Sample Question Paper - 16
Mathematics (041)
Class- XII, Session: 2021-22
TERM II

Time Allowed: 2 hours **Maximum Marks: 40**

General Instructions:

1. This question paper contains three sections – A, B and C. Each part is compulsory.
2. Section - A has 6 short answer type (SA1) questions of 2 marks each.
3. Section – B has 4 short answer type (SA2) questions of 3 marks each.
4. Section - C has 4 long answer-type questions (LA) of 4 marks each.
5. There is an internal choice in some of the questions.
6. Q 14 is a case-based problem having 2 sub-parts of 2 marks each.

Section - A

[2 Marks each]

1. Find the value of $\int_{-\pi/2}^{\pi/2}(x^3 + x\cos x + \tan^5 x + 1)\,dx$

OR

Evaluate : $\int \frac{dx}{e^x + e^{-x}}$

2. Show that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ is the solution of $y = e^{-x}(A\cos x + B\sin x)$

3. Find the projection of vector $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on the vector $\vec{b} = 2\hat{i} + 2\hat{j} + \hat{k}$.

4. If the lines $\frac{x-1}{-2} = \frac{y-4}{3p} = \frac{z-3}{4}$ and $\frac{x-2}{4p} = \frac{y-5}{2} = \frac{z-1}{-7}$ are perpendicular to each other, then find the value of p.

5. If $P(A) = 0.4$, $P(B) = 0.8$ and $P\left(\frac{B}{A}\right) = 0.6$, then $P(A \cup B)$

6. Find the probability distribution of X, the number of heads is a simultaneous toss of two coins.

Section - B

[3 Marks each]

7. Find the value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

8. Find the general solution of $\frac{dy}{dx} + y\tan x = \sec x$

OR

Solve the differential equation:

$$x\sin\left(\frac{y}{x}\right)\frac{dy}{dx}+x-y\sin\left(\frac{y}{x}\right)=0$$

Given that $x = 1$ when $y = \frac{\pi}{2}$.

9. If $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.

10. Find the shortest distance between the lines:

$$\vec{r}=(t+1)\hat{i}+(2-t)\hat{j}+(1+t)\hat{k}$$

$$\vec{r}=(2s+2)\hat{i}-(1-s)\hat{j}+(2s-1)\hat{k}.$$

OR

A plane meets the co-ordinate axes at A, B and C such that the centroid of ΔABC is the point (α, β, γ). Show that the equation of the plane is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$.

Section - C

[4 Marks each]

11. Find : $\int \frac{\sec x}{1+\operatorname{cosec} x}dx$

12. Find the area bounded by lines $x = 2y + 3$, $y - 1 = 0$ and $y + 1 = 0$.

OR

Find the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$.

13. Find the equation of plane passing through the points $A(3, 2, 1)$, $B(4, 2, -2)$ and $C(6, 5, -1)$ and hence find the value of λ for which $A(3, 2, 1)$, $B(4, 2, -2)$, $C(6, 5, -1)$ and $D(\lambda, 5, 5)$ are coplanar.

Case-Based/Data Based

14. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. Let E_1 and E_2 be the events that selecting a student with 100% attendance and selecting a student who is not regular, respectively.

Based on the above information, answer the following questions:

(i) Find the values of $P\left(\frac{A}{E_1}\right)$ and $P\left(\frac{A}{E_2}\right)$. **[2]**

(ii) What is the probability that the student has 100% attendance. **[2]**

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Solution

MATHEMATICS 041

Class 12 - Mathematics

Section - A

1. Let $I = \int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1)\,dx$

$$= \int_{-\pi/2}^{\pi/2} x^3 dx + \int_{-\pi/2}^{\pi/2} x\cos x\,dx + \int_{-\pi/2}^{\pi/2} \tan^5 x\,dx + \int_{-\pi/2}^{\pi/2} 1\cdot dx$$

1

When $f(x)$ is an even function, then,

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

and if $f(x)$ is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

$\therefore \quad I = 0 + 0 + 0 + 2\int_0^{\pi/2} 1\cdot dx$

$= 2[x]_0^{\pi/2} = \dfrac{2\pi}{2} = \pi$ **1**

OR

Let $I = \int \dfrac{dx}{e^x + e^{-x}}\,dx$

$= \int \dfrac{e^x}{e^{2x}+1}\,dx$

Also, put $e^x = t, \Rightarrow e^x dx = dt$ **1**

$\Rightarrow \quad I = \int \dfrac{dt}{1+t^2}$

$= \tan^{-1} t + C$

$= \tan^{-1}(e^x) + C$ **1**

2. Given that, $y = e^{-x}(A\cos x + B\sin x)$

On differentiating both sides w.r.t., x we get

$\dfrac{dy}{dx} = -e^{-x}(A\cos x + B\sin x) + e^{-x}(-A\sin x + B\cos x)$

$\dfrac{dy}{dx} = -y + e^{-x}(-A\sin x + B\cos x)$

½

Again, differentiating both sides w.r.t. x, we get

$\dfrac{d^2y}{dx^2} = \dfrac{-dy}{dx} + e^{-x}(-A\cos x - B\sin x) - e^{-x}(-A\sin x + B\cos x)$ ½

