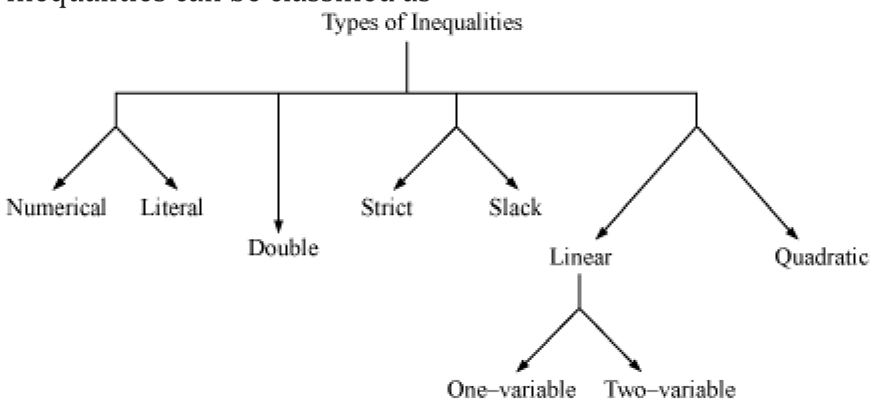
Linear Inequalities

Inequality and Its Classification

- Two real numbers or two algebraic expressions related by the symbols '<', '>', '≤' or '≥' form an **inequality**.
- For example: $6 < 26$, $3 < z + 1 \le 22$, $27 \ge s \ge 16$, $p + t > 100$
- Inequalities can be classified as



- **Numerical inequality:** Inequalities that involve numbers only are classified as numerical inequalities. For example: $87 < 117$, $19 > 17 > 8$ etc.
- **Literal inequality:** Inequalities that involve a variable on one side and a number on the other side are classified as literal inequalities. For example: $a < 6$, $18 > k$, $b \ge -27$, $21 \le m$, etc.
- **Double inequality:** Inequalities in which the variable or the numbers lie in a certain interval are known as double inequalities. For example: $x \in [-15, 8]$, $9 > 6 > 2$, $8 \le p + 1 \le 11$, etc.
- **Strict inequality:** Inequalities of the type $px + q < 0$, $px + q > 0$, $px + qy < r$, $px + qy > r$, $ax^2 + bx + c > 0$, or $ax^2 + bx + c < 0$ are classified as strict inequalities. For example: $2x < -3$, $x + 17 < 9$, $x + 3y > 14$, $2a + 5b < 8$, $2y^2 + 5y > 8$ etc.
- **Slack inequality:** Inequalities of the type $px + q \le 0$, $px + q \ge 0$, $px + qy \le r$, $px + qy \ge r$, $ax^2 + bx + c \ge 0$, or $ax^2 + bx + c \le 0$ are classified as slack inequalities. For example: $x \le 89$, $5x + 8y \le 9$, $8x + y \ge 7$, $x + 14 \ge 28$, $z^2 + 3z \le 30$ etc.
- **Linear inequality in one variable:** Inequalities of the type $px + q \ge 0$, $px + q \le 0$, $px + q > 0$, or $px + q < 0$, where $p \ne 0$, are classified as linear inequalities in one variable (here, the variable in each inequality is x). For example: $x - 23 \ge 0$, $12y < 85$, etc.
- **Linear inequality in two variables:** Inequalities of the type $px + qy + r \ge 0$, $px + qy + r \le 0$, $px + qy + r > 0$, or $px + qy + r < 0$, where $p \ne 0$ and $q \ne 0$, are classified as linear inequalities in two variables (here, the variables in each inequality are x and y). For example: $9x + y > 0$, $x + 11y \ge 13$, etc.

- **Quadratic inequality:** Inequalities of the type $ax^2 + bx + c \geq 0$, $ax^2 + bx + c \leq 0$, $ax^2 + bx + c > 0$, or $ax^2 + bx + c < 0$, where $a \neq 0$, are classified as quadratic inequalities. For example: $x^2 + 16 \geq 23$, $p^2 < 2p + 7$, etc.

  Let's now try and solve the following puzzle to check whether we have understood this concept.

  Let us now solve an example based on inequality.

