Sample Question Paper - 13
Mathematics (041)
Class- XII, Session: 2021-22
TERM II

Time Allowed: 2 hours **Maximum Marks: 40**

General Instructions:

1. This question paper contains three sections – A, B and C. Each part is compulsory.
2. Section - A has 6 short answer type (SA1) questions of 2 marks each.
3. Section – B has 4 short answer type (SA2) questions of 3 marks each.
4. Section - C has 4 long answer-type questions (LA) of 4 marks each.
5. There is an internal choice in some of the questions.
6. Q 14 is a case-based problem having 2 sub-parts of 2 marks each.

Section - A

[2 Marks each]

1. Find $\int \frac{1}{\cos^2 x(1-\tan x)^2} dx$

OR

Evaluate $\int_{-1}^{2} \frac{|x|}{x} dx.$

2. Show that the function $y = ax + 2a^2$ is a solution of the differential equation $2\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) - y = 0.$

3. If $\hat{a}, \hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors, then find value of $|2\hat{a} + \hat{b} + \hat{c}|$.

4. A line passes through the point with position vector $2\hat{i} - 3\hat{j} + 4\hat{k}$ and makes angles 60°, 120°, and 45° with x, y and z-axis respectively. Find the equation of the line in the Cartesian form.

5. Find the probability distribution of X, the number of heads in a simultaneous toss of two coins.

6. The probability that it will rain on any particular day is 50%. Find the probability that it rains only on first 4 days of the week.

Section - B

[3 Marks each]

7. Evaluate: $\int_{0}^{\frac{\pi}{4}} \log[1 + \tan x] dx.$ AI

8. Solve $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ subject to the initial condition $y(0) = 0$.

OR

Find the particular solution of the following differential equation :

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1;\ y = 0 \text{ when } x = 0$$

9. Find the area of the parallelogram whose diagonals are represented by the vectors $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$.

OR

Find λ and μ if $(\hat{i} + 3\hat{j} + 9\hat{k}) \times (3\hat{i} - \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.

10. Find the value of λ, so that the lines $\frac{1-x}{3} = \frac{7y-14}{\lambda} = \frac{z-3}{2}$ and $\frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not.

Section - C

[4 Marks each]

11. Find: $\int \frac{\sin x}{\sin^3 x + \cos^3 x} dx$.

12. Find the area of the region bounded by the curve $x^2 = 4y$ and the straight-line $x = 4y - 2$

OR

Area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$

13. Find the shortest distance between the lines $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ and $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$ **AI**

If the lines intersect find their point of intersection.

Case-Based/Data Based

14. Board exam are near by, so Mr. Sharma decided to check the preparation of the few weak students in the class. He chooses four students A, B, C and D then a problem in mathematics is given to those four students A, B, C, D. Their chances of solving the problem, respectively, are $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ and $\frac{2}{3}$. Based on the given information answer the following questions. What is the probability that:

(i) the problem will be solved? **[2]**

(ii) at most one of them solve the problem? **[2]**

Solution

MATHEMATICS 041

Class 12 - Mathematics

Section - A

1. $I = \int \frac{1}{\cos^2 x(1-\tan x)^2} dx$

Put, $1 - \tan x = y$

So that, $-\sec^2 x\, dx = dy$ **1**

$I = \int \frac{-1\, dy}{y^2} = -\int y^{-2} dy$

$= +\frac{1}{y} + c = \frac{1}{1-\tan x} + c$ **1**

OR

Let $I = \int_{-1}^{2} \frac{|x|}{x} dx$

Since, $\frac{|x|}{x} = \begin{cases} \frac{-x}{x}, & x < 0 \\ \frac{x}{x}, & x > 0 \end{cases}$

