sqrt{2})$.

21. The sum of three numbers is 6. If we multiply the third number by 3 and add second number to it we get 11. By adding first and third numbers we get double of the second number. Represent the following information mathematically and solve using matrices.

22. A factory manufactures two types of machines A and B. Each type is made of certain metal. The factory has only 480 kgs of this metal available in a day. To manufacture machine A, 10 kgs of metal is required and 20kgs is required for B. Machine A and B require 15 and 10 minutes to be painted. Painting department can use only 400 minutes in a day. The factory earns profit of 10,500 on machine A and 9000 on machine B. State as a linear programming problem and maximizes the profit.

23. Sketch the region common to the circle $x^2 + y^2 = 16$ and the parabola $x^2=6y$. Also, find the area of the region using integration.



24. A factory has two machines A and B. Past record shows that the machine A produced 60% of the items of the output and machine B produced 40% of the items of the output. Further, 2% of the items produced by machine A and 1% of the items produced by machine B were defective. One item is chosen at random. What is the probability that it was produced by machine B, given that it was defective?

25. Differentiate $\tan^{-1}\dfrac{2\sqrt{x}}{1-x}$ $w.r.t$ $\sin^{-1}\dfrac{2\sqrt{x}}{1+x}$.

26. $\displaystyle\int_0^{\pi} \frac{x\,dx}{4\cos^2 x + 9\sin^2 x}$

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Section A

1. Solution:

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 3 & 0 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ -7 & 9 \end{bmatrix}$$

2. Solution:

$$\vec{a} = 3i + 4j \text{ and } \vec{b} = 4i + 3j$$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{12+12}{5.5} = \frac{24}{25} \Rightarrow \theta = \cos^{-1}\left(\frac{24}{25}\right)$$

3. Solution:

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & b+a+c \\ 1 & c & c+a+b \end{vmatrix} (C_3 \to C_2 + C_3)$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = 0 (\because C_3 = C_1)$$

4. Solution:

$[2.1] = 2, [2.3] = 2$, thus it is not one-one.

Since it takes only integral values, hence it is not onto also.

5. Solution:

$$Consider\ (AB - BA)' = (AB)' - (BA)' = B'A' - A'B' = BA - AB = -(AB - BA)$$

Thus, AB-BA is skew symmetric.

6. Solution:

The operation * is not a binary operation as 2*3=2-3=-1$\notin \mathbb{Z}^+$.

Section B

7. Solution:

$$F(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix},\ F(y)=\begin{bmatrix}\cos y & -\sin y & 0\\ \sin y & \cos y & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\therefore F(x)F(y)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos y & -\sin y & 0\\ \sin y & \cos y & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\cos x\cos y-\sin x\sin y & -\sin x\cos y-\cos x\sin y & 0\\ \sin x\cos y+\cos x\sin y & \cos x\cos y-\sin x\sin y & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\cos(x+y) & -\sin(x+y) & 0\\ \sin(x+y) & \cos(x+y) & 0\\ 0 & 0 & 1\end{bmatrix}$$

8. Solution:

$$\frac{d}{dx}\left(\cos^{-1}\sqrt{\frac{1+x}{2}}\right)=\frac{-1}{\sqrt{1-\left(\sqrt{\frac{1+x}{2}}\right)^2}}\frac{d}{dx}\sqrt{\frac{1+x}{2}}=\frac{-1}{\sqrt{\frac{1-x}{2}}}\left(\frac{1}{2}\right)\left(\frac{1+x}{2}\right)^{-1/2}\frac{1}{2}$$

$$=\frac{-1}{2\sqrt{(1-x)(1+x)}}=\frac{-1}{2\sqrt{(1-x^2)}}$$

9. Solution:

$$f\circ g(x)=f\left(\frac{x+3}{2}\right)=2\left(\frac{x+3}{2}\right)-3=x=I_R(x)\Rightarrow f\circ g=I_R$$

$$g\circ f(x)=g\left(2x-3\right)=\frac{2x-3+3}{2}=x=I_R(x)\Rightarrow g\circ f=I_R$$

10. Solution:

$$y = x^3 - 11x + 5$$
$$\Rightarrow \frac{dy}{dx} = 3x^2 - 11$$

Also, from equation of tangent we get dy/dx=1

$\therefore 3x^2 - 11 = 1 \Rightarrow x = \pm 2$.

