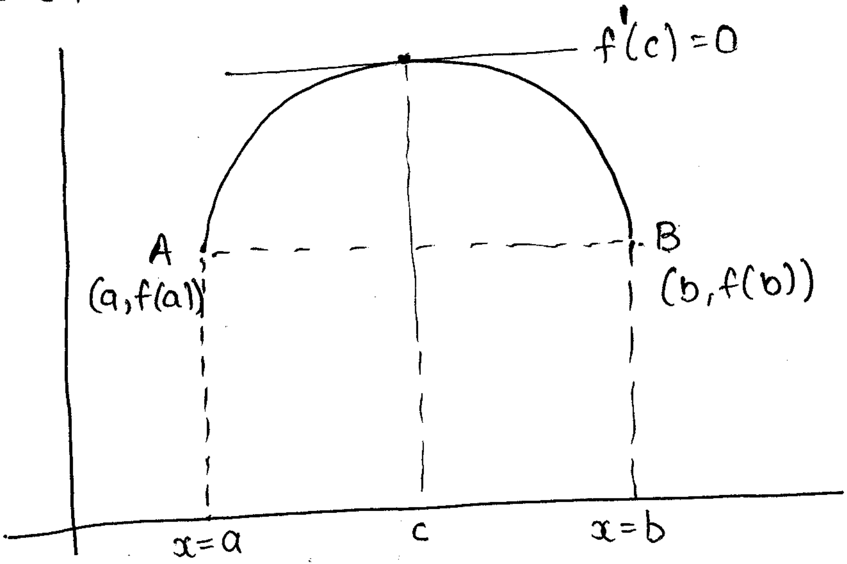
* Mean value Theorem

① Rolle's mean value theorem

(i) function $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is differentiable in (a,b)

(iii) $f(a)=f(b)$ then $\exists$ a $\cdot$ C [at least one] $\in (a,b)$ such that $f'(c)=0$.



$f'(c)$ = Slope of the tangent at point c
= Slope of $\overline{AB}$
(when lines are parallel slopes are equal).

$$f'(c) = \frac{y_2-y_1}{x_2-x_1}$$

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

$$f'(c) = 0$$

NOTE:

Every polynomial function is continuous and differentiable at $\forall x$.

Q: $f(x) = x^3(1-x)^2$ in $[0, 1]$. Hence find C.

Sol^n:- Rolle's M.V.T. ① st & ② nd condition is satisfied

③ rd

$\cancel{f'(x) = 3x^2 - 5x^4}$

$$f(x) = x^3(1 - 2x + x^2)$$

$$f(x) = x^5 - 2x^4 + x^3$$

$$f'(x) = 5x^4 - 8x^3 + 3x^2$$

$$f'(x) = f'(c) = 0$$

$$5x^4 - 8x^3 + 3x^2 = 0$$

$$5x^2 - 8x + 3 = 0 \qquad x = 0, 0$$

$$x = \frac{+8 \pm \sqrt{64 - 60}}{2 \cdot 5}$$

$$x = \frac{+8 \pm 2}{10} = 1, \frac{3}{5}$$

Total $\boxed{c = 3/5}$ $\because c \in (a, b)$

$\because c \in (0, 1)$

Q:- $f(x) = x^3 - 4x$ in $[-2, 2]$ find C?

$$f'(x) = 3x^2 - 4$$

$$f'(c) = 0$$

$$\therefore 3x^2 - 4 = 0$$

$$3x^2 = 4$$

$$x = \pm 2/\sqrt{3}$$

$\because$ $\boxed{c = \pm 2/\sqrt{3}}$

Q :- $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$; Find c

(a) $\frac{5\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\pi/2$ (d) $\pi/4$

Sol^n:- $f(x) = \frac{\sin x}{e^x}$

$f(x) = e^{-x} \cdot \sin x$

$f'(x) = -e^{-x} \cdot \sin x + e^{-x} \cdot \cos x$

$f'(c) = 0$

$\therefore -\sin x + \cos x = 0$

$\sin x = \cos x$ $\therefore \boxed{x = \pi/4}$

Q:- $f(x) = |x|$ in $[-2, 2]$. Hence find c ?