$\Rightarrow \dfrac{d^2y}{dx^2} = -\dfrac{dy}{dx} - y - \left[\dfrac{dy}{dx} + y\right]$

$\Rightarrow \dfrac{d^2y}{dx^2} = -\dfrac{dy}{dx} - y - \dfrac{dy}{dx} - y$

$\Rightarrow \dfrac{d^2y}{dx^2} = -2\dfrac{dy}{dx} - 2y$

$\Rightarrow \dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$ **Hence Proved 1**

3. $\vec{a}\cdot\vec{b} = (2\hat{i} + 3\hat{j} + 2\hat{k})\cdot(2\hat{i} + 2\hat{j} + \hat{k})$

$= 4 + 6 + 2 = 12$

$p = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ **1**

or $p = \dfrac{12}{|\vec{b}|}$ $\quad |\vec{b}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

$= \dfrac{12}{3} = 4$ **1**

4. Using formula for perpendicular condition,

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ **1**

or $\quad -8p + 6p - 28 = 0$

or $\quad -2p = 28$

$\therefore \quad p = -14$ **1**

5. Here,

$P(A) = 0.4,\ P(B) = 0.8$ and $P\left(\frac{B}{A}\right) = 0.6$

$$\because \quad P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}$$

$$\Rightarrow \quad P(B \cap A) = P\frac{B}{A}.P(A)$$

$$= 0.6 \times 0.4 = 0.24$$ **1**

$$\because \quad P(A \cup B) = P(A) + P(B) - P(B \cap A)$$

$$= 0.4 + 0.8 - 0.24$$

$$= 1.2 - 0.24 = 0.96$$ **1**

6. Let X be the number of heads.

Possible values of X are 0, 1, 2. ½

$P(x=0) = \frac{1}{4},\ P(x=1) = \frac{1}{2},\ P(x=2) = \frac{1}{4}$

The probability distribution of X is :

X	0	1	2
$P(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

1½

Section - B

7. Let

$$\text{I} = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$$

$$\Rightarrow \quad \text{I} = \int_0^1 \tan^{-1}\left(\frac{x-(1-x)}{1+x(1-x)}\right)dx$$

$$\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$\Rightarrow \quad I = \int_0^1 [\tan^{-1}x - \tan^{-1}(1-x)]dx \quad ...(i)$$ **1**

Apply,

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$ ½

$$\Rightarrow \quad I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(1-1+x)]dx$$

$$\Rightarrow \quad I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}(x)]dx \quad ...(ii)$$

½

Adding equations (i) and (ii), we obtain

$$2\text{I} = \int_0^1 [\tan^{-1}x + \tan^{-1}(1-x) - \tan^{-1}(1-x) - \tan^{-1}x]dx$$

$$\Rightarrow \quad 2\text{I} = 0$$

$$\Rightarrow \quad \text{I} = 0$$ **1**

8. Given differential equation is

$$\frac{dy}{dx} + y\tan x = \sec x$$

which is a linear differential equation

Here, $P = \tan x$, $Q = \sec x$, **1**

$$\therefore \quad \text{IF} = e^{\int \tan x dx}$$

$$= e^{\log|\sec x|}$$

$$= \sec x$$ **1**

The general solution is

$$y \cdot \sec x = \int \sec x \cdot \sec x\, dx + \text{C}$$

$$\Rightarrow \quad y \cdot \sec x = \int \sec^2 x\, dx + \text{C}$$

$$\Rightarrow \quad y \cdot \sec x = \tan x + \text{C}$$ **1**

OR

We have,

$$x\sin\left(\frac{y}{x}\right)\frac{dy}{dx} + x - y\sin\frac{y}{x} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) - x}{x\sin\left(\frac{y}{x}\right)} \quad ...(i)$$

Above differential equation is a homogeneous equation

Put $\quad y = vx$

Then, $$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad ...(ii)$$

From (i) and (ii),

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{vx.\sin\left(\frac{vx}{x}\right) - x}{x\sin\left(\frac{vx}{x}\right)}$$ **1**

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{x(v\sin v - 1)}{x\sin v}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\sin v - 1 - v\sin v}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1}{\sin v}$$

$$\Rightarrow \quad \sin v\, dv = -\frac{1}{x}dx [\text{Here } x \neq 0]$$

Now, integrating both sides

$$\Rightarrow \quad \int \sin v\, dv = -\int \frac{1}{x} dx$$

$$\Rightarrow \quad -\cos v = -\log|x| + C$$

Put, $v = \frac{y}{x}$

$$\Rightarrow \quad -\cos\left(\frac{y}{x}\right) = -\log|x| + C \quad \text{...(iii)}$$

