Chapter 5

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

❖*Mathematics is the Queen of Sciences and Arithmetic is the Queen of Mathematics. – GAUSS* ❖

5.1 Introduction

In earlier classes, we have studied linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation $x^2 + 1 = 0$ has no real solution as $x^2 + 1 = 0$ gives $x^2 = -1$ and square of every real number is non-negative. So, we need to extend the real number system to a larger system so that we can find the solution of the equation $x^2 = -1$. In fact, the main objective is to solve the equation $ax^2 + bx + c = 0$, where $D = b^2 - 4ac < 0$, which is not possible in the system of real numbers.

W. R. Hamilton (1805-1865)

5.2 Complex Numbers

Let us denote $\sqrt{-1}$ by the symbol i. Then, we have $i^2 = -1$. This means that i is a solution of the equation $x^2 + 1 = 0$.

A number of the form $a + ib$, where a and b are real numbers, is defined to be a complex number. For example, $2 + i3$, $(-1) + i\sqrt{3}$, $4 + i\left(\frac{-1}{11}\right)$ are complex numbers.

For the complex number $z = a + ib$, a is called the *real part*, denoted by Re z and b is called the *imaginary part* denoted by Im z *of the complex number z*. For example, if $z = 2 + i5$, then Re $z = 2$ and Im $z = 5$.

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.

Example 1 If $4x + i(3x - y) = 3 + i(-6)$, where x and y are real numbers, then find the values of x and y.

Solution We have

$$4x + i(3x - y) = 3 + i(-6) \quad \dots (1)$$

Equating the real and the imaginary parts of (1), we get

$$4x = 3,\ 3x - y = -6,$$

which, on solving simultaneously, give $x = \frac{3}{4}$ and $y = \frac{33}{4}$.

5.3 Algebra of Complex Numbers

In this Section, we shall develop the algebra of complex numbers.

5.3.1 ***Addition of two complex numbers*** Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the sum $z_1 + z_2$ is defined as follows:

$z_1 + z_2 = (a + c) + i(b + d)$, which is again a complex number.

For example, $(2 + i3) + (-6 + i5) = (2 - 6) + i(3 + 5) = -4 + i\,8$

The addition of complex numbers satisfy the following properties:

(i) *The closure law* The sum of two complex numbers is a complex number, i.e., $z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) *The commutative law* For any two complex numbers z_1 and z_2, $z_1 + z_2 = z_2 + z_1$

(iii) *The associative law* For any three complex numbers z_1, z_2, z_3, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

(iv) *The existence of additive identity* There exists the complex number $0 + i\,0$ (denoted as 0), called the *additive identity* or the *zero complex number,* such that, for every complex number z, $z + 0 = z$.

(v) *The existence of additive inverse* To every complex number $z = a + ib$, we have the complex number $-a + i(-b)$ (denoted as $-z$), called the *additive inverse* or *negative of* z. We observe that $z + (-z) = 0$ (the additive identity).

5.3.2 ***Difference of two complex numbers*** Given any two complex numbers z_1 and z_2, the difference $z_1 - z_2$ is defined as follows:

$$z_1 - z_2 = z_1 + (-z_2).$$

For example, $(6 + 3i) - (2 - i) = (6 + 3i) + (-2 + i) = 4 + 4i$

and $(2 - i) - (6 + 3i) = (2 - i) + (-6 - 3i) = -4 - 4i$

5.3.3 ***Multiplication of two complex numbers*** Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the product $z_1 z_2$ is defined as follows:

$z_1 z_2 = (ac - bd) + i(ad + bc)$

For example, $(3 + i5)(2 + i6) = (3 \times 2 - 5 \times 6) + i(3 \times 6 + 5 \times 2) = -24 + i28$

The multiplication of complex numbers possesses the following properties, which we state without proofs.

(i) **The closure law** The product of two complex numbers is a complex number, the product $z_1 z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) **The commutative law** For any two complex numbers z_1 and z_2,

$$z_1 z_2 = z_2 z_1.$$

(iii) **The associative law** For any three complex numbers z_1, z_2, z_3,

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

(iv) **The existence of multiplicative identity** There exists the complex number $1 + i\,0$ (denoted as 1), called the *multiplicative identity* such that $z.1 = z$, for every complex number z.