$f(x) = |x|$ is not differential at $x = -2$

$\therefore$ Rolle's M.V.T. doesn't exists (notapplicable)

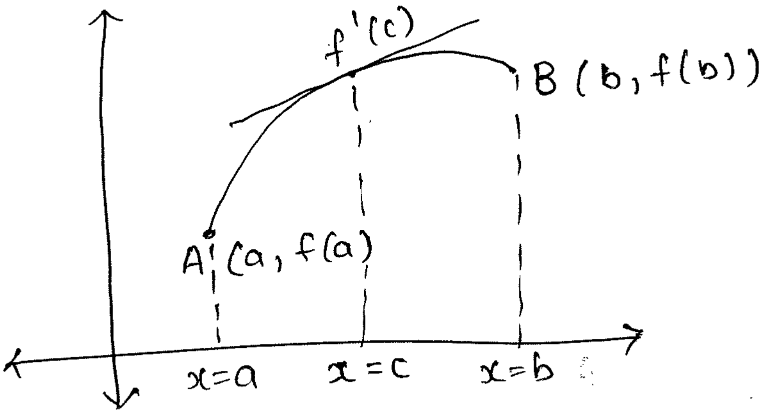
* Lagrange Value Theorem

(i) function $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable in (a, b).

(iii) $f(a) \neq f(b)$ then $\exists_a$ C [atleast one] $\in (a, b)$

such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.



$f'(c) = \text{slope of } \overline{AB}$

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

Q:- Find the value of c, $f(x) = x^3 - 6x^2 + 11x - 6$ in $[0,4]$

$f'(x) = 3x^2 - 12x + 11$

$f'(c) = 0$

$x = \frac{+12 \pm \sqrt{144-132}}{2 \cdot 3}$

$x = \frac{+12 \pm 2\sqrt{3}}{2 \cdot 3}$

$x = 2 \pm \frac{1}{\sqrt{3}}$

$c = 2 \pm 1/\sqrt{3}$

$f(0) = -6$

$f(4) = 64 - 72 + 44 - 6$

$f(4) = 30$

$f(0) \neq f(4)$

$\therefore$ Lagrange value theorem is applicable.

$f'(c) = 3$

$3x^2 - 12x + 11 = \frac{30+6}{4-0}$

$3x^2 - 12x + 11 = \frac{36}{4} = 9$

$3x^2 - 12x + 2 = 0$

$x = \frac{12 \pm \sqrt{144-24}}{2 \cdot 3} = \frac{12 \pm \sqrt{120}}{6}$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$3x^2 - 12x + 11 = \frac{30 + 6}{4 - 0}$$

$$3c^2 - 12c + 11 = \frac{\cancel{36}\,9}{\cancel{4}}$$

$$3c^2 - 12c + 11 - 9 = 0$$

$$3c^2 - 12c + 2 = 0$$

$$c = \frac{+12 \pm \sqrt{144 - 24}}{2 \cdot 3}$$

$$= \frac{12 \pm \sqrt{120}}{6}$$

$$= 2 \pm \frac{\cancel{2}\sqrt{30}}{\cancel{6}\,3}$$

$$c = 2 \pm \sqrt{\frac{10}{3}} \;//$$

* Cauchy Value Theorem

① function $f(x)$ and $g(x)$ is continuous in $[a, b]$.

② $f(x)$ & $g(x)$ is differentiable in (a, b).

③ $g'(x) \neq 0$, $\forall x \in (a, b)$

$$\frac{f'(x)}{g'(x)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Q:- $f(x) = e^{x}$, $g(x) = e^{-x}$ Find c?
in $[a, b]$

$g'(c) \neq 0$ ✓

$f'(x) = e^{x}$; $g'(x) = -e^{-x}$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{f'(c)}{g'(c)} = \frac{e^{b} - e^{a}}{e^{-b} - e^{-a}}$$

$$\frac{e^{x}}{-e^{-x}} = \frac{e^{b} - e^{a}}{e^{-b} - e^{-a}}$$

$$-e^{2x} = \frac{e^{b} - e^{a}}{\frac{1}{e^{b}} - \frac{1}{e^{a}}}$$

$$-e^{2x} = \frac{e^{b} - e^{a}}{\frac{e^{a} - e^{b}}{e^{a} \cdot e^{b}}}$$

$\therefore -e^{2x}$

A.M. $= \frac{a+b}{2}$

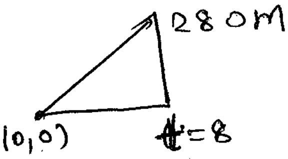
a M $= \sqrt{ab}$

H.M. $= \frac{2ab}{a+b}$

Q:- A rail engine accelerates from a stationary position for 8 sec. and travel a distance 280 m. Acc. to the MVT, the speedometer at a certain time during acceleration must read ______ km/h

