07

If a function f is differentiable in an interval I, i.e. its derivative f' exists at each point of I. Then, functions that could have possibly given function as a derivative are called anti-derivatives of the function. The formula that gives all these anti-derivatives is called the indefinite integral of the function and such process of finding anti-derivatives is called the integration or anti-differentiation. Two forms of integral are indefinite and definite integral which together constitute integral calculus.

INTEGRALS

|TOPIC 1|

Integration and Its Properties

CHAPTER CHECKLIST

- Integration and Its Properties
- Integration by Substitution
- Integration by Partial Fractions
- Integration by Parts
- Definite Integral
- Evaluation of Definite Integral by Substitution

INTEGRATION AS AN INVERSE PROCESS OF DIFFERENTIATION

Let $F(x)$ and $f(x)$ be two functions connected together such that $\frac{d}{dx}F(x) = f(x)$, then $F(x)$ is called **integral of $f(x)$** or **indefinite integral** or **anti-derivative.**

If $\frac{d}{dx}F(x) = f(x)$, then for any arbitrary constant C, $\frac{d}{dx}[F(x)+C] = f(x)$. Thus, $F(x)+C$ is also an anti-derivative of $f(x)$. Actually, there exist infinitely many anti-derivative of a function which can be obtained by choosing C arbitrarily from the set of real numbers. Hence, $\int f(x)\,dx = F(x)+C$, where C is an arbitrary constant (also called constant of integration) and symbol '$\int$' indicates the sign of integration. By varying the parameter C, one gets different anti-derivatives or integrals of the given function. The symbols/terms/phrases related to integration with their meaning are given in following table

Symbols/Terms/Phrases	Meaning
$\int f(x)\,dx$	Integral of f with respect to x
$f(x)$ in $\int f(x)\,dx$	Integrand
x in $\int f(x)\,dx$	Variable of integration
Integrate	Find the integral
An integral of f	A function F such that $F'(x) = f(x)$
Integration	The process of finding the integral
Constant of integration	Any real number C, considered as constant function

Some Standard Formulae

	Derivatives	Indefinite integrals (Anti-derivatives)
1.	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

EXAMPLE |1| Evaluate the following integrals.

(i) $\int x^6 dx$ (ii) $\int \frac{1}{x^{3/4}} dx$

(iii) $\int 5^x dx$ (iv) $\int a^{3\log_a x} dx$

Sol. (i) Let $I = \int x^6 dx$

2.	$\frac{d}{dx}(x) = 1$	$\int dx = x + C$
3.	$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\, dx = \sin x + C$
4.	$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x\, dx = -\cos x + C$
5.	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + C$
6.	$\frac{d}{dx}(-\cot x) = \text{cosec}^2 x$	$\int \text{cosec}^2 x\, dx = -\cot x + C$
7.	$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$	$\int \sec x \cdot \tan x\, dx = \sec x + C$
8.	$\frac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cdot \cot x$	$\int \text{cosec}\, x \cdot \cot x\, dx = -\text{cosec}\, x + C$
9.	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
10.	$\frac{d}{dx}(-\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$
11.	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
12.	$\frac{d}{dx}(-\cot^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$
13.	$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$
14.	$\frac{d}{dx}(-\text{cosec}^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{dx}{x\sqrt{x^2-1}} = -\text{cosec}^{-1} x + C$
15.	$\frac{d}{dx}(e^x) = e^x$	$\int e^x\, dx = e^x + C$
16.	$\frac{d}{dx}(\log\lvert x\rvert) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log\lvert x\rvert + C$
17.	$\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x, a > 0, a \neq 1$	$\int a^x dx = \frac{a^x}{\log a} + C, a > 0, a \neq 1$

Note

(i) The derivative of function is unique but integral of a function is not unique.

(ii) If two functions differ by a constant, then they have the same derivatives.

(iii) While solving an integral, constant of integration should be written, otherwise answer would be wrong.

(iv) Generally, we do not mention the interval over which the function is defined. But in a particular problem, it should be kept in mind.

The given figure shows some members of the family of curves given by $y = \sqrt{x} + C$ for different values of $C \in R$.

Properties of Indefinite Integrals

(i) The process of differentiation and integration are inverse of each other.

i.e. $\frac{d}{dx}\int f(x)dx = f(x)$ and $\int f'(x)dx = f(x) + C$

where, C is any arbitrary constant.

$$= \frac{x^{6+1}}{6+1} + C \quad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C, \text{ if } n \neq -1\right]$$

$$= \frac{x^7}{7} + C$$

(ii) Let $I = \int\left[\frac{1}{x^{3/4}}\right]dx = \int x^{-3/4}\, dx$

$$= \frac{x^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} + C \quad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1}\right]$$

$$= \frac{x^{1/4}}{1/4} + C = 4x^{1/4} + C$$

where, C is a constant of integration.

(iii) Let $I = \int 5^x dx = \frac{5^x}{\log 5} + C \quad \left[\because \int a^x dx = \frac{a^x}{\log a}\right]$

where, C is a constant of integration.

(iv) Let $I = \int a^{3\log_a x}dx = \int a^{\log_a x^3}dx \quad [\because m\log n = \log n^m]$

$$= \int x^3 dx \quad [\because a^{\log_a f(x)} = f(x)]$$

$$= \frac{x^4}{4} + C \quad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C, \text{ if } n \neq -1\right]$$

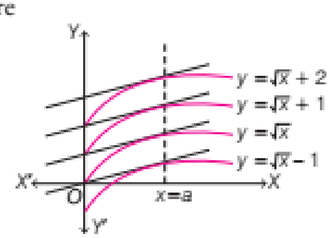
Geometrical Interpretation of Indefinite Integral

Geometrically, the statement $\int f(x)\, dx = \phi(x) + C = y$ (say) represents a family of curves. The different values of C correspond to different members of this family and the graph of these members can be obtained by shifting any one of the curves parallel to itself.

Further, the tangents to the curves at the point of intersection of a line $x = a$ with the curves are parallel.

e.g. Consider the integral of $\frac{1}{2}\sqrt{x}$,

i.e. $\int \frac{1}{2\sqrt{x}} dx = \sqrt{x} + C, C \in R$, which is represented by the following figure



EXAMPLE |3| Evaluate $\int \frac{(1+x)^2}{\sqrt{x}}dx$.

Firstly, expand numerator by using $(a+b)^2 = a^2 + b^2 + 2ab$ and then integrate it.

Sol. Let $I = \int \frac{(1+x)^2}{\sqrt{x}}dx = \int \frac{(1+x^2+2x)}{\sqrt{x}}dx$

$$= \int \frac{1}{\sqrt{x}}dx + \int \frac{x^2}{\sqrt{x}}dx + 2\int \frac{x}{\sqrt{x}}dx \text{ [by property (iv)]}$$

(ii) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.

(iii) $\int \{f(x) \pm g(x)\}\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

(iv) $\int k \cdot f(x)\, dx = k \cdot \int f(x)\, dx$, where k is any non-zero real number.

(v) Properties (iii) and (iv) can be generalised to a finite number of function, i.e. if $f_1, f_2, \ldots, f_n$ are functions and $k_1, k_2, \ldots, k_n$ are non-zero real numbers, then

$$\int [k_1\, f_1(x) + k_2\, f_2(x) + \ldots + k_n\, f_n(x)]\, dx$$
$$= k_1 \int f_1(x) dx + k_2 \int f_2(x)\, dx + \ldots + k_n \int f_n(x) dx$$

Note If more than one constant of integration is used while solving the integral, then at the end of the solution write only one constant of integration.

EXAMPLE |2| Evaluate the following integrals.

(i) $\int (\sin x + \cos x)\, dx$ (ii) $\int \sin x \left(\cot x + \dfrac{1}{\sin^3 x}\right) dx$

Sol. (i) Let $I = \int (\sin x + \cos x) dx$

$$= \int \sin x\, dx + \int \cos x\, dx$$
$$= -\cos x + C_1 + \sin x + C_2$$
$$= -\cos x + \sin x + C$$

where, $C = C_1 + C_2$.

(ii) Let $I = \int \sin x \left(\cot x + \dfrac{1}{\sin^3 x}\right) dx$

$$= \int \left(\sin x \times \cot x + \frac{\sin x}{\sin^3 x}\right) dx$$
$$= \int \left(\sin x \times \frac{\cos x}{\sin x}\right) dx + \int \frac{1}{\sin^2 x}\, dx$$
$$= \int \cos x\, dx + \int \operatorname{cosec}^2 x\, dx$$
$$= \sin x + C_1 + (-\cot x) + C_2$$
$$= \sin x - \cot x + C_1 + C_2$$
$$= \sin x - \cot x + C \qquad [\text{put } C_1 + C_2 = C]$$

where, C is a constant of integration.

INTEGRATION BY METHOD OF INSPECTION

We can find an anti-derivative of a given function by searching intuitively a function whose derivative is the given function. The search for the requisite function for finding an anti-derivative is known as integration by the method of inspection.

Note

(i) If we know one anti-derivative F of a function f, then we can write an infinite number of anti-derivatives of f by adding any constant to F.

(ii) If f is not expressible in terms of elementary functions, then it is not possible to solve integral by inspection.

e.g. We cannot find $\int e^{-x^2} dx$ by inspection, since we cannot find a function, whose derivative is e^{-x^2}.

$$= \int x^{-\frac{1}{2}} dx + \int x^{2-\frac{1}{2}} dx + 2\int x^{1-\frac{1}{2}} dx$$
$$= \int x^{-1/2} dx + \int x^{3/2} dx + 2\int x^{1/2} dx$$
$$= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{2x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C \qquad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1}\right]$$
$$= \frac{x^{1/2}}{1/2} + \frac{x^{5/2}}{5/2} + \frac{2x^{3/2}}{3/2} + C$$
$$= 2x^{1/2} + \frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + C$$
$$= 2\sqrt{x} + \frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + C$$

where, C is a constant of integration.

EXAMPLE |4| Find the anti-derivative F of f defined by $f(x) = 4x^5 - 6x$, where $F(0) = 2$.

Firstly, find the integral of $f(x)$, say $F(x)$. Further, use the condition $F(0) = 2$ to get the value of C. Finally, put the value of C in $F(x)$ and get the required result.

Sol. Given, $f(x) = 4x^5 - 6x$

On integrating both sides, we get

$$\int f(x)\, dx = \int (4x^5 - 6x)\, dx$$
$$\Rightarrow \quad F(x) = \frac{4x^{5+1}}{5+1} - \frac{6x^2}{2} + C \qquad \left[\because \int x^n\, dx = \frac{x^{n+1}}{n+1}\right]$$
$$\Rightarrow \quad F(x) = \frac{4}{6}x^6 - \frac{6}{2}x^2 + C$$
$$\Rightarrow \quad F(x) = \frac{2}{3}x^6 - 3x^2 + C \qquad \ldots(i)$$

Also, given $F(0) = 2$, therefore putting $x = 0$ in Eq. (i), we get

$$F(0) = \frac{2}{3}(0) - 3(0) + C$$
$$\Rightarrow \quad 2 = 0 - 0 + C \Rightarrow C = 2$$

Now, putting $C = 2$ in Eq. (i), we get

$$F(x) = \frac{2x^6}{3} - 3x^2 + 2$$

When a polynomial function P is integrated, then the resultant is also a polynomial whose degree is 1 more than that of P.

e.g. $\dfrac{d}{dx}(x^3) = 3x^2$ but $\int (3x^2) dx = x^3 + C$

6. Integral of a function is always discussed in an interval but derivative of a function can be discussed in an interval as well as on a point.

7. Geometrically, derivative of a function represents slope of the tangent to the corresponding curve.

On the other hand, integral of a function represents an infinite family of curves placed parallel to each other having parallel tangents at points of intersection of the curves with a line perpendicular

EXAMPLE |5| Write an anti-derivative of $3x^2 + 4x^3$ by the method of inspection.

Sol. Now, let us look for a function whose derivative is $3x^2 + 4x^3$.

Note that $\frac{d}{dx}(x^3 + x^4) = 3x^2 + 4x^3$

$\therefore$ An anti-derivative of $3x^2 + 4x^3$ is $x^3 + x^4$.

Comparison between Differentiation and Integration

1. Differentiation and integration both are operations on real valued functions.
2. Differentiation and integration both satisfy the property of linearity.

   i.e. $\frac{d}{dx}\{af(x) \pm bg(x)\} = a\frac{d}{dx}\{f(x)\} \pm b\frac{d}{dx}\{g(x)\}$

   and $\int [af(x) \pm bg(x)]dx = a\int f(x)dx \pm b\int g(x)dx$
3. All functions are not differentiable, similarly there are some functions which are not integrable.

   We will learn more about non-differentiable function and non-integrable function in higher classes.
4. The derivative of a function, when it exists, is a unique function. But the integral of a function is not unique. However, they are unique upto an additive constant, i.e. any two integrals of a function differ by a constant.

   e.g. If $\frac{d}{dx}(\sin x) = \cos x$. [unique]

   Then, $\int (\cos x)dx = \sin x + C, C \in R$ [not unique]
5. When a polynomial function P is differentiated, then the resultant is also a polynomial whose degree is 1 less than the degree of P.

3 If $f'(x) = x + \frac{1}{x}$, then the value of $f(x)$ is

(a) $x^2 + \log x + C$

(b) $\frac{x^2}{2} + \log|x| + C$

(c) $\frac{x}{2} + \log x + C$

(d) None of the above

4 Family of curves $y = F(x) + C$ can be represented geometrically by shifting any one of the curves ...A... to itself. Here, A refers to

(a) perpendicular (b) parallel

(c) Both (a) and (b) (d) None of these

to the axis representing the variable of integration.

8. The derivative is used for finding some physical quantities like the velocity of moving particle, when the distance travelled at any time t is known. Similarly, the integral is used for calculating the distance travelled, when velocity at time t is known.
9. Differentiation and integration both are processes involving limits.
10. The process of differentiation and integration are inverse of each other.

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 The anti-derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ equal to **[NCERT]**

(a) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ (b) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$

(c) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ (d) $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

2 If $\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$, then $f(x)$ is **[NCERT]**

(a) $x^4 + \frac{1}{x^3} - \frac{129}{8}$

(b) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(c) $x^4 + \frac{1}{x^3} + \frac{129}{8}$

(d) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

SHORT ANSWER Type I Questions

Directions (Q. Nos. 17-19) *Evaluate the following integrals.*

17 $\int \frac{(a^x + b^x)^2}{a^x b^x} dx$

18 $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$

19 $\int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}} dx$ **[NCERT Exemplar]**

Directions (Q. Nos. 20-22) *Evaluate the following integrals.*

20 $\int (e^{x\log a} + e^{a\log x} + e^{a\log a})\, dx$

VERY SHORT ANSWER Type Questions

5 Integrate $\left(\frac{2a}{\sqrt{x}}-\frac{b}{x^2}+3c\sqrt[3]{x^2}\right)$ w.r.t. x. [NCERT Exemplar]

6 Write the anti-derivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$. [Delhi 2014]

Directions (Q. Nos. 7-14) *Evaluate the following integrals.*

7 $\int(3\operatorname{cosec}^2 x-5x+\sin x)dx$

8 $\int\frac{x^3+5x^2-4}{x^2}dx$ [NCERT]

9 $\int\frac{3x}{3x-1}dx$ [All India 2017C]

10 $\int\frac{2\cos x}{\sin^2 x}dx$ [All India 2011C]

11 $\int\frac{1}{\sin^2 x\cdot\cos^2 x}dx$ [Delhi 2014C]

12 Evaluate $\int\frac{\cos 2x+2\sin^2 x}{\cos^2 x}dx$. [CBSE 2018]

13 $\int\frac{\sin^2 x-\cos^2 x}{\sin x\cos x}dx$ [All India 2017]

14 $\int\frac{2+3\cos x}{\sin^2 x}dx$

15 Write the value of $\int\frac{(1-\sin x)}{\cos^2 x}dx$. [All India 2011C]

16 Write the value of $\int\frac{2-3\sin x}{\cos^2 x}dx$. [Delhi 2011]

21 $\int\frac{x^4}{x^2+1}dx$

22 $\int\frac{(x^3+8)(x-1)}{x^2-2x+4}dx$

23 If $\frac{dy}{dx}=\cos x+\sec^2 x$ and $x=y=0$, then find y.

24 If $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$ such that $f(2)=0$. Then, find $f(x)$. [NCERT]

25 Verify the following, using the concept of integration as an anti-derivative.

$$\int\frac{x^3}{x+1}dx=x-\frac{x^2}{2}+\frac{x^3}{3}-\log|x+1|+C$$

[NCERT Exemplar]

HINTS & SOLUTIONS

1. (c) **Hint** Anti-derivative of $\sqrt{x}+\frac{1}{\sqrt{x}}=\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$

$$=\frac{2}{3}x^{3/2}+2x^{1/2}+C$$

2. (a) **Hint** $f(x)=\int\left(4x^3-\frac{3}{x^4}\right)$

$$=x^4+\frac{1}{x^3}+C$$

Now, $f(2)=0\Rightarrow C=\frac{-129}{8}$

3. (b) **Hint** $f(x)=\int f'(x)\,dx=\int\left(x+\frac{1}{x}\right)dx$

$$=\int\frac{1}{\cos^2 x}dx=\int\sec^2 x\,dx$$

$$=\tan x+C$$

4. (b) Different values of C correspond to different members of this family and the graph of these members can be obtained by shifting any one of the curves parallel to itself.

5. Let $I=\int\left(\frac{2a}{\sqrt{x}}-\frac{b}{x^2}+3c\sqrt[3]{x^2}\right)dx$

$$=\int\frac{2a}{\sqrt{x}}dx-\int\frac{b}{x^2}dx+\int 3c\sqrt[3]{x^2}\,dx$$

$$=2a\int x^{-1/2}dx-b\int x^{-2}dx+3c\int x^{2/3}\,dx$$

$$=2a\left[\frac{x^{(-1/2)+1}}{(-1/2)+1}\right]-b\left[\frac{x^{-2+1}}{-2+1}\right]+3c\left[\frac{x^{(2/3)+1}}{(2/3)+1}\right]+C$$

$$\left[\because\int x^n\,dx=\frac{x^{n+1}}{n+1}\right]$$

$$=2a\left[\frac{x^{1/2}}{1/2}\right]-b\left[\frac{x^{-1}}{(-1)}\right]+3c\left[\frac{x^{5/3}}{5/3}\right]+C$$

$$=4a\sqrt{x}+\frac{b}{x}+\frac{9c}{5}x^{5/3}+C$$

6. **Hint** $3\int\sqrt{x}dx+\int\frac{1}{\sqrt{x}}dx=3\int x^{1/2}dx+\int x^{-1/2}dx.$

[**Ans.** $2(x^{3/2}+x^{1/2})+C$]

7. Let $I=\int(3\operatorname{cosec}^2 x-5x+\sin x)\,dx$

13. Solve as Question 11. [**Ans.** $\log|\sec x\cdot\operatorname{cosec} x|+C$]

14. Let $I=\int\frac{2+3\cos x}{\sin^2 x}dx=\int\left(\frac{2}{\sin^2 x}+\frac{3\cos x}{\sin^2 x}\right)dx$

$$=2\int\frac{1}{\sin^2 x}dx+3\int\frac{\cos x}{\sin^2 x}dx$$

$$=2\int\operatorname{cosec}^2 x\,dx+3\int\left(\frac{\cos x}{\sin x}\right)\cdot\frac{1}{\sin x}dx$$

$$=2\int\operatorname{cosec}^2 x\,dx+3\int\cot x\cdot\operatorname{cosec} x\,dx$$

$$=2(-\cot x)+3(-\operatorname{cosec} x)+C$$

$$=-2\cot x-3\operatorname{cosec} x+C$$

15. **Hint** Firstly, write the given integral as $\int(\sec^2 x-\tan x\sec x)dx$ and then integrate separately.

[**Ans.** $\tan x-\sec x+C$]

16. Solve as Question 15. [**Ans.** $2\tan x-3\sec x+C$]

17. Let $I=\int\frac{(a^x+b^x)^2}{a^x\cdot b^x}dx=\int\left(\frac{a^{2x}+b^{2x}+2a^x\cdot b^x}{a^x\cdot b^x}\right)dx$

$$[\because(a+b)^2=a^2+b^2+2ab]$$

$$=\int\left(\frac{a^{2x}}{a^x\cdot b^x}+\frac{b^{2x}}{a^x\cdot b^x}+\frac{2a^x\cdot b^x}{a^x\cdot b^x}\right)dx$$

$$= 3\int \text{cosec}^2 x\, dx - 5\int x\, dx + \int \sin x\, dx$$

$$= 3(-\cot x) - \frac{5x^2}{2} - \cos x + C$$

$$= -3\cot x - \frac{5x^2}{2} - \cos x + C$$

8. **Hint** Firstly, write the given integral as $\int\left((x+5)-\frac{4}{x^2}\right)dx$ and then integrate separately. $\left[\textbf{Ans. } \frac{x^2}{2}+5x+\frac{4}{x}+C\right]$

9. Let $I = \int \frac{3x}{3x-1}dx = \int \frac{(3x-1)+1}{3x-1}dx$

$$= \int dx + \int \frac{dx}{3x-1} = x + \frac{\log|3x-1|}{3} + C$$

10. **Hint** Write the given integral as $2\int \cot x \,\text{cosec}\, x\, dx$ and then integrate it. [Ans. $-2\,\text{cosec}\, x + C$]

11. Let $I = \int \frac{1}{\sin^2 x \cdot \cos^2 x}dx = \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cdot \cos^2 x}dx$

$$= \int \sec^2 x\, dx + \int \text{cosec}^2 x\, dx$$

$$= \tan x - \cot x + C$$

12. Let $I = \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\, dx$

$$= \int \frac{1-2\sin^2 x + 2\sin^2 x}{\cos^2 x}\, dx \quad [\because \cos 2A = 1-2\sin^2 A]$$

$$= \int \left(\frac{a^x}{b^x}+\frac{b^x}{a^x}+2\right)dx$$

$$= \int\left(\frac{a}{b}\right)^x dx + \int\left(\frac{b}{a}\right)^x dx + 2\int dx$$

$$= \frac{\left(\frac{a}{b}\right)^x}{\log\left(\frac{a}{b}\right)} + \frac{\left(\frac{b}{a}\right)^x}{\log\left(\frac{b}{a}\right)} + 2x + C \quad \left[\because \int a^x dx = \frac{a^x}{\log a}\right]$$

18. **Hint** Write $x^3 - x^2 + x - 1 = x^2(x-1) + 1(x-1)$
$= (x^2+1)(x-1)$ $\left[\text{Ans. } \frac{x^3}{3} + x + C\right]$

19. Let $I = \int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}dx = \int \frac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}}dx$

$$= \int \frac{x^6 - x^5}{x^4 - x^3}dx \quad [\because e^{\log f(x)} = f(x)]$$

$$= \int \frac{x^5(x-1)}{x^3(x-1)}dx = \int x^2 dx = \frac{x^3}{3} + C$$

20. Let $I = \int (e^{x\log a} + e^{a\log x} + e^{a\log a})dx$

$$= \int (e^{\log a^x} + e^{\log x^a} + e^{\log a^a})dx \quad [\because m\log n = \log n^m]$$

$$= \int (a^x + x^a + a^a)dx \quad [\because e^{\log f(x)} = f(x)]$$

$$= \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + a^a x + C \quad \left[\because \int a^x\, dx = \frac{a^x}{\log a}\right]$$

21. Let $I = \int \frac{x^4}{x^2+1}dx = \int \frac{x^4 - 1 + 1}{x^2+1}dx$

$$= \int \frac{x^4-1}{x^2+1}dx + \int \frac{1}{x^2+1}dx$$

$$= \int\left(\frac{(x^2-1)(x^2+1)}{x^2+1} + \frac{1}{x^2+1}\right)dx$$

$$[\because a^4 - b^4 = (a^2-b^2)(a^2+b^2)]$$

$$= \int (x^2-1)dx + \int \frac{1}{x^2+1}dx$$

$$= \int x^2 dx - \int dx + \int \frac{dx}{x^2+1}$$

$$= \frac{x^3}{3} - x + \tan^{-1} x + C \quad \left[\because \int \frac{dx}{x^2+1} = \tan^{-1} x\right]$$

22. Let $I = \int \frac{(x^3+8)(x-1)}{x^2-2x+4}dx$

$$= \int \frac{(x+2)(x^2+2^2-2x)(x-1)}{(x^2-2x+4)}\, dx$$

$$= \int (x+2)(x-1)\, dx = \int (x^2+x-2)\, dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} - 2x + C$$

23. Given, $\frac{dy}{dx} = \cos x + \sec^2 x$

On integrating both sides, we get

$$\int \frac{dy}{dx}\, dx = \int (\cos x + \sec^2 x)\, dx$$

$$\Rightarrow \quad \int dy = \int (\cos x + \sec^2 x)\, dx$$

$$\Rightarrow \quad y = \int \cos x\, dx + \int \sec^2 x\, dx$$

$$\Rightarrow \quad y = \sin x + \tan x + C \quad \text{...(i)}$$

Also, given $x = y = 0$

On putting $x = y = 0$ in Eq. (i), we get

$$0 = \sin 0 + \tan 0 + C$$

$$\Rightarrow \quad 0 = 0 + 0 + C$$

$$\Rightarrow \quad C = 0$$

On putting $C = 0$ in Eq. (i), we get

$$y = \sin x + \tan x + 0$$

$$\Rightarrow \quad y = \sin x + \tan x$$

24. **Hint** Firstly integrate, further to find C, put $x = 2$ and use the condition $f(2) = 0$.

$\left[\text{Ans. } x^4 + \frac{1}{x^3} - \frac{129}{8}\right]$

25. Consider, $\frac{d}{dx}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \log|x+1| + C\right)$

$$= 1 - \frac{2x}{2} + \frac{3x^2}{3} - \frac{1}{x+1}$$

$$= 1 - x + x^2 - \frac{1}{x+1}$$

$$= \frac{(x+1)(x^2-x+1)-1}{x+1} = \frac{(x^3+1)-1}{x+1} = \frac{x^3}{x+1}$$

$$[\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)]$$

Hence, $x - \frac{x^2}{2} + \frac{x^3}{3} - \log|x+1| + C = \int \frac{x^3}{x+1}\, dx$

|TOPIC 2|

Integration by Substitution

In previous topic, we discussed the integrals of those functions which are in standard forms. But integrals of certain functions cannot be obtained directly, if they are not in one of the standard forms given in previous topic. For such integrals we use the following method to solve

(i) Integration by substitution
(ii) Integration by partial fractions
(iii) Integration by parts

Here, we consider the method of integration by substitution.

METHOD OF SUBSTITUTION

The method of reducing a given integral into one of the standard integrals by a proper substitution is called method of substitution. To evaluate an integral of the type $\int f\{g(x)\} \cdot g'(x)\, dx$, we substitute $g(x) = t$, so that $g'(x)\, dx = dt$.

EXAMPLE |1| Evaluate $\int \frac{e^{\tan^{-1} x}}{1 + x^2}\, dx$. **[All India 2011]**

Sol. Let $I = \int \frac{e^{\tan^{-1} x}}{1 + x^2}\, dx$

Here, integrand has differentiation of $\tan^{-1} x$, so we substitute for $\tan^{-1} x$.

Now, put $\tan^{-1} x = t$

$\Rightarrow \frac{1}{1 + x^2} dx = dt$ [differentiating both sides w.r.t. x]

Now, $I = \int e^t\, dt$

Thus, $I = \int e^t\, dt = e^t + C$

On putting $t = \tan^{-1} x$, we get

$I = e^{\tan^{-1} x} + C$

EXAMPLE |2| Evaluate $\int \cos 6x \sqrt{1 + \sin 6x}\, dx$. **[NCERT]**

Sol. Let $I = \int \cos 6x \sqrt{1 + \sin 6x}\, dx$

Now, put $1 + \sin 6x = t$

$\Rightarrow \cos 6x \cdot 6\, dx = dt \Rightarrow \cos 6x\, dx = \frac{dt}{6}$

$\therefore \quad I = \frac{1}{6} \int \sqrt{t}\, dt = \frac{1}{6} \frac{t^{1/2 + 1}}{\frac{1}{2} + 1} + C$

$= \frac{1}{6} \cdot \frac{t^{3/2}}{\frac{3}{2}} + C = \frac{1}{6} \times \frac{2}{3} t^{3/2} + C$

Method to Find Integration by Substitution

Suppose given integral is $I = \int f(x)\, dx$, where $f(x)$ is not in any standard form.

Then, we use the following steps

I. Identify the term for which we use substitution say $g(x)$.

II. Take suitable substitution to reduce the integral into standard form, say $g(x) = t$.

III. Now, integrate with respect to t and then put $t = g(x)$ to get the required value.

Note It is often important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand.

Sol. Let $I = \int \frac{dx}{1 + \tan x} = \int \frac{dx}{1 + \frac{\sin x}{\cos x}}$

$= \int \frac{\cos x}{\cos x + \sin x}\, dx = \int \frac{2 \cos x\, dx}{2(\cos x + \sin x)}$

$= \frac{1}{2} \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\cos x + \sin x}\, dx$

[adding and subtracting $(\sin x)$ in numerator]

$= \frac{1}{2} \int 1\, dx + \int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx$

Now, put $\cos x + \sin x = t \Rightarrow (-\sin x + \cos x)\, dx = dt$

Then, we get

$I = \frac{1}{2} \int 1\, dx + \int \frac{dt}{t} = \frac{x}{2} + \log|t| + C$

$= \frac{x}{2} + \log|\cos x + \sin x| + C$ [$\because t = \cos x + \sin x$]

Some Standard Formulae

Some standard formulae for integrals involving trigonometric functions are given below. These formulae are obtained by using substitution technique.

(i) $\int \tan x\, dx = -\log|\cos x| + C = \log|\sec x| + C$

(ii) $\int \cot x\, dx = \log|\sin x| + C$

(iii) $\int \sec x\, dx = \log|\sec x + \tan x| + C$

(iv) $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$

Proof

$$= \frac{1}{9}(1 + \sin 6x)^{3/2} + C \qquad [\because t = 1 + \sin 6x]$$

EXAMPLE |3| Integrate the following function w.r.t. x.

$$\frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}$$

[NCERT]

Sol. Let $I = \int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} dx$

Now, put $\tan \sqrt{x} = t$

$$\Rightarrow \quad \sec^2 \sqrt{x} \cdot \frac{1}{2\sqrt{x}} dx = dt \Rightarrow \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx = 2dt$$

$$\therefore \quad I = \int t^4 (2dt) = 2\int t^4 dt$$

$$= 2 \cdot \frac{t^5}{5} + C = \frac{2}{5} \tan^5 \sqrt{x} + C$$

EXAMPLE |4| Evaluate $\int \frac{dx}{1 + \tan x}$.

Given integrand is not in standard form, so firstly convert in $\sin x$ and $\cos x$ form. Further, adjust them and use suitable substitution to integrate easily.

Put $\sec x + \tan x = t$

$\Rightarrow$ $(\sec x \tan x + \sec^2 x)\, dx = dt$

$\therefore$ $I = \int \frac{1}{t} dt = \log |t| + C$

$\Rightarrow$ $I = \log |\sec x + \tan x| + C \quad [\because t = \sec x + \tan x]$

(iv) Firstly, multiplying numerator and denominator by $(\operatorname{cosec} x + \cot x)$. Further, put $\operatorname{cosec} x + \cot x = t$ and then solve as part (iii).

Some Important Deductions

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$ and n is a rational number.

(ii) $\int \sin(ax+b)\, dx = \frac{-\cos(ax+b)}{a} + C$

(iii) $\int \cos(ax+b)\, dx = \frac{\sin(ax+b)}{a} + C$

(iv) $\int \tan(ax+b) dx = -\frac{1}{a} \log|\cos(ax+b)| + C$

$= \frac{1}{a} \log|\sec(ax+b)| + C$

(v) $\int \cot(ax+b)\, dx = \frac{1}{a} \log|\sin(ax+b)| + C$

(vi) $\int \sec(ax+b)\, dx = \frac{1}{a} \log|\sec(ax+b) + \tan(ax+b)| + C$

(vii) $\int \operatorname{cosec}(ax+b)\, dx = \frac{1}{a} \log|\operatorname{cosec}(ax+b) - \cot(ax+b)| + C$

(viii) $\int \sec(ax+b) \tan(ax+b)\, dx = \frac{1}{a} \sec(ax+b) + C$

(i) Let $I = \int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$

Now, put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$\Rightarrow$ $\sin x\, dx = -dt$

$$\therefore \quad I = -\int \frac{1}{t} dt = -\log|t| + C$$

$$= -\log|\cos x| + C \qquad [\because t = \cos x]$$

$$= \log|(\cos x)^{-1}| + C$$

$$= \log|\sec x| + C \qquad \left[\because \frac{1}{\cos x} = \sec x\right]$$

(ii) Put $\sin x = t$ and then solve same as above.

(iii) Let $I = \int \sec x\, dx$

On multiplying numerator and denominator by $(\sec x + \tan x)$, we get

$$I = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx$$

$$I = \int \frac{(\sec^2 x + \sec x \tan x)}{(\sec x + \tan x)} dx$$

(ii) Let $I = \int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}}$

$$= \int \frac{(\sqrt{x+a} - \sqrt{x+b})}{(\sqrt{x+a} + \sqrt{x+b})(\sqrt{x+a} - \sqrt{x+b})} dx$$

[rationalising denominator]

$$= \int \frac{\sqrt{x+a} - \sqrt{x+b}}{(\sqrt{x+a})^2 - (\sqrt{x+b})^2} dx$$

$$= \int \frac{(\sqrt{x+a} - \sqrt{x+b})}{(x+a) - (x+b)} dx$$

$$= \frac{1}{a-b} \int [(x+a)^{1/2} - (x+b)^{1/2}] dx$$

$$= \frac{1}{a-b} \left[\frac{2}{3}(x+a)^{3/2} - \frac{2}{3}(x+b)^{3/2}\right] + C$$

$$= \frac{2}{3(a-b)} [(x+a)^{3/2} - (x+b)^{3/2}] + C$$

Integration Using Trigonometric Identities

When the integrand involves some trigonometric functions, then some known trigonometric identities are used to evaluate integral easily.

IMPORTANT TRIGONOMETRIC IDENTITIES

Some useful trigonometric identities are given below

1. $\sin 2x = 2 \sin x \cos x = \frac{2 \tan x}{1 + \tan^2 x}$

(ix) $\int \text{cosec}\,(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\text{cosec}\,(ax+b)+C$

(x) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+C$

(xi) $\int \text{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)+C$

(xii) $\int e^{(ax+b)}\,dx = \frac{e^{(ax+b)}}{a}+C$ (xiii) $\int a^{mx+b}\,dx = \frac{a^{mx+b}}{m\log_e a}+C$

Note Above integral can be derived by substituting $ax+b=t$ and $dx=\frac{1}{a}dt$.

EXAMPLE |5| Evaluate the following integrals.

