EXERCISE 5.7

Find the second order derivatives of the following functions given in Exercises 1 to 10.

Q No. 1 $\frac{dy}{dx} = \frac{d}{dx}(x^2+3x+2) = 2x+3$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(2x+3) = 2$$

Q No. 2 $\frac{d^2}{dx^2}(x^{20}) = \frac{d}{dx}\left(\frac{d}{dx}(x^{20})\right) = \frac{d}{dx}(20x^{19}) = 20\frac{d}{dx}(x^{19})$

$$= 20\times 19x^{18} = 380x^{18}$$

Q No. 3 Let $y = x\cos x$.

$$\frac{dy}{dx} = x\cdot(-\sin x) + \cos x.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-x\sin x + \cos x\right)$$

$$= -\left[x\cdot\cos x + \sin x\cdot 1\right] - \sin x. = -x\cos x - 2\sin x.$$

Q No. 4 $y = \log x$

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}. \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Q No. 5 $y = x^3\log x$

$$\frac{dy}{dx} = x^3\cdot\frac{1}{x} + \log x\cdot 3x^2 = x^2\left[1+3\log x\right]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(x^2(1+3\log x)\right) = x^2\cdot\frac{d}{dx}(1+3\log x) + (1+3\log x)\frac{d}{dx}(x^2)$$

$$= x^2\left(0+\frac{3}{x}\right) + (1+3\log x)\cdot 2x = 3x+2x+6x\log x$$

$$= x\left[5+6\log x\right]$$

QNo. 6: $y = e^x \sin 5x$

$$\therefore \frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\sin 5x) + \sin 5x \cdot \frac{d}{dx}(e^x)$$

$$= e^x(5 \cdot \cos 5x) + \sin 5x \cdot e^x = e^x(5\cos 5x + \sin 5x)$$

$$\therefore \frac{d^2y}{dx^2} = e^x \cdot \frac{d}{dx}(5\cos 5x + \sin 5x) + (5\cos 5x + \sin 5x)\frac{d}{dx}(e^x)$$

$$= e^x\left[-25\sin 5x + \cos 5x \cdot 5\right] + \left[5\cos 5x + \sin 5x\right] \cdot e^x$$

$$= e^x\left[-25\sin 5x + 5\cos 5x + 5\cos 5x + \sin 5x\right]$$

$$= e^x\left[10\cos 5x - 24\sin 5x\right]$$

QNo. 7: $y = e^{6x}\cos 3x$.

$$\frac{dy}{dx} = e^{6x} \cdot \frac{d}{dx}(\cos 3x) + \cos 3x \cdot \frac{d}{dx}(e^{6x})$$

$$= e^{6x}(-3\sin 3x) + \cos 3x \cdot e^{6x} \cdot 6$$

$$= e^{6x}\left[-3\sin 3x + 6\cos 3x\right]$$

$$\therefore \frac{d^2y}{dx^2} = e^{6x}\frac{d}{dx}\left[-3\sin 3x + 6\cos 3x\right] + \left[-3\sin 3x + \cos 3x \times 6\right]\frac{d}{dx}e^{6x}$$

$$= e^{6x}\left[-9\cos 3x - 18\sin 3x\right] + \left[-3\sin 3x + 6\cos 3x\right] \cdot e^{6x} \cdot 6$$

$$= e^{6x}\left[-9\cos 3x - 18\sin 3x - 18\sin 3x + 36\cos 3x\right]$$

$$= e^{6x}\left[27\cos 3x - 36\sin 3x\right]$$

QNo. 8: $y = \tan^{-1}x$.

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(1+x^2)(0) - 1 \cdot (0+2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$

Q.No 9 Let $y = \log(\log x)$

$$\frac{dy}{dx} = \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x \log x}$$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x \log x}\right) = \frac{(x \log x)(0) - 1 \cdot \left(x \cdot \frac{1}{x} + \log x \cdot 1\right)}{x^2 (\log x)^2}$$

$$= \frac{-(1 + \log x)}{(x \log x)^2}$$

Q.No 10. $y = \sin(\log x)$

$$\frac{dy}{dx} = \cos(\log x) \cdot \frac{1}{x} = \frac{\cos(\log x)}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{x \cdot \left(-\sin(\log x) \cdot \frac{1}{x}\right) - \cos(\log x) \cdot 1}{x^2}$$

$$= \frac{-\left[\sin(\log x) + \cos(\log x)\right]}{x^2}.$$

Q.No 11. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

$$\frac{dy}{dx} = -5\sin x - 3\cos x$$

$$\frac{d^2y}{dx^2} = -5\cos x - 3(-\sin x) = -5\cos x + 3\sin x. = -y$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