  Example 1: State true or false for the following statements:

1. **The inequality $9x^2 + 5x < 0$ is a quadratic inequality.**
2. **The inequality $7p + 3q > 2$ is a linear inequality in one variable and a slack inequality.**
3. **The inequality $8 \geq p + q \geq 2$ is a double linear inequality in two variables.**
4. **The inequality $2k + 1 \leq 8$ is a numerical inequality.**
5. **The inequality $s > 100$ is not a strict inequality.**

Solution:

1. True.
2. False. The inequality $7p + 3q > 2$ is a linear inequality in two variables.
3. True.
4. False. The inequality $2k + 1 \leq 8$ is not a numerical inequality as it involves a variable k.
5. False. The inequality $s > 100$ is a strict inequality.

Expressing Given Situations Mathematically as Linear Inequations

- While converting a word problem into an inequality, the following points are kept in mind:

- If the word 'at least' is used in the word problem, then we use the sign '≥'.
- If the word 'at most' is used in the word problem, then we use the sign '≤'.
- If x lies between a and b, then we use $a < x < b$.
- If the word 'minimum' is used in the word problem, then we use the sign '≥'.
- If the word 'maximum' is used in the word problem, then we use the sign '≤'.
- If the word 'greater than' or 'more than' is used in the word problem, then we use the sign '>'.
- If the word 'less than' is used in the word problem, then we use the sign '<'.

Example: Suppose Ishaan goes to a market and buys x shirts and y pairs of trousers for himself. The maximum amount that he can spend is Rs 3000. The price of each shirt is Rs 200 and the price of each pair of trousers is Rs 700. Now, this information needs to be represented mathematically.
The cost of each shirt is given to be Rs 200. Hence, the total cost of x shirts is Rs $200x$.
Also, the cost of each pair of trousers is given to be Rs 700. Hence, the total cost of y pairs of

trousers is Rs $700y$.
Thus, the total cost of x shirts and y pairs of trousers is Rs $200x + 700y$.
Since the maximum amount to be spent by Ishaan is Rs 3000, the total cost of shirts and trousers can either be equal to or less than Rs 3000.
This implies that the required inequality is $200x + 700y \leq 3000$

Example 1: A shopkeeper sells some toy-trucks and toy-cars every day. He wants to earn a profit of at least Rs 6500 each week. For this, he must sell 20 trucks and 30 cars every week or he must sell 50 trucks and 10 cars every week. Express this situation mathematically.

Solution:

Let the profit on each toy-truck be Rs x and the profit on each toy-car be Rs y.

It is given that to earn a profit of at least Rs 6500, the shopkeeper must sell 20 trucks and 30 cars every week. Hence, the total profit earned by the shopkeeper by selling 20 trucks and 30 cars is Rs $(20x + 30y)$. Thus,

$20x + 30y \geq 6500$

It is also given that to earn a profit of at least Rs 6500, the shopkeeper must sell 50 trucks and 10 cars every week. Hence, the total profit earned by the shopkeeper by selling 50 trucks and 10 cars is Rs $(50x + 10y)$. Thus,

$50x + 10y \geq 6500$

Hence, the given situation can be expressed mathematically by the following two inequalities.

$20x + 30y \geq 6500$

and

$50x + 10y \geq 6500$

Example 2: Amaan goes to a shop to buy some cassettes and some CDs. The money that Amaan can spend on buying these items is Rs 800. The cost of each cassette is Rs 50 and the cost of each CD is Rs 150. Express this situation with the help of an inequality.

Solution:

Let the number of cassettes and CDs bought by Amaan be x and y respectively.

It is given that the cost of each cassette is Rs 50. Hence, the cost of x number of cassettes is Rs $50x$.

It is also given that the cost of each CD is Rs 150. Hence, the cost of y number of CD is Rs $150y$.

Thus, the total amount spent by Amaan on buying cassettes and CDs is Rs $(50x + 150y)$.

It is given that Amaan has a total of Rs 800. Hence, the total cost of x cassettes and y CDs should be less than or equal to Rs 800.

Thus, the required inequality is $50x + 150y \leq 800$

Example 3: In a locality, 5000 people went to the polling booth to vote for one party amongst party A and party B. However, due to some reason, few votes got rejected. When the results were announced, it was found that party B won the elections by 400 votes. Express this situation mathematically.

Solution:

Let the number of votes in favour of party A be x.

It is given that the number of votes in favour of the party B was 400 more than that for party A.

Thus, the number of votes in favour of the party B is $x + 400$.

It is also given that a total of 5000 people casted their votes. However, votes by few people got rejected.

Thus, the total number of people who casted their votes in favour of party A and B should be less than 5000.

Hence, required inequality is $x + x + 400 < 5000$.

Solving Linear Inequalities in One Variable

- Any **solution of an inequality in one variable** is a value of the variable that makes it a true statement.
- The set of numbers consisting of all the solutions of an inequality is known as the **solution set** of an inequality.