(i) $\int \sqrt{ax+b}\,dx$ (ii) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$ **[NCERT]**

Sol. (i) Let $I = \int \sqrt{ax+b}\,dx = \int (ax+b)^{1/2}\,dx$

$$= \frac{(ax+b)^{(1/2)+1}}{\{(1/2)+1\}\cdot a}+C = \frac{2}{3}\frac{(ax+b)^{3/2}}{a}+C$$

$$= \frac{2}{3a}(ax+b)^{3/2}+C$$

10. $\cot\frac{x}{2} = \text{cosec}\,x + \cot x = \sqrt{\frac{1+\cos x}{1-\cos x}}$

$$= \frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$$

11. $\sin(x+y) = \sin x\cos y + \cos x\sin y$
12. $\sin(x-y) = \sin x\cos y - \cos x\sin y$
13. $\cos(x+y) = \cos x\cos y - \sin x\sin y$
14. $\cos(x-y) = \cos x\cos y + \sin x\sin y$
15. $\tan(x+y) = \frac{\tan x+\tan y}{1-\tan x\tan y}$
16. $\tan(x-y) = \frac{\tan x-\tan y}{1+\tan x\tan y}$
17. $2\sin A\cos B = \sin(A+B)+\sin(A-B)$
18. $2\cos A\sin B = \sin(A+B)-\sin(A-B)$
19. $2\cos A\cos B = \cos(A+B)+\cos(A-B)$
20. $2\sin A\sin B = \cos(A-B)-\cos(A+B)$
21. $\sin A+\sin B = 2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}$
22. $\sin A-\sin B = 2\cos\frac{A+B}{2}\cdot\sin\frac{A-B}{2}$
23. $\cos A+\cos B = 2\cos\frac{A+B}{2}\cdot\cos\frac{A-B}{2}$
24. $\cos A-\cos B = -2\sin\frac{A+B}{2}\cdot\sin\frac{A-B}{2}$
25. **Inverse trigonometric functions**

(i) $\sin^{-1}(\sin x) = x$ (ii) $\cos^{-1}(\cos x) = x$

(iii) $\tan^{-1}(\tan x) = x$ (iv) $\text{cosec}^{-1}(\text{cosec}\,x) = x$

(v) $\sec^{-1}(\sec x) = x$ (vi) $\cot^{-1}(\cot x) = x$

2. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$

$$= 1-2\sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

3. $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$
4. $\cot 2x = \frac{\cot^2 x-1}{2\cot x}$
5. $\sin 3x = -\sin^3 x + 3\cos^2 x\sin x = -4\sin^3 x + 3\sin x$
6. $\cos 3x = \cos^3 x - 3\sin^2 x\cos x = 4\cos^3 x - 3\cos x$
7. $\tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}$
8. $\cot 3x = \frac{3\cot x - \cot^3 x}{1-3\cot^2 x}$
9. $\tan\frac{x}{2} = \text{cosec}\,x - \cot x = \sqrt{\frac{1-\cos x}{1+\cos x}}$

$$= \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

$$= x\cos a - \sin a\log|\sin(x+a)| + C_1 + a\cos a$$

$$= x\cos a - \sin a\log|\sin(x+a)| + C,$$

where, $C = C_1 + a\cos a$

EXAMPLE |7| Evaluate $\int \frac{1}{\cos(x-\alpha)\cos(x-\beta)}\,dx.$

Sol. Let $I = \int \frac{1}{\cos(x-\alpha)\cos(x-\beta)}\,dx$

$$= \int \frac{1}{\cos(x-\alpha)\cos(x-\beta)} \times \frac{\sin(\beta-\alpha)}{\sin(\beta-\alpha)}\,dx$$

[multiplying numerator and denominator by $\sin(\beta-\alpha)$]

$$= \frac{1}{\sin(\beta-\alpha)}\int \frac{\sin[(x-\alpha)-(x-\beta)]}{\cos(x-\alpha)\cos(x-\beta)}\,dx$$

[adding and subtracting x from numerator]

$$= \frac{1}{\sin(\beta-\alpha)}\int \frac{\sin(x-\alpha)\cdot\cos(x-\beta)-\cos(x-\alpha)\cdot\sin(x-\beta)}{\cos(x-\alpha)\cos(x-\beta)}\,dx$$

[$\because \sin(A-B) = \sin A\cos B - \cos A\sin B$]

$$= \frac{1}{\sin(\beta-\alpha)}\int\left[\frac{\sin(x-\alpha)\cos(x-\beta)}{\cos(x-\alpha)\cos(x-\beta)} - \frac{\cos(x-\alpha)\sin(x-\beta)}{\cos(x-\alpha)\cos(x-\beta)}\right]dx$$

$$= \frac{1}{\sin(\beta-\alpha)}\int[\tan(x-\alpha)-\tan(x-\beta)]\,dx$$

$$= \frac{1}{\sin(\beta-\alpha)}[-\log|\cos(x-\alpha)| + \log|\cos(x-\beta)|] + C$$

[$\because \int \tan x\,dx = -\log|\cos x|$]

$$= \frac{1}{\sin(\beta-\alpha)}\log\left|\frac{\cos(x-\beta)}{\cos(x-\alpha)}\right| + C$$

$\left[\because \log m - \log n = \log\frac{m}{n}\right]$

EXAMPLE |6| Evaluate the integral $\int \frac{\sin x}{\sin (x+a)}\, dx$.

Sol. Let $I = \int \frac{\sin x}{\sin (x+a)} dx$

Now, put $x + a = t \Rightarrow dx = dt$

$\therefore \quad I = \int \frac{\sin (t-a)}{\sin t} dt$

$= \int \frac{\sin t \cos a - \cos t \sin a}{\sin t} dt$

$[\because \sin (A - B) = \sin A \cos B - \cos A \sin B]$

$= \cos a \int dt - \sin a \int \cot t \, dt$

$= (\cos a)\, t - \sin a \log |\sin t| + C_1$

$= (x + a) \cos a - \sin a \log |\sin (x+a)| + C_1$

$[\because t = x + a]$

Now, putting $\cos \alpha + \cot x \sin \alpha = t$

$\Rightarrow -\operatorname{cosec}^2 x \sin \alpha \, dx = dt \Rightarrow \operatorname{cosec}^2 x \, dx = \frac{-dt}{\sin \alpha}$

$\therefore \quad I = \frac{-1}{\sin \alpha} \int \frac{dt}{\sqrt{t}} = \frac{-1}{\sin \alpha} [2\sqrt{t}] + C$

$= \frac{-1}{\sin \alpha} [2\sqrt{\cos \alpha + \cot x \sin \alpha}] + C$

$= \frac{-2}{\sin \alpha} \sqrt{\cos \alpha + \frac{\cos x \sin \alpha}{\sin x}} + C$

$= \frac{-2}{\sin \alpha} \sqrt{\frac{\sin x \cos \alpha + \cos x \sin \alpha}{\sin x}} + C$

$= -\frac{2}{\sin \alpha} \sqrt{\frac{\sin (x+\alpha)}{\sin x}} + C$

$[\because \sin (A + B) = \sin A \cos B + \cos A \sin B]$

EXAMPLE |9| Evaluate $\int \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right) dx$.

Use trigonometric formulae to convert the given integral into $\int \tan^{-1}\left(\tan \frac{x}{2}\right) dx$ and then integrate.

Sol. Let $I = \int \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right) dx = \int \tan^{-1}\left(\frac{2\sin^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}}\right) dx$

$\left[\because \sin^2 x = \frac{1-\cos 2x}{2} \text{ and } \sin x = 2\sin \frac{x}{2} \cos \frac{x}{2}\right]$

$= \int \tan^{-1}\left(\tan \frac{x}{2}\right) dx = \int \frac{x}{2} dx \qquad [\because \tan^{-1}(\tan \theta) = \theta]$

$= \frac{x^2}{2 \cdot 2} + C = \frac{x^2}{4} + C$

EXAMPLE |10| Evaluate $\int \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) dx$.

EXAMPLE |8| Evaluate $\int \frac{1}{\sqrt{\sin^3 x \sin (x+\alpha)}}\, dx$. **[NCERT]**

Sol. Let $I = \int \frac{1}{\sqrt{\sin^3 x \sin (x+\alpha)}} dx$

$= \int \frac{1}{\sqrt{\sin^3 x (\sin x \cos \alpha + \cos x \sin \alpha)}} dx$

$[\because \sin (A + B) = \sin A \cos B + \cos A \sin B]$

$= \int \frac{1}{\sqrt{\sin^4 x (\cos \alpha + \cot x \sin \alpha)}} dx$

$= \int \frac{1}{\sin^2 x \sqrt{\cos \alpha + \cot x \sin \alpha}} dx$

$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos \alpha + \cot x \sin \alpha}} dx$

Some Standard Integrals and Methods to Evaluate it

Integral	Methods
$\int \sin^p x\, dx$ or $\int \cos^p x\, dx$ where, $p \le 3$	To evaluate, we express $\sin^p x$ or $\cos^p x$ in terms of sines and cosines of multiples of x. For which we use the following trigonometrical identities (i) $\sin^2 x = \frac{1-\cos 2x}{2}$ (ii) $\cos^2 x = \frac{1+\cos 2x}{2}$ (iii) $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$ (iv) $\cos^3 x = \frac{\cos 3x + 3\cos x}{4}$
$\int \sin px \cos qx\, dx$, $\int \sin px \sin qx\, dx$, $\int \cos px \cos qx\, dx$	To evaluate these type of integral, firstly multiply and divide by 2 and then use the following trigonometrical identities $2\sin A \cos B = \sin (A+B) + \sin (A-B)$ $2\cos A \sin B = \sin (A+B) - \sin (A-B)$ $2\cos A \cos B = \cos (A+B) + \cos (A-B)$ $2\sin A \sin B = \cos (A-B) - \cos (A+B)$
$\int \tan^p x \sec^{2q} x\, dx$ or $\int \cot^p x \operatorname{cosec}^{2q} x\, dx$ where, p and $q \in N$	To evaluate these type of integral, firstly write the given integral as $\int \tan^p x (\sec^2 x)^{q-1} \cdot \sec^2 x\, dx$ or $\int \cot^p x (\operatorname{cosec}^2 x)^{q-1} \cdot \operatorname{cosec}^2 x\, dx$ and then put $\tan x = t$ (or $\cot x = t$)
$\int \tan^{2p+1} x \sec^{2q+1} x\, dx$ or $\int \cot^{2p+1} x \operatorname{cosec}^{2q+1} x\, dx$,	To evaluate these type of integral, firstly write the given integral as $\int (\tan^2 x)^p (\sec^2 x)^q \sec x \tan x\, dx$ or

EXAMPLE |11| Evaluate $\int \sin^2 x\, dx$.

Sol. Let $I = \int \sin^2 x\, dx$

$$= \int \frac{1-\cos 2x}{2} dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$$

$$= \frac{1}{2}\int (1-\cos 2x) dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C$$

EXAMPLE |12| Evaluate $\int \cos^4 x\, dx$.

Sol. Let $I = \int \cos^4 x\, dx = \int (\cos^2 x)^2 dx$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \qquad \left[\because \cos^2 x = \frac{1+\cos 2x}{2}\right]$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1+\cos 2(2x)}{2}\right) dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\int \left(\frac{2 + 4\cos 2x + 1 + \cos 4x}{2}\right) dx$$

$$= \frac{1}{8}\int (3 + 4\cos 2x + \cos 4x) dx$$

$$= \frac{1}{8}\left(3x + 4\cdot\frac{\sin 2x}{2} + \frac{\sin 4x}{4}\right) + C$$

$$= \frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

EXAMPLE |13| Evaluate $\int \cos 2x \cos 4x \cos 6x\, dx$. **[NCERT]**

There is no standard integral formula for given integral. So, we simplify the integral by using the formula $2\cos A \cos B = [\cos(A+B) + \cos(A-B)]$ and then integrate it.

Sol. Let $I = \int \cos 2x \cdot \cos 4x \cdot \cos 6x\, dx$

$$= \frac{1}{2}\int (2\cos 4x \cdot \cos 2x)\cdot \cos 6x\, dx$$

$$= \frac{1}{2}\int [\cos(4x+2x) + \cos(4x-2x)]\cdot \cos 6x\, dx$$

$$[\because 2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)]$$

$$= \frac{1}{2}\int (\cos 6x + \cos 2x)\cdot \cos 6x\, dx$$

$$= \frac{1}{2}\int (\cos^2 6x + \cos 6x \cdot \cos 2x) dx$$

$$= \frac{1}{4}\int (2\cos^2 6x + 2\cos 6x \cdot \cos 2x) dx$$

$$= \frac{1}{4}\int (1 + \cos 12x + \cos 8x + \cos 4x) dx$$

$$\left[\begin{array}{l}\because \cos^2\theta = \dfrac{1+\cos 2\theta}{2} \text{ and} \\ 2\cos x \cos y = \cos(x+y) + \cos(x-y)\end{array}\right]$$

$$= \frac{1}{4}\left(x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right) + C$$

$$\left[\because \int \cos ax\, dx = \frac{1}{a}\sin ax\right]$$

$$= \frac{x}{4} + \frac{1}{48}\sin 12x + \frac{1}{32}\sin 8x + \frac{1}{16}\sin 4x + C$$

EXAMPLE |14| Evaluate $\int \tan^2 x \sec^4 x\, dx$. **[NCERT Exemplar]**

Firstly, write $\sec^4 x$ as $\sec^2 x \cdot \sec^2 x$ and use the formula $\sec^2 x = 1 + \tan^2 x$, further put $\tan x = t$ and then integrate it.

Sol. Let $I = \int \tan^2 x \sec^4 x\, dx = \int \tan^2 x \sec^2 x \sec^2 x\, dx$

$$= \int \tan^2 x (1 + \tan^2 x) \sec^2 x\, dx \qquad [\because \sec^2 x = 1 + \tan^2 x]$$

Now, put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\therefore \quad I = \int t^2 (1 + t^2)\, dt = \int (t^2 + t^4) dt$$

$$= \frac{t^3}{3} + \frac{t^5}{5} + C$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C \qquad [\because t = \tan x]$$

EXAMPLE |15| Evaluate $\int \tan^3 x \sec^3 x\, dx$.

Sol. Let $I = \int \tan^3 x \sec^3 x\, dx = \int \tan^2 x \sec^2 x \cdot \tan x \sec x\, dx$

$$= \int (\sec^2 x - 1)\sec^2 x \cdot \tan x \sec x\, dx$$

$$[\because \tan^2 x = \sec^2 x - 1]$$

Now, put $\sec x = t \Rightarrow \sec x \tan x\, dx = dt$

$$\therefore \quad I = \int (t^2 - 1)\cdot t^2\, dt = \int (t^4 - t^2)\, dt = \frac{t^5}{5} - \frac{t^3}{3} + C$$

$$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C \qquad [\because t = \sec x]$$

EXAMPLE |16| Evaluate $\int \sin^3 x \cos^5 x\, dx$.

Sol. Let $I = \int \sin^3 x \cos^5 x\, dx$

Here, the powers of both $\sin x$ and $\cos x$ are odd. So, either put $\sin x = t$ or $\cos x = t$.

Now, put $\cos x = t$

$$\Rightarrow \quad -\sin x\, dx = dt \Rightarrow dx = \frac{-dt}{\sin x}$$

$$\therefore \quad I = \int \sin^3 x \cdot t^5 \left(\frac{-dt}{\sin x}\right)$$

$$= -\int \sin^2 x\, t^5\, dt = -\int (1-\cos^2 x)\, t^5 dt$$

$$= -\int (1-t^2)\, t^5\, dt \qquad [\because \cos x = t]$$

$$= -\int (t^5 - t^7)\, dt = \int (t^7 - t^5)\, dt$$

$$= \frac{t^8}{8} - \frac{t^6}{6} + C = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C$$

EXAMPLE |17| Evaluate $\int \sec^{4/3} x \operatorname{cosec}^{8/3} x\, dx$.

Sol. Let $I = \int \sec^{4/3} x \operatorname{cosec}^{8/3} x\, dx$

$$= \int \frac{dx}{\cos^{4/3} x \cdot (\sin^{8/3} x)} = \int \cos^{-4/3} x \sin^{-8/3} x\, dx$$

Here, $-\left(\frac{4}{3} + \frac{8}{3}\right) = -4$, which is an negative even integer.

So, divide both numerator and denominator by $\cos^4 x$.

$$\therefore \quad I = \int \frac{\frac{\cos^{-4/3} x \sin^{-8/3} x}{\cos^4 x}}{\frac{1}{\cos^4 x}} dx$$

$$= \int \frac{\cos^4 x \cos^{-4/3} x \sin^{-8/3} x}{\cos^4 x} dx$$

$$= \int \frac{\cos^{8/3} x \sin^{-8/3} x}{\cos^4 x} dx = \int \frac{\sec^4 x}{\tan^{8/3} x} dx$$

$$= \int \frac{\sec^2 x \cdot \sec^2 x}{\tan^{8/3} x} dx = \int \frac{\sec^2 x \cdot (1 + \tan^2 x)}{\tan^{8/3} x} dx$$

Now, put $\tan x = t$

$\Rightarrow \quad \sec^2 x\, dx = dt$

$$\therefore \quad I = \int \frac{1 + t^2}{t^{8/3}} dt = \int (t^{-8/3} + t^{-2/3})\, dt$$

$$= -\frac{3}{5} t^{-5/3} + 3t^{1/3} + C$$

$$= -\frac{3}{5} \tan^{-5/3} x + 3 \tan^{1/3} x + C \qquad [\because t = \tan x]$$

SOME SPECIAL INTEGRALS

Here, we discussed some standard formulae with their proof and the methods to solve some other standard integrals with the help of these formulae.

(i) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$

(ii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$

(iii) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

(iv) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$

(v) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$

(vi) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$

Proof

(i) Let $I = \int \frac{dx}{x^2 - a^2}$

$$= \int \frac{1}{(x-a)(x+a)} \times \frac{2a}{2a} dx$$

[multiplying numerator and denominator by $2a$]

$$= \frac{1}{2a} \int \frac{(x+a) - (x-a)}{(x-a)(x+a)} dx$$

$$= \frac{1}{2a} \int \left[\frac{1}{(x-a)} - \frac{1}{(x+a)} \right] dx$$

$$= \frac{1}{2a} \left[\int \frac{1}{(x-a)} dx - \int \frac{1}{(x+a)} dx \right]$$

$$= \frac{1}{2a} [\log |x-a| - \log |x+a|] + C$$

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

(ii) Similarly, we can prove second formula.

(iii) Now, put $x = a \tan\theta$ and $dx = a \sec^2 \theta\, d\theta$ in $\int \frac{dx}{x^2 + a^2}$, we get

$$\int \frac{dx}{x^2 + a^2} = \int \frac{a \sec^2 \theta}{a^2 \tan^2 \theta + a^2} d\theta$$

$$= \int \frac{a \sec^2 \theta}{a^2 (\sec^2 \theta)} d\theta \quad [\because \sec^2 \theta = \tan^2 \theta + 1]$$

$$= \frac{1}{a} \int d\theta = \frac{1}{a} \theta + C$$

$$\Rightarrow \quad \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$\left[\because x = a \tan\theta \Rightarrow \tan\theta = \frac{x}{a} \Rightarrow \theta = \tan^{-1} \frac{x}{a} \right]$$

Similarly, we can prove other formulae by using following substitutions

(iv) Put $x = a \sec\theta$

(v) Put $x = a \sin\theta$

(vi) Put $x = a \tan\theta$

Some Standard Substitutions Which are Useful in Evaluating Integrals

	Expression	Substitution
1.	$a^2 - x^2$ or $\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
2.	$a^2 + x^2$ or $\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $a\cot\theta$
3.	$x^2 - a^2$ or $\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\operatorname{cosec}\theta$
4.	$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$
5.	$\sqrt{\frac{x-\alpha}{\beta - x}}$ or $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$
6.	$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$	$x = a\sin^2\theta$ or $x = a\cos^2\theta$
7.	$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$	$x = a\tan^2\theta$ or $x = a\cot^2\theta$

EXAMPLE |18| Find the following integrals.

(i) $\int \frac{dx}{x^2 - 16}$ (ii) $\int \frac{dx}{\sqrt{9 - 25x^2}}$

Sol. (i) Let $I = \int \frac{dx}{x^2 - 16} = \int \frac{dx}{x^2 - 4^2}$

$$= \frac{1}{2(4)} \log\left|\frac{x-4}{x+4}\right| + C$$

$$= \frac{1}{8}\log\left|\frac{x-4}{x+4}\right| + C$$

(ii) Let $I = \int \frac{dx}{\sqrt{9 - 25x^2}} = \int \frac{dx}{\sqrt{25\left(\frac{9}{25} - x^2\right)}}$

$$= \frac{1}{5}\int \frac{dx}{\sqrt{\left(\frac{3}{5}\right)^2 - x^2}} = \frac{1}{5}\sin^{-1}\left(\frac{x}{(3/5)}\right) + C$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

Integral of the Type $\int \frac{dx}{ax^2 + bx + c}$

Suppose given integral is of the form $\int \frac{dx}{ax^2 + bx + c}$, then to evaluate such integrals, we use the following steps

I. Firstly, take a common from denominator to make coefficient of x^2 unity, i.e. $I = \int \frac{dx}{a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)}$

II. Add and subtract $(b/2a)^2$ from denominator and try to write denominator in the form $X^2 \pm k^2$ or $k^2 - X^2$ where, $X^2 = \left(x \pm \frac{b}{2a}\right)^2$ and $k^2 = \left(\frac{c}{a} \mp \frac{b^2}{4a^2}\right)$.

III. Substitute $x \pm \frac{b}{2a} = t$ and reduce the integral obtained in step II into one of the form $\frac{1}{a}\int \frac{dt}{t^2 \pm k^2}$ or $\frac{1}{a}\int \frac{dt}{k^2 - t^2}$.

EXAMPLE |19| Evaluate $\int \frac{1}{9x^2 + 6x + 5}\,dx$.

Sol. Let $I = \int \frac{1}{9x^2 + 6x + 5}\,dx = \frac{1}{9}\int \frac{1}{x^2 + \frac{2}{3}x + \frac{5}{9}}\,dx$

[take 9 common from denominator]

$$= \frac{1}{9}\int \frac{1}{x^2 + 2\cdot\frac{1}{3}\cdot x + \left(\frac{1}{3}\right)^2 + \frac{5}{9} - \left(\frac{1}{3}\right)^2}\,dx$$

[here $a = 9$ and $b = 6$, adding and subtracting $\left(\frac{b}{2a}\right)^2 = \left(\frac{6}{2\times 9}\right)^2 = \left(\frac{1}{3}\right)^2$ from denominator]

$$= \frac{1}{9}\int \frac{1}{x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 + \frac{5}{9} - \frac{1}{9}}\,dx$$

$$= \frac{1}{9}\int \frac{1}{\left(x + \frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2}\,dx$$

$$= \frac{1}{9}\left(\frac{1}{2/3}\right)\tan^{-1}\left(\frac{x + 1/3}{2/3}\right) + C$$

$\left[\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right) + C$$

Integral of the Type $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$

To evaluate such integrals, we do same steps as previous integral. The only difference is that here reduced integral (obtained after step II) will be of the form $\frac{1}{a}\int \frac{dt}{\sqrt{t^2 \pm k^2}}$ or $\frac{1}{a}\int \frac{dt}{\sqrt{k^2 - t^2}}$, which can be integrated by using suitable formulae.

Note If we have integral of the form $\int \frac{dx}{(x \pm c)^2 \pm a^2}$ or $\int \frac{dx}{a^2 - (x \pm c)^2}$ or $\int \frac{dx}{\sqrt{(x \pm c)^2 \pm a^2}}$ or $\int \frac{dx}{\sqrt{a^2 - (x \pm c)^2}}$, then to evaluate it we can directly replace x by $x \pm c$ in the corresponding standard formulae.

e.g. $\int \frac{dx}{\sqrt{3^2 - (x+2)^2}} = \sin^{-1}\left(\frac{x+2}{3}\right) + C$

$$\left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C\right]$$

EXAMPLE |20| Evaluate $\int \frac{dx}{\sqrt{3 - x + x^2}}$.

Sol. Let $I = \int \frac{dx}{\sqrt{3 - x + x^2}} = \int \frac{dx}{\sqrt{x^2 - x + 3}}$

$$= \int \frac{dx}{\sqrt{x^2 - 2 \times x \times \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 3}}$$

$$\left[\text{here } a = 1, b = -1, \text{ adding and subtracting } \left(\frac{b}{2a}\right)^2 = \left(\frac{-1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \text{ from denominator}\right]$$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{1}{2}\right)^2 + 3 - \frac{1}{4}}} = \int \frac{dx}{\sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{11}{4}}}$$

Now, put $x - \frac{1}{2} = t \Rightarrow dx = dt$

$$\therefore \ I = \int \frac{dt}{\sqrt{t^2 + \left(\frac{\sqrt{11}}{2}\right)^2}} = \log\left|t + \sqrt{t^2 + \left(\frac{\sqrt{11}}{2}\right)^2}\right| + C$$

$$\left[\because \int \frac{dx}{\sqrt{x^2 + a^2}} = \log|x + \sqrt{x^2 + a^2}|\right]$$

$$= \log\left|\left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2}\right| + C \quad \left[\because t = x - \frac{1}{2}\right]$$

Hence, $I = \log\left|\left(x - \frac{1}{2}\right) + \sqrt{3 - x + x^2}\right| + C.$

EXAMPLE |21| Evaluate $\int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}} dx$.

Sol. Let $I = \int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}} dx$

Now, put $e^x = t \Rightarrow e^x dx = dt$

$$\therefore \quad I = \int \frac{dt}{\sqrt{5 - 4t - t^2}} = \int \frac{dt}{\sqrt{-(t^2 + 4t - 5)}}$$

$$= \int \frac{dt}{\sqrt{-[(t+2)^2 - 5 - 4]}}$$

$$\left[\text{adding and subtracting } \left(\frac{b}{2a}\right)^2 = \left(\frac{4}{2 \times 1}\right)^2 = (2)^2 \text{ from denominator}\right]$$

$$= \int \frac{dt}{\sqrt{9 - (t+2)^2}} = \int \frac{dt}{\sqrt{(3)^2 - (t+2)^2}}$$

$$= \sin^{-1}\left(\frac{t+2}{3}\right) + C \qquad \left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}\right]$$

$$= \sin^{-1}\left(\frac{e^x + 2}{3}\right) + C \qquad [\because t = e^x]$$

Integral of the Types $\int \frac{px + q}{ax^2 + bx + c} dx$ and $\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$

To evaluate such integrals, we firstly write the numerator as

$$px + q = A\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + B = A(2ax + b) + B$$

Then, find A and B by comparing the coefficients of like powers of x from both sides. Now, put the resultant value of $(px + q)$ in given integral and then given integral is reduced to one of the known forms which can be integrate easily.

EXAMPLE |22| Evaluate $\int \frac{x + 2}{2x^2 + 6x + 5} dx$. **[Delhi 2015C]**

Sol. Let $(x + 2) = A \frac{d}{dx}(2x^2 + 6x + 5) + B$

$$\Rightarrow \quad (x + 2) = A(4x + 6) + B \qquad \text{...(i)}$$

$$\Rightarrow \quad (x + 2) = 4Ax + (6A + B)$$

On equating the coefficients of x and constant term from both sides, we get $1 = 4A$ and $6A + B = 2$

$$\Rightarrow \ A = \frac{1}{4} \text{ and } 6 \cdot \frac{1}{4} + B = 2 \ \Rightarrow \ A = \frac{1}{4} \text{ and } B = 2 - \frac{3}{2}$$

$$\Rightarrow \ A = \frac{1}{4} \text{ and } B = \frac{1}{2}$$

$\therefore$ From Eq. (i), we get $(x + 2) = \frac{1}{4}(4x + 6) + \frac{1}{2}$

Now, the given integral can be written as

$$I = \int \frac{(x+2)}{2x^2 + 6x + 5} = \int \frac{\frac{1}{4}(4x + 6) + \frac{1}{2}}{2x^2 + 6x + 5} dx$$

$$= \frac{1}{4} \int \frac{4x + 6}{2x^2 + 6x + 5} dx + \frac{1}{2} \int \frac{dx}{2x^2 + 6x + 5}$$

$$\Rightarrow \quad I = \frac{1}{4} I_1 + \frac{1}{2} I_2 \qquad \text{...(ii)}$$

where, $I_1 = \int \frac{4x+6}{2x^2+6x+5}\,dx$

and $\quad I_2 = \int \frac{dx}{2x^2+6x+5}$

Now, consider $I_1 = \int \frac{4x+6}{2x^2+6x+5}\,dx$

Now, put $2x^2+6x+5 = t \Rightarrow (4x+6)\,dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \log|t| + C_1 = \log\left|2x^2+6x+5\right| + C_1$

and $\quad I_2 = \int \frac{dx}{2x^2+6x+5} = \frac{1}{2}\int \frac{dx}{x^2+3x+\frac{5}{2}}$

$$= \frac{1}{2}\int \frac{dx}{x^2 + 2\cdot\frac{3}{2}\cdot x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{5}{2}}$$

$\left[\text{adding and subtracting } \left(\frac{3}{2}\right)^2 \text{ from denominator}\right]$

$$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\cdot\frac{1}{\left(\frac{1}{2}\right)}\tan^{-1}\left(\frac{x+\frac{3}{2}}{\frac{1}{2}}\right) + C_2$$

$\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$

$$= \frac{1}{2}\cdot 2\tan^{-1}(2x+3) + C_2 = \tan^{-1}(2x+3) + C_2$$

Now, substitute the value of I_1 and I_2 in Eq. (ii). Then, the given integral becomes

$$I = \frac{1}{4}\log|2x^2+6x+5| + \frac{1}{2}\tan^{-1}(2x+3) + \frac{1}{4}C_1 + \frac{1}{2}C_2$$

$$= \frac{1}{4}\log\left|2x^2+6x+5\right| + \frac{1}{2}\tan^{-1}(2x+3) + C$$

where, $C = \frac{1}{4}C_1 + \frac{1}{2}C_2$.

EXAMPLE |23| Evaluate $\int \frac{x+3}{\sqrt{5-4x-2x^2}}\,dx$.

[All India 2015C]

Sol. Let $\quad x + 3 = A\frac{d}{dx}(5-4x-2x^2) + B$

$\Rightarrow \quad x+3 = A(-4-4x) + B \quad$ ...(i)

$\Rightarrow \quad x+3 = -4A - 4Ax + B$

$\Rightarrow \quad x+3 = -4Ax + (B-4A)$

On equating the coefficient of x and constant term from both sides, we get $1 = -4A$ and $3 = B - 4A$

$\Rightarrow \quad A = -\frac{1}{4}$ and $B = 3 + 4\times\left(-\frac{1}{4}\right)$

$\Rightarrow \quad A = -\frac{1}{4}$ and $B = 2$

$\therefore$ From Eq. (i), we get $(x+3) = -\frac{1}{4}(-4-4x) + 2$

Now, the given integral can be written as

$$I = \int \frac{x+3}{\sqrt{5-4x-2x^2}}\,dx = \int \frac{-\frac{1}{4}(-4-4x)+2}{\sqrt{5-4x-2x^2}}\,dx$$

$$= -\frac{1}{4}\int \frac{(-4-4x)}{\sqrt{5-4x-2x^2}}\,dx + 2\int \frac{dx}{\sqrt{5-4x-2x^2}}$$

$$= -\frac{1}{4}I_1 + 2I_2 \quad \text{...(ii)}$$

where, $I_1 = \int \frac{(-4-4x)}{\sqrt{5-4x-2x^2}}\,dx$ and $I_2 = \int \frac{dx}{\sqrt{5-4x-2x^2}}$

Consider, $I_1 = \int \frac{(-4-4x)}{\sqrt{5-4x-2x^2}}\,dx$

Now, put $5-4x-2x^2 = t \Rightarrow (-4-4x)\,dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}}\,dt = 2\sqrt{t} + C_1$

$$= 2\sqrt{5-4x-2x^2} + C_1$$

and $\quad I_2 = \int \frac{dx}{\sqrt{5-4x-2x^2}} = \int \frac{dx}{\sqrt{(-2)\left(x^2+2x-\frac{5}{2}\right)}}$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{-\left(x^2+2\cdot x\cdot 1 + 1^2 - 1^2 - \frac{5}{2}\right)}}$$

[adding and subtracting $(1)^2$ from denominator]

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{-\left[(x+1)^2 - \left(\sqrt{\frac{7}{2}}\right)^2\right]}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2 - (x+1)^2}}$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{7/2}}\right) + C_2 \quad \left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left[\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right] + C_2$$

Now, putting the values of I_1 and I_2 in Eq. (ii), we get

$$I = -\frac{1}{4}\times 2\sqrt{5-4x-2x^2} - \frac{1}{4}C_1 + 2\cdot\frac{1}{\sqrt{2}}\sin^{-1}\left[\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right] + 2C_2$$

$$= -\frac{1}{2}\sqrt{5-4x-2x^2} + \sqrt{2}\sin^{-1}\left[\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right] + C$$

where, $C = -\frac{1}{4}C_1 + 2C_2$

Some Standard Integrals and Substitutions for Them

Integral	Substitution
$\int \frac{dx}{a \pm b\cos x}$	Put $\cos x = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$, then put $\tan \frac{x}{2} = t$
$\int \frac{dx}{a \pm b\sin x}$	Put $\sin x = \frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$, then put $\tan \frac{x}{2} = t$
$\int \frac{dx}{a\sin x + b\cos x}$, $\int \frac{dx}{a\sin x + b\cos x + c}$	(i) Put $\sin x = \frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$ and $\cos x = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$ (ii) Replace $1+\tan^2 \frac{x}{2}$ in the numerator by $\sec^2 \frac{x}{2}$. (iii) Put $\tan \frac{x}{2} = t$ and integrate it.
$\int \frac{dx}{a\sin x + b\cos x}$	Put $a = r\cos\theta$ and $b = r\sin\theta$, where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
$\int \frac{dx}{a + b\sin^2 x}$, $\int \frac{dx}{a + b\cos^2 x}$, $\int \frac{dx}{a\cos^2 x + b\sin^2 x}$, $\int \frac{dx}{(a\sin x + b\cos x)^2}$ and $\int \frac{dx}{a + b\sin^2 x + c\cos^2 x}$	(i) Divide numerator and denominator by $\cos^2 x$. (ii) Reduce $\sec^2 x$ in denominator as $1+\tan^2 x$. (iii) Put $\tan x = t$ and proceed for perfect square.
$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$	(i) Write $a\sin x + b\cos x = A\frac{d}{dx}(c\sin x + d\cos x) + B(c\sin x + d\cos x)$ (ii) Obtain the values of A and B by equating the coefficients of $\sin x$ and $\cos x$ on both sides. (iii) Put the value of $a\sin x + b\cos x$ in the given integral and integrate it.
$\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx$	(i) Write $a\sin x + b\cos x + c = A\frac{d}{dx}(p\sin x + q\cos x + r) + B(p\sin x + q\cos x + r) + C$ (ii) Obtain the values of A, B and C by equating the coefficients of $\sin x$, $\cos x$ and constant term. (iii) Put the value of $a\sin x + b\cos x + c$ in the given integral and integrate it.

Integral	Substitution
$\int \frac{x^2 \pm 1}{x^4 + \lambda x^2 + 1}\,dx$, $\lambda \in R$	Divide numerator and denominator by x^2 and make a perfect square as $\left(x \mp \frac{1}{x}\right)^2$ in denominator and substitute $x \mp \frac{1}{x} = t$.

If the given integral is of the form $\int \frac{dx}{x^4 + \lambda x^2 + 1}$, $\lambda \in R$, then first write the given integral as $\int \frac{dx}{x^4 + \lambda x^2 + 1} = \frac{1}{2}\int \frac{2\,dx}{x^4 + \lambda x^2 + 1}$

$= \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+\lambda x^2+1}\,dx = \frac{1}{2}\left\{\int \frac{(x^2+1)}{x^4+\lambda x^2+1}\,dx - \int \frac{x^2-1}{x^4+\lambda x^2+1}\,dx\right\}$, further integrate easily.

Integral	Substitution
$\int \frac{dx}{(ax+b)\sqrt{px+q}}$, $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$	Put $\sqrt{px+q} = t$
$\int \frac{dx}{(px+q)(\sqrt{ax^2+bx+c})}$	Put $px + q = \frac{1}{t}$
$\int \frac{dx}{(px^2+q)\sqrt{ax^2+b}}$	Put $x = \frac{1}{t}$ and then put $\sqrt{a+bt^2} = u$

EXAMPLE |24| Evaluate $\int \frac{dx}{1-3\sin x}$.

Use the substitution $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, further put $\tan\frac{x}{2} = t$ and then integrate it.

Sol. Let $I = \int \frac{dx}{1-3\sin x}$

$$= \int \frac{dx}{1-3\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} \qquad \left[\because \sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)}{\left(\tan^2\frac{x}{2} - 6\tan\frac{x}{2} + 1\right)}\,dx = \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2} - 6\tan\frac{x}{2} + 1}\,dx$$

$$\left[\because \sec^2\frac{x}{2} - \tan^2\frac{x}{2} = 1 \Rightarrow \sec^2\frac{x}{2} = 1 + \tan^2\frac{x}{2}\right]$$

Now, put $\tan\frac{x}{2} = t$

$$\Rightarrow \sec^2\frac{x}{2}\cdot\frac{1}{2}\,dx = dt \Rightarrow \sec^2\frac{x}{2}\,dx = 2\,dt$$

$$\therefore \quad I = \int \frac{2dt}{t^2 - 6t + 1} = 2\int \frac{dt}{t^2 - 6t + 9 + 1 - 9}$$

$$\left[\text{adding and subtracting} \left(\frac{b}{2a}\right)^2 = \left(\frac{-6}{2}\right)^2 = 9 \text{ in denominator}\right]$$

$$= 2\int \frac{dt}{(t-3)^2 - 8} = 2\int \frac{dt}{(t-3)^2 - (2\sqrt{2})^2}$$

$$= \frac{2 \times 1}{2 \times 2\sqrt{2}} \log \left|\frac{t - 3 - 2\sqrt{2}}{t - 3 + 2\sqrt{2}}\right| + C$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right|\right]$$

$$= \frac{1}{2\sqrt{2}} \log \left|\frac{\tan\frac{x}{2} - 3 - 2\sqrt{2}}{\tan\frac{x}{2} - 3 + 2\sqrt{2}}\right| + C \qquad \left[\because \tan\frac{x}{2} = t\right]$$

EXAMPLE |25| Evaluate $\int \frac{dx}{\sqrt{3} \sin x + \cos x}$.