(v) **The existence of multiplicative inverse** For every non-zero complex number $z = a + ib$ or $a + bi (a \neq 0, b \neq 0)$, we have the complex number

$\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$ (denoted by $\frac{1}{z}$ or z^{-1}), called the *multiplicative inverse* of z such that

$z.\frac{1}{z} = 1$ (the multiplicative identity).

(vi) **The distributive law** For any three complex numbers z_1, z_2, z_3,

(a) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

(b) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

5.3.4 ***Division of two complex numbers*** Given any two complex numbers z_1 and z_2, where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined by

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$$

For example, let $z_1 = 6 + 3i$ and $z_2 = 2 - i$

Then $$\frac{z_1}{z_2} = \left((6+3i) \times \frac{1}{2-i}\right) = (6+3i)\left(\frac{2}{2^2+(-1)^2} + i\frac{-(-1)}{2^2+(-1)^2}\right)$$

$$= (6+3i)\left(\frac{2+i}{5}\right) = \frac{1}{5}\left[12-3+i(6+6)\right] = \frac{1}{5}(9+12i)$$

5.3.5 *Power of i* we know that

$$i^3 = i^2 i = (-1)\, i = -i\,, \qquad i^4 = \left(i^2\right)^2 = (-1)^2 = 1$$

$$i^5 = \left(i^2\right)^2 i = (-1)^2\, i = i\,, \qquad i^6 = \left(i^2\right)^3 = (-1)^3 = -1\text{, etc.}$$

Also, we have $\quad i^{-1} = \frac{1}{i}\times\frac{i}{i} = \frac{i}{-1} = -i, \qquad i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{-i}\times\frac{i}{i} = \frac{i}{1} = i, \qquad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

In general, for any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

5.3.6 *The square roots of a negative real number*

Note that $i^2 = -1$ and $(-i)^2 = i^2 = -1$

Therefore, the square roots of – 1 are $i, -i$. However, by the symbol $\sqrt{-1}$, we would mean i only.

Now, we can see that i and $-i$ both are the solutions of the equation $x^2 + 1 = 0$ or $x^2 = -1$.

Similarly $\qquad \left(\sqrt{3}\,i\right)^2 = \left(\sqrt{3}\right)^2 i^2 = 3\,(-1) = -3$

$$\left(-\sqrt{3}\,i\right)^2 = \left(-\sqrt{3}\right)^2 i^2 = -3$$

Therefore, the square roots of –3 are $\sqrt{3}\, i$ and $-\sqrt{3}\, i$.

Again, the symbol $\sqrt{-3}$ is meant to represent $\sqrt{3}\, i$ only, i.e., $\sqrt{-3} = \sqrt{3}\, i$.

Generally, if a is a positive real number, $\sqrt{-a} = \sqrt{a}\,\sqrt{-1} = \sqrt{a}\, i$,

We already know that $\sqrt{a}\times\sqrt{b} = \sqrt{ab}$ for all positive real number a and b. This result also holds true when either $a > 0, b < 0$ or $a < 0, b > 0$. What if $a < 0, b < 0$? Let us examine.

Note that

$i^2 = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)\,(-1)}$ (by assuming $\sqrt{a}\times\sqrt{b} = \sqrt{ab}$ for all real numbers)

$= \sqrt{1} = 1$, which is a contradiction to the fact that $i^2 = -1$.

Therefore, $\sqrt{a}\times\sqrt{b} \neq \sqrt{ab}$ if both a and b are negative real numbers.

Further, if any of a and b is zero, then, clearly, $\sqrt{a}\times\sqrt{b} = \sqrt{ab} = 0$.

5.3.7 *Identities* We prove the following identity

$(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2$, for all complex numbers z_1 and z_2.

Proof We have, $(z_1 + z_2)^2 = (z_1 + z_2)\,(z_1 + z_2)$,

$= (z_1 + z_2)\,z_1 + (z_1 + z_2)\,z_2$ (Distributive law)

$= z_1^2 + z_2z_1 + z_1z_2 + z_2^2$ (Distributive law)

$= z_1^2 + z_1z_2 + z_1z_2 + z_2^2$ (Commutative law of multiplication)

$= z_1^2 + 2z_1z_2 + z_2^2$

Similarly, we can prove the following identities:

(i) $(z_1 - z_2)^2 = z_1^2 - 2\,z_1\,z_2 + z_2^2$

(ii) $(z_1 + z_2)^3 = z_1^3 + 3\,z_1^2 z_2 + 3z_1z_2^2 + z_2^3$

(iii) $(z_1 - z_2)^3 = z_1^3 - 3z_1^2 z_2 + 3z_1z_2^2 - z_2^3$

(iv) $z_1^2 - z_2^2 = (z_1 + z_2)\,(z_1 - z_2)$

In fact, many other identities which are true for all real numbers, can be proved to be true for all complex numbers.