$= \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$ **1**

$\therefore$ $I = \int_{-1}^{0} (-1) dx + \int_{0}^{2} (1) dx$

$= [-x]_{-1}^{0} + [x]_{0}^{2}$

$= -[0-(-1)] + (2-0)$

$= -1 + 2$ **1**

$= 1$

2. $y = ax + 2a^2 \Rightarrow \frac{dy}{dx} = a$

From L.H.S. of differential equation

$= 2\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) - y$ **1**

$= 2(a)^2 + x(a) - (ax + 2a^2) = 0$

$=$ R.H.S. **1**

3. Given $\hat{a}, \hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors, *i.e.*,

$\hat{a}.\hat{b} = \hat{b}.\hat{c} = \hat{c}.\hat{a} = 0.$ ...(i)

and $|\hat{a}| = |\hat{b}| = |\hat{c}| = 1$ ...(ii)

Now,

$|2\hat{a}+\hat{b}+\hat{c}|^2 = (2\hat{a}+\hat{b}+\hat{c}).(2\hat{a}+\hat{b}+\hat{c})$

$= 4(\hat{a}.\hat{a}) + 2(\hat{a}.\hat{b}) + 2(\hat{a}.\hat{c})$

$+2(\hat{b}.\hat{a}) + (\hat{b}.\hat{b}) + (\hat{b}.\hat{c})$

$+2(\hat{c}.\hat{a}) + (\hat{c}.\hat{b}) + (\hat{c}.\hat{c})$ **1**

[$\because$ Dot product is distributive over addition]

$= 4(|\hat{a}|^2) + 2(0) + 2(0)$

$+2(0) + |\hat{b}|^2 + (0)$

$+2(0) + (0) + |\hat{c}|^2$

$\because \vec{a}.\vec{a} = |\vec{a}|^2$

$= 4(1) + 1 + 1 = 6$

$\therefore |2\hat{a}+\hat{b}+\hat{c}| = \sqrt{6}$ **1**

4. D-Cosines of line are $\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}$ ½

Equation of line is :

$\frac{x-2}{\frac{1}{2}} = \frac{y+3}{\frac{-1}{2}} = \frac{z-4}{\frac{1}{\sqrt{2}}}$ **1**

or $2x - 4 = -2y - 6 = \sqrt{2}(z-4)$ ½

5. Let X be the number of heads.

Possible values of X are 0, 1, 2

$P(X = 0) = \frac{1}{4}, \; P(X = 1) = \frac{1}{2}, \; P(X = 2) = \frac{1}{4}.$ **1**

The probability distribution of X is :

X	0	1	2
P(X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

1

6. P(rain on any particular day)

$$= 50\% = \frac{50}{100} = \frac{1}{2} \quad \mathbf{1}$$

P(rain on first four days of week)

$$= \left(\frac{1}{2}\right)^4 \left(1 - \frac{1}{2}\right)^3$$

$$= \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^7 = \frac{1}{128} \quad \mathbf{1}$$

Commonly Made Error

- Some students forget to take the probability for not raining on the remaining three days and give the answer as $\frac{1}{16}$.

Answering Tip

- Read the question twice to sort out the hidden elements in it.

Section - B

7. $$I = \int_0^{\frac{\pi}{4}} \log[1 + \tan x]dx \quad ...(i)$$

Apply the property $\int_0^a f(x) = \int_0^a f(a-x)$ **1**

$$I = \int_0^{\frac{\pi}{4}} \log\left[1 + \frac{1-\tan x}{1+\tan x}\right]dx$$

or $$I = \int_0^{\frac{\pi}{4}} \log\left[\frac{2}{1+\tan x}\right]dx$$

or $$I = \int_0^{\frac{\pi}{4}} [\log 2 - \log(1+\tan x)]dx \quad ...(ii)$$ **1**

Adding eqn. (i) and (ii), we get

or $$2I = \int_0^{\frac{\pi}{4}} \log 2dx = \frac{\pi}{4}\log 2$$

So, $$I = \frac{\pi}{8}\log 2.$$ **1**

8. Given differential equation can be written as:

$$\frac{dy}{dx} + \frac{2x}{1+x^2} \cdot y = \frac{4x^2}{1+x^2},$$

Comparing with

$$\frac{dy}{dx} + Py = Q,$$

$$\Rightarrow \quad P = \frac{2x}{1+x^2}, \; Q = \frac{4x^2}{1+x^2} \quad ½$$

I.F. (Integrating factor)