Sol. Let $I = \int \frac{dx}{\sqrt{3} \sin x + \cos x}$

Now, put $\sqrt{3} = r \cos \theta$ and $1 = r \sin \theta$.

Then, $r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3 + 1} = 2$

and $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\left[\because \text{the type of integral } \int \frac{dx}{a\sin x + b\cos x}, \text{ put } a = r\cos\theta \text{ and } b = r\sin\theta, \text{ where } r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}\left(\frac{b}{a}\right)\right]$

$$\therefore \quad I = \int \frac{dx}{r \cos \theta \sin x + r \sin \theta \cos x}$$

$$= \frac{1}{r} \int \frac{dx}{\sin(x + \theta)} = \frac{1}{2} \int \operatorname{cosec}(x + \theta)\, dx \qquad [\because r = 2]$$

$$= \frac{1}{2} \log |\operatorname{cosec}(x + \theta) - \cot(x + \theta)| + C$$

$$= \frac{1}{2} \log \left|\frac{1 - \cos(x + \theta)}{\sin(x + \theta)}\right| + C$$

$$= \frac{1}{2} \log \left|\frac{2\sin^2\left(\frac{x+\theta}{2}\right)}{2\sin\left(\frac{x+\theta}{2}\right)\cos\left(\frac{x+\theta}{2}\right)}\right| + C$$

$$\left[\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2} \text{ and } \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$$

$$= \frac{1}{2} \log \left|\tan\left(\frac{x+\theta}{2}\right)\right| + C$$

$$= \frac{1}{2} \log \left|\tan\left(\frac{x}{2} + \frac{\pi}{12}\right)\right| + C \qquad \left[\because \theta = \frac{\pi}{6}\right]$$

EXAMPLE |26| Evaluate $\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$.

Sol. Let $I = \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

On dividing numerator and denominator by $\cos^2 x$, we get

$$I = \int \frac{\left(\frac{1}{\cos^2 x}\right)}{\left(\frac{a^2 \sin^2 x}{\cos^2 x} + \frac{b^2 \cos^2 x}{\cos^2 x}\right)} dx$$

$$= \int \frac{\sec^2 x}{a^2 \tan^2 x + b^2} dx = \int \frac{\sec^2 x}{(a \tan x)^2 + b^2} dx$$

Now, put $a \tan x = t$

$\Rightarrow \quad a \sec^2 x\, dx = dt \Rightarrow \sec^2 x\, dx = \frac{dt}{a}$

$$\therefore \quad I = \frac{1}{a}\int \frac{dt}{t^2 + b^2}$$

$$= \frac{1}{a} \cdot \frac{1}{b} \tan^{-1}\frac{t}{b} + C \qquad \left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{ab} \tan^{-1}\frac{t}{b} + C$$

$$= \frac{1}{ab} \tan^{-1}\left(\frac{a \tan x}{b}\right) + C \qquad [\because t = a \tan x]$$

EXAMPLE |27| Evaluate $\int \frac{4 \sin x + 5 \cos x}{5 \sin x + 4 \cos x} dx$.

It is an integration of the form $\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx$. So, write $4\sin x + 5\cos x = A\frac{d}{dx}(5\sin x + 4\cos x) + B(5\sin x + 4\cos x)$

Sol. Let $I = \int \frac{4 \sin x + 5 \cos x}{5 \sin x + 4 \cos x} dx$

Now, let $4 \sin x + 5 \cos x = A\frac{d}{dx}(5\sin x + 4\cos x) + B(5\sin x + 4\cos x)$

$$= A(5 \cos x - 4 \sin x) + B(5 \sin x + 4 \cos x)$$

$\Rightarrow 4 \sin x + 5\cos x = (5A + 4B)\cos x + (5B - 4A) \sin x$

On comparing the coefficients of $\sin x$ and $\cos x$, we get

$$5A + 4B = 5 \text{ and } 5B - 4A = 4$$

On solving these two equations, we get

$$A = \frac{9}{41} \text{ and } B = \frac{40}{41}$$

$$\therefore \quad I = \int \frac{\left(\frac{9}{41}(5 \cos x - 4 \sin x) + \frac{40}{41}(5 \sin x + 4 \cos x)\right)}{(5 \sin x + 4 \cos x)} dx$$

$$= \frac{9}{41} \int \frac{5 \cos x - 4 \sin x}{5 \sin x + 4 \cos x} dx + \frac{40}{41} \int dx$$

Now, put $5\sin x + 4\cos x = t$

$\Rightarrow \quad (5\cos x - 4\sin x)\,dx = dt$

$$\therefore \quad I = \frac{9}{41}\int \frac{dt}{t} + \frac{40}{41}x$$

$$= \frac{9}{41}\log|t| + \frac{40}{41}x + C$$

$$= \frac{40}{41}x + \frac{9}{41}\log|5\sin x + 4\cos x| + C$$

$[\because t = 5\sin x + 4\cos x]$

EXAMPLE |28| Evaluate $\int \frac{x^2 - 1}{x^4 + x^2 + 1}\,dx$.

It is an integration of the form $\int \frac{x^2 \pm 1}{x^4 + \lambda x^2 + 1}dx$. So, divide numerator and denominator by x^2 and make a perfect square as $\left(x \mp \frac{1}{x}\right)^2$ in the denominator and then substitute $x \mp \frac{1}{x} = t$

Sol. Let $I = \int \frac{x^2 - 1}{x^4 + x^2 + 1}\,dx$

$$\Rightarrow \quad I = \int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}}\,dx$$

[dividing numerator and denominator by x^2]

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x^2 + \frac{1}{x^2}\right) + 1}\,dx = \int \frac{\left(1 - \frac{1}{x^2}\right)dx}{\left(x^2 + \frac{1}{x^2} + 2\right) + 1 - 2}$$

[adding and subtracting 2 from denominator]

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)dx}{\left(x + \frac{1}{x}\right)^2 - 1}$$

Now, put $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2}\right)dx = dt$

$$\therefore \quad I = \int \frac{dt}{t^2 - 1^2} = \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$\Rightarrow \quad I = \frac{1}{2}\log\left|\frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1}\right| + C = \frac{1}{2}\log\left|\frac{x^2 + 1 - x}{x^2 + 1 + x}\right| + C$$

$$\left[\because t = x + \frac{1}{x}\right]$$

EXAMPLE |29| Evaluate $\int \frac{dx}{(x-1)\sqrt{2x+3}}$.

It is an integral of the form $\int \frac{dx}{(ax+b)\sqrt{px+q}}$, so put $\sqrt{px+q} = t$ and then integrate it.

Sol. Let $I = \int \frac{dx}{(x-1)\sqrt{2x+3}}$

Now, put $\sqrt{2x+3} = t$

$$\Rightarrow \quad 2x + 3 = t^2 \Rightarrow x = \frac{t^2 - 3}{2}$$

Then, $dx = \frac{(2t - 0)}{2}dt = t\,dt$

$$\therefore \quad I = \int \frac{t\,dt}{\left(\frac{t^2 - 3}{2} - 1\right)\cdot t} = \int \frac{dt}{\frac{t^2 - 3 - 2}{2}}$$

$$= 2\int \frac{dt}{t^2 - 5} = 2\int \frac{dt}{t^2 - (\sqrt{5})^2}$$

$$= 2 \times \frac{1}{2\sqrt{5}}\log\left|\frac{t - \sqrt{5}}{t + \sqrt{5}}\right| + C$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$= \frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{2x+3} - \sqrt{5}}{\sqrt{2x+3} + \sqrt{5}}\right| + C \qquad [\because t = \sqrt{2x+3}]$$

EXAMPLE |30| Evaluate $\int \frac{dx}{x\sqrt{ax - x^2}}$. **[NCERT]**

It is an integral of the form $\int \frac{dx}{(px+q)\sqrt{ax^2 + bx + c}}$, so put $px + q = \frac{1}{t}$ and then integrate it.

Sol. Let $I = \int \frac{dx}{x\sqrt{ax - x^2}}$

Now, put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\,dt$

$$\therefore \quad I = \int \frac{\left(-\frac{1}{t^2}\right)dt}{\frac{1}{t}\sqrt{\frac{a}{t} - \frac{1}{t^2}}} = \int \frac{-dt}{t^2 \cdot \frac{1}{t}\frac{\sqrt{at - 1}}{t}} = \int \frac{-dt}{\sqrt{at - 1}}$$

$$= -\int (at - 1)^{-\frac{1}{2}}\,dt = -\frac{(at-1)^{-\frac{1}{2}+1}}{a\cdot\left(-\frac{1}{2}+1\right)} + C$$

$$= \frac{-2}{a}(at - 1)^{\frac{1}{2}} + C = -\frac{2}{a}\sqrt{a\cdot\frac{1}{x} - 1} + C \qquad \left[\because t = \frac{1}{x}\right]$$

$$= \frac{-2}{a}\sqrt{\frac{a - x}{x}} + C$$

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 $\int \left(\frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\right) dx$ equal to **[NCERT]**

(a) $10^x - x^{10} + C$

(b) $10^x + x^{10} + C$

(c) $(10^x - x^{10})^{-1} + C$

(d) $\log|10^x + x^{10}| + C$

2 $\int \frac{1}{e^x + e^{-x}} dx$ is equal to **[NCERT]**

(a) $\tan^{-1} e^x + C$ (b) $\tan^{-1} e^{-x} + C$

(c) $\log (e^x - e^{-x}) + C$ (d) $\log (e^x + e^{-x}) + C$

3 $\int \frac{x^9}{(4x^2 + 1)^6} dx$ is equal to **[NCERT Exemplar]**

(a) $\frac{1}{5x}\left(4 + \frac{1}{x^2}\right)^{-5} + C$ (b) $\frac{1}{5}\left(4 + \frac{1}{x^2}\right)^{-5} + C$

(c) $\frac{1}{10x}(1 + 4)^{-5} + C$ (d) $\frac{1}{10}\left(\frac{1}{x^2} + 4\right)^{-5} + C$

4 $\int \frac{2x^3 - 1}{x + x^4} dx$ is equal to

(a) $\log (x^4 + x) + C$ (b) $\log \left(\frac{x^3 + 1}{x}\right) + C$

(c) $\frac{1}{2} \log \left(x^2 + \frac{1}{x^2}\right) + C$ (d) None of these

5 $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ is equal to **[NCERT]**

(a) $\frac{-1}{\sin x + \cos x} + C$ (b) $\log |\sin x + \cos x| + C$

(c) $\log |\sin x - \cos x| + C$ (d) $\frac{1}{(\sin x + \cos x)^2} + C$

VERY SHORT ANSWER Type Questions

Directions (Q. Nos. 6-17) *Evaluate the following integrals.*

6 $\int (ax + b)^3 \, dx$ **[All India 2011]**

7 $\int \frac{3ax}{b^2 + c^2x^2} dx$ **[NCERT Exemplar]**

8 $\int \frac{(\log x)^2}{x} dx$ **[NCERT; All India 2011]**

9 $\int \frac{1}{x(1 + \log x)} dx$ **[Delhi 2017C]**

10 $\int \cos^{-1}(\sin x) dx$ **[Delhi 2014]**

11 $\int \sin^{-1}(\cos x) dx$ **[NCERT Exemplar]**

12 $\int \frac{\cos^{-1} x}{\sqrt{1 - x^2}} dx$

13 $\int \frac{(1 + \cos x)}{x + \sin x} dx$ **[NCERT Exemplar]**

14 $\int \frac{\sin 4x}{\cos 2x} dx$

15 $\int \sin (ax + b) \cos (ax + b) dx$ **[NCERT]**

16 $\int \frac{x^3}{\sqrt{1 - x^8}} dx$ **[NCERT]**

17 $\int \frac{e^x}{\sqrt{9 - e^{2x}}} dx$

SHORT ANSWER Type I Questions

Directions (Q. Nos. 18-34) *Evaluate the following integrals.*

18 $\int \frac{x}{\sqrt{32 - x^2}} dx$ **[All India 2017C]**

19 $\int \frac{\sqrt{1 + x^2}}{x^4} dx$ **[NCERT Exemplar]**

20 $\int \frac{\sin (2\tan^{-1} x)}{(1 + x^2)} dx$

21 $\int \sin x \sin(\cos x) dx$

22 $\int \frac{(x + 1)(x + \log x)^2}{x} dx$ **[NCERT]**

23 $\int \frac{x^3 \sin (\tan^{-1} x^4)}{1 + x^8} dx$ **[NCERT]**

24 $\int \cos^3 x \, e^{\log \sin x} \, dx$ **[NCERT]**

25 $\int \frac{2\cos x - 3 \sin x}{6 \cos x + 4 \sin x} dx$ **[NCERT]**

26 $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ **[NCERT]**

27 $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx$

28 $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$ **[NCERT]**

29 $\int 2^{2^{2^x}} \, 2^{2^x} \, 2^x \, dx$

30 $\int \frac{\cos x}{\sqrt{8-\sin^2 x}}\,dx$ [All India 2017C]

31 $\int \frac{\sin 2x \cos 2x}{\sqrt{9-\cos^4 (2x)}}\,dx$ [NCERT]

32 $\int \frac{dx}{x^2+4x+8}$ [Delhi 2017]

33 $\int \frac{dx}{5-8x-x^2}$ [All India 2017]

34 $\int \frac{1}{\sqrt{x^2-4x}}\,dx$ [Delhi 2017C]

SHORT ANSWER Type II Questions

Directions (Q. Nos. 35-64) *Evaluate the following integrals.*

35 $\int \frac{(x^4-x)^{1/4}}{x^5}\,dx$ [NCERT]

36 $\int \frac{dx}{x(x^3+8)}$ [All India 2013]

37 $\int \frac{1}{x^2(x^4+1)^{3/4}}\,dx$ [NCERT]

38 $\int \frac{x^2-x+2}{x^2+1}\,dx$

39 $\int \frac{x}{\sqrt{x}+1}\,dx$ [NCERT Exemplar]

40 $\int \frac{\sin x - x\cos x}{x(x+\sin x)}\,dx$ [Delhi 2016C]

41 $\int \frac{\sqrt{\tan x}}{\sin x \cdot \cos x}\,dx$ [NCERT]

42 $\int \frac{\cos x - \cos 2x}{1-\cos x}\,dx$

43 $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\,dx$ [All India 2013]

44 $\int \tan^{-1}\sqrt{\frac{1+\cos x}{1-\cos x}}\,dx$

45 $\int \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}\,dx$ [NCERT]

46 $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x}\,dx$ [NCERT Exemplar; Delhi 2014C]

47 $\int \sin x \sin 2x \sin 3x\,dx$ [NCERT; Delhi 2012]

48 $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$ [NCERT]

49 $\int \frac{2x-3}{\sqrt{x^2+4}}\,dx$

50 $\int \sqrt{\frac{1+x}{1-x}}\,dx,\ x \neq 1$ [NCERT Exemplar]

51 $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}},\ \beta > \alpha$ [NCERT Exemplar]

52 $\int \frac{x+5}{3x^2+13x-10}\,dx$ [Delhi 2017C]

53 $\int \frac{x+7}{3x^2+25x+28}\,dx$ [Delhi 2017C]

54 $\int \frac{x+3}{x^2-2x-5}\,dx$

55 $\int \frac{x^3}{x^4+3x^2+2}\,dx$ [NCERT Exemplar]

56 $\int \frac{(3\sin x-2)\cos x}{13-\cos^2 x-7\sin x}\,dx$ [Delhi 2017]

57 $\int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx$ [All India 2014]

58 $\int \frac{x+3}{\sqrt{5-4x+x^2}}\,dx$ [Delhi 2017C]

59 $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}\,dx$ [All India 2011]

60 $\int \frac{x+2}{\sqrt{4x-x^2}}\,dx$

61 $\int \frac{\sin x}{\sin 3x}\,dx$

62 $\int \frac{dx}{2\sin^2 x + 5\cos^2 x}$ [NCERT Exemplar]

63 $\int \frac{1}{1-\cot x}\,dx$

64 $\int \frac{x^2+1}{x^4+1}\,dx$ [Delhi 2011C]

LONG ANSWER Type Questions
[6 Marks]

Directions (Q. Nos. 65-71) *Evaluate the following integrals.*

65 $\int \frac{dx}{(x+1)\sqrt{x^2-1}}$

66 $\int \frac{(\cos 5x + \cos 4x)}{1-2\cos 3x}\,dx$ [NCERT Exemplar]

67 $\int \cot^{-1}\left[\frac{\sqrt{1+\cos 2x}+\sqrt{1-\cos 2x}}{\sqrt{1+\cos 2x}-\sqrt{1-\cos 2x}}\right]dx$

68 $\int \sqrt{\tan\theta}\, dx$

69 $\int \frac{1}{\cos^4 x + \sin^4 x} dx$ **[All India 2014]**

70 $\int [\sqrt{\cot x} + \sqrt{\tan x}]\, dx$ **[All India 2014]**

71 $\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$ **[All India 2014]**

HINTS & SOLUTIONS

1. (d) **Hint** Put $x^{10} + 10^x = t$

$\Rightarrow (10x^9 + 10^x \log_e 10)\, dx = dt$

2. (a) **Hint** Let $I = \int \frac{1}{e^x + e^{-x}} dx = \int \frac{dx}{e^x + \frac{1}{e^x}} = \int \frac{e^x\, dx}{(e^x)^2 + 1}$

Now put $e^x = t$.

3. (d) **Hint** Let $I = \int \frac{x^9}{(4x^2+1)^6} = \int \frac{x^9}{x^{12}\left(4 + \frac{1}{x^2}\right)^6} dx$

$$= \int \frac{dx}{x^3\left(4 + \frac{1}{x^2}\right)^6}$$

Now put $4 + \frac{1}{x^2} = t \Rightarrow \frac{-2}{x^3} dx = dt$

4. (b) **Hint** Let $I = \int \frac{2x^3 - 1}{x^4 + x} dx = \int \frac{\frac{2x^3}{x^2} - \frac{1}{x^2}}{x^2 + \frac{1}{x}} dx$

$$= \int \frac{2x - \frac{1}{x^2}}{x^2 + \frac{1}{x}} dx.$$

Now, put $x^2 + \frac{1}{x} = t \Rightarrow \left(2x - \frac{1}{x^2}\right) d = dt$

5. (b) **Hint** Let $I = \int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$

$$= \int \frac{\cos^2 x - \sin^2 x}{(\sin x + \cos x)^2} dx = \int \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

Now, put $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\, dx = dt$

6. Let $I = \int (ax+b)^3 dx$

Put $ax + b = t \Rightarrow a\, dx = dt \Rightarrow dx = \frac{1}{a} dt$

$\therefore \quad I = \frac{1}{a}\int t^3\, dt = \frac{t^4}{4a} + C = \frac{(ax+b)^4}{4 \cdot a} + C \quad [\because t = ax + b]$

$$= \frac{(ax+b)^4}{4a} + C$$

7. Let $I = \int \frac{3ax}{b^2 + c^2x^2} dx$

Now, put $b^2 + c^2x^2 = t \Rightarrow 2c^2x\, dx = dt$

$\Rightarrow x\, dx = \frac{dt}{2c^2}$

$\therefore \quad I = \frac{3a}{2c^2}\int \frac{dt}{t} = \frac{3a}{2c^2} \log|t| + C$

$= \frac{3a}{2c^2} \log|b^2 + c^2x^2| + C$

$= \frac{3a}{2c^2} \log(b^2 + c^2x^2) + C \quad [\because b^2 + c^2x^2 > 0]$

8. Let $I = \int \frac{(\log x)^2}{x} dx$

Now, put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$\therefore \quad I = \int t^2 dt = \frac{t^3}{3} + C = \frac{(\log x)^3}{3} + C \quad [\because t = \log x]$

9. **Hint** Put $1 + \log x = t$. [**Ans.** $\log|1 + \log x| + C$]

10. Let $I = \int \cos^{-1}(\sin x) dx$

$\Rightarrow I = \int \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right] dx \quad \left[\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta\right]$

$= \int \left(\frac{\pi}{2} - x\right) dx \quad [\because \cos^{-1}(\cos\theta) = \theta]$

$= \frac{\pi}{2}\int dx - \int x\, dx = \frac{\pi}{2}x - \frac{x^2}{2} + C$

11. Solve as Question 10. $\left[\textbf{Ans. } \frac{\pi}{2}x - \frac{x^2}{2} + C\right]$

12. Let $I = \int \frac{\cos^{-1} x}{\sqrt{1-x^2}} dx$

Now, put $\cos^{-1} x = t$

$\Rightarrow \quad \frac{-1}{\sqrt{1-x^2}} dx = dt \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = -dt$

$\therefore \quad I = \int t(-dt) = -\int t\, dt$

$= \frac{-t^2}{2} + C = \frac{-(\cos^{-1} x)^2}{2} + C \quad [\because t = \cos^{-1} x]$

13. Let $I = \int \frac{(1 + \cos x)}{(x + \sin x)} dx$

Now, put $x + \sin x = t \Rightarrow (1 + \cos x) dx = dt$

$\therefore \quad I = \int \frac{1}{t} dt = \log|t| + C$

$= \log|x + \sin x| + C \quad [\because t = x + \sin x]$

14. Let $I = \int \frac{\sin 4x}{\cos 2x} dx = \int \frac{\sin[2 \cdot (2x)]}{\cos 2x} dx$

$= \int \frac{2\sin 2x \cdot \cos 2x}{\cos 2x} dx \quad [\because \sin 2x = 2\sin x \cdot \cos x]$

$= 2\int \sin 2x\, dx = -\frac{2\cos 2x}{2} + C = -\cos 2x + C$

15. **Hint** Put $\sin(ax+b)=t$ $\left[\text{Ans. } \frac{1}{2a}\cdot\sin^2(ax+b)+C\right]$

16. **Hint** (i) Write the given integral as $\int \frac{x^3}{\sqrt{1-(x^4)^2}}dx$

(ii) Put $x^4=t$ and then use the formula

$$\int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+C \quad \left[\text{Ans. } \frac{1}{4}\sin^{-1}(x^4)+C\right]$$

17. Let $I=\int \frac{e^x}{\sqrt{9-e^{2x}}}dx=\int \frac{e^x}{\sqrt{(3)^2-(e^x)^2}}dx$

Now, put $e^x=t \Rightarrow e^x\,dx=dt$

$$\therefore \quad I=\int \frac{dt}{\sqrt{(3)^2-t^2}}=\sin^{-1}\frac{t}{3}+C=\sin^{-1}\left(\frac{e^x}{3}\right)+C$$

$$\left[\because \int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right) \text{and } t=e^x\right]$$

18. Let $I=\int \frac{x}{\sqrt{32-x^2}}dx$

Put $32-x^2=t \Rightarrow -2x\,dx=dt$

$$\Rightarrow \quad x\,dx=\frac{-1}{2}dt$$

$$\text{Now,} \quad I=-\frac{1}{2}\int\frac{dt}{\sqrt{t}}=\frac{-1}{2}\cdot 2\sqrt{t}+C$$

$$=-\sqrt{t}+C$$

$$=-\sqrt{32+x^2}+C$$

19. Let $I=\int \frac{\sqrt{1+x^2}}{x^4}dx=\int\frac{\sqrt{1+x^2}}{x}\cdot\frac{1}{x^3}dx$

$$=\int\sqrt{\frac{1+x^2}{x^2}}\cdot\frac{1}{x^3}dx=\int\sqrt{\frac{1}{x^2}+1}\cdot\frac{1}{x^3}dx$$

Now, putting $1+\frac{1}{x^2}=t^2$

$$\Rightarrow \frac{-2}{x^3}dx=2t\,dt \Rightarrow -\frac{1}{x^3}dx=t\,dt$$

$$\therefore \quad I=\int -t^2dt=-\frac{t^3}{3}+C$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}+C$$

20. **Hint** Put $\tan^{-1}x=t$, then $I=\int \sin 2t\,dt$.

$\left[\text{Ans. } \frac{-[\cos 2(\tan^{-1}x)]}{2}+C\right]$

21. **Hint** Put $\cos x=t$. [**Ans.** $\cos(\cos x)+C$]

22. **Hint** Put $x+\log x=t \Rightarrow \left(1+\frac{1}{x}\right)dx=dt$

$\left[\text{Ans. } \frac{1}{3}(x+\log x)^3+C\right]$

23. **Hint** Put $\tan^{-1}x^4=t \Rightarrow \frac{1}{1+x^8}\cdot 4x^3dx=dt$

$\left[\text{Ans. } -\frac{1}{4}\cos(\tan^{-1}x^4)+C\right]$

24. **Hint** $e^{\log f(x)}=f(x)$, therefore $e^{\log\sin x}=\sin x$ and then put $\cos x=t$. $\left[\text{Ans. } -\frac{1}{4}\cos^4 x+C\right]$

25. **Hint** Firstly, write the given integral as

$$\frac{1}{2}\int\frac{2\cos x-3\sin x}{3\cos x+2\sin x}dx$$

and then put $3\cos x+2\sin x=t$.

$\left[\text{Ans. } \frac{1}{2}\log|2\sin x+3\cos x|+C\right]$

26. **Hint** Firstly, write
$\cos 2x=\cos^2x-\sin^2x=(\cos x+\sin x)(\cos x-\sin x)$
and then put $\sin x+\cos x=t$
[**Ans.** $\log|\sin x+\cos x|+C$]

27. **Hint** Use the formula, $1+\sin 2x=(\cos x+\sin x)^2$
[**Ans.** $x+C$]

28. Let $I=\int\frac{\cos 2x+2\sin^2x}{\cos^2x}dx$

$$=\int\frac{1-2\sin^2x+2\sin^2x}{\cos^2x}dx \quad [\because \cos 2x=1-2\sin^2x]$$

$$=\int\frac{1}{\cos^2x}dx=\int\sec^2x\,dx=\tan x+C$$

29. Let $I=\int 2^{2^{2^x}}\,2^{2^x}\,2^x\,dx$

Now, put $2^{2^{2^x}}=t \Rightarrow 2^{2^{2^x}}\cdot 2^{2^x}\cdot 2^x\cdot(\log 2)^3\,dx=dt$

$$\therefore \quad I=\int\frac{dt}{(\log 2)^3}=\frac{1}{(\log 2)^3}t+C$$

$$=\frac{2^{2^{2^x}}}{(\log 2)^3}+C \qquad \left[\because t=2^{2^{2^x}}\right]$$

30. **Hint** Firstly, put $\sin x=t$.

$\left[\text{Ans. } \sin^{-1}\left(\frac{\sin x}{2\sqrt{2}}\right)+C\right]$

31. Let $I=\int\frac{\sin 2x\cos 2x}{\sqrt{9-\cos^4(2x)}}dx=\int\frac{\sin 2x\cos 2x}{\sqrt{3^2-[\cos^2(2x)]^2}}dx$

Now, put $\cos^2(2x)=t$

$$\Rightarrow 2\cos(2x)[-\sin(2x)]\cdot 2\,dx=dt$$

$$\Rightarrow \sin 2x\cdot\cos 2x\,dx=-\frac{dt}{4}$$

$$\therefore \quad I=-\frac{1}{4}\int\frac{dt}{\sqrt{3^2-t^2}}=-\frac{1}{4}\sin^{-1}\left(\frac{t}{3}\right)+C$$

$$=-\frac{1}{4}\sin^{-1}\left(\frac{\cos^2 2x}{3}\right)+C$$

32. Similar as Example 19.

$$\left[\text{Ans. } \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+C\right]$$

33. Let $I=\int\frac{dx}{5-8x-x^2}=\int\frac{dx}{5-2\cdot4\cdot x-x^2-(4)^2+(4)^2}$

$$=\int\frac{dx}{5+16-[x^2+(4)^2+2\cdot4\cdot x}$$

$$=\int\frac{dx}{21-(x+4)^2}=\int\frac{dx}{(\sqrt{21})^2-(x+4)^2}$$

$$=\frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right|+C$$

$$\left[\because\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C\right]$$

34. Let $I=\int\frac{dx}{\sqrt{x^2-4x}}=\int\frac{dx}{\sqrt{x^2-2\cdot2\cdot x+2^2-2^2}}$

$$=\int\frac{dx}{\sqrt{(x-2)^2-2^2}}$$

$$=\log|(x-2)+\sqrt{(x-2)^2-2^2}|+C$$

$$=\log|(x-2)+\sqrt{x^2-4x}|+C$$

35. Let $I=\int\frac{(x^4-x)^{1/4}}{x^5}dx=\int\frac{\left[x^4\left(1-\frac{1}{x^3}\right)\right]^{1/4}}{x^5}dx$

[taking x^4 common from numerator]

$$=\int\frac{(x^4)^{1/4}\left(1-\frac{1}{x^3}\right)^{1/4}}{x^5}dx=\int\frac{x\left(1-\frac{1}{x^3}\right)^{1/4}}{x^5}dx$$

$$=\int\frac{\left(1-\frac{1}{x^3}\right)^{1/4}}{x^4}dx$$

Now, put $1-\frac{1}{x^3}=t\Rightarrow\frac{3}{x^4}dx=dt$

$\Rightarrow\quad\frac{dx}{x^4}=\frac{1}{3}dt\Rightarrow I=\frac{1}{3}\int t^{1/4}\,dt$

$$=\frac{1}{3}\,\frac{t^{5/4}}{5/4}+C=\frac{4}{15}t^{5/4}+C$$

$$=\frac{4}{15}\left(1-\frac{1}{x^3}\right)^{5/4}+C\qquad\left[\because t=1-\frac{1}{x^3}\right]$$

36. Let $I=\int\frac{dx}{x(x^3+8)}=\int\frac{x^2}{x^3(x^3+8)}dx$

[multiplying numerator and denominator by x^2]

Now, put $x^3+8=t$

$\Rightarrow 3x^2\,dx=dt\;\Rightarrow\;x^2\,dx=\frac{dt}{3}$

$$\therefore I=\frac{1}{3}\int\frac{dt}{(t-8)\,t}=\frac{1}{3}\int\frac{dt}{t^2-8t}=\frac{1}{3}\int\frac{dt}{t^2-8t+(4)^2-(4)^2}$$

[adding and subtracting $(4)^2$ from denominator]

$$=\frac{1}{3}\int\frac{dt}{(t-4)^2-(4)^2}$$

$$=\frac{1}{3}\times\frac{1}{2\times4}\log\left|\frac{t-4-4}{t-4+4}\right|+C$$

$$=\frac{1}{24}\log\left|\frac{t-8}{t}\right|+C=\frac{1}{24}\log\left|\frac{x^3}{x^3+8}\right|+C$$

$[\because t=x^3+8]$

37. Let $I=\int\frac{1}{x^2(x^4+1)^{3/4}}dx=\int\frac{1}{x^2\left\{x^4\left(1+\frac{1}{x^4}\right)\right\}^{3/4}}dx$

$$=\int\frac{1}{x^2\cdot x^3\left(1+\frac{1}{x^4}\right)^{3/4}}dx=\int\frac{1}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}dx$$

Now, put $1+\frac{1}{x^4}=t\Rightarrow-\frac{4}{x^5}dx=dt\Rightarrow\frac{1}{x^5}dx=-\frac{dt}{4}$

$\therefore\quad I=\frac{1}{-4}\int\frac{1}{t^{3/4}}dt=-\frac{1}{4}\left[\frac{t^{1/4}}{1/4}\right]+C$

$=-(1+x^{-4})^{1/4}+C\qquad[\because t=1+x^{-4}]$

38. **Hint** Write the given integral as

$$\int\left(1-\frac{x}{x^2+1}+\frac{1}{x^2+1}\right)dx$$

$$\left[\text{Ans. } x-\frac{1}{2}\log|x^2+1|+\tan^{-1}x+C\right]$$

39. Let $I=\int\frac{x}{\sqrt{x}+1}dx$

Now, put $\sqrt{x}=t\;\Rightarrow\;\frac{1}{2\sqrt{x}}dx=dt\Rightarrow dx=2\sqrt{x}\,dt$

$\Rightarrow\;dx=2t\,dt$

$\therefore\quad I=2\int\frac{t^2\cdot t}{t+1}dt=2\int\frac{t^3}{t+1}dt=2\int\frac{t^3+1-1}{t+1}dt$

[adding and subtracting 1 from numerator]

$$=2\int\frac{(t+1)(t^2-t+1)}{t+1}dt-2\int\frac{1}{t+1}dt$$

$[\because a^3+b^3=(a+b)(a^2-ab+b^2)]$

$$=2\int(t^2-t+1)\,dt-2\int\frac{1}{t+1}dt$$

$$=2\left[\frac{t^3}{3}-\frac{t^2}{2}+t-\log|(t+1)|\right]+C$$

$$=2\left[\frac{x\sqrt{x}}{3}-\frac{x}{2}+\sqrt{x}-\log|(\sqrt{x}+1)|\right]+C$$

$[\because t=\sqrt{x}]$

40. Let $I=\int \frac{\sin x - x\cos x}{x(x+\sin x)}dx$

$$=\int \frac{(\sin x + x)-(x\cos x + x)}{x(x+\sin x)}dx$$

$$=\int \frac{dx}{x}-\int \frac{1+\cos x}{x+\sin x}dx=\log|x|-\int \frac{1+\cos x}{x+\sin x}dx$$

Put $x+\sin x = t \Rightarrow (1+\cos x)\,dx = dt$

$\therefore \quad I=\log|x|-\int \frac{dt}{t}=\log|x|-\log|t|+C$

$$=\log\left|\frac{x}{t}\right|+C$$

$$=\log\left|\frac{x}{x+\sin x}\right|+C$$

41. Let $I=\int \frac{\sqrt{\tan x}}{\sin x\cdot\cos x}dx$

On multiplying numerator and denominator by $\sqrt{\tan x}$, we get

$$I=\int \frac{\sqrt{\tan x}}{\sin x\cdot\cos x}\times\frac{\sqrt{\tan x}}{\sqrt{\tan x}}dx$$

$$=\int \frac{\tan x}{\sin x\cdot\cos x\sqrt{\tan x}}dx$$

$$=\int \frac{\sin x/\cos x}{\sin x\cdot\cos x\sqrt{\tan x}}dx=\int \frac{\sec^2 x}{\sqrt{\tan x}}dx$$

Now, put $\tan x = t$

$\Rightarrow \quad \sec^2 x\,dx = dt$

$$\therefore \quad I=\int \frac{dt}{\sqrt{t}}=\int t^{-1/2}dt=\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C=\frac{t^{1/2}}{1/2}+C$$

$$=2t^{1/2}+C=2\sqrt{\tan x}+C \qquad [\because t=\tan x]$$

42. **Hint** By using the formula, $\cos 2x = 2\cos^2 x - 1$, given integral reduces to

$$\int \frac{-(2\cos^2 x-\cos x-1)dx}{1-\cos x}$$

$$=\int \frac{(2\cos x+1)(1-\cos x)}{(1-\cos x)}dx$$

[Ans. $2\sin x + x + C$]