Example 2 Express the following in the form of $a + bi$:

(i) $(-5i)\left(\frac{1}{8}i\right)$ (ii) $(-i)\,(2i)\left(-\frac{1}{8}i\right)^3$

Solution (i) $(-5i)\left(\frac{1}{8}i\right) = \frac{-5}{8}i^2 = \frac{-5}{8}(-1) = \frac{5}{8} = \frac{5}{8} + i0$

(ii) $(-i)\,(2i)\left(-\frac{1}{8}i\right)^3 = 2\times\frac{1}{8\times8\times8}\times i^5 = \frac{1}{256}\left(i^2\right)^2\,i = \frac{1}{256}i$.

Example 3 Express $(5 - 3i)^3$ in the form $a + ib$.

Solution We have, $(5 - 3i)^3 = 5^3 - 3 \times 5^2 \times (3i) + 3 \times 5\,(3i)^2 - (3i)^3$

$$= 125 - 225i - 135 + 27i = -10 - 198i.$$

Example 4 Express $\left(-\sqrt{3}+\sqrt{-2}\right)\left(2\sqrt{3}-i\right)$ in the form of $a + ib$

Solution We have, $\left(-\sqrt{3}+\sqrt{-2}\right)\left(2\sqrt{3}-i\right) = \left(-\sqrt{3}+\sqrt{2}\,i\right)\left(2\sqrt{3}-i\right)$

$$= -6+\sqrt{3}i+2\sqrt{6}i-\sqrt{2}\,i^2 = \left(-6+\sqrt{2}\right)+\sqrt{3}\left(1+2\sqrt{2}\right)i$$

5.4 The Modulus and the Conjugate of a Complex Number

Let $z = a + ib$ be a complex number. Then, the modulus of z, denoted by $|z|$, is defined to be the non-negative real number $\sqrt{a^2+b^2}$, i.e., $|z| = \sqrt{a^2+b^2}$ and the conjugate of z, denoted as $\bar{z}$, is the complex number $a - ib$, i.e., $\bar{z} = a - ib$.

For example, $|3+i| = \sqrt{3^2+1^2} = \sqrt{10}$, $|2-5i| = \sqrt{2^2+(-5)^2} = \sqrt{29}$,

and $\overline{3+i} = 3-i$, $\overline{2-5i} = 2+5i$, $\overline{-3i-5} = 3i-5$

Observe that the multiplicative inverse of the non-zero complex number z is given by

$$z^{-1} = \frac{1}{a+ib} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2} = \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$$

or $$z\,\bar{z} = |z|^2$$

Furthermore, the following results can easily be derived.

For any two compex numbers z_1 and z_2, we have

(i) $|z_1 z_2| = |z_1||z_2|$ (ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ provided $|z_2| \neq 0$

(iii) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$ (iv) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ (v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ provided $z_2 \neq 0$.

Example 5 Find the multiplicative inverse of $2 - 3i$.

Solution Let $z = 2 - 3i$

Then $\bar{z} = 2 + 3i$ and $|z|^2 = 2^2 + (-3)^2 = 13$

Therefore, the multiplicative inverse of $2 - 3i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

The above working can be reproduced in the following manner also,

$$z^{-1} = \frac{1}{2-3i} = \frac{2+3i}{(2-3i)(2+3i)}$$

$$= \frac{2+3i}{2^2-(3i)^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

Example 6 Express the following in the form $a + ib$

(i) $\dfrac{5+\sqrt{2}i}{1-\sqrt{2}i}$ (ii) i^{-35}

Solution (i) We have, $\dfrac{5+\sqrt{2}i}{1-\sqrt{2}i} = \dfrac{5+\sqrt{2}i}{1-\sqrt{2}i} \times \dfrac{1+\sqrt{2}i}{1+\sqrt{2}i} = \dfrac{5+5\sqrt{2}i+\sqrt{2}i-2}{1-\left(\sqrt{2}i\right)^2}$

$$= \frac{3+6\sqrt{2}i}{1+2} = \frac{3(1+2\sqrt{2}i)}{3} = 1+2\sqrt{2}i.$$

(ii) $i^{-35} = \dfrac{1}{i^{35}} = \dfrac{1}{\left(i^2\right)^{17} i} = \dfrac{1}{-i} \times \dfrac{i}{i} = \dfrac{i}{-i^2} = i$

EXERCISE 5.1

Express each of the complex number given in the Exercises 1 to 10 in the form $a + ib$.