$$= e^{\int Pdx} = e^{\int \frac{2x}{1+x^2}dx}$$

$$= e^{\log(1+x^2)} = 1 + x^2 \quad ½$$

$\therefore$ General solution is :

$$y(1 + x^2) = \int \frac{4x^2}{1+x^2} \cdot (1+x^2)dx + C \quad \mathbf{1}$$

or $$y \cdot (1 + x^2) = \frac{4x^3}{3} + C \quad ½$$

Putting $x = 0$ and $y = 0$, we get $C = 0$

$\therefore$ Solution is:

$$y = \frac{4x^3}{3(1+x^2)} \quad ½$$

OR

Given equation can be written as

$$\int \frac{dy}{2e^{-y} - 1} = \int \frac{dx}{x+1}$$

$$\Rightarrow \quad \int \frac{e^y}{2 - e^y}dy = \int \frac{dx}{x+1} \quad ½$$

$$\Rightarrow -\log|2 - e^y| + \log c = \log|x+1| \quad \mathbf{1}$$

$$\Rightarrow \quad (2 - e^y)(x+1) = c$$

When $x = 0, \; y = 0 \Rightarrow c = 1$ **1**

$\therefore$ The required solution is $(2 - e^y)(x+1) = 1$ ½

9. The vector equation for diagonals are $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$

Now, $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= -2\hat{i} + 4\hat{j} + 4\hat{k} \quad \mathbf{1}$$

$$|\vec{a} \times \vec{b}| = \sqrt{4+16+16} = 6 \quad \mathbf{1}$$

Area of the parallelogram

$$= \frac{|\vec{a} \times \vec{b}|}{2} = 3 \text{ sq. units.} \quad \mathbf{1}$$

Commonly Made Error

- Mostly students use the formula to find the area of parallelogram when sides are given.

Answering Tip

- Clarify the concept of finding area of parallelogram whose diagonals are vectors.

OR

$$(\hat{i}+3\hat{j}+9\hat{k})\times(3\hat{i}-\lambda\hat{j}+\mu\hat{k}) = 0$$

or $$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 3 & 9\\ 3 & -\lambda & \mu\end{vmatrix} = 0$$

or $\hat{i}(3\mu+9\lambda)-\hat{j}(\mu-27)+\hat{k}(-\lambda-9) = 0$ **1**

or $3\mu+9\lambda = 0$ ...(i)

or $\mu-27 = 0$ ...(ii)

or $-\lambda-9 = 0$ ...(iii)

1

From eqn. (ii) and (iii),

$\mu = 27$

and $\lambda = -9$ **1**

10. Given lines are :

$$\frac{x-1}{-3}=\frac{y-2}{\left(\frac{\lambda}{7}\right)}=\frac{z-3}{2} \text{ and } \frac{x-1}{\left(\frac{-3\lambda}{7}\right)}=\frac{y-5}{1}=\frac{z-6}{-5}$$

½

As lines are perpendicular,

$$(-3)\left(\frac{-3\lambda}{7}\right)+\left(\frac{\lambda}{7}\right)(1)+2(-5)=0 \Rightarrow \lambda=7$$

½

So, lines are

$$\frac{x-1}{-3}=\frac{y-2}{1}=\frac{x-3}{2} \text{ and } \frac{x-1}{-3}=\frac{y-5}{1}=\frac{z-6}{-5}$$

½

Consider

$$\Delta=\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$$

$$=\begin{vmatrix}0 & 3 & 3\\ -3 & 1 & 2\\ -3 & 1 & -5\end{vmatrix}=-63$$ **1**

Since, as $\Delta \neq 0 \Rightarrow$ lines are not intersecting. ½

Section - C

11. Let

$$I=\int\frac{\sin x}{\sin^3 x+\cos^3 x}dx$$

(On dividing Nr. and Dr. by $\cos^3 x$)