1

Also, given that $x = 1$, when $y = \frac{\pi}{2}$

$$\Rightarrow \quad -\cos\left(\frac{\pi}{2}\right) = -\log 1 + C$$

$$C = 0$$

$$\Rightarrow \quad -\cos\left(\frac{y}{x}\right) + \log|x| = 0$$

Therefore $\log|x| = \cos\left(\frac{y}{x}\right)$ is the required solution. **1**

9. Given that, $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ are two adjacent sides of a parallelogram.

Let us suppose $\vec{d_1}$ and $\vec{d_2}$ are two diagonals of parallelogram.

Then, $\vec{d_1} = \vec{a} + \vec{b}$

$= \hat{i} + 2\hat{j} + 3\hat{k} + 2\hat{i} + 4\hat{j} - 5\hat{k}$

$= 3\hat{i} + 6\hat{j} - 2\hat{k}$ ½

and $\vec{d_2} = \vec{b} - \vec{a}$

$= 2\hat{i} + 4\hat{j} - 5\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k}$

$= \hat{i} + 2\hat{j} - 8\hat{k}$ ½

Now, unit vector parallel to $\vec{d_1}$,

$$\hat{d}_1 = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{|\sqrt{9 + 36 + 4}|}$$

$$= \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{|\sqrt{49}|} \quad ½$$

$$\hat{d}_1 = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{|7|} \quad ½$$

$$\hat{d}_1 = \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$$

And unit vector parallel to $\vec{d_2}$,

$$\hat{d}_2 = \frac{\hat{i} + 2\hat{j} - 8\hat{k}}{|\sqrt{1 + 4 + 64}|}$$

$$= \frac{\hat{i} + 2\hat{j} - 8\hat{k}}{\sqrt{69}} \quad 1$$

$$\hat{d}_2 = \frac{1}{\sqrt{69}}\hat{i} + \frac{2}{\sqrt{69}}\hat{j} - \frac{8}{\sqrt{69}}\hat{k}$$

Commonly Made Error

- Instead of finding the parallel vectors, some students take the cross product to find the perpendicular vector.

Answering Tip

- Practice problems based on parallel and perpendicular vectors.

10. Equations of lines can be written as :

$$\vec{r} = \vec{a_1} + t\vec{b_1}$$

$$\Rightarrow \quad \vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + t(\hat{i} - \hat{j} + \hat{k});$$

$$\vec{r} = \vec{a_2} + s\vec{b_2}$$

$$\Rightarrow \quad \vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + s(2\hat{i} + \hat{j} + 2\hat{k})$$

Here, $\vec{a_1} = \hat{i} + 2\hat{j} + \hat{k}$, and $\vec{b_1} = \hat{i} - \hat{j} + \hat{k}$,

Also $\vec{a_2} = 2\hat{i} - \hat{j} - \hat{k}$, and $\vec{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$ **1**

Then, $\vec{a_2} - \vec{a_1} = \hat{i} - 3\hat{j} - 2\hat{k}$,

and, $$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$= -3\hat{i} + 3\hat{k} \quad 1$$

∴ Shortest distance

$$= \left|\frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}\right|$$

$$= \left|\frac{-3 - 6}{\sqrt{9 + 9}}\right|$$

$$= \frac{3}{\sqrt{2}} \text{ or } \frac{3\sqrt{2}}{2} \text{ units} \quad 1$$

OR

Since, the equation of the plane having intercept a, b and c on the three co-ordinate axes is:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad 1$$

Here, the co-ordinates of A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

The centroid of ΔABC is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$. ½

Equating $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$ to (α, β, γ) we get $a = 3\alpha$, $b = 3\beta$ and $c = 3\gamma$ ½

Thus, the equation of the plane is

$$\frac{x}{3\alpha}+\frac{y}{3\beta}+\frac{z}{3\gamma} = 1$$

or $$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma} = 3$$ **1**

Section - C

11. $\int \frac{\sec x}{1+\operatorname{cosec} x}dx$

$$= \int \frac{\sin x}{\cos x(1+\sin x)}dx$$

$$= \int \frac{\sin x \cos x}{(1+\sin x)^2(1-\sin x)}dx$$

Put, $[\sin x = t \text{ or } \cos x dx = dt]$

$$= \int \frac{t}{(1+t)^2(1-t)}dt$$ **1**

Let $\frac{t}{(1+t)^2(1-t)} = \frac{A}{1+t}+\frac{B}{(1+t)^2}+\frac{C}{1-t}$

or $t = A(1+t)(1-t)+B(1-t)+C(1+t)^2$

Put $t = -1$, $-1 = -2B$, *i.e.*, $B = -\frac{1}{2}$.