Solving y=x-11 for y we get the possible points are (2,-9),(-2,-13).

But (-2,-13) does not lie on the curve, hence required point is (2,-9).

11. Solution:

$$Let\ x = \cos 2\theta \Rightarrow \theta = \frac{1}{2}\cos^{-1} x$$
$$\therefore \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$
$$Now, \frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}} = \frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}} = \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$$
$$\tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right) = \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \tan^{-1}(\tan(\frac{\pi}{4}-\theta)) = \frac{\pi}{4}-\theta$$
$$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$$

12. Solution:

$\vec{a} = 5i + 2j - 4k, \overline{N} = 2i + 3j - k$

The equation of a plane is given by $(\vec{r} - \vec{a}).\overline{N} = 0$

$\Rightarrow \left[\vec{r} - (5i + 2j - 4k)\right].(2i + 3j - k) = 0$

Transforming into Cartesian form we get, [(x-5)i+(y-2)j+(z+4)k].(2i+3j-k)=0, i.e. 2x+3y-z=20.

13. Solution:

(A) Let A denote the event that problem is solved by A and let B denote the event that problem is solved by B.

$\therefore P(A)=1/2, P(B)=1/3, P(\overline{A})=1-1/2=1/2, P(\overline{B})=2/3$

P(Problem is solved)= 1- P(Problem is not solved)=1- $P(\overline{A}\overline{B})=1-(1/2)(2/3)=2/3$

(b) P(exactly one of them solves the problem)= $P(\overline{A}B or A\overline{B})=(1/2)(2/3)+(1/2)(1/3)=1/2$

14. Solution:

Since f is constant for x<2, x>10, f(x) is continuous for x<2, x>10.

At x=2,

$$\lim_{x\to 2^-}(f(x))=\lim_{x\to 2^-}(5)=5$$
$$\lim_{x\to 2^+}(f(x))=\lim_{x\to 2^-}(ax+b)=2a+b$$
$$\therefore 2a+b=5$$

At x=10,

$$\lim_{x\to 10^-}(f(x))=\lim_{x\to 10^-}(ax+b)=10a+b$$
$$\lim_{x\to 10^+}(f(x))=\lim_{x\to 2^-}(21)=21$$
$$\therefore 10a+b=21$$

Solving the above two equations we get a=2, b=1.

15. Solution:

$$\frac{dT}{dt}=-c(T-S)$$
$$\int\frac{dT}{T-S}=\int -cdt$$
$$\therefore \log(T-S)=-ct+k$$
$$\Rightarrow e^{-ct+k}=T-S$$

Putting the condition T(0)=40, we get $(40-S)e^{-ct}=T-S$.

Measures to control global warming:

(1) Planting more trees
(2) Car pools to prevent emission of carbon dioxide which in turn causes global warming.

16. Solution:

$\because \vec{a}\times\vec{b}=\vec{c}$ *and* $\vec{b}\times\vec{c}=\vec{a}\Rightarrow \vec{a}\perp\vec{c},\vec{b}\perp\vec{c},\ \vec{a}\perp\vec{b},\vec{a}\perp\vec{c}\ \Rightarrow \vec{a},\vec{b},\vec{c}$ are mutually perpendicular.