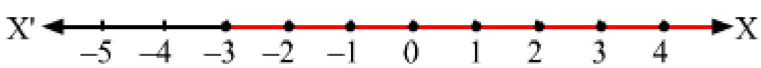
- The rules that need to be followed to solve an inequality are:

- Equal numbers can be added to (or subtracted from) both sides of an inequality without affecting the sign of the inequality.
- Both sides of an inequality can be multiplied (or divided) with the same positive number. However, when both sides are multiplied or divided by a negative number, then the sign of the inequality is reversed.

  - As we saw in the video, the solution set of the inequation $-4x - 6 < x + 9$ came out to be $x \geq -3$. Now, the solution set might be taken from real numbers or whole numbers or integers or any other set of numbers. The set from which the values of the variables (involved in the inequation) are chosen is called the **replacement set**. We may take any set as the replacement set. For example: **N**, **Z**,{−4, −3, −2}can be taken as the replacement set.
  - Depending upon the replacement set, we get the solution set as

Replacement Set	Solution Set
N	**N**
Z	{−3, −2, −1, 0, 1, 2 ...}
{−4, −3, −2}	{−3, −2}

  - We can also represent the solution of $-4x - 6 < x + 9$, $x \in$ **Z** on a number line as follows:

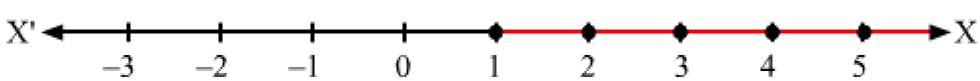


  - **Combining Inequations:**
    Let us consider two inequations $7x + 14 \geq 21$ and $-9x > -36$; where $x \in$ **R**.
    Now $7x + 14 \geq 21 \Rightarrow 7x \geq 21 - 14$
    $\Rightarrow 7x \geq 7$
    $\Rightarrow x \geq 1$

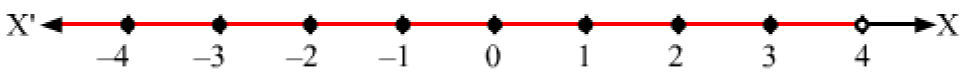
    Also, $-9x > -36$

$$\Rightarrow \frac{-9x}{-9} < \frac{-36}{-9}$$
$$\Rightarrow x < 4$$

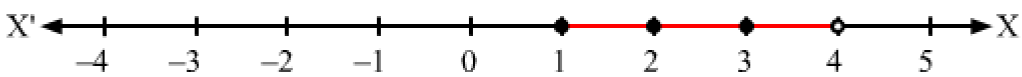
Now graph for $x \geq 1$:



Also, graph for *x* < 4 will be:



Thus, the graph of solution set of *x* ≥ 1 and *x* < 4 will be:



Let us go through following examples to understand the concept better.

Example 1: Solve $\frac{a}{3}-7\geq\frac{3a+1}{2}$**.**

Solution:

$$\frac{a}{3}-7\geq\frac{3a+1}{2}$$
$$\Rightarrow\frac{a-21}{3}\geq\frac{3a+1}{2}$$
$$\Rightarrow 2(a-21)\geq 3(3a+1)$$
$$\Rightarrow 2a-42\geq 9a+3$$
$$\Rightarrow 2a-42-2a\geq 9a+3-2a$$
$$\Rightarrow -42\geq 7a+3$$
$$\Rightarrow -42-3\geq 7a+3-3$$
$$\Rightarrow -45\geq 7a$$
$$\Rightarrow a\leq\frac{-45}{7}$$

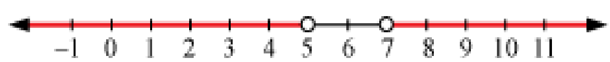
Thus, all real numbers that are less than or equal to $\frac{-45}{7}$ are solutions of the given inequality i.e., $a \in \left(-\infty,\frac{-45}{7}\right]$.