$$=\int\frac{\tan x\sec^2 x}{\tan^3 x+1}dx$$

On substituting $\tan x = t$ and $\sec^2 x\, dx = dt$, we get

$$I=\int\frac{t}{t^3+1}dt$$ **1**

$$=\int\frac{t}{(t+1)(t^2-t+1)}dt$$

$$=-\frac{1}{3}\int\frac{1}{t+1}dt+\frac{1}{3}\int\frac{t+1}{t^2-t+1}dt$$

$$=-\frac{1}{3}\log|t+1|+\frac{1}{6}\int\frac{(2t-1)+3}{t^2-t+1}dt$$

$$=-\frac{1}{3}\log|t+1|+\frac{1}{6}\int\frac{2t-1}{t^2-t+1}dt+\frac{1}{2}\int\frac{1}{t^2-t+1}dt$$ **1**

$$=-\frac{1}{3}\log|t+1|+\frac{1}{6}\log|t^2-t+1|+\frac{1}{2}\int\frac{1}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt$$

$$=-\frac{1}{3}\log|t+1|+\frac{1}{6}\log|t^2-t+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)$$

$$=-\frac{1}{3}\log|\tan x+1|+\frac{1}{6}\log|\tan^2 x-\tan x+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right)+C$$ **2**

12. As $x^2 = 4y$ and $x = 4y-2$

So, $x^2 = x+2$

$x^2-x-2 = 0$

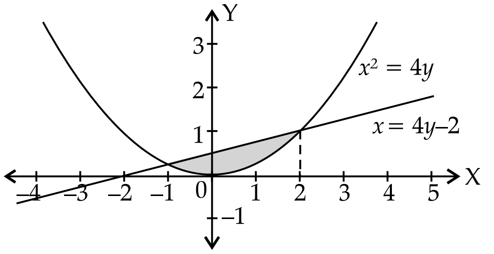
$(x-2)(x+1) = 0$

$x = -1, 2$

For $x=-1$, $y=\frac{1}{4}$ and for $x = 2$, $y = 1$ **1**

Points of intersection are $(-1, \frac{1}{4})$ and $(2, 1)$.

Graphs of parabola $x^2 = 4y$ and $x = 4y - 2$ are shown in the following figure:



1

$$A = \int_{-1}^{2}\left[\frac{x+2}{4} - \frac{x^2}{4}\right]dx$$

$$= \frac{1}{4}\left[\left(\frac{(x+2)^2}{2}\right) - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\frac{16}{3} - \frac{5}{6}\right]$$

$$= \frac{9}{8} \text{ sq. units}$$

$$= 1\frac{1}{8} \text{ sq. units}$$ **2**

OR

We have $y = 0$, $y = x$ and the circle $x^2 + y^2 = 32$ in the first quadrant.

Solving $y = x$ with the circle

$$x^2 + x^2 = 32$$

$$x^2 = 16$$

$$x = 4 \quad \text{(In the first quadrant)}$$

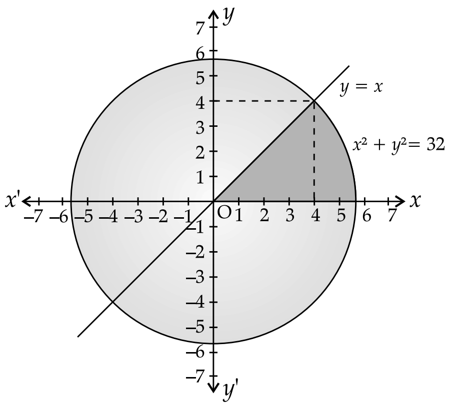
When $x = 4$, $y = 4$ for the point of intersection of the circle with the x-axis.

Put $y = 0$ in circle

$$x^2 + 0 = 32$$

$$x = \pm 4\sqrt{2}$$

So, the circle intersects the x-axis at $(\pm 4\sqrt{2}, 0)$. **1**



1

From the above figure, area of the shaded region,

$$A = \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 +$$

$$\left[\frac{x}{2}\sqrt{(4\sqrt{2})^2 - x^2} + \frac{(4\sqrt{2})^2}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left[\frac{16}{2}\right] + \left[\begin{array}{l} 0 + 16\sin^{-1}1 - \frac{4}{2}\sqrt{(4\sqrt{2})^2 - 16} \\ -16\sin^{-1}\frac{4}{4\sqrt{2}} \end{array}\right]$$

