Differentiating one Function w. r. t. Another Function

Q.1. Differentiate $\sin^{-1} x$ w. r. t. $\cos^{-1} \sqrt{(1 - x^2)}$.

Solution : 1

Let $u = \sin^{-1} x$, $v = \cos^{-1} \sqrt{(1 - x^2)}$,then we have to find du/dv.

Now, $du/dx = 1/\sqrt{(1 - x^2)}$,

In $v = \cos^{-1} \sqrt{(1 - x^2)}$, let us put $x = \sin \theta$,

Then $v = \cos^{-1} \sqrt{(1 - \sin 2\theta)} = \cos -1 (\cos \theta) = \theta = \sin^{-1} x$.

Therefore $dv/dx = 1/\sqrt{(1 - x^2)}$,

Hence , du/dv = (du/dx)/(dv/dx)

$= \{1/\sqrt{(1 - x^2)}\}/\{1/\sqrt{(1 - x^2)}\}$

Q.2. Differentiate $\sin^{-1} \{(1 - x)/(1 + x)\}$w. r. t. $\sqrt{x}$.

Solution : 2

Let $u = \sin^{-1} \{(1 - x)/(1 + x)\}$,

Then , $du/dx = 1/\sqrt{[1 - \{(1 - x)/(1 + x)\}2]} . [\{(1 + x)(- 1) - (1 - x)(1)\}/(1 + x)^2]$

$= (1 + x)/\sqrt{\{(1 + x)^2 - (1 - x)^2\}}. (- 2)/(1 + x)^2$

$= (- 1)/\{(1 + x) \sqrt{x}\}$;

And let $v = \sqrt{x}$, $dv/dx = 1/(2\sqrt{x})$;

Therefore , du/dv = (du/dx)/(dv/dx)

$= [(- 1) /\{(1 + x) \sqrt{x}\}]/[1/(2\sqrt{x})] = [(- 1) / \{(1 + x)\sqrt{x}\}] \times [(2\sqrt{x})/1]$

$= - 2 / (1 + x)$.

Q.3. Find dy/dx when $x = \cos^{-1} 1/\sqrt{(1 + t^2)}$, $y = \sin^{-1} t/\sqrt{(1 + t^2)}$

Solution : 3

$x = \cos^{-1} 1/(1 + t^2)$, Put $t = \tan\theta$; then $\theta = \tan^{-1} t$.

Therefore , $x = \cos^{-1} 1/\sqrt{(1 + \tan^2\theta)}$

$= \cos^{-1} (1/\sec\theta)$

$= \cos^{-1} (\cos\theta)$

$= \theta = \tan^{-1} t$.

Hence , $dx/dt = 1/(1 + t^2)$.

Also $y = \sin^{-1} \{t/\sqrt{(1 + t^2)}\}$

$= \sin^{-1} \{\tan\theta/\sqrt{(1 + \tan^2\theta)}\}$

$= \sin^{-1} (\tan\theta/\sec\theta)$

$= \sin^{-1} (\sin\theta) = \theta = \tan^{-1} t$.

Hence , $dy/dt = 1/(1 + t^2)$.

Therefore , $dy/dx = (dy/dt)/(dx/dt)$

$= [\{1/(1 + t^2)\}/\{1/(1 + t^2)\}]$

$= 1$.

Q.4. Differentiate : $\tan^{-1} \{2x/(1 - x^2)\}$ w. r. t. $\sin^{-1} \{2x/(1 + x^2)\}$.

Solution : 4

Let $u = \tan^{-1} \{2x/(1 - x^2)\}$ and $v = \sin^{-1} \{2x/(1 + x^2)\}$;

Then we have to find du/dv.

Now , put $x = \tan\theta$, then $u = \tan^{-1} \{2\tan\theta/(1 - \tan^2\theta)\}$

Or, $u = \tan^{-1} (\tan^{2}\theta) = 2\theta = 2\tan^{-1} x$

Therefore, du/dx = $2/(1 + x^{2})$.

And v = $\sin^{-1} \{2\tan \theta/(1 + \tan^{2}\theta)\}$

Or, v = $\sin^{-1} (\sin 2\theta) = 2\theta = 2\tan^{-1} x$

Therefore, dv/dx = $2/(1 + x^{2})$.

Hence , du/dv = {(du/dx)/(dv/dx)}

= $[\{2/(1 + x^{2})\}/\{2/(1 + x^{2})\}] = 1$.

Q.5. Differentiate $\tan^{-1} x/\sqrt{(1 - x^{2})}$ w. r. t. $\sec^{-1} 1/(2x^{2} - 1)$.

Solution : 5

Let u = $\tan^{-1} x/\sqrt{(1 - x^{2})}$ and v = $\sec^{-1} 1/(2x^{2} - 1)$,

Then we have to find du/dv .

Now , put x = cos θ , then u

= $\tan^{-1} \{\cos\theta/\sqrt{(1 - \cos^{2}\theta)}$

=$\tan^{-1} (\cos\theta/\sin\theta)$

= $\tan^{-1} (\cot\theta)$

= $\tan^{-1} \{\tan (\pi/2 - \theta)\} = \pi/2 - \theta = \pi/2 - \cos -1 x$.

Therefore , du/dx = $-1/\sqrt{(1 - x^{2})}$

And v = $\sec^{-1} \{1/(2 \cos 2\theta - 1)\}$

= $\sec^{-1} (1/\cos 2\theta) = 2\theta = 2 \cos -1 x$

Therefore , dv/dx = $2/\sqrt{(1 - x^{2})}$

Hence , du/dv = {(du/dx)/(dv/dx)}

= $\{- 1/\sqrt{(1 - x^{2}}\}/ \{2/\sqrt{(1 - x^{2})}\} = - 1/2$.

Q.6. Differentiate $\tan^{-1} [2x/(1 - x^{2})]$ w. r. t. $\tan^{-1} x$.

Solution : 6

Let u = tan $^{-1}$ [2x/(1 – x^2)] and v = tan $^{-1}$ x , the we have to find du/dv .

Putting x = tan θ in these we get u = tan −1 [2 tanθ/(1 – $tan^2θ$)]

= tan $^{-1}$ [$ta^{n2}θ$] = 2θ

And v = tan $^{-1}$ (tanθ) = θ

Then du/dθ = 2 and dv/dθ = 1 => du/dv = (du/dθ)/(dv/dθ) = 2/1 = 2 .

Q.7. Differentiate sin $^{-1}$ [2x/(1 + x^{-1})] w. r. t. tan $^{-1}$ x.

Solution : 7

Let u = sin $^{-1}$ [2x/(1 + x^2)] and v = tan $^{-1}$ x then we have to find du/dv .

Putting x = tan θ in these we get u = sin^{-1} [2tanθ/(1 + $tan^2θ$)]

= sin $^{-1}$ [$sin^2θ$] = 2θ

And v = tan^{-1} (tanθ) = θ

Then du/dθ = 2 and dv/dθ = 1 => du/dv = (du/dθ)/(dv/dθ) = 2/1 = 2 .

Q.8. Find the derivative of sin x^2 with respect to x^3.

Solution : 8

Let u = sin x^2 and v = x^3,

du/dx = d/dx[sin x^2] = 2x cos x^2 ; dv/dx = d/dx[x^3] = 3x^2,

du/dv = (du/dx)/(dv/dx) = (2x cos x^2)/(3x^2) = (2 cos x^2)/(3x)