Solⁿ:-

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$



$$\frac{\text{Speed}}{\text{Time}} = \frac{s(8)-s(0)}{8-0} = \frac{280-0}{8-0} = 35\ m/s$$

1 km = 1000 m   1 hr = 3600 s

10^{-3} km = 1 m $\frac{1}{3600}$ hr = 1 s

$$= \frac{35 \times 36\cancel{00}}{10\cancel{00} \times 1}$$

$$= 126\ km/hr.$$

EE-2017

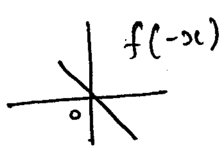
Q:- Let $g(x) = \begin{cases} -x & , x \le 1 \\ x+1 & , x > 1 \end{cases}$ & $f(x) = \begin{cases} 1-x & , x \le 0 \\ x^2 & , x > 0 \end{cases}$

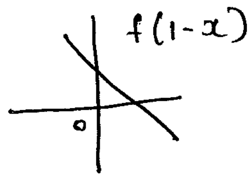
Consider the composition of $fog(x) = f(g(x))$. The no. of discontinuity in $fog(x)$ present in the interval $(-\infty, 0)$ is ______.

Solⁿ:- At $x = 0$ $f(1-x)$

$-(1-x)$

$x-1 \Rightarrow$ It is continuous $\boxed{0}$





NOTE:

When the function is continous then their composition is always continuous.

Q:- The tangent of a curve represented by $y = x \cdot \ln x$ is required to have 45° inclination with X-axis. The co-ordinate point of tangent will be ___

Sol^n:-

$$y = x \ln x$$

with X-axis,
So $y = 0$.

$$\frac{dy}{dx} = \frac{x}{x} + \ln x$$

$$\tan\theta = 1 + \ln x$$

$$\tan 45^\circ = 1 + \ln x$$

$$1 = 1 + \ln x$$

$$\ln x = 0$$

$(1, 0)$.

$\boxed{x = 1}$ $\boxed{y = 0}$

Q:- The function $f(x) = 1 - x^2 + x^3$ is defined in $[-1, 1]$. The value of x in $(-1, 1)$ for which M.V.T. is satisfied. (a) $1/2$ (b) $2/3$ (c) 1 (d) $-1/3$

Sol^n:-

$$f(-1) = 1 - 1 - 1 = -1$$

$$f(1) = 1 - 1 + 1 = 1$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$-2x + 3x^2 = \frac{+1 + 1}{1 + 1}$$

$x = -1/3 \in (-1, 1)$

$$3x^2 - 2x = +1$$

$$3x^2 - 2x - 1 = 0$$

$$x = \frac{+2 \pm \sqrt{4 + 12}}{6} = \frac{+2 \pm 4}{6} = 1, -1/3$$

*Taylor's Mean value theorem

- Let $f(x)$ be a function defined such that

(i) $f(x)$ is continuous in $[a, a+h]$.

(ii) $f^{n-1}(x)$ is differentiable in $(a, a+h)$

(iii) then $\exists$ a value $\theta \in (0,1)$

such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta \cdot h)$$

This is a Taylor Mean value theorem with langrange's form remainder.

Taylor M.V.T. has which form? Langrange's form

$$R_n = \frac{h^n}{n!} f^n(a+\theta \cdot h)$$

$n=1$

$$\boxed{f(a+h) = f(a) + hf'(a+\theta \cdot h)}$$ Langrange form

To form a series a function should have n^{th} derivative [derivative of all order]

Transidential $\rightarrow e^x \rightarrow \pi$ (continue no. | no exact value is found)

If $x = a+h$

$h = x-a$, $n \rightarrow \infty$

$$\boxed{f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots}$$

Taylor series

If $a=0$ we can get Maclaurin series

$$\boxed{f(x) = f(0) + (x) f'(0) + \frac{x^2 f''(0)}{2!} + \dots\dots}$$

* Expansions:-

① e^x at $x=0$

$f(x) = e^x$, $f(0) = e^0 = 1$

$f'(x) = e^x$, $f'(0) = e^0 = 1$

$f''(x) = e^x$, $f''(0) = e^0 = 1$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\dots$$