$$= 8 + \left[\frac{16\pi}{2} - 2\sqrt{16} - 16\frac{\pi}{4}\right]$$

$$= 8 + [8\pi - 8 - 4\pi]$$

$$= 4\pi \text{ sq. units}$$ **2**

13. We have

$$a_1 = 3\hat{i} + 2\hat{j} - 4\hat{k}$$

$$b_1 = \hat{i} + 2\hat{j} + 2\hat{k}$$

$$a_2 = 5\hat{i} - 2\hat{j}$$

$$b_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\overrightarrow{a_2} - \overrightarrow{a_1} = 2\hat{i} - 4\hat{j} + 4\hat{k}$$ ½

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 3 & 2 & 6 \end{vmatrix}$$

$$= \hat{i}(12-4) - \hat{j}(6-6) + \hat{k}(2-6)$$

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = 8\hat{i} + 0\hat{j} - 4\hat{k} = 8\hat{i} - 4\hat{k}$$ **1**

$$\because (\overrightarrow{b_1} \times \overrightarrow{b_2}).(\overrightarrow{a_2} - \overrightarrow{a_1}) = 16 - 16 = 0$$ ½

$\therefore$ The lines are intersecting and the shortest distance between the lines is 0.

Now for point of intersection

$$3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k}) = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

$$\Rightarrow \quad 3 + \lambda = 5 + 3\mu \quad ...(i)$$

$$2 + 2\lambda = -2 + 2\mu \quad ...(ii)$$

$$-4 + 2\lambda = 6\mu$$ **1**

Solving (i) and (ii) we get $\mu = -2$ and $\lambda = -4$

Substituting in equation of line we get

$$\vec{r} = 5i - 2j + (-2)(3\hat{i} + 2\hat{j} - 6\hat{k})$$
$$= -\hat{i} - 6\hat{j} + 12\hat{k}$$

Point of intersection is (–1, –6, 12) **1**

Case-Based/Data Based

14. Let

E be the event = A solves the problem

F be the event = B solves the problem

G be the event = C solves the problem

H be the event = D solves the problem

$$P(E) = \frac{1}{3} \Rightarrow P(\bar{E}) = \frac{2}{3}$$
$$P(F) = \frac{1}{4} \Rightarrow P(\bar{F}) = \frac{3}{4}$$
$$P(G) = \frac{1}{5} \Rightarrow P(\bar{G}) = \frac{4}{5}$$
$$P(H) = \frac{2}{3} \Rightarrow P(\bar{H}) = \frac{1}{3}$$

(i) The required probability

$$= P(E \cup F \cup G \cup H)$$
$$= 1 - P(\bar{E}) \times P(\bar{F}) \times P(\bar{G}) \times P(\bar{H})$$
$$= 1 - \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{1}{3}$$
$$= \frac{13}{15}$$ **2**

(ii) The required probability

$$= P(\bar{E}) \times P(\bar{F}) \times P(\bar{G}) \times P(\bar{H}) + P(E) \times P(\bar{F}) \times P(\bar{G}) \times P(\bar{H}) + P(\bar{E}) \times P(F) \times P(\bar{G}) \times P(\bar{H}) + P(\bar{E}) \times P(\bar{F}) \times P(G) \times P(\bar{H}) + P(\bar{E}) \times P(\bar{F}) \times P(\bar{G}) \times P(H)$$
$$= \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{1}{3} + \frac{1}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{1}{3} + \frac{2}{3} \times \frac{1}{4} \times \frac{4}{5} \times \frac{1}{3} + \frac{2}{3} \times \frac{3}{4} \times \frac{1}{5} \times \frac{1}{3} + \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{2}{3}$$
$$= \frac{2}{15} + \frac{1}{15} + \frac{2}{45} + \frac{1}{30} + \frac{4}{15}$$
$$= \frac{7}{15} + \frac{1}{30} + \frac{2}{45}$$
$$= \frac{42 + 3 + 4}{90} = \frac{49}{90}$$ **2**

❑❑