$$|\vec{a}\times\vec{b}|=|\vec{c}|,|\vec{b}\times\vec{c}|=|\vec{a}|\Rightarrow|\vec{a}||\vec{b}|\sin 90=|\vec{c}|\ and\ |\vec{b}||\vec{c}|\sin 90=|\vec{a}|\left(\because \vec{a},\vec{b},\vec{c}\ are\ mutually\ perpendicular\right)$$

$$\Rightarrow|\vec{a}||\vec{b}|=|\vec{c}|\ and\ |\vec{b}||\vec{c}|=|\vec{a}|$$

$$\Rightarrow|\vec{b}||\vec{c}||\vec{b}|=|\vec{c}|\Rightarrow|\vec{b}|^2|\vec{c}|=|\vec{c}|\Rightarrow|\vec{b}|=1\Rightarrow|\vec{a}|=|\vec{c}|$$

17. Solution:

$$\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx=\int e^x\left(\frac{1}{x}\right)dx-\int e^x\left(\frac{1}{x^2}\right)dx$$

$$=\frac{1}{x}\int e^x dx-\int\left(\frac{d}{dx}\left(\frac{1}{x}\right)\int e^x dx\right)dx-\int e^x\left(\frac{1}{x^2}\right)dx$$

$$=\frac{1}{x}e^x-\int\frac{-1}{x^2}e^x-\int e^x\left(\frac{1}{x^2}\right)dx$$

(using integration by parts)

$$=\frac{1}{x}e^x+\int\frac{1}{x^2}e^x-\int e^x\left(\frac{1}{x^2}\right)dx$$

$$=\frac{1}{x}e^x$$

18. Solution:

Projection vector of $\vec{a}$ along $\vec{b}$ is given by $\left(\frac{\vec{a}.\vec{b}}{|b|^2}\right)\vec{b}$.

$$\vec{a}=2i+3j-3k,\vec{b}=5j-k$$

$$\left(\frac{\vec{a}.\vec{b}}{|b|^2}\right)\vec{b}=\left(\frac{(2i+3j-3k).(5j-k)}{\left(\sqrt{26}\right)^2}\right)5j-k=\frac{9}{13}(5j-k)$$

19. Solution:

$$x_1=-3,y_1=1,z_1=5,x_2=-1,y_2=2,z_2=5$$
$$a_1=-3,b_1=1,c_1=5,a_2=-1,b_2=2,c_2=5,$$

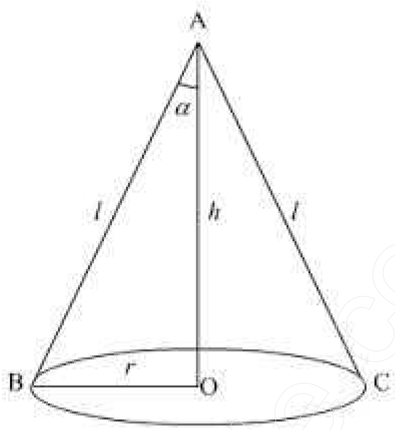
The lines are coplanar iff $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$

Substituting the values, we get the value of determinant as 0.

Hence, the lines are co-planar.

Section C

20. Solution:



Let l,h,r, α denote the slant height, height, radius and semi-vertical angle of the cone respectively.

$$\sin\alpha = \frac{r}{l} \Rightarrow r = l\sin\alpha, \cos\alpha = \frac{h}{l} \Rightarrow h = l\cos\alpha$$

$$Volume = V = \frac{\pi}{3}r^2h = \frac{\pi}{3}l^3\sin^2\alpha\cos\alpha$$

$$\therefore \frac{dV}{d\alpha} = \frac{\pi}{3}l^3[2\sin\alpha\cos^2\alpha + \sin^2\alpha(-\sin\alpha)] = \frac{\pi}{3}l^3[\sin\alpha(2\cos^2\alpha - \sin^2\alpha)]$$

$$\frac{dV}{d\alpha} = 0 \Rightarrow \alpha = \tan^{-1}(\sqrt{2})$$

$$\frac{d^2V}{d\alpha^2} = \frac{\pi}{3}l^3[\cos\alpha(2\cos^2\alpha - \sin^2\alpha) + \sin\alpha(4\cos\alpha(-\sin\alpha) - 2\sin\alpha\cos\alpha)] = \frac{\pi}{3\cos^3\alpha}l^3[2 - 7\tan^2\alpha]$$

$$If\ \tan\alpha = \sqrt{2} \Rightarrow \frac{d^2V}{d\alpha^2} < 0$$

Thus, for maximum volume $\tan\alpha = \sqrt{2}$.