1. $(5i)\left(-\frac{3}{5}i\right)$ **2.** $i^9 + i^{19}$ **3.** i^{-39}

4. $3(7 + i7) + i\,(7 + i7)$ **5.** $(1 - i) - (-1 + i6)$

6. $\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$ **7.** $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$

8. $(1 - i)^4$ **9.** $\left(\frac{1}{3}+3i\right)^3$ **10.** $\left(-2-\frac{1}{3}i\right)^3$

Find the multiplicative inverse of each of the complex numbers given in the Exercises 11 to 13.

11. $4 - 3i$ **12.** $\sqrt{5}+3i$ **13.** $-i$

14. Express the following expression in the form of $a + ib$:

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}\,i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

5.5 Argand Plane and Polar Representation

We already know that corresponding to each ordered pair of real numbers (x, y), we get a unique point in the XY-plane and vice-versa with reference to a set of mutually perpendicular lines known as the x-axis and the y-axis. The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point $P(x, y)$ in the XY-plane and vice-versa.

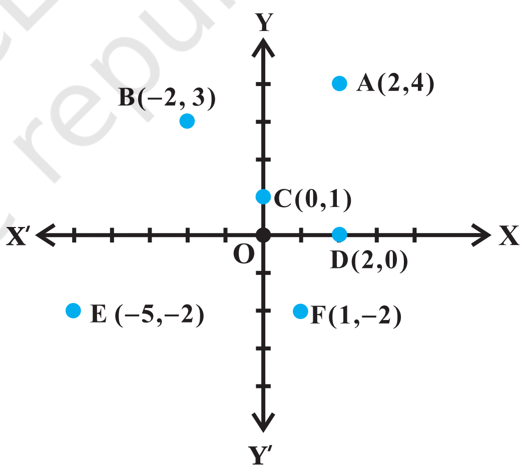

Fig 5.1

Some complex numbers such as $2 + 4i, -2 + 3i, 0 + 1i, 2 + 0i, -5 - 2i$ and $1 - 2i$ which correspond to the ordered pairs (2, 4), (– 2, 3), (0, 1), (2, 0), (–5, –2), and (1, – 2), respectively, have been represented geometrically by the points A, B, C, D, E, and F, respectively in the Fig 5.1.

The plane having a complex number assigned to each of its point is called the *complex plane* or the *Argand plane*.

Obviously, in the Argand plane, the modulus of the complex number $x + iy = \sqrt{x^2 + y^2}$ is the distance between the point P(x, y) and the origin O (0, 0) (Fig 5.2). The points on the x-axis corresponds to the complex numbers of the form $a + i\,0$ and the points on the y-axis corresponds to the complex numbers of the form

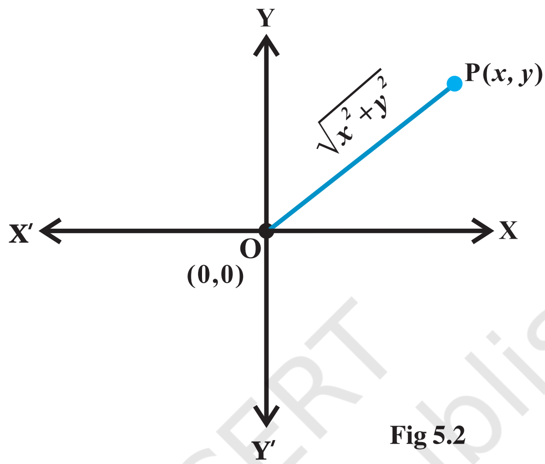

Fig 5.2

$0 + i\,b$. The x-axis and y-axis in the Argand plane are called, respectively, the *real axis* and the *imaginary axis*.

The representation of a complex number $z = x + iy$ and its conjugate $z = x - iy$ in the Argand plane are, respectively, the points P (x, y) and Q (x, $-y$).

Geometrically, the point (x, $-y$) is the mirror image of the point (x, y) on the real axis (Fig 5.3).