Example 2: Show the solution of the inequality $\frac{6z}{5}-3>\frac{3z-9}{2}>6$ **on a number line.**

Solution:

$$\frac{6z}{5}-3>\frac{3z-9}{2}>6$$
$$\Rightarrow \frac{6z-15}{5}>\frac{3z-9}{2}>6$$
$$\Rightarrow 2(6z-15)>5(3z-9)>6\times 5\times 2$$
$$\Rightarrow 12z-30>15z-45>60$$
$$\Rightarrow 12z-30>15z-45 \text{ and } 15z-45>60$$
$$\Rightarrow 12z-30-12z>15z-45-12z \text{ and } 15z-45+45>60+45$$
$$\Rightarrow -30>3z-45 \text{ and } 15z>105$$
$$\Rightarrow -30+45>3z-45+45 \text{ and } \frac{15z}{15}>\frac{105}{15}$$
$$\Rightarrow 15>3z \text{ and } z>7$$
$$\Rightarrow \frac{15}{3}>\frac{3z}{3} \text{ and } z>7$$
$$\Rightarrow 5>z \text{ and } z>7$$

Thus, all real numbers which are less than 5 and greater than 7 are the solutions of the given inequality. Thus, the solution of the given inequality can be represented on a number line as



Example 3: Ram, an electrician, cuts a piece of wire in such a manner that the length of the longer piece is three times the length of the shorter piece. What are the possible lengths of the shorter piece if the length of the longer piece is at least 2 cm less than four times the length of the shorter piece?

Solution:

Let the length of the shorter piece be x cm.

Also, the length of the longer piece is three times the length of the shorter piece. Hence, the length of the longer piece is $3x$ cm.

Now, the length of the longer piece is at least 2 cm less than four times the length of the shorter piece, then

$3x \geq 4x - 2$

$\Rightarrow 3x - 4x \geq 4x - 2 - 4x$

$\Rightarrow -x \geq -2$

$\Rightarrow x \leq 2$

Thus, the length of the shorter wire should be less than or equal to 2 cm.

Solving a System of Linear Inequalities in Two Variables

A system of linear inequalities in two variables i.e., linear inequalities of the type $px + qy + r \geq 0$, $px + qy + r \leq 0$, $px + qy + r > 0$, or $px + qy + r < 0$, where $p \neq 0$ and $q \neq 0$, can be solved by using the graphical method.

Example 1: Solve the inequality $2y + 3x > 120$ graphically.

Solution:

The graph of linear equation $2y + 3x = 120$ is given as dotted line in the graph shown below.

This line divides the *xy*-plane in two half-planes.

Select a point (not on the line) that lies in one of the half-planes to determine whether the point satisfies the given inequality or not.

We select the point as (0, 0).

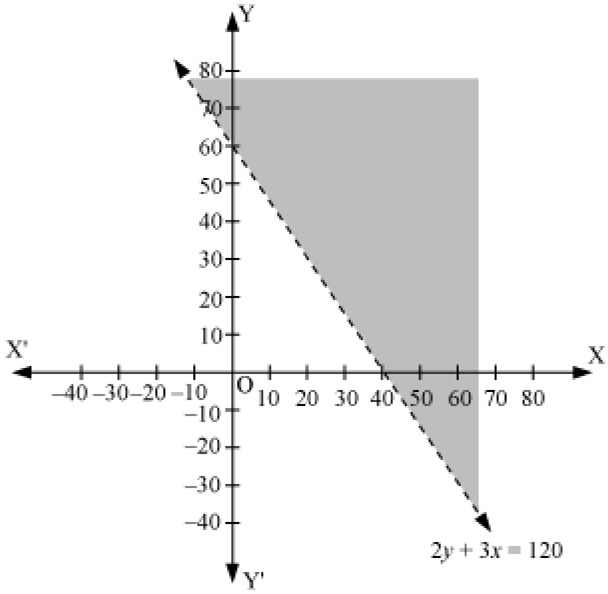
It is observed that

$2(0) + 3(0) > 120$ or $0 > 120$, which is false

Therefore, the lower half-plane is not the solution region of the given inequality. Also, it is clear that any point on the line does not satisfy the given inequality.

Thus, the solution region of the given inequality is the half-plane that doesn't contain the point (0, 0), excluding the line.

The solution region can be shown by a shaded region as



Example 2: Solve the following system of linear inequalities graphically.

$x + 5y \leq 25$

$3x + 4y > 60$

$x > 10$

Solution:

$x + 5y \leq 25$… (1)

$3x + 4y > 60$… (2)

$x > 10$… (3)

The graph of the lines $x + 5y = 25$, $3x + 4y = 60$ and $x = 10$ are drawn in the figure below.

Inequality (1) represents the region below the line $x + 5y = 25$ (including the line $x + 5y = 25$); inequality (2) represents the region above the line $3x + 4y = 60$ (excluding the line $3x + 4y = 60$) and inequality (3) represents the region on the right of the line $x = 10$ (excluding the line $x = 10$).

Hence, the solution of the given system of linear inequalities is represented by the common shaded region including the points that lie on the line $x + 5y = 25$ and it can be shown as