Q.No 12 If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Sol:

$$y = \cos^{-1} x \implies x = \cos y.$$

$$\Rightarrow \frac{dx}{dy} = -\sin y. \quad \text{or} \quad \frac{dy}{dx} = -\frac{1}{\sin y} = -\operatorname{cosec} y. \quad -(1)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(-\operatorname{cosec} y) = -(-\operatorname{cosec} y \cdot \cot y)\frac{dy}{dx} = \operatorname{cosec} y \cdot \cot y\,(-\operatorname{cosec} y)$$

$$= -\operatorname{cosec} y \cdot \cot y \quad \left(\text{using } (1)\right)$$

Q No. 13. If $y = 3\cos(\log x) + 4\sin(\log x)$ Show that

$$x^2 y_2 + x y_1 + y = 0$$

$$\therefore \frac{dy}{dx} = -3\sin(\log x)\cdot\frac{1}{x} + 4\cos(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow \quad x y_1 = -3\sin(\log x) + 4.\cos(\log x)$$

Again differentiating both sides w.r.t x.

$$x \cdot y_2 + y_1 \cdot 1 = -3\cos(\log x)\cdot\frac{1}{x} - 4\sin(\log x)\cdot\frac{1}{x}.$$

$$\Rightarrow x^2 y_2 + x y_1 = -\left[3\cos(\log x) + 4\sin(\log x)\right]$$

$$\Rightarrow x^2 y_2 + x y_1 = -y. \Rightarrow x^2 y_2 + x y_1 + y = 0.$$

Q No 14. If $y = Ae^{mx} + Be^{nx}$, show that

$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$$

Sol.. $$\frac{dy}{dx} = Ae^{mx}.m + Be^{nx}.n. \quad \cdots\cdots(1)$$

$$\Rightarrow \frac{d^2y}{dx^2} = A.m\,e^{mx}.m + B.n\,e^{nx}.n$$

$$= m^2 A e^{mx} + n^2 B e^{nx} \quad \cdots\cdots(2)$$

and $$y = Ae^{mx} + Be^{nx} \quad \cdots\cdots(3)$$

Using (1), (2), (3) we get

$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = m^2Ae^{mx} + n^2Be^{nx} - m^2Ae^{mx}$$
$$- mnBe^{nx} - mnAe^{mx} - n^2Be^{nx}$$
$$+ mnAe^{mx} + mnBe^{nx}$$

$$= 0$$

Hence proved.

Q No 15 If $y = 500e^{7x} + 600e^{-7x}$, Show that $\frac{d^2y}{dx^2} = 49y$.

Given $y = 500e^{7x} + 600e^{-7x}$

$$\frac{dy}{dx} = 500e^{7x}.7 + 600e^{-7x}(-7x)$$

$$\frac{d^2y}{dx^2} = 3500e^{7x}.7 - 4200e^{-7x}(-7)$$

$$= 49(500e^{7x} + 600e^{-7x}) = 49y.$$

$$\therefore \frac{d^2y}{dx^2} = 49y.$$

Q No 16 If $e^y(x+1) = 1$ Show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

Sol

$$e^y = \frac{1}{x+1} \Rightarrow e^y\frac{dy}{dx} = -\frac{1}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{e^y}\left(-\frac{1}{(x+1)^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x+1}{(x+1)^2} = -\frac{1}{x+1} \quad \left[\because e^y = \frac{1}{x+1}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left(-\frac{1}{(x+1)^2}\right) = \left(\frac{1}{(x+1)^2}\right) = \left(-\frac{1}{x+1}\right)^2 = \left(\frac{dy}{dx}\right)^2$$

$$\therefore \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2.$$

Q No 17 If $y = (\tan^{-1}x)^2$ show that $(x^2+1)^2y_2 + 2x(x^2+1)y_1 = 2$

$$\Rightarrow \frac{dy}{dx} = 2\tan^{-1}x \cdot \frac{1}{1+x^2} \quad \text{or} \quad (1+x^2)y_1 = 2\cdot\tan^{-1}x.$$

$$\Rightarrow (1+x^2)y_2 + y_1(0+2x) = \frac{2}{1+x^2}$$

$$\Rightarrow (1+x^2)^2y_2 + 2x(1+x^2)y_1 = 2$$

Hence proved.

#