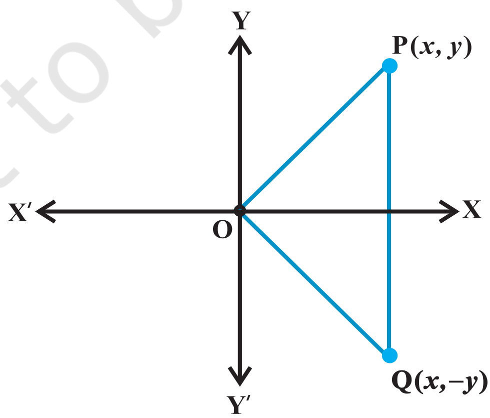

Fig 5.3

5.5.1 ***Polar representation of a complex number*** Let the point P represent the non-zero complex number $z = x + iy$. Let the directed line segment OP be of length r and θ be the angle which OP makes with the positive direction of x-axis (Fig 5.4).

We may note that the point P is uniquely determined by the ordered pair of real numbers (r, θ), called the *polar coordinates of the point* P. We consider the origin as the pole and the positive direction of the x axis as the initial line.

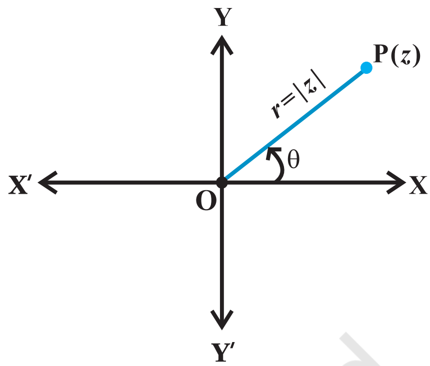

Fig 5.4

We have, $x = r \cos \theta$, $y = r \sin \theta$ and therefore, $z = r (\cos \theta + i \sin \theta)$. The latter is said to be the *polar form of the complex number*. Here $r = \sqrt{x^2 + y^2} = |z|$ is the modulus of z and θ is called the argument (or amplitude) of z which is denoted by arg z.

For any complex number $z \neq 0$, there corresponds only one value of θ in $0 \leq \theta < 2\pi$. However, any other interval of length 2π, for example $-\pi < \theta \leq \pi$, can be such an interval. We shall take the value of θ such that $-\pi < \theta \leq \pi$, called **principal argument** of z and is denoted by arg z, unless specified otherwise. (Figs. 5.5 and 5.6)

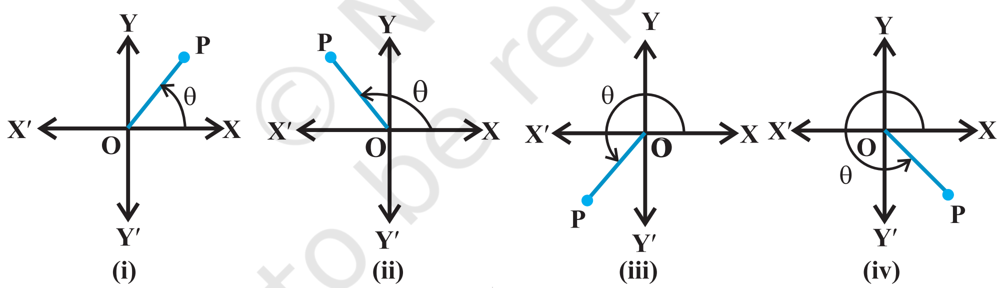

Fig 5.5 $(0 \leq \theta < 2\pi)$

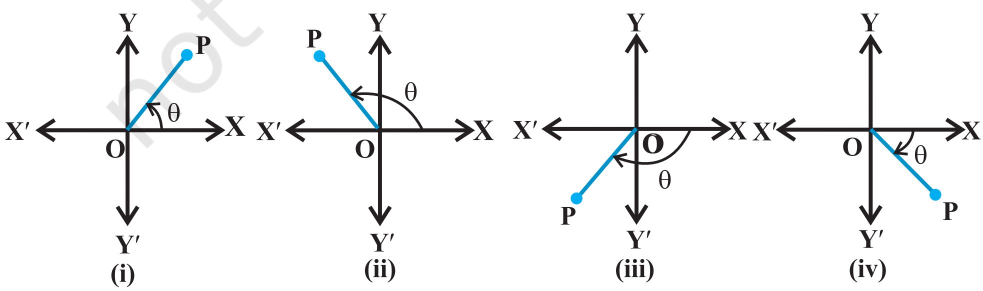

Fig 5.6 $(-\pi < \theta \leq \pi)$

Example 7 Represent the complex number $z = 1 + i\sqrt{3}$ in the polar form.