② $f(x) = \dfrac{x}{1+x}$ about $x=0$

$f(x) = \dfrac{x}{1+x}$, $f(0) = 0$

$f'(x) = \dfrac{(1+x)\cdot 1 - x}{(1+x)^2} = \dfrac{1}{(1+x)^2}$, $f'(0) = 1$

$f''(x) = \dfrac{(1+x^2)0 - 1\cdot 2(1+x)}{(1+x)^4}$

$f''(x) = \dfrac{-2}{(1+x)^3}$, $f''(0) = -2$

$f'''(x) = \dfrac{-2(1+x)^3 0 + 2\cdot 3(1+x)^2}{(1+x)^6}$

$f'''(x) = \dfrac{6}{(1+x)^4}$, $f'''(0) = 6$

$$f(x) = x - \frac{2x^2}{2!} + \frac{x^3 \cdot 6}{3!} \dots$$

$$f(x) = x - x^2 + x^3 - x^4 + \dots$$

Q:- If $f(x) = x^2$. Find expansion/series about $x = 1$

(a) $1 - x + x^2$

(b) $1 + x^2$

(c) x^2

(d) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Sol^n:- $f(x) = x^2$, $f(1) = 1$

$f'(x) = 2x$, $f'(1) = 2$

$f''(x) = 2$, $f''(1) = 2$

$f(x) = \cancel{0 + x \cdot 0 + x^2}\ 1 + x \cdot 2 + \frac{2x^2}{2!} + 0$

$= 1 + 2x + x^2$

Q:- $f(x) = \cos^2 x$ about $x = 0$. Find the co-efficient of x^2.

Sol^n:- $f(x) = \cos^2 x$, $f(0) = 1$

$f'(x) = 2\cos x(-\sin x) = -\sin 2x$, $f'(0) = 0$

$f''(x) = -\cos 2x \cdot 2$, $f''(0) = -2$

$f(x) = 1 + \frac{x \cdot 0}{1!} - \frac{2x^2}{2!} + \dots$

$f(x) = 1 + 0 - 1x^2 + \dots$

-1

Q:- Find the co-efficient of $(x-2)^4$ in function $f(x) = e^x$ about $x = 2$.

Sol^n:- $f(x) = e^x$, $f(2) = e^2$

$f'(x) = e^x$, $f'(2) = e^2$

$f''(x) = e^x$, $f''(2) = e^2$

$f'''(x) = e^x \quad , f'''(2) = e^2$

$$e^2 + e^2(x-2) + \frac{e^2(x-2)^2}{2!} + \frac{e^2(x-2)^3}{3!} + \frac{e^2}{4!}(x-2)^4 + \ldots$$

Q:- Find the quadratic approximation of $f(x) = x^3 - 3x^2 - 5$ at $x = 0$.

$\underline{\underline{Sol^n}}$:- $f(x) = x^3 - 3x^2 - 5 \quad , f(0) = -5$

$f'(x) = 3x^2 - 6x \quad , f'(0) = 0$

$f''(x) = 6x - 6 \quad , f''(0) = -6$

$f'''(x) = 6 \quad , f'''(0) = 6$

$f^{iv}(x) = 0$

$\therefore \cancel{f(x) = -5 + \frac{x \cdot 0}{1!}} + \cancel{x^2}$

$$\frac{-6}{2} = -3$$

$$f(x) = -5 + \frac{x \cdot 0}{1!} + \frac{x^2}{2!}(-6) + \frac{x^3 \cdot 6}{3!} + 0$$

$f(x) = x^3 - 3x^2 - 5$

Q:- Find linear approximation around $x = 2$ for a function $f(x) = e^{-x}$.

$\boxed{\text{NOTE:}}$

In linear approximation we take only two terms.

$\underline{Sol^n}$:- $f(x) = e^{-x} \quad , f(2) = e^{-2}$

$f'(x) = -e^{-x} \quad , f'(2) = -e^{-2}$

$f''(x) = +e^{-x} \quad , f''(2) = +e^{-2}$

$f'''(x) = -e^{-x} \quad , f'''(2) = -e^{-2}$

$$f(x) = f(2) + (x-2)f'(2) + (x-2)^2 \cdot f''(2) + \ldots$$

$$= e^{-2} + (x-2)(-e^{-2}) + \cancel{\frac{(x-2)^2 \cdot e^{-2}}{2!}} + \cancel{\frac{(x-2)^3 e^{-2}}{3!}} + \ldots$$

$$= e^{-2} + -xe^{-2} + 2e^{-2} + \cancel{\frac{-4x \cdot e^{-2}}{2!}} + \cancel{\frac{x^2 e^{-2}}{2}} + \cancel{\frac{4e^{-2}}{2!}} +$$

$$= \underline{(3-x)e^{-2}}$$

Q:- In Taylor series expansion $e^x + \sin x$ at $x = \pi$

The co-efficient $(x-\pi)^2$ is ___.