44. Let $I=\int \tan^{-1}\sqrt{\frac{1+\cos x}{1-\cos x}}\,dx=\int \tan^{-1}\sqrt{\frac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}}}\,dx$

$$\left[\because \cos^2\frac{x}{2}=\frac{1+\cos x}{2} \text{ and } \sin^2\frac{x}{2}=\frac{1-\cos x}{2}\right]$$

$$=\int \tan^{-1}\sqrt{\cot^2\frac{x}{2}}\,dx=\int \tan^{-1}\cdot\cot\frac{x}{2}dx$$

$$=\int \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]dx \quad \left[\because \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta\right]$$

$$=\int\left(\frac{\pi}{2}-\frac{x}{2}\right)dx \qquad [\because \tan^{-1}(\tan\theta)=\theta]$$

$$=\frac{\pi}{2}x-\frac{x^2}{2\times 2}+C=\frac{\pi}{2}x-\frac{x^2}{4}+C$$

45. $\int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx=\int \frac{\{(\sin^4 x)^2-(\cos^4 x)^2\}}{1-2\sin^2 x\cos^2 x}dx$

$$=\int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{1-2\sin^2 x\cos^2 x}dx$$

$$[\because a^2-b^2=(a-b)(a+b)]$$

$$=\int \frac{\left[(\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x)\{(\sin^2 x)^2+(\cos^2 x)^2\}\right]}{1-2\sin^2 x\cos^2 x}dx$$

$$=\int \frac{\left[(\sin^2 x-\cos^2 x)1\{(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cdot\cos^2 x\}\right]}{1-2\sin^2 x\cos^2 x}dx$$

$$[\because \sin^2 x+\cos^2 x=1 \text{ and } a^2+b^2=(a+b)^2-2ab]$$

$$=\int \frac{-\cos 2x\,[1-2\sin^2 x\cdot\cos^2 x]}{1-2\sin^2 x\cos^2 x}dx$$

$$[\because \cos 2x=\cos^2 x-\sin^2 x]$$

$$=-\int \cos 2x\,dx=-\frac{\sin 2x}{2}+C$$

46. Let $I=\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}dx$

$$\Rightarrow \quad I=\int \frac{(\sin^2 x)^3+(\cos^2 x)^3}{\sin^2 x\cos^2 x}dx$$

$$=\int \frac{(\sin^2 x+\cos^2 x)^3-3\sin^2 x\cos^2 x(\sin^2 x+\cos^2 x)}{\sin^2 x\cos^2 x}dx$$

$$[\because a^3+b^3=(a+b)^3-3ab(a+b)]$$

$$=\int \frac{(1)^3-3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}dx \qquad [\because \sin^2 x+\cos^2 x=1]$$

$$=\int\left[\frac{\sin^2 x}{\sin^2 x\cos^2 x}+\frac{\cos^2 x}{\sin^2 x\cos^2 x}\right]dx-3\int 1dx$$

$$=\int(\sec^2 x+\operatorname{cosec}^2 x)\,dx-3\int 1dx$$

$$=\int \sec^2 x\,dx+\int \operatorname{cosec}^2 x\,dx-3\int 1\,dx$$

$$=\tan x-\cot x-3x+C$$

47. Similar as Example 13.

$$\left[\text{Ans. } -\frac{\cos 4x}{16}+\frac{\cos 6x}{24}-\frac{\cos 2x}{8}+C\right]$$

48. Let $I=\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$.

Now, putting $\sqrt{x}=\cos t$

$$x=\cos^2 t \Rightarrow dx=2\cos t(-\sin t)\,dt$$

$$\therefore \quad I=\int\sqrt{\frac{1-\cos t}{1+\cos t}}(-2\sin t\cos t)\,dt$$

$$\Rightarrow I=-2\int\sqrt{\frac{2\sin^2\frac{t}{2}}{2\cos^2\frac{t}{2}}}\cdot 2\sin\frac{t}{2}\cos\frac{t}{2}\cdot\cos t\,dt$$

$$\left[\because 1-\cos x=2\sin^2\frac{x}{2}, 1+\cos x=2\cos^2\frac{x}{2} \text{ and } \sin 2t=2\sin t\cos t\right]$$

$$=-4\int\sin^2\frac{t}{2}\cdot\cos t\,dt$$

$$=-4\int\frac{1-\cos t}{2}\cdot\cos t\,dt \qquad \left[\because \sin^2\frac{\theta}{2}=\frac{1-\cos\theta}{2}\right]$$

$$=-2\int(\cos t-\cos^2 t)\,dt$$

$$=-2\int\left(\cos t-\frac{1+\cos 2t}{2}\right)dt$$

$$=-2\sin t+t+\frac{1}{2}\sin 2t+C$$

$$=-2\sin t+t+\frac{1}{2}\times 2\sin t\cdot\cos t+C$$

$$=-2\sqrt{1-\cos^2 t}+t+\frac{1}{2}[2\sqrt{1-\cos^2 t}\cdot\cos t]+C$$

$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}+C$$

$$[\because \cos^2 t=x]$$

$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x-x^2}+C$$

49. Let $I=\int\frac{2x-3}{\sqrt{x^2+4}}dx=\int\frac{2x}{\sqrt{x^2+4}}dx-3\int\frac{dx}{\sqrt{x^2+4}}$

Put $x^2+4=t$ in first integral, then $2x\,dx=dt$

$$\therefore I=\int\frac{dt}{\sqrt{t}}-3\int\frac{dx}{\sqrt{x^2+(2)^2}}=\int t^{-1/2}dt-3\int\frac{dx}{\sqrt{x^2+(2)^2}}$$

$$=\frac{t^{1/2}}{1/2}-3\log\left|x+\sqrt{x^2+4}\right|+C$$

$$\left[\because \int\frac{dx}{\sqrt{x^2+a^2}}=\log|x+\sqrt{x^2+a^2}|\right]$$

$$=2\sqrt{t}-3\log\left|x+\sqrt{x^2+4}\right|+C$$

$$=2\sqrt{x^2+4}-3\log\left|x+\sqrt{x^2+4}\right|+C \quad [\because t=x^2+4]$$

50. **Hint** $I=\int\sqrt{\frac{1+x}{1-x}}\,dx$

$$=\int\sqrt{\frac{(1+x)(1+x)}{(1-x)(1+x)}}\,dx$$

$$=\int\frac{1+x}{\sqrt{1-x^2}}dx$$

$$=\int\frac{dx}{\sqrt{1-x^2}}+\int\frac{x}{\sqrt{1-x^2}}dx$$

Put $1-x^2=t$ in second integral

$[\text{Ans. } \sin^{-1}(x)-\sqrt{1-x^2}+C]$

51. Let $I=\int\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

Now, put $x-\alpha=t^2 \Rightarrow dx=2t\,dt$

$$\therefore \quad I=\int\frac{2t\,dt}{\sqrt{t^2\cdot(\beta-(t^2+\alpha))}}=\int\frac{2t\,dt}{t\cdot\sqrt{(\beta-\alpha)-t^2}}$$

$$=2\int\frac{dt}{\sqrt{(\beta-\alpha)-t^2}} \qquad [\because x=t^2+\alpha]$$

$$=2\int\frac{dt}{\sqrt{k^2-t^2}}, \text{ where } k^2=\beta-\alpha$$

$$=2\sin^{-1}\left(\frac{t}{k}\right)+C \qquad \left[\because \int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$=2\sin^{-1}\left(\sqrt{\frac{x-\alpha}{\beta-\alpha}}\right)+C$$

$$[\because k=\sqrt{\beta-\alpha} \text{ and } t=\sqrt{x-\alpha}]$$

52. Let $I=\int\frac{(x+5)}{3x^2+13x-10}dx$

$$=\int\frac{(x+5)}{(x+5)(3x-2)}dx$$

$$=\int\frac{dx}{3x-2}=\frac{1}{3}\log|3x-2|+C$$

53. Similar as Question 52.

Ans. $\frac{1}{3}\log|3x+4|+C$

54. Similar as Example 22.

$$\left[\text{Ans. } \frac{1}{2}\log|x^2-2x-5|+\frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right|+C\right]$$

55. **Hint** (i) Write the given integral as $\int\frac{x^2\cdot x}{(x^2)^2+3x^2+2}dx$

(ii) Put $x^2=t$, then above integral reduces to $\frac{1}{2}\int\frac{t}{t^2+3t+2}dt$

(iii) Similar as Example 22. $\left[\text{Ans. } \log\left|\frac{x^2+2}{\sqrt{x^2+1}}\right|+C\right]$

56. Let $I = \int \frac{(3\sin x - 2)\cos x}{13 - \cos^2 x - 7\sin x}\,dx$

$$= \int \frac{(3\sin x - 2)\cos x}{13 - (1 - \sin^2 x) - 7\sin x}\,dx$$

$$= \int \frac{(3\sin x - 2)\cos x}{12 + \sin^2 x - 7\sin x}\,dx$$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

Then, $I = \int \frac{(3t-2)}{12 + t^2 - 7t}\,dt = \int \frac{3t-2}{t^2 - 7t + 12}\,dt$

Let $3t - 2 = A\frac{d}{dt}(t^2 - 7t + 12) + B$

$\Rightarrow \quad 3t - 2 = A(2t - 7) + B$

$\Rightarrow \quad 3t - 2 = 2At - 7A + B$

On equating the coefficient of t and constant term from both sides, we get

$$2A = 3 \Rightarrow A = \frac{3}{2} \text{ and } -7A + B = -2$$

$\Rightarrow \quad -7 \times \frac{3}{2} + B = -2$

$\Rightarrow \quad B = \frac{21}{2} - 2 = \frac{21-4}{2} = \frac{17}{2}$

$\therefore \quad 3t - 2 = \frac{3}{2}(2t - 7) + \frac{17}{2}$

$\therefore \quad I = \int \frac{3t-2}{t^2 - 7t + 12}\,dt$

$$= \int \frac{\frac{3}{2}(2t-7)}{t^2 - 7t + 12}\,dt + \int \frac{17/2}{t^2 - 7t + 12}\,dt$$

$$= \frac{3}{2}\int \frac{2t-7}{t^2 - 7t + 12}\,dt + \frac{17}{2}\int \frac{1}{t^2 - 7t + 12}\,dt$$

$$= I_1 + I_2 \qquad \text{...(i)}$$

where, $I_1 = \frac{3}{2}\int \frac{2t-7}{t^2 - 7t + 12}\,dt$

Put $t^2 - 7t + 12 = t_1 \Rightarrow (2t - 7)\,dt = dt_1$

Then, $I_1 = \frac{3}{2}\int \frac{dt_1}{t_1} = \frac{3}{2}\log|t_1| + C_1$

$$= \frac{3}{2}\log|t^2 - 7t + 12| + C_1$$

$$= \frac{3}{2}\log|\sin^2 x - 7\sin x + 12| + C_1$$

and $I_2 = \frac{17}{2}\int \frac{1}{t^2 - 7t + 12}\,dt$

$$= \frac{17}{2}[\log(t-4) - \log(t-3)]$$

$$= \frac{17}{2}\log\left|\frac{t-4}{t-3}\right|$$

$$= \frac{17}{2}\log\left|\frac{\sin x - 4}{\sin x - 3}\right| + C_2$$

$\therefore$ From Eq. (i), we get

$$\int \frac{(3\sin x - 2)\cos x}{13 - \cos^2 x - 7\sin x}\,dx$$

$$= \frac{3}{2}\log|\sin^2 x - 7\sin x + 12| + \frac{17}{2}\log\left|\frac{\sin x - 4}{\sin x - 3}\right| + C$$

57. Similar as Example 23.

$$\left[\textbf{Ans. } \sqrt{x^2 + 5x + 6} - \frac{1}{2}\log\left|\left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 6}\right| + C\right]$$

58. Similar as Example 23.

Ans. $\sqrt{5 - 4x + x^2} + 5\log|(x-2) + \sqrt{5 - 4x + x^2}| + C$

59. **Hint** Firstly, write the given integral as

$\int \frac{6x + 7}{\sqrt{x^2 - 9x + 20}}\,dx$ and then similar as Example 23.

$$\left[\textbf{Ans. } 6\sqrt{x^2 - 9x + 20} + 34\log\left|\left(x - \frac{9}{2}\right) + \sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}\right| + C\right]$$

60. **Hint** Firstly, put $x + 2 = A\frac{d}{dx}(4x - x^2) + B$ and then similar as Example 23.

$$\left[\textbf{Ans. } -\sqrt{4x - x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C\right]$$

61. **Hint** Firstly, write the given integral as

$\int \frac{\sin x}{\sin 3x}\,dx = \int \frac{\sin x\,dx}{3\sin x - 4\sin^3 x} = \int \frac{dx}{3 - 4\sin^2 x}$,

divide numerator and denominator by $\cos^2 x$ and then put $\tan x = t$

$$\left[\textbf{Ans. } \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3} + 2\tan x}{\sqrt{3} - 2\tan x}\right| + C\right]$$

62. Similar as Example 26. $\left[\textbf{Ans. } \frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{\sqrt{2}\tan x}{\sqrt{5}}\right) + C\right]$

63. Let $I = \int \frac{1}{1 - \cot x}\,dx = \int \frac{1}{1 - \frac{\cos x}{\sin x}}\,dx$

$$= \int \frac{\sin x}{\sin x - \cos x}\,dx \qquad \text{...(i)}$$

Let $\sin x = A\frac{d}{dx}(\sin x - \cos x) + B(\sin x - \cos x)$

$\Rightarrow \sin x = A(\cos x + \sin x) + B(\sin x - \cos x) \qquad \text{...(ii)}$

On equating the coefficients of like terms $\sin x$ and $\cos x$ from both sides, we get

$$A + B = 1 \qquad \text{...(iii)}$$

and $\quad A - B = 0 \qquad \text{...(iv)}$

On adding Eqs. (iii) and (iv), we get $2A = 1 \Rightarrow A = \frac{1}{2}$

On putting the value of A in Eq. (iv), we get

$$\frac{1}{2} - B = 0 \Rightarrow B = \frac{1}{2}$$

Now, from Eqs. (i) and (ii), we get

$$I = \int \frac{\frac{1}{2}\frac{d}{dx}(\sin x - \cos x) + \frac{1}{2}(\sin x - \cos x)}{\sin x - \cos x} dx$$

$$\Rightarrow \quad I = \int \frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(\sin x - \cos x)}{\sin x - \cos x} dx$$

$$= \frac{1}{2}\int \frac{\cos x + \sin x}{\sin x - \cos x} dx + \frac{1}{2}\int \frac{\sin x - \cos x}{\sin x - \cos x} dx$$

Put $\sin x - \cos x = t$ in first integral.

Then, $(\cos x + \sin x)\, dx = dt$

$$\therefore \quad I = \frac{1}{2}\int \frac{dt}{t} + \frac{1}{2}\int dx = \frac{1}{2}\log|t| + \frac{1}{2}x + C$$

$$= \frac{1}{2}\log|\sin x - \cos x| + \frac{1}{2}x + C$$

$[\because t = \sin x - \cos x]$

64. **Hint** It is an integration of the form $\int \frac{x^2 \pm 1}{x^4 + \lambda x^2 + 1} dx$.

So, divide numerator and denominator by x^2 and make a perfect square as $\left(x \mp \frac{1}{x}\right)^2$ in the denominator and then substitute $x \mp \frac{1}{x} = t$

$\left[\textbf{Ans. } \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right) + C\right]$

65. Similar as Example 30. $\left[\textbf{Ans. } \sqrt{\frac{x-1}{x+1}} + C\right]$

66. Let $I = \int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx = \int \frac{2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}}{1 - 2\left(2\cos^2\frac{3x}{2} - 1\right)} dx$

$$\left[\because \cos C + \cos D = 2\cos\frac{C+D}{2}\cdot\cos\frac{C-D}{2} \text{ and } \cos x = 2\cos^2\left(\frac{x}{2}\right) - 1\right]$$

$$\Rightarrow I = \int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}}{3 - 4\cos^2\frac{3x}{2}} dx = -\int \frac{2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}}{4\cos^2\frac{3x}{2} - 3} dx$$

$$= -\int \frac{2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}\cdot\cos\frac{3x}{2}}{4\cos^3\frac{3x}{2} - 3\cos\frac{3x}{2}} dx$$

$\left[\text{multiplying and dividing by } \cos\frac{3x}{2}\right]$

$$= -\int \frac{2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}\cdot\cos\frac{3x}{2}}{\cos 3\cdot\left(\frac{3x}{2}\right)} dx$$

$[\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta]$

$$= -\int 2\cos\frac{3x}{2}\cdot\cos\frac{x}{2} dx$$

$$= -\int \left\{\cos\left(\frac{3x}{2} + \frac{x}{2}\right) + \cos\left(\frac{3x}{2} - \frac{x}{2}\right)\right\} dx$$

$[\because 2\cos A \cos B = \cos(A+B) + \cos(A-B)]$

$$= -\int(\cos 2x + \cos x)\, dx$$

$$= -\left[\frac{\sin 2x}{2} + \sin x\right] + C = -\frac{1}{2}\sin 2x - \sin x + C$$

67. Let $I = \int \cot^{-1}\left(\frac{\sqrt{1+\cos 2x} + \sqrt{1-\cos 2x}}{\sqrt{1+\cos 2x} - \sqrt{1-\cos 2x}}\right) dx$

$$= \int \cot^{-1}\left(\frac{\sqrt{2\cos^2 x} + \sqrt{2\sin^2 x}}{\sqrt{2\cos^2 x} - \sqrt{2\sin^2 x}}\right) dx$$

$\left[\because \cos^2 x = \frac{1+\cos 2x}{2} \text{ and } \sin^2 x = \frac{1-\cos 2x}{2}\right]$

$$= \int \cot^{-1}\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right) dx$$

$$= \int \tan^{-1}\left[\frac{1}{\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)}\right] dx \quad \left[\because \cot^{-1} x = \tan^{-1}\frac{1}{x}\right]$$

$$= \int \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) dx$$

$$= \int \tan^{-1}\left(\frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}\right) dx$$

[dividing numerator and denominator by $\cos x$]

$$= \int \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right) dx$$

$$= \int \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\cdot\tan x}\right) dx$$

$$= \int \tan^{-1}\left[\tan\left(\frac{\pi}{4} - x\right)\right] dx$$

$\left[\because \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}\right]$

$$= \int \left(\frac{\pi}{4} - x\right) dx \quad [\because \tan^{-1}(\tan\theta) = \theta]$$

$$= \frac{\pi}{4}x - \frac{x^2}{2} + C$$

68. Let $I = \int \sqrt{\tan\theta}\, d\theta$

Now, put $\sqrt{\tan\theta} = t \Rightarrow \tan\theta = t^2$

Then, $\sec^2\theta\, d\theta = 2t\, dt$

$$\Rightarrow \quad d\theta = \frac{2t}{\sec^2\theta} dt = \frac{2t}{(1 + \tan^2\theta)} dt = \frac{2t}{(1 + t^4)} dt$$

$$\therefore I=\int t\cdot\frac{2t}{(1+t^4)}\,dt=\int\frac{2t^2}{(1+t^4)}\,dt=\int\frac{(t^2+1)+(t^2-1)}{t^4+1}\,dt$$

[adding and subtracting 1 from numerator]

$$=\int\frac{t^2+1}{t^4+1}\,dt+\int\frac{t^2-1}{t^4+1}\,dt$$

$$=\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt+\int\frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt$$

$$=\int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}\,dt+\int\frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-2}\,dt$$

Now, put $t-\frac{1}{t}=u$ and $t+\frac{1}{t}=v$

$$\Rightarrow\quad\left(1+\frac{1}{t^2}\right)dt=du\text{ and }\left(1-\frac{1}{t^2}\right)dt=dv$$

$$\therefore\quad I=\int\frac{du}{u^2+(\sqrt{2})^2}+\int\frac{dv}{v^2-(\sqrt{2})^2}$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}+\frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right|+C$$

$$\left[\because\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}\text{ and }\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{t-\frac{1}{t}}{\sqrt{2}}\right]+\frac{1}{2\sqrt{2}}\log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|+C$$

$$\left[\because u=t-\frac{1}{t}\text{ and }v=t+\frac{1}{t}\right]$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{t^2-1}{\sqrt{2}\,t}\right]+\frac{1}{2\sqrt{2}}\log\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right|+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{\tan\theta-1}{\sqrt{2\tan\theta}}\right]+\frac{1}{2\sqrt{2}}\log\left|\frac{\tan\theta-\sqrt{2\tan\theta}+1}{\tan\theta+\sqrt{2\tan\theta}+1}\right|+C$$

$$[\because t=\sqrt{\tan\theta}]$$

69. Let $I=\int\frac{1}{\cos^4x+\sin^4x}\,dx$

On dividing numerator and denominator by $\cos^4 x$, we get

$$I=\int\frac{\sec^4x}{1+\tan^4x}\,dx\;\Rightarrow\;I=\int\frac{(\sec^2x)(\sec^2x)}{1+\tan^4x}\,dx$$

$$\Rightarrow\;I=\int\frac{\sec^2x\,(1+\tan^2x)}{1+\tan^4x}\,dx\qquad[\because 1+\tan^2x=\sec^2x]$$

Now, put $\tan x=t\;\Rightarrow\;\sec^2x\,dx=dt$

$$\therefore\qquad I=\int\frac{1+t^2}{1+t^4}\,dt$$

Again, dividing numerator and denominator by t^2, we get

$$I=\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+2-2}\,dt=\int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}\,dt$$

[adding and subtracting 2 from denominator]

Again, putting $t-\frac{1}{t}=u\Rightarrow\left(1+\frac{1}{t^2}\right)dt=du$, we get

$$I=\int\frac{du}{u^2+(\sqrt{2})^2}\Rightarrow I=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C$$

$$\left[\because\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$\Rightarrow\;I=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+C\qquad\left[\because u=t-\frac{1}{t}\right]$$

$$\Rightarrow\;I=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right)+C$$

$$\Rightarrow\;I=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2x-1}{\sqrt{2}\tan x}\right)+C\qquad[\because t=\tan x]$$

70. **Hint** $I=\int[\sqrt{\cot x}+\sqrt{\tan x}]dx=\int\sqrt{\tan x}\,(1+\cot x)\,dx$

Now, put $\tan x=t^2\;\Rightarrow\;\sec^2x\,dx=2t\,dt$

$$\Rightarrow\qquad dx=\frac{2t\,dt}{1+t^4}\;\Rightarrow\;I=\int t\left[1+\frac{1}{t^2}\right]\frac{2t}{(1+t^4)}dt$$

$$\Rightarrow\qquad I=2\int\frac{t^2+1}{t^4+1}\,dt$$

Further, solve as Question 64.

$$\left[\textbf{Ans. } I=\sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)+C\right]$$

71. **Hint** Let $I=\int\frac{1}{\sin^4x+\sin^2x\cos^2x+\cos^4x}\,dx$

On dividing numerator and denominator by $\cos^4 x$, we get

$$I=\int\frac{\sec^4x}{\tan^4x+\tan^2x+1}\,dx=\int\frac{(\sec^2x)(\sec^2x)}{\tan^4x+\tan^2x+1}\,dx$$

Now, put $\tan x=t\Rightarrow\sec^2x\,dx=dt$

$$[\because\sec^2x=1+\tan^2x=1+t^2]$$

$$\therefore\qquad I=\int\frac{1+t^2}{t^4+t^2+1}\,dt\qquad[\because t=\tan x]$$

Further, solve as Example 28.

$$\left[\textbf{Ans. } I=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan^2x-1}{\sqrt{3}\tan x}\right)+C\right]$$

|TOPIC 3|

Integration by Partial Fractions

Sometimes, an integral of the form $\int \frac{P(x)}{Q(x)} dx$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$, also $Q(x)$ has only linear and quadratic factors is given to us, if we cannot integrate it directly or by previous methods, then we use the partial fractions.

For this, firstly we have to know partial fraction decomposition which is given below

PARTIAL FRACTION DECOMPOSITION

It is always possible to write the integrand of the form $\frac{P(x)}{Q(x)}$, where $P(x), Q(x)$ are polynomials in x and $Q(x) \neq 0$ as a sum of simpler rational functions by a method which is known as partial fraction decomposition. Each such fraction is called a partial fraction and it have a simplest factor of $Q(x)$. In this method, we use the following steps

I. Suppose the given integral is in the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$. Then, firstly check that it is a proper fraction or improper fraction.

II. If $\frac{P(x)}{Q(x)}$ is a proper fraction, then we go to next step directly. If $\frac{P(x)}{Q(x)}$ is an improper fraction, then we divide $P(x)$ by $Q(x)$, so that $\frac{P(x)}{Q(x)}$ is expressed in the form of $T(x) + \frac{P_1(x)}{Q(x)}$, where $T(x)$ is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is a proper rational function.

III. Now, the decomposition of proper fraction $\frac{P(x)}{Q(x)}$ or $\frac{P_1(x)}{Q(x)}$ into the partial fractions depends mainly upon the nature of the factors of $Q(x)$.

According to nature of factors of $Q(x)$, corresponding form of the partial fraction is given below.

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px \pm q}{(x \pm a)(x \pm b)}, a \neq b$	$\frac{A}{x \pm a} + \frac{B}{x \pm b}$
2.	$\frac{px \pm q}{(x \pm a)^2}$	$\frac{A}{(x \pm a)} + \frac{B}{(x \pm a)^2}$
3.	$\frac{px^2 \pm qx \pm r}{(x \pm a)(x \pm b)(x \pm c)}$	$\frac{A}{(x \pm a)} + \frac{B}{(x \pm b)} + \frac{C}{(x \pm c)}$
4.	$\frac{px^2 \pm qx \pm r}{(x \pm a)(x \pm b)^2}$	$\frac{A}{(x \pm a)} + \frac{B}{(x \pm b)} + \frac{C}{(x \pm b)^2}$
5.	$\frac{px^2 \pm qx \pm r}{(x \pm a)^2 (x \pm b)}$	$\frac{A}{(x \pm a)} + \frac{B}{(x \pm a)^2} + \frac{C}{(x \pm b)}$
6.	$\frac{px^2 \pm qx \pm r}{(x \pm a)^3}$	$\frac{A}{(x \pm a)} + \frac{B}{(x \pm a)^2} + \frac{C}{(x \pm a)^3}$
7.	$\frac{px^2 \pm qx \pm r}{(x \pm a)(x^2 \pm bx \pm c)}$	$\frac{A}{(x \pm a)} + \frac{Bx + C}{x^2 \pm bx \pm c}$ where, $x^2 \pm bx \pm c$ cannot be factorised further.

Using the above form of rational function, write $\frac{P(x)}{Q(x)}$ or $\frac{P_1(x)}{Q(x)}$ in suitable form of partial fraction and assume it Eq. (i).

IV. Now, multiply both sides of Eq. (i) by $Q(x)$ and assume this as Eq. (ii), then for finding the values of constants A, B, C, etc., we use the following two methods

Method 1 Contains factor of the form $(x \pm a)$, $(x \pm b)$, $(x \pm c)$, then put $(x \pm a), (x \pm b), (x \pm c)$ equal to zero to find values of x and then put these values of x in Eq. (ii) to get the required values of A, B, C, etc.

Method 2 Simplify LHS of Eq. (ii) to convert it into a polynomial in x and then equate coefficients of x^3, x^2, x and constant terms from both sides to get equations in A, B, C, ... etc. On solving these equations, we get values of A, B and C.

V. Now, put the values of $A, B, C, \ldots$ etc., in Eq. (i) and get the required partial fraction form.

Note A function of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials in x and $g(x) \neq 0$ is called a rational function.

If degree of $f(x)$ < degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called proper rational function. If degree of $f(x) \geq$ degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called an improper rational function.

EXAMPLE |1| Resolve $\frac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6}$ into partial fractions.

Sol. Here, degree of numerator > degree of denominator.

So, it is an improper fraction.

Now, divide numerator by denominator, we get

$$\frac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6} = (x - 1) + \frac{(-x + 4)}{x^2 - 5x + 6} \quad \ldots\text{(i)}$$

Now, $\frac{-x + 4}{x^2 - 5x + 6} = \frac{-x + 4}{(x - 2)(x - 3)}$

$[\because x^2 - 5x + 6 = x^2 - 2x - 3x + 6 = (x - 2)(x - 3)]$

So, let $\frac{-x + 4}{x^2 - 5x + 6} = \frac{A}{x - 2} + \frac{B}{x - 3} \quad \ldots\text{(ii)}$

[from fraction table, it is of the form 1]

$\Rightarrow \; -x + 4 = A(x - 3) + B(x - 2) \quad \ldots\text{(iii)}$

[multiplying both sides by $x^2 - 5x + 6$]

On putting $x - 3 = 0$, i.e. $x = 3$ in Eq. (iii), we get

$$-3 + 4 = A(0) + B(3 - 2) \Rightarrow B = 1$$

Again, putting $x - 2 = 0$, i.e. $x = 2$ in Eq. (iii), we get

$$-2 + 4 = A(2 - 3) + B(0) \Rightarrow A = -2$$

Now, putting the values of A and B in Eq. (ii), we get

$$\frac{-x + 4}{x^2 - 5x + 6} = \frac{-2}{x - 2} + \frac{1}{x - 3}$$

Hence, $\frac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6} = x - 1 - \frac{2}{x - 2} + \frac{1}{x - 3}$

[from Eq. (i)]

Method of Solving Integral by Partial Fractions

Suppose given integral is of the form $\int \frac{P(x)}{Q(x)}\,dx$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$. Then, to evaluate such integrals by partial fraction, we firstly take the given integrand $\frac{P(x)}{Q(x)}$ and decompose it into suitable partial fraction form by above method and then integrate each term by using suitable method to get the required answer.

EXAMPLE |2| Evaluate $\int \frac{x}{(x-1)^2(x+2)}\,dx.$ **[NCERT]**

Firstly, write the given integrand into partial fraction form and then integrate.

Sol. Let $I = \int \frac{x}{(x-1)^2(x+2)}\,dx$

Here, integrand is a proper rational function.

So, by using the form of partial fraction, we write

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)} \quad \ldots\text{(i)}$$

$$\Rightarrow \frac{x}{(x-1)^2(x+2)} = \frac{A(x-1)(x+2) + B(x+2) + C(x-1)^2}{(x-1)^2(x+2)}$$

$$\Rightarrow x = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \quad \ldots\text{(ii)}$$

On putting $x = 1$ in Eq. (ii), we get

$$1 = A(1-1)(1+2) + B(1+2) + C(1-1)^2$$

$$\Rightarrow \quad 1 = 3B \Rightarrow B = 1/3$$

On putting $x = -2$ in Eq. (ii), we get

$$-2 = A(-2-1)(-2+2) + B(-2+2) + C(-2-1)^2$$

$$\Rightarrow \quad -2 = 0 + 0 + 9C \Rightarrow C = \frac{-2}{9}$$

On putting $x = 0$ in Eq. (ii), we get

$$0 = A(0-1)(0+2) + B(0+2) + C(0-1)^2$$

$$\Rightarrow \quad 0 = -2A + 2B + C$$

$$\Rightarrow \quad 0 = -2A + 2 \cdot \frac{1}{3} - \frac{2}{9} \quad \left[\because B = \frac{1}{3}, C = -\frac{2}{9}\right]$$

$$\Rightarrow \quad 0 = -2A + \frac{2}{3} - \frac{2}{9}$$

$$\Rightarrow \quad 0 = -2A + \frac{6-2}{9}$$

$$\Rightarrow \quad 2A = 4/9 \Rightarrow A = 2/9$$

Now, putting the values of A, B and C in Eq. (i), we get

$$\frac{x}{(x-1)^2(x+2)} = \frac{2/9}{(x-1)} + \frac{1/3}{(x-1)^2} + \frac{-2/9}{(x+2)}$$

$$\therefore I = \int \frac{x}{(x-1)^2(x+2)}\,dx$$

$$= \frac{2}{9}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{dx}{(x-1)^2} - \frac{2}{9}\int \frac{dx}{x+2}$$

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\int (x-1)^{-2}\,dx - \frac{2}{9}\log|x+2| + C$$

$$= \frac{2}{9}\log|x-1| + \frac{1}{3} \cdot \frac{(x-1)^{-1}}{(-1)} - \frac{2}{9}\log|x+2| + C$$

$$= \frac{2}{9}\log|x-1| - \frac{1}{3(x-1)} - \frac{2}{9}\log|x+2| + C$$

$$= \frac{2}{9}\left(\log\left|\frac{x-1}{x+2}\right|\right) - \frac{1}{3(x-1)} + C$$

$$\left[\because \log m - \log n = \log\frac{m}{n}\right]$$

EXAMPLE |3| Evaluate $\int \frac{x}{(x^2+1)(x-1)} dx.$

[NCERT; All India 2015C]

Sol. Let $I = \int \frac{x}{(x^2+1)(x-1)} dx$

Here, integrand $= \frac{x}{(x^2+1)(x-1)} = \frac{P(x)}{Q(x)}$ (say)

where, degree of $P(x)$ is less than degree of $Q(x)$, so it is a proper fraction.

Now, let $\frac{x}{(x^2+1)(x-1)} = \frac{A}{(x-1)} + \frac{Bx+C}{x^2+1}$ ...(i)

On multiplying both sides of Eq. (i) by $(x^2+1)(x-1)$, we get

$$x = A(x^2+1) + (Bx+C)(x-1) \quad ...(ii)$$

On putting $x = 1$ in Eq. (ii), we get

$$1 = A(1+1) + (B+C)(0) \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

On equating the coefficients of x^2 and x from both sides of Eq. (ii), we get

$$0 = A + B \text{ and } 1 = -B + C$$

$$\Rightarrow \quad 0 = \frac{1}{2} + B \Rightarrow B = -\frac{1}{2} \quad \left[\because A = \frac{1}{2}\right]$$

and $\quad 1 = -\left(-\frac{1}{2}\right) + C \Rightarrow C = 1 - \frac{1}{2} = \frac{1}{2}$

From Eq. (i), we get

$$\frac{x}{(x^2+1)(x-1)} = \frac{1}{2(x-1)} + \frac{-\frac{1}{2}x + \frac{1}{2}}{x^2+1} \quad ...(iii)$$

Now, putting partial fraction form of integrand from Eq. (iii) in given integral, we get

$$I = \int \frac{1}{2(x-1)} dx + \frac{1}{2}\int \frac{(-x+1)}{x^2+1} dx$$

$$= \frac{1}{2}\log|x-1| + \frac{1}{2}\left[-\int \frac{x}{x^2+1} \times \frac{2}{2} dx + \int \frac{1}{x^2+1} dx\right]$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\int \frac{2x}{x^2+1} dx + \frac{1}{2}\int \frac{1}{x^2+1} dx$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1} x + C$$

$\left[\begin{array}{l}\text{for evaluating } \int \frac{2x}{x^2+1} dx, \text{ put } x^2+1 = t \Rightarrow 2x\,dx = dt, \\ \text{then } \int \frac{2x}{x^2+1} dx = \int \frac{1}{t} dt = \log|t| = \log|x^2+1|\end{array}\right]$

EXAMPLE |4| Find $\int \frac{2\cos x}{(1-\sin x)(2-\cos^2 x)} dx.$

[All India 2019]

Sol. Let $I = \int \frac{2\cos x}{(1-\sin x)(2-\cos^2 x)} dx$

$$= \int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)} dx$$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore \quad I = \int \frac{2}{(1-t)(1+t^2)} dt$$

Now, let $\frac{2}{(1-t)(1+t^2)} = \frac{A}{1-t} + \frac{Bt+C}{1+t^2}$

$$\Rightarrow \quad 2 = A(1+t^2) + (Bt+C)(1-t) \quad ...(i)$$

Putting $t = 1$ in Eq. (i), we get

$$2 = 2A \Rightarrow A = 1$$

Putting $t = 0$ in Eq. (i), we get

$$2 = A + C \Rightarrow 2 = 1 + C \Rightarrow C = 1$$

Putting $t = -1$ in Eq. (i), we get

$$2 = 2A + (-B+C)(2) \Rightarrow 2 = 2 - 2B + 2$$

$$\Rightarrow \quad 2B = 2 \Rightarrow B = 1$$

$$\therefore I = \int \frac{1}{1-t} dt + \int \frac{t+1}{1+t^2} dt$$

$$= \int \frac{1}{1-t} dt + \frac{1}{2}\int \frac{2t}{1+t^2} dt + \int \frac{1}{1+t^2} dt$$

$$= -\log(1-t) + \frac{1}{2}\log(1+t^2) + \tan^{-1} t + C$$

$$= -\log(1-\sin x) + \frac{1}{2}\log(1+\sin^2 x) + \tan^{-1}(\sin x) + C$$

EXAMPLE |5| Evaluate $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx.$

Sol. Firstly, convert it into simple form by putting $x^2 = t$,

i.e. write $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(t+1)(t+2)}{(t+3)(t+4)} = \frac{t^2+3t+2}{t^2+7t+12}$

Now, degree of numerator and denominator is same, so it can be written as $1 - \frac{4t+10}{t^2+7t+12}$.