Solution Let $1 = r\cos\theta$, $\sqrt{3} = r\sin\theta$

By squaring and adding, we get

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = 4$$

i.e., $r = \sqrt{4} = 2$ (conventionally, $r > 0$)

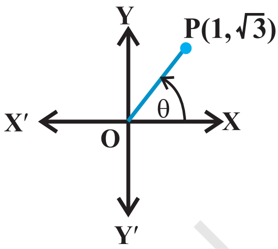

Fig 5.7

Therefore, $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$, which gives $\theta = \frac{\pi}{3}$

Therefore, required polar form is $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

The complex number $z = 1 + i\sqrt{3}$ is represented as shown in Fig 5.7.

Example 8 Convert the complex number $\frac{-16}{1 + i\sqrt{3}}$ into polar form.

Solution The given complex number $\frac{-16}{1+i\sqrt{3}} = \frac{-16}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$

$$= \frac{-16\left(1-i\sqrt{3}\right)}{1-\left(i\sqrt{3}\right)^2} = \frac{-16\left(1-i\sqrt{3}\right)}{1+3} = -4\left(1-i\sqrt{3}\right) = -4 + i4\sqrt{3} \text{ (Fig 5.8).}$$

Let $-4 = r\cos\theta$, $4\sqrt{3} = r\sin\theta$

By squaring and adding, we get

$$16 + 48 = r^2\left(\cos^2\theta + \sin^2\theta\right)$$

which gives $r^2 = 64$, i.e., $r = 8$

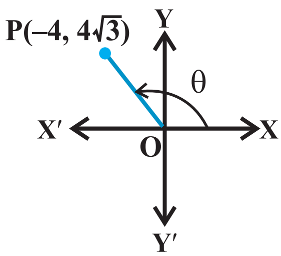

Fig 5.8

Hence $\cos\theta = -\frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Thus, the required polar form is $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

EXERCISE 5.2

Find the modulus and the arguments of each of the complex numbers in Exercises 1 to 2.

1. $z = -1 - i\sqrt{3}$ **2.** $z = -\sqrt{3} + i$

Convert each of the complex numbers given in Exercises 3 to 8 in the polar form:

3. $1 - i$ **4.** $-1 + i$ **5.** $-1 - i$

6. -3 **7.** $\sqrt{3} + i$ **8.** i

5.6 Quadratic Equations

We are already familiar with the quadratic equations and have solved them in the set of real numbers in the cases where discriminant is non-negative, i.e., ≥ 0,

Let us consider the following quadratic equation:

$ax^2 + bx + c = 0$ with real coefficients a, b, c and $a \neq 0$.

Also, let us assume that the $b^2 - 4ac < 0$.

Now, we know that we can find the square root of negative real numbers in the set of complex numbers. Therefore, the solutions to the above equation are available in the set of complex numbers which are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{4ac - b^2}\, i}{2a}$$

Note At this point of time, some would be interested to know as to how many roots does an equation have? In this regard, the following theorem known as the *Fundamental theorem of Algebra* is stated below (without proof).

"A polynomial equation has at least one root."

As a consequence of this theorem, the following result, which is of immense importance, is arrived at:

"A polynomial equation of degree n has n roots."

Example 9 Solve $x^2 + 2 = 0$

Solution We have, $x^2 + 2 = 0$

or $x^2 = -2$ i.e., $x = \pm\sqrt{-2} = \pm\sqrt{2}\, i$

Example 10 Solve $x^2 + x + 1 = 0$

Solution Here, $b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = 1 - 4 = -3$

Therefore, the solutions are given by $x = \dfrac{-1 \pm \sqrt{-3}}{2 \times 1} = \dfrac{-1 \pm \sqrt{3}i}{2}$

Example 11 Solve $\sqrt{5}x^2 + x + \sqrt{5} = 0$

Solution Here, the discriminant of the equation is

$$1^2 - 4 \times \sqrt{5} \times \sqrt{5} = 1 - 20 = -19$$

Therefore, the solutions are

$$\frac{-1 \pm \sqrt{-19}}{2\sqrt{5}} = \frac{-1 \pm \sqrt{19}i}{2\sqrt{5}}.$$

EXERCISE 5.3

Solve each of the following equations:

1. $x^2 + 3 = 0$ **2.** $2x^2 + x + 1 = 0$ **3.** $x^2 + 3x + 9 = 0$

4. $-x^2 + x - 2 = 0$ **5.** $x^2 + 3x + 5 = 0$ **6.** $x^2 - x + 2 = 0$

7. $\sqrt{2}x^2 + x + \sqrt{2} = 0$ **8.** $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

9. $x^2 + x + \dfrac{1}{\sqrt{2}} = 0$ **10.** $x^2 + \dfrac{x}{\sqrt{2}} + 1 = 0$

Miscellaneous Examples

Example 12 Find the conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.