Sol^n:- $f(x) = e^x + \sin x$, $f(\pi) = e^\pi$

$f'(x) = e^x + \cos x$, $f'(\pi) = e^\pi - 1$

$f''(x) = e^x - \sin x$, $f''(\pi) = e^\pi$

$$f(x) = e^\pi + \frac{(x-\pi)(e^\pi - 1)}{1!} + \frac{(x-\pi)^2 \cdot e^\pi}{2!}$$

Q:-

The tangent of the curve represented by $x \cdot \ln x$ is required to have 45° inclination with X-axis. The co-ordinate of tangent point will be ___.

(a) $(1,0)$ (c) $(0,1)$

(b) $(1,1)$ (d) $(\sqrt{2}, \sqrt{2})$

Sol^n:-

$\tan 45^\circ = 1$

$y = x \ln x$

$\frac{dy}{dx} = \ln x + x \cdot 1/x$

$\frac{dy}{dx} = 1 + \ln x$

$1 = 1 + \ln x$

$\ln x = 0$

$x = 1$, $y = 0$ (x-axis

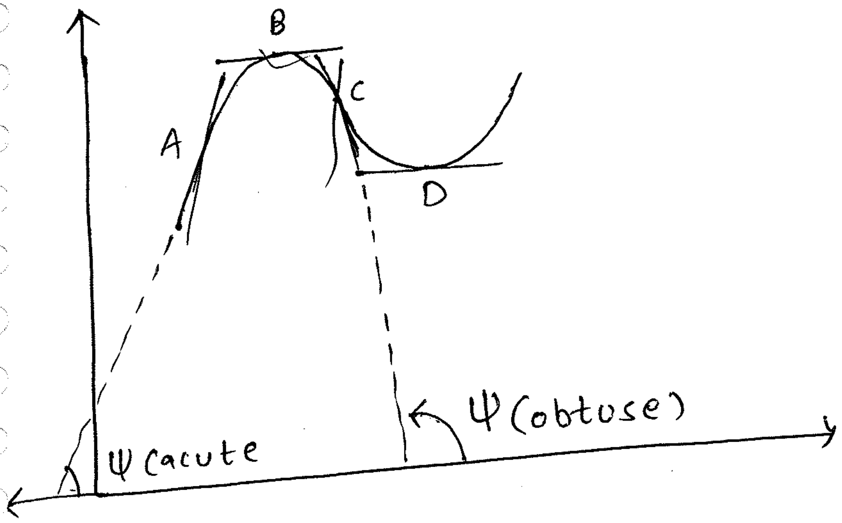
$\underline{(1,0)}$

* Increasing and Decreasing Function

If $y = f(x)$, $y \rightarrow$ increases then $x \rightarrow$ increases

If $y \rightarrow$ increases then $x \rightarrow$ decreases.

If $\frac{dy}{dx} = 0 \Rightarrow$ stationary function at that point.



- If $y = f(x)$ if y increasing as x increases it is called increasing function as at point a at x.

- On the contrary. if y decreasing as x increases it is called decreasing function as at point c

- Angle ψ = acute i.e. $\frac{dy}{dx} = +ve$ and function is increasing

- If angle ψ = obtuse, i.e. $\frac{dy}{dx} = -ve$ and function is decreasing

- If derivative is 0 as at point B and D, then y is neither increasing nor decreasing we say that function is stationary.

*Concavity, Convexity and point of inflection.

- If the portion of a curve on both sides of a point, however small it may be lies above a tangent as at point D then the curve is said to be concave upwards at B, where $\frac{d^2y}{dx^2} = +ve$.

- If the portion of a Curve on both sides of a point lies below the tangent as at point B then the curve is said to be convex upwards at B, where $\frac{d^2y}{dx^2} = -ve$.

- If the two portions of a curve lies on different sides of tangent as at point C then the point C is said to be point of inflection. At the point of inflexion, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$.