[dividing numerator by denominator]

Thus, $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \frac{(4t+10)}{t^2+7t+12}$

$$= 1 - \frac{4t+10}{(t+3)(t+4)} \quad ...(i)$$

$[\because t^2 + 7t + 12 = t^2 + 4t + 3t + 12 = (t+3)(t+4)]$

Now, consider $\frac{4t+10}{(t+4)(t+3)} = \frac{A}{(t+4)} + \frac{B}{(t+3)}$

$$\Rightarrow \quad \frac{4t+10}{(t+3)(t+4)} = \frac{A(t+3)+B(t+4)}{(t+4)(t+3)}$$

$$\Rightarrow \quad 4t + 10 = At + 3A + Bt + 4B$$

$$\Rightarrow \quad 4t + 10 = t(A+B) + (3A+4B)$$

On comparing the coefficients of t and constant term from both sides, we get

$$A + B = 4$$

$\Rightarrow \quad 3A + 3B = 12$ [multiplying both sides by 3] ...(ii)

and $3A + 4B = 10$ ...(iii)

On subtracting Eq. (iii) from Eq. (ii), we get

$$-B = 2 \Rightarrow B = -2$$

Then, from Eq. (iii), we get

$$3A + 4(-2) = 10 \Rightarrow 3A = 18 \Rightarrow A = 6$$

$$\therefore \quad \frac{4t+10}{(t+4)(t+3)} = \frac{6}{t+4} - \frac{2}{t+3}$$

On putting this value in Eq. (i), we get

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \left[\frac{6}{t+4} - \frac{2}{t+3}\right]$$

$$= 1 - \left(\frac{6}{(x^2+4)} - \frac{2}{(x^2+3)}\right) \quad [\because t = x^2]$$

Now, given integral,

$$I = \int 1 dx - \int \left(\frac{6}{x^2+2^2} - \frac{2}{x^2+(\sqrt{3})^2}\right) dx$$

$$= x - 6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right) + 2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right) + C$$

$$\left[\because \int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= x - 3\tan^{-1}\frac{x}{2} + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

TOPIC PRACTICE 3

OBJECTIVE TYPE QUESTIONS

1 $\int \frac{x}{(x-1)(x-2)} dx$ equals **[NCERT]**

(a) $\log\left|\frac{(x-1)^2}{x-2}\right| + C$

(b) $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

(c) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right|$

(d) $\log|(x-1)(x-2)| + C$

2 $\int \frac{dx}{x(x^2+1)}$ equals **[NCERT]**

(a) $\log|x| - \frac{1}{2}\log(x^2+1) + C$

(b) $\log|x| + \frac{1}{2}\log(x^2+1) + C$

(c) $-\log|x| + \frac{1}{2}\log(x^2+1) + C$

(d) $\frac{1}{2}\log|x| + \log(x^2+1) + C$

3 If $\int \frac{\sin x}{\cos x(1+\cos x)} dx = f(x) + C$, then $f(x)$ is equal to

(a) $\log\left|\frac{1+\cos x}{\cos x}\right|$

(b) $\log\left|\frac{\cos x}{1+\cos x}\right|$

(c) $\log\left|\frac{\sin x}{1+\sin x}\right|$

(d) $\log\left|\frac{1+\sin x}{\sin x}\right|$

SHORT ANSWER Type II Questions

Directions (Q. Nos. 4-19) *Evaluate the following integrals.*

4 Find $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)} dx$. **[CBSE 2018]**

5 $\int \frac{e^x}{(1+e^x)(2+e^x)} dx$ **[NCERT]**

6 $\int \frac{2x}{(x^2+1)(x^2+3)} dx$ **[NCERT; Delhi 2011]**

7 $\int \frac{(3\sin\phi - 2)\cos\phi}{5-\cos^2\phi - 4\sin\phi} d\phi$ **[NCERT; Delhi 2016, 2013C]**

8 $\int \frac{dx}{\sin x + \sin 2x}$ **[Delhi 2015]**

9 $\int \frac{(2x-1)}{(x-1)(x+2)(x-3)} dx$ **[NCERT Exemplar]**

10 $\int \frac{e^x}{(e^x-1)^2(e^x+2)} dx$ **[All India 2017]**

11 $\int \frac{2x}{(x^2+1)(x^2+2)^2} dx$ **[Delhi 2017]**

12 $\int \frac{2}{(1-x)(1+x^2)} dx$ **[NCERT; Delhi 2012]**

13 $\int \frac{2x}{(x^2+1)(x^4+4)} dx$ **[Delhi 2017]**

14 $\int \frac{dx}{x(x^3+1)}$ **[All India 2013]**

15 $\int \frac{1}{(x^4-1)} dx$

16 $\int \frac{x^2+x+1}{(x^2+1)(x+2)} dx$ **[All India 2019, 2016, 2015]**

17 $\int \frac{x^2}{(x-1)(x^2+1)} dx$ **[Delhi 2017C]**

18 $\int \frac{x^4}{(x-1)(x^2+1)} dx$ **[NCERT]**

LONG ANSWER Type Question

19 $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$ **[All India 2017]**

20 Evaluate $\int \frac{x^2}{(x^2+4)(x^2+9)}dx$. **[Delhi 2013]**

HINTS & SOLUTIONS

1. (b) Let $\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$\Rightarrow \quad x = A(x-2) + B(x-1)$

Putting $x=1$, we get $A=-1$

and putting $x=2$, we get $B=2$

Thus, $\int \frac{x}{(x-1)(x-2)} = \int \frac{-1}{x-1}dx + 2\int \frac{dx}{x-2}$

$= -\log|x-1| + 2\log|x-2| + C$

$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$

2. (a) Let $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$\Rightarrow \quad 1 = A(x^2+1) + (Bx+C)x$

$\Rightarrow \quad A+B=0,\ C=0$ and $A=1$

On solving these equations, we get

$A=1,\ B=-1$ and $C=0$

$\therefore \quad \frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}$

$\Rightarrow \int \frac{1}{x(x^2+1)}dx = \int\left\{\frac{1}{x} - \frac{x}{x^2+1}dx\right\}$

$= \log|x| - \frac{1}{2}\log(x^2+1) + C$

3. (a) Let $I = \int \frac{\sin x}{\cos x(1+\cos x)}dx$

Put $\cos x = t \Rightarrow -\sin x\,dx = dt$

$\therefore\ I = \int \frac{-dt}{t(1+t)} = -\left[\frac{1}{t} - \frac{1}{t+1}\right]dt$

$= -[\log|t| - \log|t+1|] + C = \log\left|\frac{\cos x+1}{\cos x}\right| + C$

4. Let $I = \int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

Put $\sin x = t$, then $\cos x\,dx = dt$

$\therefore \quad I = \int \frac{2dt}{(1-t)(1+t^2)}$ ...(i)

Now, let $\frac{2}{(1-t)(1+t^2)} = \frac{A}{1-t} + \frac{Bt+C}{1+t^2}$

$\Rightarrow \quad 2 = (1+t^2)A + (1-t)(Bt+C)$

$\Rightarrow \quad 2 = (1+t^2)A + (Bt+C-Bt^2-Ct)$

$\Rightarrow \quad 2 = t^2(A-B) + t(B-C) + (A+C)$

On comparing the coefficients of like powers of t, we get

$A-B=0;\ B-C=0$ and $A+C=2$

$\Rightarrow \quad A=B;\ B=C$ and $A+C=2$

$\Rightarrow \quad A=B=C=1$

$\therefore \quad \frac{2}{(1-t)(1+t^2)} = \frac{1}{1-t} + \frac{1+t}{1+t^2}$

Now, from Eq. (i), we get

$I = \int\left(\frac{1}{1-t} + \frac{1+t}{1+t^2}\right)dt = \int\frac{dt}{1-t} + \int\frac{1}{1+t^2}dt + \frac{1}{2}\int\frac{2t}{1+t^2}dt$

$= \frac{\log|1-t|}{(-1)} + \tan^{-1}t + \frac{1}{2}\log|1+t^2| + C$

$= \frac{1}{2}\log|1+\sin^2 x| - \log|1-\sin x| + \tan^{-1}(\sin x) + C$ $[\because t = \sin x]$

$= \tan^{-1}(\sin x) + \log\left|\frac{\sqrt{1+\sin^2 x}}{1-\sin x}\right| + C$

$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right) \text{ and } n\log m = \log m^n\right]$

5. **Hint** Substitute $e^x = t$ and $e^x\,dx = dt$, then given integral reduces to $\int\frac{dt}{(1+t)(2+t)}$. Further, use partial fractions and then integrate. $\left[\textbf{Ans. } \log\left|\frac{1+e^x}{2+e^x}\right| + C\right]$

6. **Hint** Firstly, put $x^2 = t \Rightarrow 2x\,dx = dt$ and then

$I = \int\frac{dt}{(t+1)(t+3)}$. $\left[\textbf{Ans. } \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C\right]$

7. **Hint** (i) Write the given integral as $\int\frac{(3\sin\phi-2)\cos\phi}{\sin^2\phi - 4\sin\phi + 4}d\phi$

(ii) On substituting $\sin\phi = t$ and $\cos\phi\,d\phi = dt$, then above integral reduces to $\int\frac{(3t-2)}{(t-2)^2}dt$.

(iii) Now, use partial fraction and then integrate.

$\left[\textbf{Ans. } 3\log|2-\sin\phi| + \frac{4}{2-\sin\phi} + C\right]$

8. Let $I = \int\frac{dx}{\sin x + \sin 2x} = \int\frac{dx}{\sin x + 2\sin x\cos x}$

$[\because \sin 2x = 2\sin x\cos x]$

$= \int\frac{dx}{\sin x(1+2\cos x)} = \int\frac{\sin x\,dx}{\sin^2 x(1+2\cos x)}$

[multiplying numerator and denominator by $\sin x$]

$= \int\frac{\sin x\,dx}{(1-\cos^2 x)(1+2\cos x)}$

Now, put $\cos x = t \Rightarrow -\sin x\, dx = dt \Rightarrow \sin x\, dx = -dt$

$$I = \int \frac{-dt}{(1-t^2)(1+2t)} = \int \frac{-dt}{(1-t)(1+t)(1+2t)} \quad \text{...(i)}$$

Now, let

$$\frac{1}{(1-t)(1+t)(1+2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1+2t} \quad \text{...(ii)}$$

$$\Rightarrow 1 = (1+t)(1+2t)A + (1-t)(1+2t)B + (1-t)(1+t)C \quad \text{...(iii)}$$

On putting $t = -1$ in Eq. (iii), we get

$$1 = (2)(-1)B \Rightarrow B = -\frac{1}{2}$$

On putting $t = 1$ in Eq. (iii), we get

$$1 = 2\cdot(3)A \Rightarrow A = \frac{1}{6}$$

On putting $t = -\frac{1}{2}$ in Eq. (iii), we get

$$1 = \left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)C \Rightarrow 1 = \left(\frac{3}{2}\times\frac{1}{2}\right)C \Rightarrow C = \frac{4}{3}$$

$$\therefore I = -\left[\int \frac{A}{1-t}dt + \int \frac{B}{1+t}dt + \int \frac{C}{1+2t}dt\right]$$

[using Eqs. (i) and (ii)]

$$= -\left[\frac{1}{6}\int \frac{dt}{1-t} + \left(-\frac{1}{2}\right)\int \frac{dt}{1+t} + \frac{4}{3}\int \frac{dt}{1+2t}\right]$$

$$= -\left[\frac{1}{6}\frac{\log|1-t|}{-1} - \frac{1}{2}\log|1+t| + \frac{4}{3}\frac{\log|1+2t|}{2}\right] + C$$

$$= \frac{1}{6}\log|1-t| + \frac{1}{2}\log|1+t| - \frac{2}{3}\log|1+2t| + C$$

$$= \frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log|1+\cos x| - \frac{2}{3}\log|1+2\cos x| + C \quad [\because t = \cos x]$$

9. **Hint** $\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$

$$\left[\textbf{Ans. } \log\left|\frac{\sqrt{x-3}}{(x-1)^{1/6}(x+2)^{1/3}}\right| + C\right]$$

10. Let $I = \int \frac{e^x dx}{(e^x-1)^2(e^x+2)}$

Put $e^x = t \Rightarrow e^x dx = dt$

$$\therefore \quad I = \int \frac{dt}{(t-1)^2(t+2)}$$

Now, let $\frac{1}{(t-1)^2(t+2)} = \frac{A}{(t-1)} + \frac{B}{(t-1)^2} + \frac{C}{(t+2)}$

$$\Rightarrow 1 = A(t-1)(t+2) + B(t+2) + C(t-1)^2$$

$$\Rightarrow 1 = A(t^2+t-2) + B(t+2) + C(t^2-2t+1)$$

$$\Rightarrow 1 = t^2(A+C) + t(A+B-2C) - 2A + 2B + C$$

On comparing the coefficient of t^2, t and the constant term from the both sides, we get

$A + C = 0$, $A + B - 2C = 0$ and $-2A + 2B + C = 1$

$$\Rightarrow \quad A = -C, \quad \text{...(i)}$$

$$A + B = 2C \quad \text{...(ii)}$$

and $-2A + 2B + C = 1$ ...(iii)

On substituting $A = -C$ in Eqs. (ii) and (iii), we get

$$-C + B = 2C \Rightarrow B = 3C \quad \text{...(iv)}$$

and $2C + 2B + C = 1$

$$\Rightarrow 2C + 2(3C) + C = 1 \quad \text{[from Eq. (iv)]}$$

$$\Rightarrow 9C = 1 \Rightarrow C = \frac{1}{9}$$

Now, from Eqs. (i) and (iv), we get

$$A = \frac{-1}{9}, B = \frac{1}{3}$$

Now, $I = \int\left[-\frac{1}{9}\frac{1}{(t-1)} + \frac{1}{3}\frac{1}{(t-1)^2} + \frac{1}{9}\frac{1}{(t+2)}\right]dt$

$$= -\frac{1}{9}\int\frac{dt}{t-1} + \frac{1}{3}\int\frac{dt}{(t-1)^2} + \frac{1}{9}\int\frac{dt}{t+2}$$

$$= -\frac{1}{9}\log|t-1| + \frac{1}{3}\cdot\frac{(t-1)^{-2+1}}{(-2+1)} + \frac{1}{9}\log|t+2| + C$$

$$= \frac{1}{9}\log\left|\frac{t+2}{t-1}\right| - \frac{1}{3}\cdot\frac{1}{(t-1)} + C$$

$$= \frac{1}{9}\log\left|\frac{e^x+2}{e^x-1}\right| - \frac{1}{3}\frac{1}{(e^x-1)} + C \quad [\text{putting } t = e^x]$$

11. Solve as Question 10.

$$\left[\textbf{Ans. } \log|x^2+1| - \log|x^2+2| + \frac{1}{x^2+2} + C\right]$$

12. Let $I = \int \frac{2}{(1-x)(1+x^2)}dx$

Decompose the rational function into partial fraction.

Let $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$

$$\Rightarrow \frac{2}{(1-x)(1+x^2)} = \frac{A(1+x^2)+(Bx+C)(1-x)}{(1-x)(1+x^2)}$$

$$\Rightarrow \quad 2 = A(1+x^2) + (Bx+C)(1-x) \quad \text{...(i)}$$

On putting $x = 1$ in Eq. (i), we get

$$2 = A(1+1) + 0 \Rightarrow 2A = 2 \Rightarrow A = 1$$

On putting $x = 0$ in Eq. (i), we get

$$2 = A(1+0) + (0+C)(1-0) \Rightarrow 2 = A + C \quad \text{...(ii)}$$

On putting the value of A in Eq. (ii), we get

$$2 = 1 + C \Rightarrow C = 1$$

Now, putting $x = -1$ in Eq. (i), we get

$$2 = A(1+1) + (-B+C)(1+1)$$

$$\Rightarrow \quad 2 = 2A - 2B + 2C \quad \text{...(iii)}$$

On putting the values of A and C in Eq. (iii), we get

$$2 = 2 - 2B + 2 \Rightarrow B = 1$$

$$\therefore \quad \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{1\cdot x+1}{1+x^2}$$

Now, $I = \int \frac{2}{(1-x)(1+x^2)} dx = \int \left(\frac{1}{1-x} + \frac{x+1}{x^2+1} \right) dx$

$= \int \frac{1}{1-x} dx + \int \frac{x+1}{x^2+1} dx$

$= \int \frac{dx}{1-x} + \int \frac{x}{x^2+1} dx + \int \frac{dx}{x^2+1}$

Put $1 + x^2 = t$ in second integral, then

$2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$

$\therefore \quad I = \int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dt}{t} + \int \frac{dx}{x^2+1}$

$= -\log|1-x| + \frac{1}{2}\log|t| + \tan^{-1} x + C$

$= \log\left|\frac{1}{1-x}\right| + \frac{1}{2}\log|1+x^2| + \tan^{-1} x + C$

$\left[\because -\log x = \log\frac{1}{x}\right]$

13. Solve as Question 12.

$\left[\textbf{Ans. } \frac{1}{5}\log|x^2+1| - \frac{1}{5}\left[\frac{1}{2}\log|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x^2}{2}\right)\right] + C\right]$

14. Let $I = \int \frac{dx}{x(x^3+1)}$, $I = \int \frac{dx}{x(x+1)(x^2-x+1)}$

$[\because a^3 + b^3 = (a+b)(a^2+b^2-ab)]$

Decompose the rational function into partial fraction.

Again, let

$$\frac{1}{x(x+1)(x^2-x+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1}$$

$\Rightarrow \quad 1 = A(x+1)(x^2-x+1) + Bx(x^2-x+1) + (Cx+D)x(x+1)$

$\Rightarrow \quad 1 = A(x^3-x^2+x+x^2-x+1) + B(x^3-x^2+x) + (Cx+D)(x^2+x)$

$\Rightarrow \quad 1 = A(x^3+1) + B(x^3-x^2+x) + (Cx^3+Dx^2+Cx^2+Dx)$

$\Rightarrow 1 = (A+B+C)x^3 + (-B+D+C)x^2 + (B+D)x + A$

On comparing the coefficients of different powers of x from both sides, we get

$A+B+C=0$ ...(i)

$-B+D+C=0$ ...(ii)

$B+D=0$ ...(iii)

and $A=1$ ...(iv)

From Eqs. (ii) and (iii), we get

$C-2B=0$ ...(v)

From Eqs. (i) and (iv), we get

$B+C=-1$ ...(vi)

From Eqs. (v) and (vi), we get

$B = -\frac{1}{3}$ and $C = -\frac{2}{3} \Rightarrow D = \frac{1}{3}$ [using Eq. (iii)]

Now, $I = \int \frac{dx}{x(x^3+1)} = \int \left[\frac{1}{x} - \frac{1}{3(x+1)} + \frac{-\frac{2x}{3}+\frac{1}{3}}{x^2-x+1}\right] dx$

$\Rightarrow \quad I = \int \frac{dx}{x} - \frac{1}{3}\int \frac{dx}{x+1} + \frac{1}{3}\int \frac{1-2x}{x^2-x+1} dx$

$\Rightarrow \quad I = \log|x| - \frac{1}{3}\log|x+1| - \frac{1}{3}\int \frac{2x-1}{x^2-x+1} dx$

Now, put $t = x^2 - x + 1 \Rightarrow dt = (2x-1)dx$

$\therefore \quad I = \log|x| - \frac{1}{3}\log|x+1| - \frac{1}{3}\int \frac{dt}{t}$

$= \log|x| - \frac{1}{3}\log|x+1| - \frac{1}{3}\log|t| + C$

$= \log|x| - \frac{1}{3}\log|x+1| - \frac{1}{3}\log|x^2-x+1| + C$

[put $t = x^2 - x + 1$]

$= \log|x| - \frac{1}{3}\log|(x+1)(x^2-x+1)| + C$

$[\because \log m + \log n = \log mn]$

$= \log|x| - \frac{1}{3}\log|x^3+1| + C$

$= \log|x| - \log|x^3+1|^{1/3} + C = \log\frac{|x|}{|x^3+1|^{1/3}} + C$

$\left[\because \log m - \log n = \log\frac{m}{n}\right]$

15. **Hint** (i) Firstly, write the given integral as $\int \frac{dx}{(x-1)(x+1)(x^2+1)}$.

(ii) Let $\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$

$\left[\textbf{Ans. } \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1} x + C\right]$

16. Let $I = \int \frac{x^2+x+1}{(x^2+1)(x+2)} dx$

Using partial fraction method, we get

$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$ ...(i)

$\Rightarrow \quad x^2 + x + 1 = A(x^2+1) + (Bx+C)(x+2)$

$\Rightarrow \quad x^2 + x + 1 = x^2(A+B) + x(2B+C) + (A+2C)$

On comparing the coefficients of x^2, x and constant terms from both sides, we get

$A+B=1$ ...(ii)

$2B+C=1$ ...(iii)

and $A+2C=1$ ...(iv)

On substituting the value of B from Eq. (ii) in Eq. (iii), we get

$2(1-A) + C = 1 \Rightarrow 2 - 2A + C = 1$

$\Rightarrow \quad 2A - C = 1$ ...(v)

Now, solving Eqs. (iv) and (v), we get

$C = \frac{1}{5}$ and $A = \frac{3}{5}$

On putting the value of A in Eq. (ii), we get

$B = 1 - \frac{3}{5} = \frac{2}{5}$

Thus, from Eq. (i), we get

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{3}{5}\cdot\frac{1}{(x+2)} + \frac{1}{5}\cdot\frac{(2x+1)}{x^2+1}$$

$$\Rightarrow \quad I = \frac{3}{5}\int\frac{dx}{x+2} + \frac{1}{5}\int\frac{(2x+1)}{x^2+1}dx$$

$$= \frac{3}{5}\int\frac{dx}{x+2} + \frac{1}{5}\int\frac{2x}{x^2+1}dx + \frac{1}{5}\int\frac{dx}{x^2+1}$$

$$= \frac{3}{5}\log|x+2| + \frac{1}{5}\log|x^2+1| + \frac{1}{5}\tan^{-1}x + C$$

$$\left[\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C \text{ and } \int\frac{f'(x)}{f(x)}dx = \log|f(x)|+C\right]$$

17. Solve as Question 16.

$$\left[\textbf{Ans. } \frac{1}{2}\log|x-1| + \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C\right]$$

18. **Hint** Firstly, write the given integral as

$$\int\left[(x+1)+\left(\frac{1}{x^3-x^2+x-1}\right)\right]dx = \int(x+1)\,dx + \int\frac{dx}{(x-1)(x^2+1)}$$

and then solve as Question 12.

$$\left[\textbf{Ans. } \frac{x^2}{2} + x + \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1}x + C\right]$$

19. Let $I = \int\frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$

$$= \int\frac{\cos\theta}{(4+\sin^2\theta)[5-4(1-\sin^2\theta)]}d\theta$$

$$= \int\frac{\cos\theta}{(4+\sin^2\theta)(5-4+4\sin^2\theta)}d\theta$$

$$= \int\frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)}d\theta$$

Put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

Then, $I = \int\frac{dt}{(4+t^2)(1+4t^2)}$ ...(i)

Again, let $\frac{1}{(4+t^2)(1+4t^2)} = \frac{A}{4+t^2} + \frac{B}{1+4t^2}$ ...(ii)

[by partial fraction]

At $t=0$, $\frac{A}{4} + \frac{B}{1} = \frac{1}{4\times1} \Rightarrow A + 4B = 1$ ...(iii)

At $t=1$, $\frac{A}{5} + \frac{B}{5} = \frac{1}{5\times5} \Rightarrow 5A + 5B = 1$ ...(iv)

On solving Eqs. (iii) and (iv), we get

$$A = -\frac{1}{15} \text{ and } B = \frac{4}{15}$$

On putting $A = -\frac{1}{15}$ and $B = \frac{4}{15}$ in Eq. (ii), we get

$$\frac{1}{(4+t^2)(1+4t^2)} = \frac{-\frac{1}{15}}{4+t^2} + \frac{\frac{4}{15}}{1+4t^2}$$

$$\Rightarrow \quad \frac{1}{(4+t^2)(1+4t^2)} = \frac{-1}{15(4+t^2)} + \frac{4}{15(1+4t^2)}$$

On integrating both sides w.r.t. t, we get

$$\int\frac{1}{(4+t^2)(1+4t^2)}dt = \frac{-1}{15}\int\frac{1}{4+t^2}dt + \frac{4}{15}\int\frac{1}{1+4t^2}dt$$

$$= \frac{-1}{15}\int\frac{1}{2^2+t^2} + \frac{4}{15\times4}\int\frac{1}{\left(\frac{1}{2}\right)^2+t^2}dt$$

$$= \frac{-1}{15}\cdot\frac{1}{2}\tan^{-1}\frac{t}{2} + \frac{1}{15}\cdot\frac{1}{1/2}\tan^{-1}\frac{t}{1/2} + C$$

$$\left[\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C\right]$$

$$= \frac{-1}{30}\tan^{-1}\frac{\sin\theta}{2} + \frac{2}{15}\tan^{-1}2\sin\theta + C$$

[put $t = \sin\theta$]

20. Let $I = \int\frac{x^2}{(x^2+4)(x^2+9)}dx = \frac{1}{2}\int\frac{2x^2+4+9-4-9}{(x^2+4)(x^2+9)}dx$

[adding and subtracting 4 and 9 from numerator]

$$= \frac{1}{2}\int\frac{x^2+4}{(x^2+4)(x^2+9)}dx + \frac{1}{2}\int\frac{x^2+9}{(x^2+4)(x^2+9)}dx - \frac{1}{2}\int\frac{13}{(x^2+4)(x^2+9)}dx$$

$$= \frac{1}{2}\int\frac{dx}{x^2+9} + \frac{1}{2}\int\frac{dx}{x^2+4} - \frac{1}{2}\int\frac{13\,dx}{(x^2+4)(x^2+9)}$$

$$= \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{2}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\cdot\frac{13}{5}\int\left(\frac{1}{(x^2+4)} - \frac{1}{(x^2+9)}\right)dx$$

$$\left[\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) - \frac{13}{10}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{13}{10}\cdot\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) - \frac{13}{20}\tan^{-1}\left(\frac{x}{2}\right) + \frac{13}{30}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$= \tan^{-1}\left(\frac{x}{3}\right)\left(\frac{1}{6}+\frac{13}{30}\right) + \tan^{-1}\left(\frac{x}{2}\right)\left(\frac{1}{4}-\frac{13}{20}\right) + C$$

$$= \tan^{-1}\left(\frac{x}{3}\right)\left(\frac{5+13}{30}\right) + \tan^{-1}\left(\frac{x}{2}\right)\left(\frac{5-13}{20}\right) + C$$

$$= \frac{18}{30}\tan^{-1}\left(\frac{x}{3}\right) - \frac{8}{20}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) - \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C$$

|TOPIC 4|

Integration by Parts

Let u and v be two differentiable functions of a single variable x, then the integral of the product of these two functions

= Ist function × Integral of the IInd function
− Integral of [Derivatives of Ist function × Integral of the IInd function]

i.e. $$\int \underset{\text{I}}{u} \cdot \underset{\text{II}}{v}\, dx = u\int v\, dx - \int \left(\frac{d}{dx}(u)\int v\, dx\right) dx$$

If in the product, two functions are of different types, then take that function as first function (i.e. u) which comes first in word ILATE, where

- I : Inverse trigonometric function. e.g. $\sin^{-1} x$
- L : Logarithmic function. e.g. $\log x$
- A : Algebraic function. e.g. $1, x, x^2$
- T : Trigonometric function. e.g. $\sin x, \cos x$
- E : Exponential function. e.g. e^x

Note

(i) If the integrand contains a logarithmic or an inverse trigonometric function and the second function is not given, we take second function as 1. e.g. In the integral of $\int \sin^{-1} x\, dx$, we take second function as unity (i.e. 1).

(ii) Integration by parts is not applicable in all cases. For instance, the method does not work for $\int \sqrt{x} \sin x\, dx$. The reason is that there does not exist any function whose derivative is $\sqrt{x}\sin x$.

Method to Find Integration by Parts

Let the given integration is of the form $I = \int u \cdot v\, dx$, where u and v are the functions of x. Then, to evaluate such integrals, we use the following steps

I. Firstly, choose the Ist and IInd functions with the help of ILATE, i.e. take that function as I function which comes Ist in ILATE and take other function as IInd function.

II. Now, integrate by using integration by parts, i.e.

$$\int \underset{\text{I}}{u} . \underset{\text{II}}{v}\, dx = \underbrace{u\int v\, dx}_{\text{I Integral}} - \underbrace{\int \left(\frac{d}{dx}(u)\int v\, dx\right) dx}_{\text{II Integral}}$$

III. From step II, we get one of the three possible cases

(i) If IInd integral is in simple form, then integrate it by using appropriate method.

(ii) If IInd integral is the product of two functions, then again use Steps I and II.

(iii) If IInd integral is same as the given integral, then put Ist in place of IInd integral. Finally, simplify it and get the required result.

EXAMPLE |1| Evaluate $\int x\cos x\, dx$.

Sol. Let $I = \int x \cos x\, dx$

Here, we see that integrand is the product of algebraic and trigonometric functions and algebraic function comes first in the word ILATE, so we consider it as Ist function and trigonometric function as IInd function.

$$\therefore \quad I = \int \underset{\text{I}}{x} \cos \underset{\text{II}}{x}\, dx$$

$$= x\int \cos x\, dx - \int\left(\frac{d}{dx}(x)\int \cos x\, dx\right)dx$$

$$\left[\because \int \underset{\text{I}}{u} \cdot \underset{\text{II}}{v}\, dx = u\int v\, dx - \int\left(\frac{d}{dx}(u)\int v\, dx\right)dx\right]$$

$$= x \sin x - \int 1 \cdot \sin x\, dx$$

$$= x \sin x - \int \sin x\, dx$$

$$= x \sin x + \cos x + C$$

Note Suppose $I = \int x\cos x\, dx = x(\sin x + k) - \int (\sin x + k)\, dx$

$$= x(\sin x + k) - \int \sin x\, dx - \int k\, dx$$

$$= x(\sin x + k) - \cos x - kx + C$$

$$= x\sin x - \cos x + C$$

which shows that adding a constant to the integral of the IInd function is unnecessary. So, as the result is concerned, we conclude that, to write the constant of integration at the last part of the integral while integrating.

EXAMPLE |2| Evaluate $\int e^x \cos x\, dx$.

Firstly, apply integration by parts to integrate it. Now, if we again get IInd integral as product of two functions, then again apply integration by parts.

Sol. Let $I = \int e^x \cos x\, dx$ ... (i)

By using the order of functions in ILATE, take $\cos x$ as Ist function and e^x as IInd function.

$$\therefore \quad I = \int \underset{\text{I}}{\cos x}\, \underset{\text{II}}{e^x}\, dx$$

$$\Rightarrow \quad I = \cos x \int e^x\, dx - \int\left[\frac{d}{dx}(\cos x)\int e^x\, dx\right]dx$$

$$\left[\because \int \underset{\text{I}}{u} \cdot \underset{\text{II}}{v}\, dx = u\int v\, dx - \int\left(\frac{d}{dx}(u)\int v\, dx\right)dx\right]$$

$$\Rightarrow \quad I = \cos x\, e^x - \int(-\sin x)\, e^x\, dx$$

$\Rightarrow \quad I = e^x \cos x + \int \underset{\text{I}}{\sin x}\, \underset{\text{II}}{e^x}\, dx$

[$\because$ integrand is the product of two functions, so we again choose Ist and IInd functions by using ILATE]

$$\Rightarrow I = e^x \cos x + \sin x \int e^x dx - \int \left(\frac{d}{dx}(\sin x) \int e^x\, dx \right) dx$$

$$\left[\because \int u \cdot v\, dx = u \int v\, dx - \int \left(\frac{d}{dx}(u) \int v\, dx \right) dx \right]$$

$$\Rightarrow I = e^x \cos x + \sin x\, e^x - \int \cos x\, e^x\, dx$$

$$\Rightarrow I = e^x \cos x + e^x \sin x - I + C_1 \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow 2I = e^x (\cos x + \sin x) + C_1$$

$$\Rightarrow \quad I = \frac{e^x}{2}(\cos x + \sin x) + C, \text{ where } C = \frac{C_1}{2}$$

Note Above integral can also be solved by taking e^x as Ist function and $\cos x$ as IInd function.

EXAMPLE |3| Evaluate $\int \log|1 + x^2|\, dx$.

Here, we are unable to guess a function, whose derivative is $\log(1 + x^2)$. So, take $\log(1 + x^2)$ as the Ist function and the constant function 1 as the IInd function, then integrate it by parts.

Sol. Let $I = \int \log|1 + x^2| dx = \int \underset{\text{I}}{\log|1 + x^2|} \cdot \underset{\text{II}}{1}\, dx$

$$= \log|1 + x^2| \int 1\, dx - \int \left[\frac{d}{dx} \log|1 + x^2| \int 1\, dx \right] dx$$

[using integration by parts]

$$= \log|1 + x^2| \cdot x - \int \frac{1}{1 + x^2}(2x) \cdot x\, dx$$

$$= x \log|1 + x^2| - 2\int \frac{x^2}{1 + x^2} dx$$

$$= x \log|1 + x^2| - 2\int \frac{1 + x^2 - 1}{1 + x^2} dx$$

$$= x \log|1 + x^2| - 2\int \left(1 - \frac{1}{1 + x^2} \right) dx$$

$$= x \log|1 + x^2| - 2\int dx + 2\int \frac{dx}{1 + x^2} + C$$

$$= x \log|1 + x^2| - 2x + 2\tan^{-1} x + C$$

EXAMPLE |4| $\int \frac{\log x}{(x + 1)^2}\, dx$.

[All India 2015]

Sol. Let $I = \int \frac{\log x}{(x + 1)^2} dx = \int \underset{\text{I}}{\log x} \cdot \underset{\text{II}}{\frac{1}{(x + 1)^2}}\, dx$

On applying integration by parts, we get

$$I = \log x \cdot \int \frac{dx}{(x + 1)^2} - \int \frac{d}{dx}(\log x) \cdot \left(\int \frac{dx}{(x + 1)^2} \right) dx$$

$$\left[\because \int \underset{\text{I}}{u} \cdot \underset{\text{II}}{v}\, dx = u \int v\, dx - \int \left(\frac{d}{dx}(u) \int v\, dx \right) dx \right]$$

$$= \log x \cdot \frac{(-1)}{x + 1} + \int \frac{1}{x(x + 1)} dx$$

$$\left[\because \int \frac{1}{(x + 1)^2} dx = -\frac{1}{(x + 1)} \right]$$

$$= \frac{-\log x}{x + 1} + I_1, \text{ where } I_1 = \int \frac{dx}{x(x + 1)} \text{ (say)} \qquad \text{...(i)}$$

Now, using partial fraction method, consider

$$\frac{1}{x(x + 1)} = \frac{A}{x} + \frac{B}{x + 1}$$

$$\Rightarrow \qquad 1 = A(x + 1) + Bx$$

On putting $x = 0$, we get $A = 1$

Again, putting $x = -1$, we get $B = -1$

$$\therefore \quad I_1 = \int \left(\frac{1}{x} - \frac{1}{x + 1} \right) dx = \int \frac{1}{x} dx - \int \frac{dx}{x + 1}$$

$$= \log|x| - \log|x + 1| + C \qquad \text{...(ii)}$$

Now, from Eqs. (i) and (ii), we get

$$I = \frac{-\log x}{x + 1} + \log|x| - \log|x + 1| + C$$

$$= \frac{-\log x}{x + 1} + \log\left| \frac{x}{x + 1} \right| + C \quad \left[\because \log m - \log n = \log \frac{m}{n} \right]$$

SOME MORE SPECIAL TYPES OF INTEGRALS

Here, we will discuss some more special types of integrals, which can be proved by using integration by parts and directly used to evaluate the given integrals.

(i) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left| x + \sqrt{x^2 - a^2} \right| + C$

(ii) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left| x + \sqrt{x^2 + a^2} \right| + C$

(iii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

Proof

(i) Let $I = \int \underset{\text{I}}{\sqrt{x^2 - a^2}} \times \underset{\text{II}}{1}\, dx \qquad \text{...(i)}$

$$= \sqrt{x^2 - a^2} \int 1 dx - \int \left[\frac{d}{dx}(\sqrt{x^2 - a^2}) \int 1 dx \right] dx$$

$$\left[\because \int \underset{\text{I}}{u} \cdot \underset{\text{II}}{v}\, dx = u \int v dx - \int \left(\frac{d}{dx}(u) \int v dx \right) dx \right]$$

$$= \sqrt{x^2 - a^2} \times x - \int \left[\frac{1}{2\sqrt{x^2 - a^2}} \times 2x \right] x \, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}} dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}} dx$$

[adding and subtracting a^2]

$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2} \, dx - \int \frac{a^2 \, dx}{\sqrt{x^2 - a^2}}$$

$$\Rightarrow \quad I = x\sqrt{x^2 - a^2} - I - a^2 \int \frac{1 \, dx}{\sqrt{x^2 - a^2}}$$

$$\Rightarrow 2I = x\sqrt{x^2 - a^2} - a^2 [\log |x + \sqrt{x^2 - a^2}|]$$

$$\left[\because \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log |x + \sqrt{x^2 - a^2}| \right]$$

$$\Rightarrow \quad I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \quad [\log |x + \sqrt{x^2 - a^2}|] + C$$

Similarly, we can prove other formulae given above.

EXAMPLE |5| Integrate the following functions.