Put, $t = 1$,

$1 = 4C$, i.e., $C = \frac{1}{4}$ Put $t = 0$,

$0 = A+B+C$, which gives $A = \frac{3}{4}$

Therefore the required integral

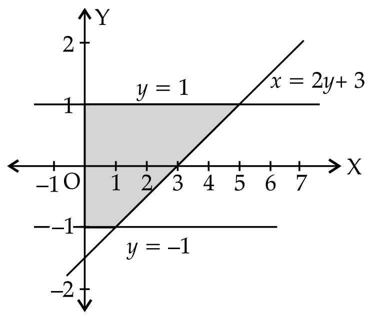
$$= \frac{3}{4}\int \frac{1}{1+t}dt + \frac{-1}{2}\int \frac{1}{(1+t)^2}dt + \frac{1}{4}\int \frac{1}{(1-t)}dt$$ **1**

$$= \frac{3}{4}\log|1+t| + \frac{-1}{2}\times\frac{-1}{1+t} - \frac{1}{4}\log|1+t| + c$$

$$= \frac{3}{4}\log|1+\sin x| - \frac{1}{2}\times\frac{1}{1+\sin x} - \frac{1}{4}\log|1-\sin x| + c$$

$$= \frac{1}{4}\log\frac{|1+\sin x|^3}{|1-\sin x|} - \frac{1}{2}.\frac{1}{1+\sin x} + c$$ **2**

12.



2

From the figure, area of the shaded region,

$$A = \int_{-1}^{1}(2y+3)dy$$

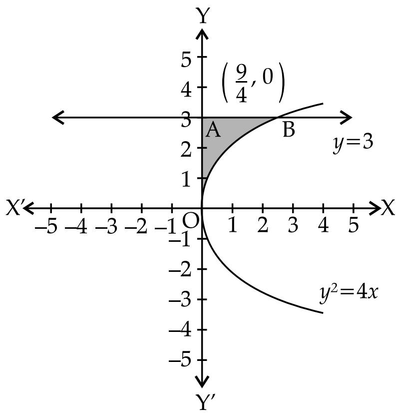
$$= \left[y^2+3y\right]_{-1}^{1}$$

$$= [1+3-1+3]$$

$$= 6 \text{ sq. units}$$ **2**

OR

The area bounded by the curve, $y^2 = 4x$, y-axis, and $y = 3$ is represented as :



2

Area of $OAB = \int_0^3 x dy$

$$= \int_0^3 \frac{y^2}{4}dy$$

$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3$$

$$= \frac{1}{12}\times 27$$

$$= \frac{9}{4} \text{ sq. units}$$ **2**

13. A plane which passes through $A(3, 2, 1)$, $B(4, 2, -2)$ and $C(6, 5, -1)$ is

$$\begin{vmatrix} x-3 & y-2 & z-1 \\ 4-3 & 2-2 & -2-1 \\ 6-3 & 5-2 & -1-1 \end{vmatrix} = 0 \quad \textbf{1}$$

$$\Rightarrow \begin{vmatrix} x-3 & y-2 & z-1 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

$$\Rightarrow (x-3)(0+9)-(y-2)(-2+9) +(z-1)(3-0) = 0$$

$$\Rightarrow 9x-7(y-2)+3(z-1) = 0$$

$$\Rightarrow 9(x-3)\ 9x-7y+3z = 16 \quad \textbf{1}$$

Thus, plane passing through point A, B and C is $9x - 7y + 3z = 16$

Now, given A, B, C and $D(\lambda, 5, 5)$ are coplanar.

So, D lies on the plane passing through A, B and C **1**

$$\therefore \quad 9\lambda-7(5)+3(5) = 16$$

$$\Rightarrow \quad 9\lambda = 36$$

$$\Rightarrow \quad \lambda = 4 \quad \textbf{1}$$

Case-Based/Data Based

14. Let E_1 : Selecting a student with 100% attendance

E_2 : Selecting a student who is not regular

A : selected student attains A grade.

$$P(E_1) = \frac{30}{100} \text{ and } P(E_2) = \frac{70}{100}$$

(i) $P\left(\frac{A}{E_1}\right) = \frac{70}{100}$ and $P\left(\frac{A}{E_2}\right) = \frac{10}{100}$ **2**

(ii) $$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\cdot P\left(\frac{A}{E_1}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_1}\right)}$$

$$= \frac{\frac{30}{100}\times\frac{70}{100}}{\frac{30}{100}\times\frac{70}{100}+\frac{70}{100}\times\frac{10}{100}} = \frac{3}{4} \quad \textbf{2}$$

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