21. Solution:

Let the three numbers be x,y,z. We can formulate the above as the mathematical problem:

$x + y + z = 6$

$y + 3z = 11$

$x - 2y + z = 0$

$$Let\ A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$|A| = 9, A^{-1} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}, X = A^{-1}b = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore x = 1, y = 2, z = 3$

22. Solution:

Suppose the factory produces x units of machine A and y units of machine B.

Then, Profit Z= 10,500x + 9000y

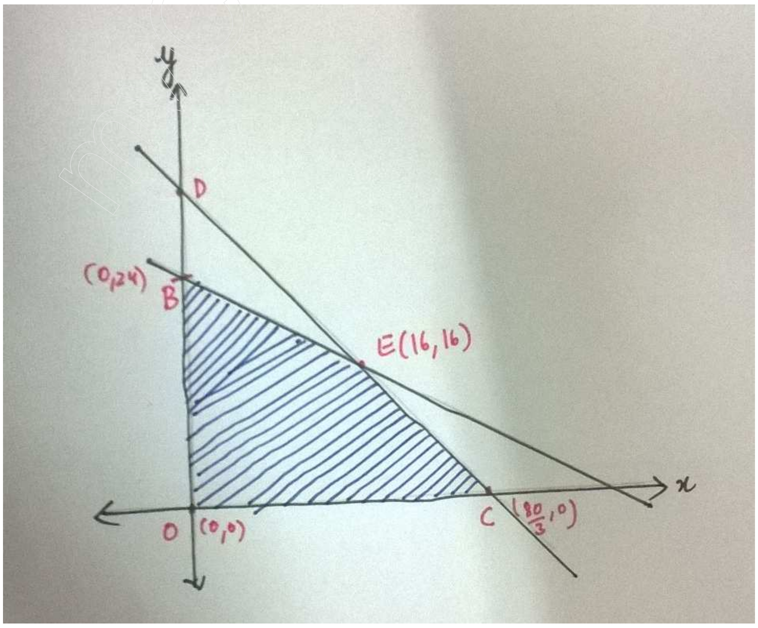
The mathematical formulation of the problem is as follows:

Max Z= 10,500x+9000y

s.t $10x+20y \leq 480$, $x+2y \leq 48$ (metal constraint)

$15x+10y \leq 400$, $3x+2y \leq 80$ (painting constraint)

$x \geq 0$, $y \geq 0$



We graph the above inequalities. The feasible region is as shown in the figure. We observe the feasible region is bounded and the corner points are O,B,E and C. The co-ordinates of the corner points are (0,0), (0,24),(16,16),(80/3,0).

Corner Point	Z=10,500x + 9000y
(0,0)	0
(0,24)	2,16,000
(16,16)	**3,12,000**
(80/3,0)	2,80,000

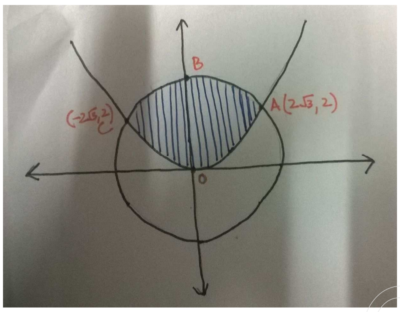
Thus profit is maximized by producing 16 units each of machine A and B.

23. Solution:

The point of intersection of the two curves:

$$x^2 + y^2 = 16, x^2 = 6y$$
$$y^2 \quad 6y \quad 16 \quad 0$$
$$y \quad 2, \; 8$$

Rejecting y=-8, we get $x = \pm 2\sqrt{3}$.