(i) $\sqrt{1 - 4x^2}$ (ii) $\sqrt{1 + \frac{x^2}{9}}$

Sol. (i) Let $I = \int \sqrt{1 - 4x^2} \, dx = \int \sqrt{4\left(\frac{1}{4} - x^2\right)} \, dx$

$$= \int 2\sqrt{\left(\frac{1}{2}\right)^2 - x^2} \, dx$$

$$= 2\int \sqrt{\left(\frac{1}{2}\right)^2 - x^2} dx$$

$$= 2\left[\frac{x}{2}\sqrt{\left(\frac{1}{2}\right)^2 - x^2} + \frac{\left(\frac{1}{2}\right)^2}{2} \sin^{-1}\left(\frac{x}{\left(\frac{1}{2}\right)} \right) \right] + C$$

$$\left[\because \int \sqrt{a^2 - x^2} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C \right]$$

$$= x\sqrt{\frac{1}{4} - x^2} + \frac{1}{4}\sin^{-1}(2x) + C$$

$$= \frac{x}{2}\sqrt{1 - 4x^2} + \frac{1}{4}\sin^{-1}(2x) + C$$

(ii) Let $I = \int \sqrt{1 + \frac{x^2}{9}} \, dx = \int \frac{\sqrt{9 + x^2}}{3} dx = \frac{1}{3}\int \sqrt{3^2 + x^2} \, dx$

$$= \frac{1}{3}\left[\frac{x}{2}\sqrt{3^2 + x^2} + \frac{3^2}{2}\log |x + \sqrt{3^2 + x^2}| \right] + C$$

$$\left[\because \int \sqrt{a^2 + x^2} = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C \right]$$

$$= \frac{x}{6}\sqrt{9 + x^2} + \frac{3}{2}\log |x + \sqrt{9 + x^2}| + C$$

Method to Evaluate Integrals of the Form $\int \sqrt{ax^2 + bx + c} \, dx$

Let $I = \int \sqrt{ax^2 + bx + c} \, dx$

I. Firstly, take a common from integrand to make coefficient of x^2 unity,

i.e. $I = \int \sqrt{a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)} \, dx$

II. Add and subtract $\left(\frac{b}{2a}\right)^2$ from integrand inside the square root and convert it in the form $\sqrt{X^2 \pm k^2}$ or $\sqrt{k^2 - X^2}$, where $X^2 = \left(x \pm \frac{b}{2a}\right)^2$ and $k^2 = \left(\frac{c}{a} \mp \frac{b^2}{4a^2}\right)$.

III. Substitute $x \pm \frac{b}{2a} = t$ and reduce the integral obtained from step II into one of the form

$$\int \sqrt{t^2 \pm k^2} dt \text{ or } \int \sqrt{k^2 - t^2} dt$$

Then, apply suitable formula to integrate.

EXAMPLE |6| Evaluate $\int \sqrt{x^2 - 8x + 7} \, dx$.

Sol. Let $I = \int \sqrt{x^2 - 8x + 7} \, dx$

Here, the coefficient of x^2 is unity.

$\therefore I = \int \sqrt{x^2 - 8x + 7} \, dx = \int \sqrt{x^2 - 8x + 7 + 4^2 - (4)^2} \, dx$

[here, $a = 1$ and $b = -8$, adding and subtracting $\left(\frac{b}{2a}\right)^2 = \left(\frac{-8}{2}\right)^2 = (4)^2$, under the square root]

$$= \int \sqrt{(x - 4)^2 + 7 - 16} \, dx = \int \sqrt{(x - 4)^2 - (3)^2} \, dx$$

Now, put $x - 4 = t \Rightarrow dx = dt$

$$\therefore \quad I = \int \sqrt{t^2 - 3^2}\, dt$$
$$= \frac{t}{2}\sqrt{t^2 - 3^2} - \frac{(3)^2}{2} \log \left| t + \sqrt{t^2 - 3^2} \right| + C$$
$$\left[\because \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}|\right]$$
$$= \frac{(x-4)}{2}\sqrt{(x-4)^2 - (3)^2} - \frac{(3)^2}{2} \log |x - 4 + \sqrt{(x-4)^2 - (3)^2}| + C$$
$$= \frac{x-4}{2}\sqrt{x^2 - 8x + 7} - \frac{9}{2} \log |(x-4) + \sqrt{x^2 - 8x + 7}| + C$$

EXAMPLE |7| Evaluate $\int \sqrt{x^2 + 4x + 6}\, dx$. **[NCERT]**

Here, to convert the integrand of the form $\sqrt{ax^2 + bx + c}$ into standard integrand of the form $\sqrt{x^2 \pm a^2}$ by adding and subtracting $\left(\frac{b}{2a}\right)^2$ under the square root and then use suitable formula to integrate it.

Sol. Let $I = \int \sqrt{x^2 + 4x + 6}\, dx = \int \sqrt{x^2 + 4x + 2^2 + 6 - 4}\, dx$
$$= \int \sqrt{(x+2)^2 + (\sqrt{2})^2}\, dx$$
$$= \frac{(x+2)}{2}\sqrt{x^2 + 4x + 6} + \frac{2}{2} \log |(x+2) + \sqrt{x^2 + 4x + 6}| + C$$
$$\left[\because \int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}|\right]$$
$$\Rightarrow I = \frac{x+2}{2}\sqrt{x^2 + 4x + 6} + \log|(x+2) + \sqrt{x^2 + 4x + 6}| + C$$

EXAMPLE |8| Evaluate $\int \sqrt{1 + 3x - x^2}\, dx$. **[NCERT]**

Sol. Let $I = \int \sqrt{1 + 3x - x^2}\, dx = \int \sqrt{-(x^2 - 3x - 1)}\, dx$
$$= \int \sqrt{-\left(x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 1\right)}\, dx$$
$$\left[\text{adding and subtracting } \left(\frac{b}{2a}\right)^2 = \left(\frac{3}{2}\right)^2\right]$$
$$= \int \sqrt{-\left\{\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 1\right\}}\, dx$$
$$= \int \sqrt{-\left\{\left(x - \frac{3}{2}\right)^2 - \left(\frac{9+4}{4}\right)\right\}}\, dx$$
$$= \int \sqrt{-\left\{\left(x - \frac{3}{2}\right)^2 - \frac{13}{4}\right\}}\, dx = \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}\, dx$$
$$= \frac{\left(x - \frac{3}{2}\right)}{2}\sqrt{1 + 3x - x^2} + \frac{13}{4 \times 2} \sin^{-1}\left\{\frac{\left(x - \frac{3}{2}\right)}{\frac{\sqrt{13}}{2}}\right\} + C$$
$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + C\right]$$
$$= \frac{2x-3}{4}\sqrt{1 + 3x - x^2} + \frac{13}{8} \sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + C$$

Integral of the Type $\int e^x [f(x) + f'(x)]dx$

In this type of integral, integrand is the product of two functions. One is in exponential form and second function is the sum of two functions in which one is derivative of other function. Then, to evaluate such integrals, we directly use the following formula
$$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$$

Proof It can be proved by using integration by parts as given below

Let $\quad I = \int e^x [f(x) + f'(x)]\, dx$
$$= \int e^x f(x) dx + \int \underset{\text{I}}{e^x} \underset{\text{II}}{f'(x)} dx$$
$$= \int e^x f(x)\, dx + e^x \int f'(x) dx - \int \left[\frac{d}{dx}(e^x) \int f'(x)\right] dx$$
[using integration by parts]
$$= \int e^x f(x)\, dx + e^x f(x) - \int e^x f(x)\, dx + C$$
$$= e^x f(x) + C$$

Note Sometimes, it may be possible that the IInd function is not in standard form i.e. not in the form $f(x) + f'(x)$. In that case, we try to make it in standard form, if it is possible, otherwise we use some another ways to integrate it.

EXAMPLE |9| Evaluate $\int e^x (\sin x + \cos x)\, dx$.

Sol. Let $I = \int e^x (\sin x + \cos x)\, dx$

and $f(x) = \sin x$, then $f'(x) = \cos x$

So, the given integral is of the form
$$I = \int e^x [f(x) + f'(x)]\, dx$$

We know that
$$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$$
$$\therefore \quad I = e^x \sin x + C$$

EXAMPLE |10| Evaluate $\int \frac{x-3}{(x-1)^3} e^x \, dx$.

Here, the integrand $\frac{x-3}{(x-1)^3}$ is not in the form of $e^x [f(x)+f'(x)]$, so we firstly convert it in the form of $e^x [f(x)+f'(x)]$ and then simplify it.

Sol. Let $I = \int \frac{x-3}{(x-1)^3} e^x \, dx = \int \frac{x-1-2}{(x-1)^3} e^x \, dx$

$$= \int e^x \left(\frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3} \right) dx$$

$$= \int e^x \left(\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right) dx \quad \text{...(i)}$$

Now, let $f(x) = \frac{1}{(x-1)^2} \Rightarrow f'(x) = \frac{-2}{(x-1)^3}$

Then, Eq. (i) becomes of the form

$$I = \int e^x [f(x) + f'(x)] \, dx$$

Also, we know that

$$\int e^x [f(x) + f'(x)] \, dx = e^x f(x) + C$$

Hence, $I = \frac{e^x}{(x-1)^2} + C$.

TOPIC PRACTICE 4

OBJECTIVE TYPE QUESTIONS

1 If $\int x \sin x \, dx = -x \cos x + \alpha$, then α is equal to

(a) $\sin x + C$ (b) $\cos x + C$

(c) $-\sin x + C$ (d) None of these

2 $\int \frac{x + \sin x}{1 + \cos x} dx$ is equal to **[NCERT Exemplar]**

(a) $\log|1 + \cos x| + C$ (b) $\log|x + \sin x| + C$

(c) $x - \tan\frac{x}{2} + C$ (d) $x \cdot \tan\frac{x}{2} + C$

3 $\int \sqrt{1 + x^2} \, dx$ is equal to **[NCERT]**

(a) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|x + \sqrt{1+x^2}\right| + C$

(b) $\frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$

(c) $\frac{2}{3}x(1+x^2)^{\frac{3}{2}} + C$

(d) $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2 \log|x + \sqrt{1+x^2}|$

4 $\int e^x \{f(x) + f'(x)\} \, dx$ is equal to

(a) $e^x f(x) + C$ (b) $e^x + f(x) + C$

(c) $2e^x f(x) + C$ (d) $e^x - f(x) + C$

5 $\int \sin(\log x) + \cos(\log x) \, dx$ equals **[NCERT]**

(a) $x \sin(\log x) + C$

(b) $x \cos(\log x) + C$

(c) $\frac{1}{x}\cos(\log x) + C$

(d) $\frac{1}{x}\sin(\log x) + C$

VERY SHORT ANSWER Type Questions

Directions (Q. Nos. 6-8) *Evaluate the following integrals.*

6 $\int x \cdot e^x \, dx$ **[NCERT]**

7 $\int x \log x \, dx$ **[NCERT]**

8 $\int x \sec^2 x \, dx$

SHORT ANSWER Type I Questions

Directions (Q. Nos. 9-12) *Evaluate the following integrals.*

9 $\int (x^2 + 1) \log x \, dx$ **[NCERT]**

10 $\int \sqrt{x^2 - 2x} \, dx$ **[Delhi 2017C]**

11 $\int \sqrt{2x - x^2} \, dx$ **[Delhi 2017C]**

12 $\int e^x \sec x (1 + \tan x) \, dx$ **[NCERT]**

SHORT ANSWER Type II Questions

Directions (Q. Nos. 13-31) *Evaluate the following integrals.*

13 $\int e^{2x} \sin x \, dx$ **[Foreign 2011]**

14 $\int \tan^{-1} x \, dx$ **[NCERT]**

15 $\int x^2 \tan^{-1} x \, dx$ **[NCERT Exemplar]**

16 $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

17 $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$ **[Delhi 2012]**

18 $\int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx$ **[All India 2014C; Foreign 2014]**

19 $\int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\,dx$ **[NCERT; All India 2014C]**

20 $\int \frac{\sqrt{x^2+1}\,[\log(x^2+1)-2\log(x)]}{x^4}\,dx$ **[NCERT; All India 2014C]**

21 $\int \frac{x^2}{(x\sin x+\cos x)^2}\,dx$ **[All India 2012C]**

22 $\int \left[\log(\log x)+\frac{1}{(\log x)^2}\right]dx$ **[NCERT]**

23 $\int \sqrt{5-2x+x^2}\,dx$ **[NCERT Exemplar]**

24 Find $\int \sqrt{3-2x-x^2}\,dx.$ **[All India 2019]**

25 $\frac{x^2-3x+1}{\sqrt{1-x^2}}\,dx$ **[Delhi 2015]**

26 $\int \frac{xe^x}{(1+x)^2}dx$ **[NCERT]**

27 $\int \frac{(x^2+1)\,e^x}{(x+1)^2}\,dx$ **[Delhi 2015C]**

28 $\int \frac{(2x-5)\,e^{2x}}{(2x-3)^3}\,dx$ **[All India 2016]**

29 $\int \frac{2+\sin 2x}{1+\cos 2x}\,e^x\,dx$ **[NCERT]**

30 $\int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$ **[Delhi 2013C]**

31 $\int \left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\,dx$

LONG ANSWER Type Questions

Directions (Q. Nos. 32-34) *Evaluate the following integrals.*

32 $\int e^{-3x}\cos^3 x\,dx$ **[NCERT Exemplar]**

33 $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$ **[NCERT Exemplar]**

34 $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

HINTS & SOLUTIONS

1. (a) Let $I=\int \underset{\text{I}}{x}\,\underset{\text{II}}{\sin x}\,dx$

$$= x\cdot(-\cos x)-\int 1\cdot(-\cos x)\,dx$$
$$= -x\cos x+\int \cos x\,dx$$
$$= -x\cos x+\sin x+C$$

2. (d) Let $I=\int \frac{x+\sin x}{1+\cos x}\,dx$

$$=\int \frac{x}{1+\cos x}\,dx+\int\frac{\sin x}{1+\cos x}\,dx$$
$$=\int \frac{x}{2\cos^2 x/2}\,dx+\int\frac{2\sin x/2\cos x/2}{2\cos^2 x/2}\,dx$$
$$=\frac{1}{2}\int x\sec^2 x/2\,dx+\int \tan x/2\,dx$$
$$=\frac{1}{2}\left[x\cdot\tan x/2\cdot 2-\int \tan\frac{x}{2}\cdot 2\,dx\right]+\int\tan\frac{x}{2}\,dx$$
$$=x\cdot\tan\frac{x}{2}+C$$

3. (a) Let $I=\int\sqrt{1+x^2}\,dx$

$$\Rightarrow\quad I=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$$
$$\left[\because \int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C\right]$$

4. (a) **Hint** This is a direct result, proved by integration by parts.

5. (a) **Hint** Put $\log x=t\Rightarrow x=e^t\Rightarrow dx=e^t\,dt$

$$\therefore\quad \int e^t(\sin t+\cos t)\,dt$$
$$=e^t\sin t+C$$
$$=x\sin(\log x)+C$$

6. Let $I=\int \underset{\text{I}}{x}\cdot\underset{\text{II}}{e^x}\,dx=x\int e^x dx-\int\left[\frac{d}{dx}x\left(\int e^x dx\right)\right]dx$

[using integration by parts]

$$=x\cdot e^x-\int 1\cdot e^x dx=x\cdot e^x-e^x+C$$

7. Let $I=\int x\log x\,dx=\int \underset{\text{I}}{(\log x)}\cdot\underset{\text{II}}{x}\,dx$

$$=\log x\int x\,dx-\int\left[\frac{d}{dx}(\log x)\left(\int x\,dx\right)\right]dx$$

[using integration by parts]

$$=\frac{x^2}{2}\cdot\log x-\int\frac{1}{x}\cdot\frac{x^2}{2}dx=\frac{x^2}{2}\log x-\frac{1}{2}\int x\,dx$$
$$=\frac{x^2}{2}\log x-\frac{1}{2}\cdot\frac{x^2}{2}+C=\frac{x^2}{2}\log x-\frac{x^2}{4}+C$$

8. Similar as Example 1. [**Ans.** $x\tan x+\log|\cos x|+C$]

9. **Hint** Integrate by parts, taking $\log x$ as Ist function and (x^2+1) as IInd function.

$$\left[\textbf{Ans.} \left(\frac{x^3}{3}+x\right)\log x - \frac{x^3}{9} - x + C\right]$$

10. Let $I = \int \sqrt{x^2-2x}\,dx = \int \sqrt{(x^2-2x+1)-1}\,dx$

$$= \int \sqrt{(x-1)^2-1^2}\,dx$$

$$= \frac{(x-1)}{2}\sqrt{x^2-2x} - \frac{1}{2}\log|(x-1)+\sqrt{x^2-2x}| + C$$

11. Let $I = \int \sqrt{2x-x^2}\,dx$

$$= \int \sqrt{-(x^2-2x)}\,dx = \int \sqrt{-(x^2-2x+1-1)}\,dx$$

$$= \int \sqrt{-((x-1)^2-1)}\,dx = \int \sqrt{1^2-(x-1)^2}\,dx$$

$$= \frac{(x-1)}{2}\sqrt{2x-x^2} + \frac{1}{2}\sin^{-1}(x-1) + C$$

12. **Hint** Write the given integral as $\int e^x(\sec x + \sec x \tan x)dx$.

$\because \frac{d}{dx}(\sec x) = \sec x \tan x$ [**Ans.** $e^x \sec x + C$]

13. Similar as Example 2. $\left[\textbf{Ans.} \frac{1}{5}e^{2x}(2\sin x - \cos x) + C\right]$

14. Let $I = \int \tan^{-1} x\,dx = \int \underset{\text{II}}{1}\cdot\underset{\text{I}}{\tan^{-1} x}\,dx$

Using integration by parts, taking $\tan^{-1} x$ as Ist function and 1 as IInd function, we get

$$I = \tan^{-1} x \cdot \int 1\,dx - \int\left[\frac{d}{dx}(\tan^{-1} x)\cdot\int 1\,dx\right]dx$$

$$= \tan^{-1} x\cdot x - \int \frac{1}{1+x^2}\cdot x\,dx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$= x\tan^{-1} x - I_1, \text{ where } I_1 = \frac{1}{2}\int\frac{2x}{1+x^2}dx \quad \text{...(i)}$$

Now, put $1+x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore I_1 = \frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\log|t| + C_1 = \frac{1}{2}\log|1+x^2| + C_1$$

$[\because t = 1+x^2]$

Now, from Eq. (i), we get

$$I = x\tan^{-1} x - \frac{1}{2}\log|1+x^2| - C_1$$

$$= x\tan^{-1} x - \frac{1}{2}\log|1+x^2| + C, \text{ where } C = -C_1$$

15. Let $I = \int \underset{\text{II}}{x^2}\cdot\underset{\text{I}}{\tan^{-1} x}\,dx$

Using integration by parts, taking $\tan^{-1} x$ as Ist function and x^2 as IInd function, we get

$$I = \tan^{-1} x\int x^2 dx - \int\left(\frac{d}{dx}(\tan^{-1} x)\int x^2 dx\right)dx$$

$$I = \tan^{-1} x\cdot\frac{x^3}{3} - \int\frac{1}{1+x^2}\cdot\frac{x^3}{3}dx$$

$$= \frac{x^3}{3}\cdot\tan^{-1} x - \frac{1}{3}\int\frac{x^3}{1+x^2}dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int\left(x - \frac{x}{x^2+1}\right)dx$$

[dividing x^3 by x^2+1]

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int x\,dx + \frac{1}{3}\int\frac{x}{x^2+1}dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{x^2}{6} + \frac{1}{6}\int\frac{2x}{x^2+1}dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{x^2}{6} + \frac{1}{6}\log|x^2+1| + C$$

$$\left[\because \int\frac{f'(x)}{f(x)}dx = \log|f(x)| + C\right]$$

16. **Hint** (i) On substituting $x = \tan\theta$ and $dx = \sec^2\theta\,d\theta$, given integral reduces to $\int 2\theta\cdot\sec^2\theta\,d\theta$

(ii) After integrating by parts and then substitute $\theta = \tan^{-1} x$

[**Ans.** $2[x\tan^{-1} x - \log(1+x^2)] + C$]

17. Let $I = \int\frac{x\sin^{-1} x}{\sqrt{1-x^2}}dx$

Now, put $x = \sin t \Rightarrow \sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$

$$\therefore \quad I = \int \underset{\text{I}}{t}\ \underset{\text{II}}{\sin t}\,dt$$

Using integration by parts, taking t as the Ist function and $\sin t$ as the IInd function, we get

$$I = t\int\sin t\,dt - \int\left[\frac{d}{dt}(t)\cdot\int\sin t\,dt\right]dt$$

$$\Rightarrow \quad I = -t\cos t - \int 1\times(-\cos t)\,dt$$

$$= -t\cos t + \int\cos t\,dt$$

$$\Rightarrow \quad I = -t\cos t + \sin t + C$$

$$\Rightarrow \quad I = -t\sqrt{1-\sin^2 t} + \sin t + C$$

$[\because \cos^2 t = 1-\sin^2 t \Rightarrow \cos t = \sqrt{1-\sin^2 t}]$

$$\therefore \quad I = -\sin^{-1} x\sqrt{1-x^2} + x + C$$

$[\because t = \sin^{-1} x \text{ and } x = \sin t]$

18. Solve as Question 17. [**Ans.** $-\sqrt{1-x^2}\cos^{-1} x - x + C$]

19. Let $I = \int\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx$

We know that $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$

$$\Rightarrow \quad \cos^{-1}\sqrt{x} = \frac{\pi}{2} - \sin^{-1}\sqrt{x}$$

$$\therefore\quad I=\int\frac{\sin^{-1}\sqrt{x}-\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}\,dx$$

$$=\int\frac{2\sin^{-1}\sqrt{x}-\frac{\pi}{2}}{\frac{\pi}{2}}\,dx=\frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x}-\frac{\pi}{2}\right)dx$$

$$=\frac{4}{\pi}\int\sin^{-1}\sqrt{x}\,dx-\int 1\,dx=\frac{4}{\pi}\int\sin^{-1}\sqrt{x}\,dx-x$$

$$\Rightarrow I=\frac{4}{\pi}I_1-x+C \qquad \text{...(i)}$$

where, $I_1=\int\sin^{-1}\sqrt{x}\,dx$

Now, put $\sqrt{x}=t\Rightarrow x=t^2\Rightarrow dx=2t\,dt$

$$\therefore\quad I_1=2\int\underset{\text{I}}{\sin^{-1}t}\cdot\underset{\text{II}}{t}\,dt$$

$$=2\left[\sin^{-1}t\cdot\frac{t^2}{2}-\int\frac{1}{\sqrt{1-t^2}}\cdot\frac{t^2}{2}\,dt\right]$$

[using integration by parts]

$$=t^2\sin^{-1}t-\int\frac{t^2}{\sqrt{1-t^2}}\,dt$$

$$=t^2\sin^{-1}t-\int\frac{-(1-t^2)+1}{\sqrt{1-t^2}}\,dt$$

[adding and subtracting 1 from numerator]

$$=t^2\sin^{-1}t+\int\sqrt{1-t^2}\,dt-\int\frac{1}{\sqrt{1-t^2}}\,dt$$

$$=t^2\sin^{-1}t+\frac{t\sqrt{1-t^2}}{2}+\frac{1}{2}\sin^{-1}t-\sin^{-1}t$$

$$=\left(t^2-\frac{1}{2}\right)\sin^{-1}t+\frac{1}{2}t\sqrt{1-t^2}$$

$$=\frac{1}{2}[(2x-1)\sin^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}] \qquad [\because t=\sqrt{x}]$$

$$=\frac{1}{2}[(2x-1)\sin^{-1}\sqrt{x}+\sqrt{x-x^2}]$$

On putting the value of I_1 in Eq. (i), we get

$$I=\frac{2}{\pi}[(2x-1)\sin^{-1}\sqrt{x}+\sqrt{x-x^2}]-x+C$$

20. Let $I=\int\frac{\sqrt{x^2+1}\,[\log(x^2+1)-2\log(x)]}{x^4}\,dx$

$$=\int\frac{\sqrt{x^2+1}\,\log\left(\frac{x^2+1}{x^2}\right)}{x^4}\,dx$$

$$\left[\because n\log m=\log m^n \text{ and } \log m-\log n=\log\frac{m}{n}\right]$$

$$=\int\frac{x\sqrt{1+\frac{1}{x^2}}\,\log\left(1+\frac{1}{x^2}\right)}{x^4}\,dx$$

$$=\int\frac{\sqrt{1+\frac{1}{x^2}}\,\log\left(1+\frac{1}{x^2}\right)}{x^3}\,dx$$

Now, put $1+\frac{1}{x^2}=t\ \Rightarrow\ \frac{-2}{x^3}\,dx=dt\ \Rightarrow\ \frac{dx}{x^3}=-\frac{dt}{2}$

$$\therefore\quad I=-\frac{1}{2}\int\underset{\text{II}}{\sqrt{t}}\,\underset{\text{I}}{\log t}\,dt$$

$$=-\frac{1}{2}\left[\log t\times\frac{t^{3/2}}{3/2}-\int\frac{t^{3/2}}{3/2}\times\frac{1}{t}\,dt\right]$$

[using integration by parts]

$$=-\frac{1}{3}[t^{3/2}\log t-\int\sqrt{t}\,dt]$$

$$=-\frac{1}{3}\left[t^{3/2}\log t-\frac{t^{3/2}}{3/2}\right]+C$$

$$=-\frac{1}{3}t^{3/2}\left[\log t-\frac{2}{3}\right]+C$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+C$$

$$\left[\because t=1+\frac{1}{x^2}\right]$$

21. Let $I=\int\frac{x^2}{(x\sin x+\cos x)^2}\,dx$

$$=\int\frac{x\cos x}{(x\sin x+\cos x)^2}\,x\sec x\,dx \qquad \text{...(i)}$$

[multiplying numerator by $\cos x$ and $\sec x$]

$$I=\int\underbrace{\frac{x\cos x}{(x\sin x+\cos x)^2}}_{\text{II}}\times\underbrace{x\sec x}_{\text{I}}\,dx$$

$$=x\sec x\,\frac{(-1)}{x\sin x+\cos x}-\int(1\cdot\sec x+x\sec x\tan x)\frac{-dx}{x\sin x+\cos x}$$

$$\left[\begin{array}{l}\therefore\ \int\frac{x\cos x}{(x\sin x+\cos x)^2}dx\\ \text{Let } x\sin x+\cos x=t\\ \Rightarrow(x\cos x+\sin x-\sin x)dx=dt\\ \text{or}\ \ x\cos x\,dx=dt\\ \quad=\int\frac{dt}{t^2}=-\frac{1}{t}=\frac{-1}{x\sin x+\cos x}\end{array}\right]$$

$$=\frac{-x\sec x}{x\sin x+\cos x}+\int\sec x\left(1+\frac{x\sin x}{\cos x}\right)\frac{dx}{x\sin x+\cos x}$$

$$=\frac{-x\sec x}{x\sin x+\cos x}+\int\sec^2x\,dx$$

$$=\frac{-\sec x}{x\sin x+\cos x}+\tan x+C$$

22. Let $I = \int \left[\log(\log x) + \frac{1}{(\log x)^2} \right] dx$

$$= \int \log(\log x)\, dx + \int \frac{dx}{(\log x)^2} = I_1 + I_2 \quad \text{...(i)}$$

where, $I_1 = \int \log(\log x)\, dx$ and $I_2 = \int \frac{dx}{(\log x)^2}$

Consider, $I_1 = \int \log(\log x)\, dx$

Let us take 1 as the IInd function, thus

$$I_1 = \int \underset{\text{I}}{\log(\log x)} \cdot \underset{\text{II}}{1}\, dx$$

$$= \log(\log x) \cdot x - \int \frac{1}{(\log x)} \cdot \frac{1}{x} \cdot x\, dx$$

[using integration by parts]

$$= x \cdot \log(\log x) - \int (\log x)^{-1} \cdot 1\, dx$$

Again, consider 1 as the IInd function and applying integration by parts, we get

$$I_1 = x \cdot \log(\log x) - \left\{ (\log x)^{-1} \cdot x - \int (-1)(\log x)^{-2} \cdot \frac{1}{x} \cdot x\, dx \right\}$$

$$= x \cdot \log(\log x) - \frac{x}{(\log x)} - \int \frac{dx}{(\log x)^2}$$

$$= x \cdot \log(\log x) - \frac{x}{(\log x)} - I_2 + C$$

Now, substituting the value of I_1 in Eq. (i), we get

$$I = x \cdot \log(\log x) - \frac{x}{(\log x)} - I_2 + C + I_2$$

$$= x \cdot \log(\log x) - \frac{x}{\log x} + C$$

23. Similar as Example 6.

$$\left[\textbf{Ans. } \frac{(x-1)}{2}\sqrt{5 - 2x + x^2} + 2\log|(x-1) + \sqrt{5 - 2x + x^2}| + C \right]$$

24. Let $I = \int \sqrt{3 - 2x - x^2}\, dx = \int \sqrt{-(x^2 + 2x - 3)}\, dx$

$$= \int \sqrt{-(x^2 + 2x + 1 - 4)}\, dx$$

$$= \int \sqrt{-\{(x+1)^2 - 2^2\}}\, dx$$

$$= \int \sqrt{2^2 - (x+1)^2}\, dx$$

Now, put $x + 1 = t \Rightarrow dx = dt$

$$\therefore I = \int \sqrt{2^2 - t^2}\, dt = \frac{1}{2}\left[t\sqrt{2^2 - t^2} + 2^2 \sin^{-1}\left(\frac{t}{2}\right) \right] + C$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a} \right) + C \right]$$

$$= \frac{1}{2}\left[(x+1)\sqrt{3 - 2x - x^2} + 4\sin^{-1}\left(\frac{x+1}{2}\right) \right] + C$$

$[\because t = x + 1]$

25. Let $I = \int \frac{x^2 - 3x + 1}{\sqrt{1 - x^2}} = (-1)\int \frac{-x^2 + 3x - 1}{\sqrt{1 - x^2}}\, dx$

$$= (-1)\int \frac{1 - x^2 + 3x - 2}{\sqrt{1 - x^2}}\, dx$$

[adding and subtracting 1 from numerator]

$$= (-1)\int \left[\frac{1 - x^2}{\sqrt{1 - x^2}} + \frac{3x - 2}{\sqrt{1 - x^2}} \right] dx$$

$$= (-1)\int \left[\sqrt{1 - x^2} + \frac{3x - 2}{\sqrt{1 - x^2}} \right] dx$$

$$= (-1)\left[\int \sqrt{1 - x^2}\, dx + \int \frac{3x - 2}{\sqrt{1 - x^2}}\, dx \right]$$

$$= (-1)[I_1 + I_2] \text{ (say)} \quad \text{...(i)}$$

Now, $I_1 = \int \sqrt{1 - x^2}\, dx = \frac{1}{2}[x\sqrt{1 - x^2} + \sin^{-1}(x)] + C_1$ ...(ii)

and $I_2 = \int \frac{3x - 2}{\sqrt{1 - x^2}}\, dx = \int \frac{3x}{1 - x^2}\, dx - 2\int \frac{dx}{\sqrt{1 - x^2}}$

$$= -\frac{3}{2}\int \frac{-2x}{\sqrt{1 - x^2}} dx - 2\int \frac{dx}{\sqrt{1 - x^2}}$$

$$= -\frac{3}{2} \times 2\sqrt{1 - x^2} - 2\sin^{-1}(x) + C_2$$

$$\left[\because \int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + C \right]$$

$$= -3\sqrt{1 - x^2} - 2\sin^{-1}(x) + C_2 \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we have

$$I = (-1)\left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1}(x) - 2\sin^{-1}(x) - 3\sqrt{1 - x^2} + C_3 \right]$$

[where, $C_3 = C_1 + C_2$]

$$= \frac{3}{2}\sin^{-1}(x) - \frac{x}{2}\sqrt{1 - x^2} + 3\sqrt{1 - x^2} + C$$

where, $C = -C_3$

26. Let $I = \int \frac{xe^x}{(1+x)^2} dx = \int \frac{(x + 1 - 1)e^x}{(1+x)^2} dx$

$$= \int \left[\frac{1 + x}{(1+x)^2} - \frac{1}{(1+x)^2} \right] e^x dx$$

$$\Rightarrow I = \int e^x \left[\frac{1}{(1+x)} - \frac{1}{(1+x)^2} \right] dx$$

Let $f(x) = \frac{1}{1+x}$. Then, $f'(x) = -\frac{1}{(1+x)^2}$

$\therefore$ Given integral is of the form

$$I = \int e^x [f(x) + f'(x)], \text{ where } f(x) = \frac{1}{1+x}$$

We know that $\int e^x[f(x)+f'(x)]dx = e^x f(x)+C$

$\therefore \quad I = e^x f(x)+C = e^x \cdot \frac{1}{(1+x)} + C$

$\Rightarrow \quad I = \frac{e^x}{(1+x)} + C$

27. Let $I = \int e^x \frac{x^2+1}{(x+1)^2}dx \Rightarrow I = \int e^x\left(1-\frac{2x}{(x+1)^2}\right)dx$

$= \int e^x\, dx - 2\int e^x \frac{x}{(x+1)^2}dx$

$\Rightarrow I = e^x - 2\int e^x \frac{x+1-1}{(x+1)^2}dx$

$\Rightarrow I = e^x - 2\int e^x\left\{\frac{1}{x+1}+\frac{-1}{(x+1)^2}\right\}dx$

$\Rightarrow I = e^x - 2\left\{\int \underset{\text{II}}{e^x}\cdot \underset{\text{I}}{\frac{1}{x+1}}dx - \int e^x \frac{1}{(x+1)^2}dx\right\}$

$I = e^x - 2\left\{\frac{1}{x+1}e^x - \int -\frac{1}{(x+1)^2}e^x dx - \int e^x \frac{1}{(x+1)^2}dx\right\}$

[using integration by part]

$\Rightarrow I = e^x - 2\left\{\begin{matrix}\frac{1}{x+1}e^x + \int e^x \cdot \frac{1\,dx}{(x+1)^2} \\ -\int e^x \cdot \frac{1}{(x+1)^2}dx\end{matrix}\right\} + C$

$\Rightarrow I = e^x - 2\frac{e^x}{x+1} + C$

28. Let $I = \int \frac{(2x-5)e^{2x}}{(2x-3)^3}dx = \int \frac{(2x-3-2)e^{2x}}{(2x-3)^3}dx$

$= \int \frac{e^{2x}}{(2x-3)^2}dx - 2\int \frac{e^{2x}}{(2x-3)^3}dx$

$= \int \underset{\text{II}}{e^{2x}}\underset{\text{I}}{(2x-3)^{-2}}dx - 2\int e^{2x}(2x-3)^{-3}dx$

$= \left[(2x-3)^{-2}\int e^{2x}dx - \int\left\{\frac{d}{dx}(2x-3)^{-2}\int e^{2x}dx\right\}dx\right]$

$- 2\int e^{2x}(2x-3)^{-3}dx$

[using integration by parts]

$= (2x-3)^{-2}\frac{e^{2x}}{2} - \int -2(2x-3)^{-3}\times 2\times\frac{e^{2x}}{2}dx$

$- 2\int e^{2x}(2x-3)^{-3}dx$

$= \frac{e^{2x}(2x-3)^{-2}}{2} + 2\int e^{2x}(2x-3)^{-3}dx$

$- 2\int e^{2x}(2x-3)^{-3}dx$

$= \frac{e^{2x}(2x-3)^{-2}}{2} + C$

29. Let $I = \int \frac{2+\sin 2x}{1+\cos 2x}e^x\, dx$

$= \int\left[\frac{2}{1+\cos 2x}+\frac{\sin 2x}{1+\cos 2x}\right]e^x dx$

$= \int\left[\frac{2}{2\cos^2 x}+\frac{2\sin x\cos x}{2\cos^2 x}\right]\cdot e^x\, dx$

$[\because 1+\cos 2x = 2\cos^2 x$ and $\sin 2x = 2\sin x\cos x]$

$= \int\left(\frac{1}{\cos^2 x}+\tan x\right)e^x\, dx = \int(\tan x+\sec^2 x)\,e^x dx$

Let $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$

$\therefore\ I = \tan x\, e^x + C$

$[\because \int[f(x)+f'(x)]e^x\, dx = e^x\, f(x)+C]$

30. Let $I = \int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$

$= \int e^{2x}\left(\frac{1-2\sin x\cos x}{2\sin^2 x}\right)dx$

$[\because 1-\cos 2x = 2\sin^2 x$ and $\sin 2x = 2\sin x\cos x]$

$= \frac{1}{2}\int e^{2x}(\text{cosec}^2 x - 2\cot x)\,dx$

$= \frac{1}{2}\int \underset{\text{I}}{e^{2x}}\,\underset{\text{II}}{\text{cosec}^2 x}\,dx - \int e^{2x}\cot x\, dx$