Solution We have , $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

$$= \frac{6+9i-4i+6}{2-i+4i+2} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i}$$

$$= \frac{48-36i+20i+15}{16+9} = \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i$$

Therefore, conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ is $\dfrac{63}{25} + \dfrac{16}{25}i$.

Example 13 Find the modulus and argument of the complex numbers:

(i) $\frac{1+i}{1-i}$, (ii) $\frac{1}{1+i}$

Solution (i) We have, $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{1-1+2i}{1+1} = i = 0 + i$

Now, let us put $0 = r \cos \theta$, $1 = r \sin \theta$

Squaring and adding, $r^2 = 1$ i.e., $r = 1$ so that

$$\cos \theta = 0, \ \sin \theta = 1$$

Therefore, $\theta = \frac{\pi}{2}$

Hence, the modulus of $\frac{1+i}{1-i}$ is 1 and the argument is $\frac{\pi}{2}$.

(ii) We have $\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{1+1} = \frac{1}{2} - \frac{i}{2}$

Let $\frac{1}{2} = r \cos \theta, -\frac{1}{2} = r \sin \theta$

Proceeding as in part (i) above, we get $r = \frac{1}{\sqrt{2}}; \ \cos\theta = \frac{1}{\sqrt{2}}, \ \sin\theta = \frac{-1}{\sqrt{2}}$

Therefore $\theta = \frac{-\pi}{4}$

Hence, the modulus of $\frac{1}{1+i}$ is $\frac{1}{\sqrt{2}}$, argument is $\frac{-\pi}{4}$.

Example 14 If $x + iy = \frac{a+ib}{a-ib}$, prove that $x^2 + y^2 = 1$.

Solution We have,

$$x + iy = \frac{(a+ib)(a+ib)}{(a-ib)(a+ib)} = \frac{a^2-b^2+2abi}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2} + \frac{2ab}{a^2+b^2}i$$

So that, $x - iy = \frac{a^2-b^2}{a^2+b^2} - \frac{2ab}{a^2+b^2}i$

Therefore,

$$x^2 + y^2 = (x + iy)(x - iy) = \frac{(a^2-b^2)^2}{(a^2+b^2)^2} + \frac{4a^2b^2}{(a^2+b^2)^2} = \frac{(a^2+b^2)^2}{(a^2+b^2)^2} = 1$$

Example 15 Find real θ such that

$$\frac{3+2i\sin\theta}{1-2i\sin\theta} \text{ is purely real.}$$

Solution We have,

$$\frac{3+2i\sin\theta}{1-2i\sin\theta} = \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$$

$$= \frac{3+6i\sin\theta+2i\sin\theta-4\sin^2\theta}{1+4\sin^2\theta} = \frac{3-4\sin^2\theta}{1+4\sin^2\theta} + \frac{8i\ \sin\theta}{1+4\sin^2\theta}$$

We are given the complex number to be real. Therefore

$$\frac{8\sin\theta}{1+4\sin^2\theta} = 0, \text{ i.e., } \sin\theta = 0$$

Thus $\theta = n\pi,\ n \in \mathbf{Z}$.

Example 16 Convert the complex number $z = \frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ in the polar form.

Solution We have, $z = \frac{i-1}{\frac{1}{2} + \frac{\sqrt{3}}{2}i}$

$$= \frac{2(i-1)}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{2\left(i+\sqrt{3}-1+\sqrt{3}i\right)}{1+3} = \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2}i$$

Now, put $\frac{\sqrt{3}-1}{2} = r\cos\theta,\ \frac{\sqrt{3}+1}{2} = r\sin\theta$

Squaring and adding, we obtain

$$r^2 = \left(\frac{\sqrt{3}-1}{2}\right)^2 + \left(\frac{\sqrt{3}+1}{2}\right)^2 = \frac{2\left(\left(\sqrt{3}\right)^2+1\right)}{4} = \frac{2\times 4}{4} = 2$$

Hence, $r = \sqrt{2}$ which gives $\cos\theta = \frac{\sqrt{3}-1}{2\sqrt{2}}, \quad \sin\theta = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Therefore, $\theta = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$ (Why?)