Shaded area= Required area=Ar(OAB)+Ar(OBC)=2 Ar(OAB)

$$Area = 2\int_0^{2\sqrt{3}} (y_1 - y_2)dx = 2\int_0^{2\sqrt{3}} \left(\sqrt{16-x^2} - \frac{x^2}{6}\right)dx$$
$$= 2\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right) - \frac{1}{6}\left(\frac{x^3}{3}\right)\Big|_0^{2\sqrt{3}}\right] = 2\left[2\sqrt{3} + \frac{8\pi}{3} - \frac{4\sqrt{3}}{3}\right] = \frac{4}{3}[4\pi + \sqrt{3}]$$

24. Solution:

Let B_1 denote the event that the item is produced by A. $P(B_1)$=60/100=.6

Let B_2 denote the event that the item is produced by B. $P(B_2)$=40/100=.4

Let E denote the event that the item is defective.

P(B2/E)=? , P(E/B1)=0.02, P(E/B2)=0.01

By Baye's theorem,

$$P(B_2/E)=\frac{P(E/B_2)P(B_2)}{P(E/B_2)P(B_2)+P(E/B_2)P(B_2)}=\frac{(1/100)(40/100)}{(2/100)(60/100)+(1/100)(40/100)}=\frac{1}{4}$$

25. Solution:

$$Let\ y=\tan^{-1}\frac{2\sqrt{x}}{1-x}$$
$$Let\ x=\tan^2\theta$$
$$\therefore y=\tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta}=\tan^{-1}(\tan 2\theta)=2\theta=2\tan^{-1}\sqrt{x}$$
$$\frac{dy}{dx}=2\frac{1}{1+\left(\sqrt{x}\right)^2}\frac{1}{2\sqrt{x}}=\frac{1}{(1+x)\sqrt{x}}$$
$$Let\ z=\sin^{-1}\frac{2\sqrt{x}}{1+x}$$
$$Let\ x=\tan^2\theta$$
$$\therefore z=\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}=\sin^{-1}(\sin 2\theta)=2\theta=2\tan^{-1}\sqrt{x}$$
$$\frac{dz}{dx}=\frac{1}{(1+x)\sqrt{x}}$$
$$\frac{dy}{dz}=\frac{dy}{dx}\div\frac{dz}{dx}=\frac{1}{(1+x)\sqrt{x}}\div\frac{1}{(1+x)\sqrt{x}}=1$$

26. Solution :

$$I=\int_0^{\pi}\frac{xdx}{4\cos^2 x+9\sin^2 x}=\int_0^{\pi}\frac{(\pi-x)dx}{4\cos^2 x+9\sin^2 x}$$

$$\therefore 2I=\pi\int_0^{\pi}\frac{dx}{4\cos^2 x+9\sin^2 x}=2\pi\int_0^{\frac{\pi}{2}}\frac{dx}{4\cos^2 x+9\sin^2 x}$$

$$=2\pi\left[\int_0^{\frac{\pi}{4}}\frac{dx}{4\cos^2 x+9\sin^2 x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{4\cos^2 x+9\sin^2 x}\right]$$

$$=2\pi\left[\int_0^{\frac{\pi}{4}}\frac{\sec^2 xdx}{4+9\tan^2 x}+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos ec^2 xdx}{4\cot^2 x+9}\right]$$

Putting tanx=t and cotx=u, we get

$$2I = 2\pi\left[\int_0^1 \frac{dt}{4+9t^2} - \int_1^0 \frac{du}{4u^2+9}\right] = 2\pi\left[\frac{1}{9}\left(\frac{3}{2}\right)\tan^{-1}\frac{t}{2/3}\Bigg|_0^1 - \frac{1}{4}\left(\frac{2}{3}\right)\tan^{-1}\frac{u}{3/2}\Bigg|_1^0\right]$$

$$= 2\pi\left[\frac{1}{6}\tan^{-1}\left(\frac{3}{2}\right) + \frac{1}{6}\tan^{-1}\left(\frac{2}{3}\right)\right] = \frac{2\pi}{6}\left(\frac{\pi}{2}\right) = \frac{\pi^2}{6}$$

$$\therefore I = \frac{\pi^2}{12}$$