$= \frac{1}{2}[-e^{2x}\cot x + \int 2e^{2x}\cot x\, dx] - \int e^{2x}\cot x\, dx$

$= -\frac{e^{2x}}{2}\cot x + \int e^{2x}\cot x\, dx - \int e^{2x}\cot x\, dx + C$

$= -\frac{e^{2x}}{2}\cot x + C$

31. Let $I = \int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$

$= \int e^{x+\frac{1}{x}}dx + \int \underset{\text{I}}{x}\left[\underset{\text{II}}{\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}}\right]dx$

$= \int e^{x+\frac{1}{x}}dx + xe^{x+\frac{1}{x}} - \int 1\times e^{x+\frac{1}{x}}dx$

$\left[\begin{array}{l}\because \int u\cdot v\, dx = u\int v\, dx - \int\left\{\frac{d}{dx}(u)\int v\, dx\right\}dx \\ I_1 = \int\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx \\ \text{put } x+\frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt \\ \therefore \int e^t\, dt = e^t = e^{x+\frac{1}{x}}\end{array}\right.$

$= xe^{x+\frac{1}{x}} + C$

32. **Hint** $I = \int e^{-3x} \cos^3 x \, dx$

$$I = \int e^{-3x} \left[\frac{\cos 3x + 3\cos x}{4} \right] dx$$

$$= \frac{1}{4} \left[\int e^{-3x} \cos 3x \, dx + 3 \int e^{-3x} \cos x \, dx \right]$$

Further, similar as Example 2.

$$\left[\text{Ans. } I = \frac{e^{-3x}}{24} [\sin 3x - \cos 3x] - \frac{9}{40} e^{-3x} \cos x + \frac{3}{40} e^{-3x} \sin x + C \right]$$

33. Let $I = \int e^{\tan^{-1} x} \left(\frac{1 + x + x^2}{1 + x^2} \right) dx$

$$= \int e^{\tan^{-1} x} \left(\frac{1 + x^2}{1 + x^2} + \frac{x}{1 + x^2} \right) dx$$

$$= \int e^{\tan^{-1} x} dx + \int \frac{x \, e^{\tan^{-1} x}}{1 + x^2} dx$$

$\Rightarrow I = I_1 + I_2$ (say)

Now, consider $I_2 = \int \frac{x \, e^{\tan^{-1} x}}{1 + x^2} dx$

Now, put $\tan^{-1} x = t \Rightarrow x = \tan t$

and $\frac{1}{1 + x^2} dx = dt$

$\therefore \quad I_2 = \int \underset{\text{I}}{\tan t} \cdot \underset{\text{II}}{e^t} \, dt$

$$= \tan t \int e^t \, dt - \int \left[\frac{d}{dt} (\tan t) \int e^t \, dt \right] dt$$

$$= \tan t \cdot e^t - \int \sec^2 t \cdot e^t \, dt + C$$

$$= \tan t \cdot e^t - \int (1 + \tan^2 t) \, e^t \, dt + C$$

$[\because \sec^2 \theta = 1 + \tan^2 \theta]$

$$\Rightarrow \quad I_2 = \tan t \cdot e^t - \int (1 + x^2) \frac{e^{\tan^{-1} x}}{1 + x^2} dx + C$$

$$\Rightarrow \quad I_2 = x \, e^{\tan^{-1} x} - \int e^{\tan^{-1} x} dx + C \quad [\text{put } t = \tan^{-1} x]$$

$$\therefore \quad I = \int e^{\tan^{-1} x} dx + x \, e^{\tan^{-1} x} - \int e^{\tan^{-1} x} dx + C$$

$$= x \, e^{\tan^{-1} x} + C$$

34. Let $I = \int \sin^{-1} \sqrt{\frac{x}{a + x}} \, dx$.

Now, put $x = a \tan^2 \theta$

$\Rightarrow \quad dx = 2 \, a \tan\theta \sec^2\theta \, d\theta$

$$\therefore \quad I = \int \sin^{-1} \sqrt{\frac{a \tan^2 \theta}{a + a \tan^2 \theta}} \, (2a \tan \theta \cdot \sec^2 \theta) \, d\theta$$

$$= 2a \int \sin^{-1} \left(\frac{\tan\theta}{\sec \theta} \right) \tan \theta \cdot \sec^2 \theta \, d\theta$$

$$= 2a \int \sin^{-1} (\sin\theta) \tan \theta \sec^2 \theta \, d\theta$$

$$= 2a \int \theta \cdot \tan \theta \sec^2 \theta \, d\theta$$

$$= 2a \left[\theta \int \tan \theta \sec^2 \theta \, d\theta - \int \left\{ \frac{d}{d\theta} \theta \int \tan \theta \sec^2 \theta \, d\theta \right\} d\theta \right]$$

$$= 2a \left[\theta \cdot \frac{\tan^2 \theta}{2} - \int 1 \cdot \frac{\tan^2 \theta}{2} \, d\theta \right]$$

$$\left[\text{put } \tan\theta = t \Rightarrow \sec^2\theta \, d\theta = dt \Rightarrow \int \tan\theta \sec^2\theta \, d\theta = \int t \, dt = \frac{t^2}{2} \right]$$

$$= a \, \theta \tan^2 \theta - a \int (\sec^2 \theta - 1) \, d\theta$$

$$= a \, \theta \cdot \tan^2 \theta - a \tan \theta + a\theta + C$$

$$= a \left[\frac{x}{a} \tan^{-1} \sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1} \sqrt{\frac{x}{a}} \right] + C$$

$$\left[\because \quad \tan^2 \theta = \frac{x}{a} \Rightarrow \theta = \tan^{-1} \left(\sqrt{\frac{x}{a}} \right) \right]$$

|TOPIC 5|

Definite Integral

An integral of the form of $\int_a^b f(x) \, dx$ is known as definite integral, where a and b are called the lower and upper limits of a definite integral. The value of definite integral is given as if it has an anti-derivative F, then its value is the difference between the values of F at the end points, i.e. $F(b) - F(a)$.

Here, $\int_a^b f(x) \, dx$ is read as 'the integral of $f(x)$ from a to b'.

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Fundamental theorem of integral calculus is a connection between indefinite integral and definite integral and it

Then, $A'(x) = f(x)$, for all $x \in [a, b]$.

Second Fundamental Theorem of Integral Calculus

Let f be a continuous function defined on the closed interval $[a, b]$ and F be an anti-derivative of f.

Then, $\int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a)$

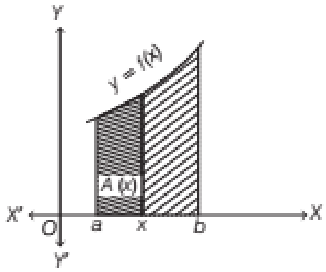
In other words, $\int_a^b f(x) \, dx$ = Value of the anti-derivative F of f at the upper limit b − Value of the same anti-derivative at the lower limit a.

Note

makes the definite integral as a practical tool for Science and Engineering.

Area Function

The space occupied by the curve $y = f(x)$ along with the axis and the given coordinates $x = a$ and $x = b$ is called area of bounded region.



Let x be any point in $[a, b]$. Then, $\int_a^x f(x)dx$ represents the area of the shaded region $A(x)$. The area of shaded region depends upon the value of x or we can say area of this shaded region is a function of x and denoted by $A(x)$. Then, function $A(x)$ is known as area function and is given by $A(x) = \int_a^x f(x)\,dx$.

First Fundamental Theorem of Integral Calculus

Let f be a continuous function defined on the closed interval $[a, b]$ and $A(x)$ be the area of function, i.e. $A(x) = \int_a^x f(x)dx$.

Sol. Let $I = \int_0^{\pi/2} \cos^2 x\, dx$

Here, $\int \cos^2 dx = \int \left(\frac{\cos 2x + 1}{2}\right) dx$ $\quad [\because \cos 2x = 2\cos^2 x - 1]$

$$= \frac{1}{2}\int \cos 2x\, dx + \frac{1}{2}\int 1\, dx$$

$$= \frac{1}{2}\left[\frac{\sin 2x}{2}\right] + \frac{1}{2}x = \frac{1}{4}\sin 2x + \frac{1}{2}x = F(x) \text{ (say)}$$

Therefore, by the fundamental theorem of integral calculus, we get

$$I = [F(x)]_0^{\pi/2} = F(\pi/2) - F(0)$$

$$= \left[\frac{1}{4}\sin 2\left(\frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right)\right] - \left[\frac{1}{4}\sin(0) + \frac{1}{2}(0)\right]$$

$$= \left[\frac{1}{4}(0) + \frac{\pi}{4}\right] - [0 + 0] = \frac{\pi}{4}$$

EXAMPLE |2| Evaluate $\int_1^2 (4x^3 - 5x^2 + 6x + 9)\, dx$. **[NCERT]**

Sol. Let $I = \int_1^2 (4x^3 - 5x^2 + 6x + 9)\, dx$

and $f(x) = 4x^3 - 5x^2 + 6x + 9$

Then, $\int f(x)dx = \int (4x^3 - 5x^2 + 6x + 9)dx$

$$= \frac{4x^4}{4} - \frac{5x^3}{3} + \frac{6x^2}{2} + 9x$$

Note

(i) The crucial operation in evaluating a definite integral is that to finding a function whose derivative is equal to the integrand. This strengthens the relationship between differentiation and integration.

(ii) In $\int_a^b f(x)\,dx$, the function f needs to be well-defined and continuous in $[a, b]$.

STEPS FOR CALCULATING THE DEFINITE INTEGRAL

Suppose given integral is $\int_a^b f(x)\, dx$, then to calculate, we use the following steps

I. Find the indefinite integral $\int f(x)\, dx$ by using suitable method. Let it be, $F(x)$, i.e. $\int f(x)\, dx = F(x)$.

II. Evaluate $[F(x)]_a^b = F(b) - F(a)$, which gives the required value.

Thus, $\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a)$.

Note There is no need to keep integration constant C because if we consider $F(x) + C$ instead of $F(x)$, we get $\int_a^b f(x)\,dx = [F(x) + C]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$

Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

EXAMPLE |1| Evaluate $\int_0^{\pi/2} \cos^2 x\, dx$.

There is no standard integration of $\cos^2 x$, therefore we convert it in standard form by using the relation $\frac{\cos 2x + 1}{2}$ and then use fundamental theorem of integral calculus.

TOPIC PRACTICE 5

OBJECTIVE TYPE QUESTIONS

1 $\int_1^{\sqrt{3}} \frac{1}{1 + x^2} dx$ is equal to **[NCERT]**

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{12}$

2 $\int_0^{2/3} \frac{1}{4 + 9x^2} dx$ is equal to **[NCERT]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{12}$

(c) $\frac{\pi}{24}$ (d) $\frac{\pi}{4}$

3 The value of $\int_0^4 (x + e^{2x})\, dx$ is

(a) $\frac{15 + e^8}{2}$ (b) $\frac{15 - e^8}{2}$

(c) $\frac{e^8 - 15}{2}$ (d) $\frac{-e^8 - 15}{2}$

4 $\int_0^{\pi/8} \tan^2(2x)\, dx$ is equal to **[Delhi 2020]**

(a) $\frac{4 - \pi}{8}$ (b) $\frac{4 + \pi}{8}$

$$= x^4 - \frac{5}{3}x^3 + 3x^2 + 9x = F(x) \text{ (say)}$$

Now, by the fundamental theorem of integral calculus, we get

$$I = [F(x)]_1^2 = F(2) - F(1)$$

$$= \left[(2)^4 - \frac{5}{3}(2)^3 + 3(2)^2 + 9(2)\right] - \left[1 - \frac{5}{3}(1)^3 + 3(1)^2 + 9(1)\right]$$

$$= \left[16 - \frac{40}{3} + 12 + 18\right] - \left[1 - \frac{5}{3} + 3 + 9\right]$$

$$= \left[46 - \frac{40}{3}\right] - \left[13 - \frac{5}{3}\right] = \left[\frac{138 - 40}{3}\right] - \left[\frac{39 - 5}{3}\right]$$

$$= \frac{98}{3} - \frac{34}{3} = \frac{64}{3}$$

EXAMPLE |3| Evaluate $\int_0^{\pi/2} e^x (\sin x - \cos x)\, dx$. **[Delhi 2014]**

Sol. Let $I = \int_0^{\pi/2} e^x (\sin x - \cos x)\, dx$

$\Rightarrow \quad I = -\int_0^{\pi/2} e^x(\cos x - \sin x)\, dx$

Now, let $f(x) = \cos x$. Then, $f'(x) = -\sin x$

By using $\int e^x [f(x) + f'(x)]\, dx = e^x f(x)$, we get

$$I = -[e^x \cos x]_0^{\pi/2}$$

$$= -e^{\pi/2} \cos \frac{\pi}{2} + e^0 \cos(0) = 0 + 1(1) = 1$$

SHORT ANSWER Type II Questions

13 Prove that $\int_1^3 \frac{1}{x^2(x+1)}\, dx = \frac{2}{3} + \log \frac{2}{3}$. **[NCERT]**

14 Evaluate $\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$.

15 Evaluate $\int_0^1 x(\tan^{-1} x)\, dx$. **[Delhi 2016C]**

16 If $f(x) = \int_0^x t \sin t\, dt$, then write the value of $f'(x)$. **[All India 2014]**

17 Evaluate $\int_0^{\pi} e^{2x} \cdot \sin\left(\frac{\pi}{4} + x\right) dx$. **[Delhi 2016]**

HINTS & SOLUTIONS

1. (d) Let $I = \int_1^{\sqrt{3}} \frac{1}{1+x^2}\, dx = [\tan^{-1} x]_1^{\sqrt{3}}$

$$= \tan^{-1}\sqrt{3} - \tan^{-1}(1) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

2. (c) Let $I = \int_0^{2/3} \frac{1}{4 + 9x^2}\, dx = \frac{1}{9}\int_0^{2/3} \frac{1}{\left(\frac{2}{3}\right)^2 + x^2}$

$$= \frac{1}{9} \cdot \frac{1}{2/3}\left[\tan^{-1}\left(\frac{x}{2/3}\right)\right]_0^{2/3}$$

(c) $\frac{4-\pi}{4}$ (d) $\frac{4-\pi}{2}$

VERY SHORT ANSWER Type Questions

5 Evaluate $\int_2^3 3^x\, dx$. **[Delhi 2017]**

6 Evaluate $\int_0^3 \frac{dx}{9 + x^2}$. **[Delhi 2014]**

7 If $\int_0^a \frac{1}{4 + x^2}\, dx = \frac{\pi}{8}$, then find the value of a. **[All India 2014]**

Directions (Q. Nos. 8-10) *Evaluate the following.*

8 $\int_0^{\pi/4} \tan x\, dx$ **[Foreign 2014]**

9 $\int_{\pi/6}^{\pi/4} \operatorname{cosec} x\, dx$ **[NCERT]**

10 $\int_2^3 [x]\, dx$, where $[x]$ is the greatest integer function.

SHORT ANSWER Type I Questions

Directions (Q. Nos. 11-12) *Evaluate the following.*

11 $\int_0^{\pi/4} \sqrt{1 + \sin 2x}\, dx$ **[NCERT Exemplar]**

12 $\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$

6. Let $I = \int_0^3 \frac{dx}{9 + x^2} \Rightarrow I = \int_0^3 \frac{dx}{x^2 + (3)^2}$

$\Rightarrow I = \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3 \quad \left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$\Rightarrow I = \frac{1}{3}\left[\tan^{-1}\left(\frac{3}{3}\right) - \tan^{-1}(0)\right]$

$$= \frac{1}{3}[\tan^{-1}(1) - 0] = \frac{1}{3}\left(\frac{\pi}{4}\right) = \frac{\pi}{12}$$

7. **Hint** $\frac{1}{2}\tan^{-1}\left(\frac{a}{2}\right) = \frac{\pi}{8}$

$\Rightarrow \quad \tan^{-1}\left(\frac{a}{2}\right) = \frac{\pi}{4} \Rightarrow \frac{a}{2} = \tan\left(\frac{\pi}{4}\right)$ **[Ans.** $a = 2$**]**

8. Let $I = \int_0^{\pi/4} \tan x\, dx = [\log|\sec x|]_0^{\pi/4}$

$$= \log\left|\sec\frac{\pi}{4}\right| - \log|\sec 0|$$

$$= \log\sqrt{2} - \log 1 = \log\sqrt{2} - 0 = \log\sqrt{2}$$

9. Let $I = \int_{\pi/6}^{\pi/4} \operatorname{cosec} x\, dx = [\log|\operatorname{cosec} x - \cot x|]_{\pi/6}^{\pi/4}$

$$= \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$$

$$= \log|\sqrt{2} - 1| - \log|2 - \sqrt{3}|$$

$$= \log\left|\frac{\sqrt{2} - 1}{2 - \sqrt{3}}\right|$$

10. **Hint** $\int_2^3 [x]\, dx = \int_2^3 2\, dx \quad [\because \text{if } 2 < x < 3, \text{ then } [x] = 2]$

$$= \frac{1}{6}[\tan^{-1}(1) - \tan^{-1}(0)] = \frac{1}{6} \times \frac{\pi}{4} = \frac{\pi}{24}$$

3. (a) Let $I = \int_0^4 (x + e^{2x})dx = \left[\frac{x^2}{2} + \frac{e^{2x}}{2}\right]_0^4$

$$= \left(8 + \frac{e^8}{2}\right) - \left(\frac{1}{2}\right) = \frac{15 + e^8}{2}$$

4. (a) Let $I = \int_0^{\pi/8} \tan^2(2x)dx$

$\Rightarrow \quad I = \int_0^{\pi/8} \{\sec^2(2x) - 1\}dx$

$\Rightarrow \quad I = \left[\frac{\tan 2x}{2} - x\right]_0^{\pi/8}$

$\Rightarrow \quad I = \left(\frac{\tan \pi/4}{2} - \frac{\pi}{8}\right) - (0 - 0) = \frac{1}{2} - \frac{\pi}{8} = \frac{4 - \pi}{8}$

5. $\int_2^3 3^x dx = \left(\frac{3^x}{\log 3}\right)_2^3 = \frac{1}{\log 3}(3^x)_2^3 = \frac{1}{\log 3}(3^3 - 3^2)$

$$= \frac{1}{\log 3}(27 - 9) = \frac{18}{\log 3}$$

$\therefore \int_1^3 \frac{1}{x^2(x+1)} dx = \int_1^3 \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right)dx$

$= \left[-\log|x| - \frac{1}{x} + \log|x+1|\right]_1^3 = \left[\log\left|\frac{x+1}{x}\right| - \frac{1}{x}\right]_1^3$

$\left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$

$= \left(\log\left|\frac{4}{3}\right| - \frac{1}{3}\right) - \left(\log\left|\frac{2}{1}\right| - \frac{1}{1}\right)$

$= \left[\log\frac{4}{3} - \log 2\right] + \left[1 - \frac{1}{3}\right] = \log\left(\frac{4}{3} \times \frac{1}{2}\right) + \frac{2}{3}$

$= \log\frac{2}{3} + \frac{2}{3}$

14. **Hint** $\int \frac{dx}{\sqrt{(x-1)(2-x)}} = \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}}$ and

use $\sin^{-1}(-x) = -\sin^{-1}x, \forall x \in [-1, 1]$. **[Ans. π]**

15. Let $I = \int_0^1 \underset{\text{II}}{x} \underset{\text{I}}{\tan^{-1}x}\, dx = \left[\tan^{-1}x \cdot \frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx\right]_0^1$

[using integration by parts]

$= \left[\frac{x^2}{2} \cdot \tan^{-1}x - \frac{1}{2}\int \frac{x^2 + 1 - 1}{1 + x^2} dx\right]_0^1$

$= \left[\frac{x^2}{2}\tan^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1 dx + \int_0^1 \frac{dx}{1 + x^2}$

[Ans. 2]

11. **Hint** Use the formula, $1 + \sin 2x = (\cos x + \sin x)^2$.

[Ans. 1]

12. **Hint** Rationalise the denominator, i.e. multiply numerator and denominator by $\sqrt{1 + x} + \sqrt{x}$.

$\left[\textbf{Ans.} \frac{4\sqrt{2}}{3}\right]$

13. To prove, $\int_1^3 \frac{1}{x^2(x+1)} dx = \frac{2}{3} + \log\frac{2}{3}$

Let $\quad \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \quad$ ...(i)

[by partial fraction]

$\Rightarrow 1 = Ax(x+1) + B(x+1) + Cx^2 \quad$ ...(ii)

On putting $x = 0, -1$ respectively in Eq. (ii), we get

$1 = B(0+1)$ and $1 = C(-1)^2$

$\Rightarrow B = 1$ and $C = 1$

On equating the coefficient of x^2 from both sides of Eq. (ii), we get

$0 = A + C \Rightarrow A = -C \Rightarrow A = -1$

17. Let $I = \int_0^{\pi} e^{2x} \sin\left(\frac{\pi}{4} + x\right)dx$

Again, let $I_1 = \int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\sin}\left(\frac{\pi}{4} + x\right)dx \quad$ ...(i)

$= \sin\left(\frac{\pi}{4} + x\right)\int e^{2x}dx - \int\left\{\frac{d}{dx}\sin\left(\frac{\pi}{4} + x\right)\int e^{2x}dx\right\}dx$

[using integration by parts]

$= \sin\left(\frac{\pi}{4} + x\right)\frac{e^{2x}}{2} - \int \cos\left(\frac{\pi}{4} + x\right)\frac{e^{2x}}{2}dx$

$= \frac{e^{2x}}{2}\sin\left(\frac{\pi}{4} + x\right) - \frac{1}{2}\int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\cos}\left(\frac{\pi}{4} + x\right)dx$

$= \frac{e^{2x}}{2}\sin\left(\frac{\pi}{4} + x\right)$

$\quad - \frac{1}{2}\left[\cos\left(\frac{\pi}{4} + x\right)\frac{e^{2x}}{2} - \int -\sin\left(\frac{\pi}{4} + x\right)\frac{e^{2x}}{2}dx\right]$

[using integration by parts]

$= \frac{e^{2x}}{2}\sin\left(\frac{\pi}{4} + x\right) - \frac{e^{2x}}{4}\cos\left(\frac{\pi}{4} + x\right)$

$\quad - \frac{1}{4}\int e^{2x}\sin\left(\frac{\pi}{4} + x\right)dx$

$\Rightarrow \quad I_1 = \frac{e^{2x}}{4}\left\{2\sin\left(\frac{\pi}{4} + x\right) - \cos\left(\frac{\pi}{4} + x\right)\right\} - \frac{1}{4}I_1$

[from Eq. (i)]

$\Rightarrow I_1 + \frac{1}{4}I_1 = \frac{e^{2x}}{4}\left\{2\sin\left(\frac{\pi}{4} + x\right) - \cos\left(\frac{\pi}{4} + x\right)\right\}$

$$= \left[\frac{1}{2}\tan^{-1}(1) - 0\right] - \frac{1}{2}(1-0) + [\tan^{-1} x]_0^1$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2} + [\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{4} = \frac{3\pi}{8} - \frac{1}{2}$$

16. We have, $f(x) = \int_0^x \underset{\text{I}}{t}\, \underset{\text{II}}{\sin t}\, dt$

$$= t\int_0^x \sin t\, dt - \int_0^x \left[\frac{d}{dt}(t)\int \sin t\, dt\right] dt$$

[using integration by parts]

$$= [t(-\cos t)]_0^x - \int_0^x (-\cos t)dt = [-t\cos t]_0^x + [\sin t]_0^x$$

$$= -x\cos x + 0 + \sin x - 0$$

$$= \sin x - x\cos x$$

Thus, $f(x) = \sin x - x\cos x$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x - \left[x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x)\right]$$

[by product rule of derivative]

$$= \cos x - [x(-\sin x) + \cos x]$$

$$= \cos x + x\sin x - \cos x = x\sin x$$

$$\Rightarrow \quad \frac{5}{4}I_1 = \frac{e^{2x}}{4}\left\{2\sin\left(\frac{\pi}{4}+x\right) - \cos\left(\frac{\pi}{4}+x\right)\right\}$$

$$\Rightarrow \quad I_1 = \frac{e^{2x}}{5}\left\{2\sin\left(\frac{\pi}{4}+x\right) - \cos\left(\frac{\pi}{4}+x\right)\right\}$$

$$\therefore \quad I = [I_1]_0^\pi = \left[\frac{e^{2x}}{5}\left\{2\sin\left(\frac{\pi}{4}+x\right) - \cos\left(\frac{\pi}{4}+x\right)\right\}\right]_0^\pi$$

$$= \frac{1}{5}\left[e^{2\pi}\left\{2\sin\left(\frac{\pi}{4}+\pi\right) - \cos\left(\frac{\pi}{4}+\pi\right)\right\} - e^0\left\{2\sin\left(\frac{\pi}{4}+0\right) - \cos\left(\frac{\pi}{4}+0\right)\right\}\right]$$

$$= \frac{1}{5}\left[e^{2\pi}\left\{-2\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right\} - e^0\left\{2\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right\}\right]$$

$$= \frac{1}{5}\left[e^{2\pi}\left\{-2\times\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right\} - 1\left\{2\times\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right\}\right]$$

$$= \frac{1}{5}\left[e^{2\pi}\left\{-\frac{1}{\sqrt{2}}\right\} - \frac{1}{\sqrt{2}}\right]$$

$$= -\frac{1}{5\sqrt{2}}[e^{2\pi} + 1]$$

|TOPIC 6|

Evaluation of Definite Integral by Substitution

There are several methods for finding the definite integral. One of the important methods for finding the definite integral is the method of substitution. To evaluate $\int_a^b f(x)dx$ by substitution, we use the following steps

I. Consider, the given integral without limits, i.e. $\int f(x)\, dx$ and substitute some part of integrand as another variable (say t), such that its differentiation exist in the integral, so that the given integral reduces to a known form.

II. Integrate the new integral with respect to the new variable without mentioning the constant of integration.

III. Replace the new variable by the original variable in the answer obtained in step II.

IV. Find the difference of the values of the answer, obtained in step III, at the upper and lower limits.

Alternate Method

This method is quicker than the previous method. To evaluate, we use the following steps:

I. Firstly, substitute some part of integrand as another variable (say t), such that its differentiation

$$I_1 = \frac{(\tan^{-1} x)^2}{2} \quad \text{...(ii)}$$

$$\Rightarrow I = \left[\frac{(\tan^{-1} x)^2}{2}\right]_0^1 = \frac{(\tan^{-1} 1)^2}{2} - \frac{(\tan^{-1} 0)^2}{2}$$

[using Eq. (ii)]

$$= \frac{\left(\frac{\pi}{4}\right)^2}{2} - 0 = \frac{\pi^2}{32}$$

EXAMPLE |2| Evaluate $\int_1^3 \frac{1}{x(1+\log x)}dx$.

Sol. Let $I = \int_1^3 \frac{dx}{x(1+\log x)}$

Now, putting $\log x = t \Rightarrow \frac{1}{x}dx = dt$

Lower limit When $x = 1$, then $t = \log 1 \Rightarrow t = 0$

Upper limit When $x = 3$, then $t = \log 3$

Now, $I = \int_0^{\log 3} \frac{dt}{(1+t)} = [\log|1+t|]_0^{\log 3}$

$$= \log|1+\log 3| - \log|1+0|$$

$$= \log|1+\log 3| - \log 1$$

$$= \log|1+\log 3| - 0 \quad [\because \log 1 = 0]$$

$$= \log|1+\log 3|$$

exist in the integral, so that given integral reduces to a known form.

II. Change the upper and lower limits corresponding to the new variable.

III. Integrate the new integral with respect to the new variable.

IV. Find the difference of the values of the answer, obtained in step III, at new upper and lower limits.

EXAMPLE |1| Evaluate $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$. **[All India 2014C]**

Sol. Let given integral be $I = \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$

and $\quad I_1 = \int \frac{\tan^{-1} x}{1+x^2} dx$

On putting $\tan^{-1} x = t$, we get $\frac{1}{1+x^2} dx = dt$

$\therefore \quad I_1 = \int t\, dt = \frac{t^2}{2} \quad$ ...(i)

Now, put $t = \tan^{-1} x$ in Eq. (i), we get

EXAMPLE |3| Evaluate $\int_0^{\pi/2} \sqrt{\sin \phi} \cos^5 \phi \, d\phi$.

Sol. Let $I = \int_0^{\pi/2} \sqrt{\sin \phi} \cos^5 \phi \, d\phi$

$$= \int_0^{\pi/2} \sqrt{\sin \phi} \cos^4 \phi \cos \phi \, d\phi$$

$$= \int_0^{\pi/2} \sqrt{\sin \phi} (1 - \sin^2 \phi)^2 \cos \phi \, d\phi$$

Now, put $\sin \phi = t \Rightarrow \cos \phi \, d\phi = dt$

Lower limit When $\phi = 0$, then $t = 0$

Upper limit When $\phi = \frac{\pi}{2}$, then $t = 1$

Now, $I = \int_0^1 \sqrt{t}(1-t^2)^2 \, dt = \int_0^1 \sqrt{t}(1+t^4-2t^2)\, dt$

$$= \int_0^1 (t^{1/2} + t^{9/2} - 2t^{5/2})\, dt$$

$$= \left[\frac{t^{3/2}}{3/2} + \frac{t^{11/2}}{11/2} - \frac{2t^{7/2}}{7/2}\right]_0^1$$

$$= \left[\frac{2}{3}t^{3/2} + \frac{2}{11}t^{11/2} - \frac{4}{7}t^{7/2}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7} = \frac{154 + 42 - 132}{11 \times 3 \times 7} = \frac{64}{231}$$

EXAMPLE |4| Evaluate $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$.

[All India 2014C; Delhi 2011]

Sol. Let $I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

We know that

$(\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2 \sin x \cos x$

$\Rightarrow (\sin x - \cos x)^2 = 1 - \sin 2x$

$[\because \sin^2 x + \cos^2 x = 1$ and $2 \sin x \cos x = \sin 2x]$

$\Rightarrow \sin 2x = 1 - (\sin x - \cos x)^2$

Now, putting $\sin x - \cos x = t \quad$ ...(i)

$\Rightarrow \quad (\cos x + \sin x)\, dx = dt$

Also, when $x = \frac{\pi}{6}$, then

$\Rightarrow \quad t = \sin \frac{\pi}{6} - \cos \frac{\pi}{6} = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2}$

When $x = \frac{\pi}{3}$, then

$$t = \sin \frac{\pi}{3} - \cos \frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$$

$$\therefore \quad I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{1}{\sqrt{1-t^2}}\, dt$$

$$= \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{1}{\sqrt{1-t^2}}\, dt = [\sin^{-1} t]_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}$$

$$= \left[\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) - \sin^{-1}\left\{-\left(\frac{\sqrt{3}-1}{2}\right)\right\}\right]$$

Special case $\int_a^a f(x)\, dx = 0$.

(iii) $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x) dx$, where $a < c < b$.

Special case If $a < c_1 < c_2 < \ldots < c_n < b$, then

$$\int_a^b f(x)\, dx = \int_a^{c_1} f(x)\, dx + \int_{c_1}^{c_2} f(x)\, dx + \int_{c_2}^{c_3} f(x)\, dx + \ldots + \int_{c_n}^{b} f(x)\, dx.$$

(iv) $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

(v) $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

[it is a particular case of property (iv)]

(vi) $\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a-x)\, dx$

(vii) $\int_0^{2a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

Special case

$$\int_0^a f(x)\, dx = \begin{cases} 2\int_0^{a/2} f(x)\, dx, & \text{if } f(a-x) = f(x) \\ 0, & \text{if } f(a-x) = -f(x) \end{cases}$$

(viii) $\int_{-a}^a f(x)\, dx$

$$= \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f \text{ is an even function, i.e. } f(-x) = f(x) \\ 0, & \text{if } f \text{ is an odd function, i.e. } f(-x) = -f(x) \end{cases}$$

$$= \left[\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)\right]$$

$$[\because \sin^{-1}(-\theta) = -\sin^{-1}\theta]$$

$$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

Properties of Definite Integrals

Some important properties which will be useful in evaluating the definite integrals are given below

(i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

This property shows that the value of a definite integral does not change if the variable is changed.

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

This property shows that when we interchange the limits, then it changes by negative sign.

$$= -\left[\frac{x^2}{2} - x\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$$

$$= -\left[\left(\frac{1^2}{2} - 1\right) - (0-0)\right] + \left[\left(\frac{4^2}{2} - 4\right) - \left(\frac{1^2}{2} - 1\right)\right]$$

$$= \frac{1}{2} + \left[4 + \frac{1}{2}\right] = 5$$

EXAMPLE |6| Evaluate

$$\int_2^5 [|x-2| + |x-3| + |x-5|]\,dx.$$

[Delhi 2013]

Sol. Let $I = \int_2^5 [|x-2| + |x-3| + |x-5|]\,dx$

Now, let us first define the given absolute functions.

Clearly, $|x-2| = \begin{cases} (x-2), & \text{if } x \ge 2 \\ -(x-2), & \text{if } x < 2 \end{cases}$,

$|x-3| = \begin{cases} (x-3), & \text{if } x \ge 3 \\ -(x-3), & \text{if } x < 3 \end{cases}$

and $|x-5| = \begin{cases} (x-5), & \text{if } x \ge 5 \\ -(x-5), & \text{if } x < 5 \end{cases}$

Now, divide the given limit at $x = 3$, i.e. write I as

$$I = \int_2^3 [|x-2| + |x-3| + |x-5|]\,dx + \int_3^5 [|x-2| + |x-3| + |x-5|]\,dx$$

[using property (iii)]

$$= \int_2^3 [(x-2) + \{-(x-3)\} + \{-(x-5)\}]\,dx + \int_3^5 [(x-2) + (x-3) + \{-(x-5)\}]\,dx$$

$$= \int_2^3 [x - 2 - x + 3 - x + 5]\,dx + \int_3^5 [x - 2 + x - 3 - x + 5]\,dx$$

$$= \int_2^3 (6-x)\,dx + \int_3^5 x\,dx = \left[6x - \frac{x^2}{2}\right]_2^3 + \left[\frac{x^2}{2}\right]_3^5$$

$$= \left\{\left(18 - \frac{9}{2}\right) - \left(12 - \frac{4}{2}\right)\right\} + \left[\frac{25}{2} - \frac{9}{2}\right]$$

$$= \frac{27}{2} - \frac{20}{2} + \frac{25}{2} - \frac{9}{2} = \frac{7}{2} + \frac{16}{2} = \frac{23}{2}$$

Note In an absolute integral function, please be careful while breaking the limits.

EXAMPLE |5| Evaluate $\int_0^4 |x-1|\,dx$.

Here, the given integrand is in the form of absolute function and we define the absolute function $|x-a|$ as $|x-a| = \begin{cases} x-a, & \text{if } x \ge a \\ -(x-a), & \text{if } x < a \end{cases}$.

By using it, we convert the given integrand in simplest form and then integrate it.

Sol. Let $I = \int_0^4 |x-1|\,dx$. Now, let us first define the given absolute function.

Clearly, $|x-1| = \begin{cases} (x-1), & \text{if } x \ge 1 \\ -(x-1), & \text{if } x < 1 \end{cases}$

Now, divide the given limit at $x = 1$, i.e. write I as

$$I = \int_0^1 |x-1|\,dx + \int_1^4 |x-1|\,dx$$

$$\left[\because \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \text{ if } a < c < b\right]$$

$$= \int_0^1 -(x-1)dx + \int_1^4 (x-1)dx$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sin^3\left(\frac{\pi}{2} - x\right)}{\sin^3\left(\frac{\pi}{2} - x\right) + \cos^3\left(\frac{\pi}{2} - x\right)}dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}dx \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^3 x + \cos^3 x}{\cos^3 x + \sin^3 x}dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} 1\,dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

EXAMPLE |8| Evaluate $\int_0^{\pi/4} \log(1 + \tan x)\,dx$.

[NCERT; All India 2015C]

Firstly, use the property (v), i.e. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ and then simplify the integrand. Further, add it into original integral and simplify the result.

Sol. Let $I = \int_0^{\pi/4} \log(1 + \tan x)\,dx \qquad \text{...(i)}$

$$\Rightarrow \quad I = \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right]dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right)dx$$

$$\left[\because \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right]$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right)dx$$

$$= \int_0^{\pi/4} \{\log 2 - \log(1 + \tan x)\}\,dx \qquad \text{...(ii)}$$

$$\left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$$

EXAMPLE |7| Evaluate $\int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx.$

Firstly, use the property (v), i.e. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ to get the new integral, then add it into original integral and simplify the result.