Hence, the polar form is

$$\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

Miscellaneous Exercise on Chapter 5

1. Evaluate: $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$.

2. For any two complex numbers z_1 and z_2, prove that
Re $(z_1 z_2)$ = Re z_1 Re z_2 – Imz_1 Imz_2.

3. Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form .

4. If $x - iy = \sqrt{\frac{a-ib}{c-id}}$ prove that $\left(x^2+y^2\right)^2 = \frac{a^2+b^2}{c^2+d^2}$.

5. Convert the following in the polar form:

(i) $\frac{1+7i}{(2-i)^2}$, (ii) $\frac{1+3i}{1-2i}$

Solve each of the equation in Exercises 6 to 9.

6. $3x^2 - 4x + \frac{20}{3} = 0$ **7.** $x^2 - 2x + \frac{3}{2} = 0$

8. $27x^2 - 10x + 1 = 0$

9. $21x^2 - 28x + 10 = 0$

10. If $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$.

11. If $a + ib = \dfrac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$.

12. Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find

(i) $\text{Re}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$, (ii) $\text{Im}\left(\dfrac{1}{z_1 \bar{z}_1}\right)$.

13. Find the modulus and argument of the complex number $\dfrac{1+2i}{1-3i}$.

14. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

15. Find the modulus of $\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}$.

16. If $(x + iy)^3 = u + iv$, then show that $\dfrac{u}{x} + \dfrac{v}{y} = 4(x^2 - y^2)$.

17. If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\dfrac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$.

18. Find the number of non-zero integral solutions of the equation $|1 - i|^x = 2^x$.

19. If $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, then show that

$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$

20. If $\left(\dfrac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m.

Summary

- A number of the form $a + ib$, where a and b are real numbers, is called a *complex number*, a is called the *real part* and b is called the *imaginary* part of the complex number.
- Let $z_1 = a + ib$ and $z_2 = c + id$. Then
  (i) $z_1 + z_2 = (a + c) + i\,(b + d)$
  (ii) $z_1\, z_2 = (ac - bd) + i\,(ad + bc)$
- For any non-zero complex number $z = a + ib$ $(a \neq 0, b \neq 0)$, there exists the complex number $\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$, denoted by $\frac{1}{z}$ or z^{-1}, called the *multiplicative inverse* of z such that $(a + ib)\left(\frac{a^2}{a^2+b^2} + i\frac{-b}{a^2+b^2}\right) = 1 + i0 = 1$
- For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$
- The conjugate of the complex number $z = a + ib$, denoted by $\bar{z}$, is given by $\bar{z} = a - ib$.
- The polar form of the complex number $z = x + iy$ is $r\,(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$ (the modulus of z) and $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$. (θ is known as the argument of z. The value of θ, such that $-\pi < \theta \leq \pi$, is called the *principal argument* of z.
- A polynomial equation of n degree has n roots.
- The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in R$, $a \neq 0$, $b^2 - 4ac < 0$, are given by $x = \frac{-b \pm \sqrt{4ac - b^2}\,i}{2a}$.

Historical Note

The fact that square root of a negative number does not exist in the real number system was recognised by the Greeks. But the credit goes to the Indian mathematician *Mahavira* (850) who first stated this difficulty clearly. "He mentions in his work '*Ganitasara Sangraha*' as in the nature of things a negative (quantity) is not a square (quantity)', it has, therefore, no square root". *Bhaskara*, another Indian mathematician, also writes in his work *Bijaganita*, written in 1150. "There is no square root of a negative quantity, for it is not a square." *Cardan* (1545) considered the problem of solving

$$x + y = 10, xy = 40.$$

He obtained $x = 5 + \sqrt{-15}$ and $y = 5 - \sqrt{-15}$ as the solution of it, which was discarded by him by saying that these numbers are 'useless'. *Albert Girard* (about 1625) accepted square root of negative numbers and said that this will enable us to get as many roots as the degree of the polynomial equation. *Euler* was the first to introduce the symbol i for $\sqrt{-1}$ and *W.R. Hamilton* (about 1830) regarded the complex number $a + ib$ as an ordered pair of real numbers (a, b) thus giving it a purely mathematical definition and avoiding use of the so called '*imaginary numbers*'.