Sol. Let $I = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx$ ...(i)

EXAMPLE |9| Evaluate $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx.$

[All India 2017C, 2017; Delhi 2016C, 2014C; Foreign 2014]

Sol. Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ ...(i)

$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} dx$

$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^{\pi} \frac{-(\pi - x)\tan x}{-\sec x - \tan x} dx$

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x} dx$$

$$= \pi \int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)} dx$$

$$= \pi \int_0^{\pi} \frac{\tan x \cdot \sec x - \tan^2 x}{\sec^2 x - \tan^2 x} dx$$

$$= \pi \left[\int_0^{\pi} \sec x \cdot \tan x\,dx - \int_0^{\pi} \tan^2 x\,dx\right]$$

$$= \pi \left\{[\sec x]_0^{\pi} - \int_0^{\pi} (\sec^2 x - 1)\,dx\right\}$$

$$= \pi\{[\sec\pi - \sec 0] - [\tan x - x]_0^{\pi}\}$$

$$= \pi\{[-1 - 1] - [(\tan\pi - \pi) - (\tan 0 - 0)]\}$$

$$= \pi\{-2 - [(-\pi) - 0]\} = \pi(\pi - 2)$$

$\Rightarrow \quad I = \frac{\pi}{2}(\pi - 2)$

EXAMPLE |10| Evaluate the integral $\int_0^{2\pi} \frac{1}{1 + e^{\sin x}} dx.$

[All India 2013]

Firstly, use the property (vi), i.e. $\int_0^{2a} f(x)\,dx = \int_0^a [(f(x) + f(2a - x)]dx$. Further, integrate and then simplify it.

Sol. Let $I = \int_0^{2\pi} \frac{1}{1 + e^{\sin x}} dx$

$\therefore \quad I = \int_0^{\pi} \left[\frac{1}{1 + e^{\sin x}} + \frac{1}{1 + e^{\sin(2\pi - x)}}\right] dx$

$$= \int_0^{\pi} \left[\frac{1}{1 + e^{\sin x}} + \frac{e^{\sin x}}{e^{\sin x} + 1}\right] dx$$

$$= \int_0^{\pi} \frac{1 + e^{\sin x}}{1 + e^{\sin x}} dx$$

$$= \int_0^{\pi} 1\,dx = [x]_0^{\pi} = \pi$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/4} \log 2\,dx$$

$$= \log 2 \int_0^{\pi/4} 1\,dx = \log 2\,[x]_0^{\pi/4} = \log 2\left(\frac{\pi}{4} - 0\right)$$

$\Rightarrow \quad 2I = \frac{\pi}{4} \log 2$

$\Rightarrow \quad I = \frac{\pi}{8} \log 2$

EXAMPLE |11| Evaluate $\int_0^{2\pi} \cos^5 x\,dx.$

Firstly, use the property (vii),

i.e. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$

Further, use again the above property and simplify it.

Sol. Let $I = \int_0^{2\pi} \cos^5 x\,dx = 2\int_0^{\pi} \cos^5 x\,dx$

$\left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)\,dx, \text{ if } f(2a - x) = f(x), \text{ here } \cos^5(2\pi - x) = \cos^5 x\right]$

$= 2 \times 0 = 0$

$\left[\because \int_0^{2a} f(x) = 0, \text{ if } f(2a - x) = -f(x), \text{ here } \cos^5(\pi - x) = -\cos^5 x\right]$

EXAMPLE |12| Evaluate the integral $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx.$

Sol. Let $I = \int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$

Here, $\quad f(x) = \sin^2 x$

Now, $\quad f(-x) = \sin^2(-x) = [\sin(-x)]^2$

$= (-\sin x)^2 = \sin^2 x = f(x)$

$[\because \sin(-\theta) = -\sin\theta]$

So, $f(x)$ is an even function.

$\therefore \quad I = \int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = 2\int_0^{\pi/2} \sin^2 x\,dx$

$\left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is an even function, here } \sin^2 x \text{ is an even function.}\right]$

$= 2\int_0^{\pi/2} \left[\frac{1 - \cos 2x}{2}\right] dx \qquad [\because \cos 2x = 1 - 2\sin^2 x]$

$= \int_0^{\pi/2} (1 - \cos 2x)\,dx = \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2}$

$$[\because \int_0^{2a} f(x)\,dx = \int_0^a \{f(x) + f(2a - x)\}\,dx]$$

$$= \int_0^{\pi}\left[\frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin x}}\right]dx$$

$$[\because \sin(2\pi - x) = -\sin x]$$

$$= \left[\frac{\pi}{2} - \frac{\sin \pi}{2}\right] - [0 - 0]$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

TOPIC PRACTICE 6

OBJECTIVE TYPE QUESTIONS

1 $\int_0^{\pi/2} \cos x e^{\sin x} dx$ is equal to **[NCERT Exemplar]**

(a) $e+1$ (b) $e-1$ (c) e (d) $-e$

2 The value of the integral $\int_{1/3}^{1} \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx$ is **[NCERT]**

(a) 6 (b) 0 (c) 3 (d) 4

3 $\int_{a+c}^{b+c} f(x)\,dx$ is equal to **[NCERT Exemplar]**

(a) $\int_a^b f(x-c)\,dx$ (b) $\int_a^b f(x+c)\,dx$

(c) $\int_a^b f(x)\,dx$ (d) $\int_{a-c}^{b-c} f(x)\,dx$

4 If $f(a+b-x) = f(x)$, then $\int_a^b x f(x)\,dx$ is equal to **[NCERT]**

(a) $\frac{a+b}{2}\int_a^b f(b-x)\,dx$ (b) $\frac{a+b}{2}\int_a^b f(b+x)\,dx$

(c) $\frac{b-a}{2}\int_a^b f(x)\,dx$ (d) $\frac{a+b}{2}\int_a^b f(x)\,dx$

5 The value of $\int_{-\pi/2}^{\pi/2}(x^3 + x\cos x + \tan^5 x + 1)dx$ is **[NCERT]**

(a) zero (b) 2 (c) π (d) 1

VERY SHORT ANSWER Type Questions

Directions (Q. Nos. 6-11) *Evaluate the following integrals.*

6 $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\,dx$ **[NCERT]**

7 $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\,dx$ **[NCERT Exemplar]**

8 $\int_{-\pi/3}^{\pi/3} \sin^3 x\,dx$

SHORT ANSWER Type I Questions

Directions (Q. Nos. 12-14) *Evaluate the following integrals.*

12 $\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx$ **[NCERT Exemplar]**

13 $\int_0^{\pi/2} \cos x\, e^{\sin x}\,dx$ **[NCERT Exemplar]**

14 $\int_0^1 \frac{dx}{e^x + e^{-x}}$ **[NCERT Exemplar]**

15 If $g(x) = \int_0^x \cos 4t\,dt$, then prove that

$$g(x) = g(x+\pi).$$

16 Evaluate $\int_0^{\pi/2} \frac{\tan^7 x}{\cot^7 x + \tan^7 x}\,dx.$

17 Evaluate $\int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x}\,dx.$

18 Show that $\int_0^a f(x)\,g(x)\,dx = 2\int_0^a f(x)\,dx$, if f and g are defined as

$$f(x) = f(a-x) \text{ and } g(x) + g(a-x) = 4.$$

19 Prove that $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx = 0.$

SHORT ANSWER Type II Questions

20 Evaluate $\int_0^1 x\cdot\sqrt{\frac{1-x^2}{1+x^2}}\,dx.$

21 Prove that $\int_0^{\pi/4} 2\tan^3 x\,dx = 1 - \log 2.$

Directions (Q. Nos. 22-29) *Evaluate the following integrals.*

22 $\int_0^{\pi/4} \frac{1}{\cos^2 x + 4\sin^2 x}\,dx$ **[Delhi 2017C]**

23 $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\,dx$ **[All India 2015C]**

9 $\int_e^{e^2} \frac{dx}{x \log x}$ [All India 2014]

10 $\int_2^4 \frac{x}{x^2+1} dx$ [All India 2014]

11 $\int_0^1 x e^{x^2} dx$ [Foreign 2014]

27 $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$ [NCERT; Delhi 2016C, 2014C; Foreign 2014]

28 $\int_0^2 (x - [x])\, dx$, where $[x]$ is the greatest integer of x.

29 $\int_0^1 x(1-x)^n dx$ [NCERT]

30 Prove that $\int_0^\pi x f(\sin x)\, dx = \pi \int_0^{\pi/2} f(\cos x)\, dx$.

Directions (Q. Nos. 31-38) *Evaluate the following integrals.*

31 $\int_{-2}^{2} \frac{x^2}{1+5^x} dx$ [All India 2016]

32 $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$ [Delhi 2016C, 2015C]

33 $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$ [Delhi 2014]

34 $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ [All India 2011]

35 $\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ [NCERT]

36 Evaluate $\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9 \sin 2x} dx$. [CBSE 2018]

37 $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ [NCERT]

38 $\int_0^1 \cot^{-1}\{1 - x + x^2\}\, dx$.

39 Show that $\int_0^\pi \log(\tan x)\, dx = 0$.

40 Evaluate $\int_0^\pi \frac{x\, dx}{1+\sin x}$. [NCERT]

41 Evaluate $\int_1^4 \{|x-1| + |x-2| + |x-4|\} dx$. [All India 2017]

42 Evaluate $\int_{-1}^{2} |x^3 - x|\, dx$. [Delhi 2020, 16; NCERT]

43 Evaluate $\int_0^{3/2} |x \cos \pi x|\, dx$. [All India 2016]

LONG ANSWER Type Questions

Directions (Q. Nos. 44-51) *Evaluate the following integrals.*

24 $\int_0^1 x(\tan^{-1} x)^2\, dx$ [NCERT Exemplar]

25 $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}} dx$ [NCERT Exemplar]

26 $\int_0^{\pi/4} \frac{dx}{\cos^3 x \sqrt{2 \sin 2x}}$ [All India 2015]

45 $\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx$

46 $\int_{-1}^{1} \frac{x^3 + |x| + 1}{x^2 + 2|x| + 1} dx$ [NCERT]

47 $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ [Delhi 2014, 2011; All India 2010C]

48 $\int_0^\pi x \log |\sin x|\, dx$ [NCERT Exemplar]

49 $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

50 $\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2\, dx$ [Delhi 2015]

51 $\int_0^{\pi/2} [2 \log(\sin x) - \log(\sin 2x)]\, dx$ [NCERT]

HINTS & SOLUTIONS

1. (b) Let $I = \int_0^{\pi/2} \cos x\, e^{\sin x} dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

When $x \to 0$, then $t \to 0$

and when $x \to \pi/2$, then $t \to 1$

$\therefore \quad I = \int_0^1 e^t dt = [e^t]_0^1 = e^1 - e^0 = e - 1$

2. (a) Let $I = \int_{1/3}^{1} \frac{(x - x^3)^{\frac{1}{3}}}{x^4} dx = \int_{1/3}^{1} \frac{(x^3)^{1/3}\left(\frac{x}{x^3} - \frac{x^3}{x^3}\right)^{1/3}}{x^4} dx$

$= \int_{1/3}^{1} \frac{\left(\frac{1}{x^2} - 1\right)^{1/3}}{x^3} dx$

Put $\frac{1}{x^2} - 1 = t \Rightarrow \frac{-2}{x^3} dx = dt$

$\Rightarrow \quad \frac{1}{x^3} dx = \frac{-1}{2} dt$

When $x = 1$, then $t = 0$ and when $x = \frac{1}{3}$, then $t = 8$

$\therefore \quad I = \frac{-1}{2} \int_8^0 t^{1/3} dt = \frac{-1}{2} \times \left(\frac{3}{4}\right) [t^{4/3}]_8^0$

$= \frac{-3}{8} [0 - (2^3)^{4/3}]$

$= \frac{-3}{8} \times (-16) = 6$

44 $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\, dx$ **[NCERT; Delhi 2011]**

4. (d) Let $I = \int_a^b x f(x)\, dx$ ...(i)

Then, by a property of definite integrals

$$I = \int_a^b (a+b-x) f(a+b-x)\, dx$$

$$= \int_a^b (a+b-x) f(x)\, dx \quad \text{...(ii)}$$

[$\because$ Given $f(a+b-x) = f(x)$]

On adding Eqs. (i) and (ii), we get

$$2I = \int_a^b (a+b) f(x)\, dx$$

$$\Rightarrow \quad I = \frac{a+b}{2} \int_a^b f(x)\, dx$$

5. (c) Let $I = \int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1) dx$

$$\Rightarrow \quad I = \int_{-\pi/2}^{\pi/2} x^3 dx + \int_{-\pi/2}^{\pi/2} x \cos x\, dx + \int_{-\pi/2}^{\pi/2} \tan^5 x dx + \int_{-\pi/2}^{\pi/2} 1 dx$$

$$\Rightarrow \quad I = 0 + 0 + 0 + 2\int_0^{\pi/2} 1 dx.$$

[$\because x^3$, $x \cos x$ and $\tan^5(x)$ are odd functions.]

$$\therefore \quad I = 2[x]_0^{\pi/2} = \frac{2\pi}{2} = \pi$$

6. Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$ ...(i)

$$\Rightarrow \quad I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx \quad \text{...(ii)}$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx$$

$$\Rightarrow 2I = \int_0^a 1\, dx = [x]_0^a \Rightarrow I = \frac{1}{2}[a-0] = \frac{a}{2}$$

7. **Hint** Use the property, $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$.

[Ans. 3]

8. Let $I = \int_{-\pi/3}^{\pi/3} \sin^3 x\, dx$

Here, $f(x) = \sin^3 x$

$\therefore \quad f(-x) = \sin^3(-x) = -\sin x^3 = -f(x)$

Thus, $f(x)$ is an odd function.

$\therefore \quad I = 0 \quad \left[\because \int_{-a}^{a} f(x) dx = 0, \text{if } f(-x) = -f(x)\right]$

9. Let $I = \int_e^{e^2} \frac{dx}{x \log x}$

Now, put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

3. (b) Putting $x = t + c$, we get

$$I = \int_a^b f(c+t)\, dt = \int_a^b f(x+c)\, dx$$

Now, $I = \int_1^2 \frac{dt}{t} = [\log t]_1^2 = \log 2 - \log 1 = \log 2$

[$\because \log 1 = 0$]

10. **Hint** On putting $1 + x^2 = t$, given integral reduces to $\frac{1}{2}\int_5^{17} \frac{dt}{t}$. $\left[\textbf{Ans. } \frac{1}{2} \log\left(\frac{17}{5}\right)\right]$

11. Let $I = \int_0^1 x e^{x^2} dx$

Now, put $x^2 = t \Rightarrow 2x\, dx = dt$

Upper limit when $x = 1$, then $t = 1^2 = 1$

Lower limit When $x = 0$, $t = 0$

$$\therefore \quad I = \int_0^1 e^t \frac{dt}{2} = \frac{1}{2}[e^t]_0^1 = \frac{1}{2}[e^1 - e^0] = \frac{1}{2}(e-1)$$

12. **Hint** On putting $1 + x^2 = t$, given integral reduces to $\frac{1}{2}\int_1^2 \frac{dt}{\sqrt{t}}$. **[Ans.** $\sqrt{2} - 1$**]**

13. **Hint** On putting $\sin x = t$, given integral reduces to $\int_0^1 e^t dt$. **[Ans.** $e - 1$**]**

14. Let $I = \int_0^1 \frac{dx}{e^x + e^{-x}} = \int_0^1 \frac{e^x}{e^{2x} + 1} dx$

Now, put $e^x = t \Rightarrow e^x dx = dt$

Upper limit When $x = 1$, then $t = e^1 = e$

Lower limit When $x = 0$, then $t = e^0 = 1$

$$\therefore \quad I = \int_1^e \frac{dt}{t^2 + 1} = [\tan^{-1} t]_1^e$$

$$= \tan^{-1} e - \tan^{-1} 1 = \tan^{-1} e - \frac{\pi}{4}$$

15. Given, $g(x) = \int_0^x \cos 4t\, dt$

Now, $g(x+\pi) = \int_0^{x+\pi} \cos 4t\, dt$

$$= \int_0^x \cos 4t\, dt + \int_x^{x+\pi} \cos 4t\, dt$$

$$= g(x) + I_1 \text{ (say)} \quad \text{...(i)}$$

Now, $I_1 = \int_x^{x+\pi} \cos 4t\, dt = \left[\frac{\sin 4t}{4}\right]_x^{x+\pi}$

$$= \frac{\sin 4(x+\pi)}{4} - \frac{\sin 4x}{4} = \frac{\sin(4\pi + 4x)}{4} - \frac{\sin 4x}{4}$$

$$= \frac{\sin 4x}{4} - \frac{\sin 4x}{4} = 0$$

Then, Eq. (i) becomes $g(x+\pi) = g(x)$.

Hence proved.

16. Let $I = \int_0^{\pi/2} \frac{\tan^7 x}{\cot^7 x + \tan^7 x} dx$ ...(i)

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\tan^7(\pi/2 - x)}{\cot^7(\pi/2 - x) + \tan^7(\pi/2 - x)} dx$$

Lower limit When $x = e$, then $t = \log e = 1$

Upper limit When $x = e^2$, then $t = \log e^2 = 2\log e = 2$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cot^7 x}{\tan^7 x + \cot^7 x}\, dx \qquad \text{...(ii)}$$

$$\left[\because \tan\left(\frac{\pi}{2} - x\right) = \cot x \text{ and } \cot\left(\frac{\pi}{2} - x\right) = \tan x\right]$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\tan^7 x + \cot^7 x}{\tan^7 x + \cot^7 x}\, dx$$

$$= \int_0^{\pi/2} 1\, dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \Rightarrow \quad I = \frac{\pi}{4}$$

17. Solve as Question 16. $\left[\textbf{Ans. } \frac{\pi}{4}\right]$

18. Let $I = \int_0^a f(x) g(x)\, dx$ ...(i)

$$\Rightarrow I = \int_0^a f(a - x)\, g(a - x)\, dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$$\Rightarrow I = \int_0^a f(x)\{4 - g(x)\}\, dx \qquad \text{...(ii)}$$

$[\because f(x) = f(a - x)$ and $g(x) + g(a - x) = 4$, given$]$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^a 4 f(x)\, dx$$

$$\Rightarrow \qquad I = 2\int_0^a f(x)\, dx$$

19. **Hint** Let $f(x) = \log\left(\frac{2 - x}{2 + x}\right)$, then

$$f(-x) = \log\left(\frac{2 + x}{2 - x}\right) = \log\left(\frac{2 - x}{2 + x}\right)^{-1} = (-1)\log\left(\frac{2 - x}{2 + x}\right)$$

$$= - f(x)$$

Thus, $f(x)$ is an odd function.

20. **Hint** On putting $x^2 = t$, given integral reduces to

$$\frac{1}{2}\int_0^1 \sqrt{\frac{1 - t}{1 + t}}\, dt = \frac{1}{2}\int_0^1 \frac{1 - t}{\sqrt{1 - t^2}} dt$$

$$= \frac{1}{2}\left\{\int_0^1 \frac{dt}{\sqrt{1 - t^2}} - \int_0^1 \frac{t}{\sqrt{1 - t^2}} dt\right\} \quad \left[\text{Ans. } \frac{\pi}{4} + \frac{1}{2}\right]$$

21. **Hint** Write $\tan^3 x = \tan^2 x \cdot \tan x = (\sec^2 x - 1)\tan x$

$= \sec^2 x \tan x - \tan x$, then given integral reduces to

$$2\int_0^{\pi/4} \sec^2 x \cdot \tan x\, dx - 2\int_0^{\pi/4} \tan x\, dx$$

22. Let $I = \int_0^{\pi/4} \frac{dx}{\cos^2 x + 4\sin^2 x} = \int_0^{\pi/4} \frac{\sec^2 x}{1 + 4\tan^2 x}\, dx$

[dividing numerator and denominator by $\cos^2 x$]

Now, put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

Lower limit When $x = 0$, then $t = 0$.

Upper limit When $x = \frac{\pi}{4}$, then $t = 1$.

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$$\therefore \qquad I = \int_0^1 \frac{dt}{1 + 4t^2} = \int_0^1 \frac{dt}{1^2 + (2t)^2}$$

$$= \left[\frac{\tan^{-1}(2t)}{2}\right]_0^1$$

$$= \frac{1}{2}[\tan^{-1} 2 - \tan^{-1} 0] = \frac{1}{2}\tan^{-1} 2$$

23. Let $I = \int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1 - x^2)^{3/2}} dx$

Now, put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$.

Lower limit When $x = 0$, then $\theta = 0$.

Upper limit When $x = \frac{1}{\sqrt{2}}$, then $\theta = \frac{\pi}{4}$.

Now, $I = \int_0^{\pi/4} \frac{\theta \cos\theta}{(1 - \sin^2\theta)^{3/2}}\, d\theta = \int_0^{\pi/4} \frac{\theta\cos\theta}{(\cos^2\theta)^{3/2}}\, d\theta$

$$= \int_0^{\pi/4} \frac{\theta\cos\theta}{\cos^3\theta}\, d\theta \quad I = \int_0^{\pi/4} \underset{\text{I}}{\theta}\, \underset{\text{II}}{\sec^2\theta}\, d\theta$$

Using integrating by parts, we have

$$I = [\theta \tan\theta]_0^{\pi/4} - \int_0^{\pi/4} \tan\theta\, d\theta$$

$$= \frac{\pi}{4}\cdot\tan\frac{\pi}{4} - 0 + [\log|\cos\theta|]_0^{\pi/4}$$

$$= \frac{\pi}{4}\cdot 1 + \log\left(\cos\frac{\pi}{4}\right) - \log(\cos 0)$$

$$= \frac{\pi}{4} + \log\frac{1}{\sqrt{2}} - \log 1 = \frac{\pi}{4} - \frac{1}{2}\log 2$$

24. Let $I = \int_0^1 x(\tan^{-1} x)^2 dx$ ...(i)

Now, put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

Upper limit When $x = 1$, $\tan\theta = 1 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

Lower limit When $x = 0$, $\tan\theta = 0 \Rightarrow \theta = \tan^{-1}(0) = 0$

Now, from Eq. (i), we have

$$I = \int_0^{\pi/4} \tan\theta \cdot \theta^2 \cdot \sec^2\theta\, d\theta = \int_0^{\pi/4} \underset{\text{I}}{\theta^2}(\underset{\text{II}}{\tan\theta \cdot \sec^2\theta})d\theta$$

$$= \left[\theta^2 \cdot \frac{\tan^2\theta}{2}\right]_0^{\pi/4} - \int_0^{\pi/4} 2\theta \cdot \frac{\tan^2\theta}{2} d\theta$$

$$\left[\because \int \tan x \sec^2 x\, dx = \frac{\tan^2 x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi^2}{16}\left(\tan\frac{\pi}{4}\right)^2 - 0\right] - \int_0^{\pi/4} \underset{\text{I}}{\theta}\cdot(\underset{\text{II}}{\sec^2\theta} - 1)d\theta$$

$$[\because \sec^2\theta - \tan^2\theta = 1]$$

$$= \frac{\pi^2}{32} - \left[\theta\cdot(\tan\theta - \theta) - \int 1\cdot(\tan\theta - \theta)d\theta\right]_0^{\pi/4}$$

$$= \frac{\pi^2}{32} - [\theta\cdot\tan\theta - \theta^2]_0^{\pi/4} + \int_0^{\pi/4}(\tan\theta - \theta)d\theta$$

$$= \frac{\pi^2}{32} - \left[\frac{\pi}{4}\tan\frac{\pi}{4} - \frac{\pi^2}{16} - 0\right] + \left[\log|\sec\theta| - \frac{\theta^2}{2}\right]_0^{\pi/4}$$

$$= \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{\pi^2}{16} + \log\sqrt{2} - \frac{\pi^2}{32} - 0 \qquad \left[\because \tan\left(\frac{\pi}{4}\right) = 1\right]$$

$$= \frac{\pi^2 - 4\pi}{16} + \log\sqrt{2}$$

25. **Hint** (i) Let

$$I = \int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}\,dx}{(1-\cos x)^{5/2}} = \int_{\pi/3}^{\pi/2} \frac{\sqrt{1-\cos^2 x}}{(1-\cos x)^{\frac{5}{2}+\frac{1}{2}}}dx$$

[multiplying numerator and denominator by $\sqrt{1-\cos x}$]

$$= \int_{\pi/3}^{\pi/2} \frac{\sin x}{(1-\cos x)^3}dx$$

(ii) Put $1 - \cos x = t$ and simplify it. $\left[\text{Ans. } \frac{3}{2}\right]$

26. Let $I = \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

$$= \int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2(2\sin x\cos x)}}$$

$[\because \sin 2x = 2\sin x \cos x]$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^3 x \cos^{1/2} x \sin^{1/2} x}$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^{7/2} x \sin^{1/2} x}$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^4 x}{\cos^{7/2-4} x \sin^{1/2} x}dx$$

[dividing numerator and denominator by $\cos^4 x$]

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^4 x}{\cos^{-1/2} x \sin^{1/2} x}dx$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2 x(1+\tan^2 x)}{\tan^{1/2} x}dx \qquad [\because \sec^2 x - \tan^2 x = 1]$$

Now, put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

Lower limit When $x = 0$, then $t = \tan 0 = 0$

Upper limit When $x = \frac{\pi}{4}$, then $t = \tan\frac{\pi}{4} = 1$

$$\therefore\ I = \frac{1}{2}\int_0^1 \left(\frac{1+t^2}{t^{1/2}}\right)dt = \frac{1}{2}\int_0^1 (t^{-1/2} + t^{3/2})\,dt$$

$$= \frac{1}{2}\left[2t^{1/2} + \frac{2}{5}t^{5/2}\right]_0^1 = (1)^{1/2} + \frac{1}{5}(1)^{5/2} - 0 = 1 + \frac{1}{5} = \frac{6}{5}$$

27. Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x}dx$

We know that,

$$(\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2\sin x\cos x$$

$$\Rightarrow (\sin x - \cos x)^2 = 1 - \sin 2x$$

$[\because \sin^2 x + \cos^2 x = 1 \text{ and } 2\sin x\cos x = \sin 2x]$

$$\Rightarrow \sin 2x = 1 - (\sin x - \cos x)^2$$

$$\therefore\ I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16(1 - (\sin x - \cos x)^2)}dx$$

Now, put $(\sin x - \cos x) = t$

$$\Rightarrow \cos x + \sin x = \frac{dt}{dx} \qquad \text{...(i)}$$

$$\Rightarrow (\cos x + \sin x)dx = dt$$

Lower limit When $x = 0$, then $t = 0 - \cos 0 = -1$

Upper limit when $x = \frac{\pi}{4}$, then $t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0$

$$\therefore\ I = \int_{-1}^0 \frac{1}{9 + 16(1-t^2)}dt$$

$$= \int_{-1}^0 \frac{1}{9+16-16t^2}dt = \int_{-1}^0 \frac{1}{25-16t^2}dt$$

$$= \frac{1}{16}\int_{-1}^0 \frac{1}{\left(\frac{5}{4}\right)^2 - t^2}dt$$

$$= \frac{1}{16}\cdot\frac{1}{2\cdot\frac{5}{4}}\left[\log\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|\right]_{-1}^0$$

$$\left[\because \int \frac{1}{a^2 - x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|\right]$$

$$= \frac{1}{40}\left[\log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^0 = \frac{1}{40}\left[\log 1 - \log\frac{1}{9}\right]$$

$$\left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$$

$$= \frac{1}{40}[\log 1 - (\log 1 - \log 9)] = \frac{1}{40}\log 9$$

28. Let $I = \int_0^2 (x - [x])\,dx = \int_0^2 x\,dx - \int_0^2 [x]\,dx$

$$= \int_0^2 x\,dx - \left[\int_0^1 0\,dx + \int_1^2 1\,dx\right]$$

$$\left[\because [x] = \begin{cases} 0, \text{ if } 0 \le x < 1 \\ 1, \text{ if } 1 \le x < 2 \end{cases}\right]$$

$$= \left[\frac{x^2}{2}\right]_0^2 - 0 + [x]_1^2 = \frac{2^2}{2} - 0 - [2-1] = 2 - 1 = 1$$

29. Let $I = \int_0^1 x(1-x)^n\,dx \Rightarrow I = \int_0^1 (1-x)\{1-(1-x)\}^n\,dx$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)\,x^n\,dx = \int_0^1 (x^n - x^{n+1})\,dx$$

$$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1 = \left[\frac{1}{n+1} - \frac{1}{n+2}\right] - 0$$

$$= \frac{(n+2)-(n+1)}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}$$

30. Let $I = \int_0^\pi x\,f(\sin x)\,dx \qquad \text{...(i)}$

$$\Rightarrow I = \int_0^\pi (\pi - x)\,f\,[\sin(\pi - x)]\,dx$$

$$= \int_0^{\pi} (\pi - x)\, f(\sin x) dx \quad ...(ii)$$

$$\left[\because \int_0^a f(x) = \int_0^a f(a - x)\, dx\right]$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \pi\, f\,(\sin x)\, dx$$

$$\Rightarrow \quad I = \frac{\pi}{2} \int_0^{\pi} f\,(\sin x)\, dx$$

Now, put $\frac{\pi}{2} - x = t \Rightarrow x = \frac{\pi}{2} - t \Rightarrow dx = -dt$

Upper limit When $x = \pi$, then $t = \frac{-\pi}{2}$.

Lower limit When $x = 0$, then $t = \frac{\pi}{2}$.

$$\therefore\ I = -\frac{\pi}{2} \int_{\pi/2}^{-\pi/2} f\left[\sin\left(\frac{\pi}{2} - t\right)\right] dt = \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} f(\cos t)\, dt$$

$$= \pi \int_0^{\pi/2} f(\cos t)\, dt \ \text{ or } \ \pi \int_0^{\pi/2} f(\cos x) dx$$

$$\left[\because \int_{-a}^{a} f(x) dx = 2\int_0^a f(x) dx\right]$$

31. Let $I = \int_{-2}^{2} \frac{x^2}{1 + 5^x}\, dx \quad ...(i)$

$$= \int_{-2}^{2} \frac{(2 - 2 - x)^2}{1 + 5^{2-2-x}}\, dx \quad \left[\because \int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx\right]$$

$$= \int_{-2}^{2} \frac{x^2}{1 + 5^{-x}}\, dx$$

$$\Rightarrow \quad I = \int_{-2}^{2} \frac{5^x}{5^x + 1} x^2\, dx \quad ...(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{-2}^{2} \left(\frac{1 + 5^x}{5^x + 1}\right) x^2\, dx = \int_{-2}^{2} x^2\, dx$$

$$\Rightarrow 2I = 2\int_0^2 x^2\, dx$$

$$\left[\because x^2 \text{ is even, so } \int_{-2}^{2} x^2\, dx = 2\int_0^2 x^2\, dx\right]$$

$$\Rightarrow \quad I = \left[\frac{x^3}{3}\right]_0^2 = \frac{1}{3}(2^3 - 0) = \frac{8}{3}$$

32. Let $I = \int_0^{\pi/2} \frac{dx}{1 + \sqrt{\tan x}}$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \quad ...(i)$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}}\, dx$$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a - x) dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \quad ...(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$$

$$\Rightarrow\ 2I = \int_0^{\pi/2} 1\, dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

33. **Hint** Use the property, $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$

Here, $a + b = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$

Now, solve as Question 32. $\left[\textbf{Ans. } \frac{\pi}{12}\right]$

34. Solve as Question 32. $\left[\textbf{Ans. } \frac{\pi}{12}\right]$

35. Let $I = \int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) dx$

$$= \int_0^{\pi/2} \log\,(4 + 3\sin x) dx - \int_0^{\pi/2} \log\,(4 + 3\cos x) dx$$

$$\left[\because \log\frac{m}{n} = \log m - \log n\right]$$

$$= \int_0^{\pi/2} \log\left[4 + 3\sin\left(\frac{\pi}{2} - x\right)\right] dx - \int_0^{\pi/2} \log\,(4 + 3\cos x) dx$$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a - x) dx\right]$$

$$= \int_0^{\pi/2} \log\,(4 + 3\cos x) dx - \int_0^{\pi/2} \log\,(4 + 3\cos x) dx$$

$\therefore\ I = 0$

36. Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{9\,(2\sin x\,\cos x) + 16} dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{-9(-2\sin x \cos x) + 16}\, dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{-9(\sin^2 x + \cos^2 x - 2\sin x \cos x - 1) + 16}\, dx$$

$$[\because \sin^2 x + \cos^2 x = 1]$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 9(\sin x - \cos x)^2}\, dx$$

$$[\because a^2 + b^2 - 2ab = (a - b)^2]$$

Put $\quad \sin x - \cos x = t$

$\Rightarrow \quad (\cos x + \sin x) dx = dt$

Also, when, $x = 0$, then $t = -1$

and when, $x = \frac{\pi}{4}$, then $t = 0$

$$\therefore \quad I = \int_{-1}^{0} \frac{dt}{25 - 9t^2} = \frac{1}{9}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{3}\right)^2 - t^2}$$

$$= \frac{1}{9} \times \frac{1}{2 \times \frac{5}{3}} \left[\log \left| \frac{\frac{5}{3} + t}{\frac{5}{3} - t} \right| \right]_{-1}^{0}$$

$$\left[\because \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| + C \right]$$

$$= \frac{1}{30} \left[\log \left| \frac{5 + 3t}{5 - 3t} \right| \right]_{-1}^{0} = \frac{1}{30} \left[\log 1 - \log \left(\frac{2}{8} \right) \right]$$

$$= \frac{1}{30} \left[0 - \log \frac{1}{4} \right] \qquad [\because \log 1 = 0]$$

$$= \frac{1}{30} [-\log 4^{-1}]$$

$$= \frac{1}{30} \log 4 \qquad [\because \log m^n = n \log m]$$

37. Let $I = \int_0^1 \tan^{-1}\left(\frac{2x - 1}{1 + x - x^2}\right) dx$

$$= \int_0^1 \tan^{-1}\left(\frac{x + (x - 1)}{1 - x(x - 1)}\right) dx$$

$$= \int_0^1 \{\tan^{-1} x + \tan^{-1}(x - 1)\}\, dx$$

$$\left[\because \tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A + B}{1 - AB}\right) \right]$$

$$\Rightarrow \quad I = \int_0^1 \{\tan^{-1} x - \tan^{-1}(1 - x)\}\, dx \qquad \text{...(i)}$$

Also, $I = \int_0^1 [\tan^{-1}(1 - x) - \tan^{-1}\{1 - (1 - x)\}]\, dx$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a - x)\, dx \right]$$

$$\Rightarrow \quad I = \int_0^1 [\tan^{-1}(1 - x) - \tan^{-1}(x)]\, dx \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = 0 \Rightarrow I = 0$$

38. Let $I = \int_0^1 \cot^{-1}\{1 - x + x^2\}\, dx = \int_0^1 \tan^{-1}\left\{\frac{1}{1 - x + x^2}\right\} dx$

$$= \int_0^1 \tan^{-1}\left\{\frac{x + (1 - x)}{1 - x + x^2}\right\} dx$$

[adding and subtracting x from numerator]

$$= \int_0^1 \tan^{-1}\left(\frac{x + (1 - x)}{1 - x(1 - x)}\right) dx$$

$$= \int_0^1 \{\tan^{-1} x + \tan^{-1}(1 - x)\}\, dx$$

$$\left[\because \tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A + B}{1 - AB}\right) \right]$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1 - x)\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}\{1 - (1 - x)\}\, dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx \right]$$

$$= \int_0^1 \tan^{-1} x + \int_0^1 \tan^{-1} x\, dx$$

$$= 2\int_0^1 \tan^{-1} x = 2\int_0^1 (\underset{\text{I}}{\tan^{-1} x})\underset{\text{II}}{1}\, dx$$

$$= 2\left\{ [\tan^{-1} x \cdot x]_0^1 - \int_0^1 \frac{1}{1 + x^2} \cdot x\, dx \right\}$$

[using integration by parts]

$$= 2\, [\tan^{-1}(1) - 0] - \int_0^1 \frac{2x}{1 + x^2} dx$$

$$= 2 \cdot \frac{\pi}{4} - [\log(1 + x^2)]_0^1$$

$$= \frac{\pi}{2} - [\log 2 - \log 1] = \frac{\pi}{2} - \log 2 \qquad [\because \log 1 = 0]$$

39. Let $I = \int_0^{\pi} \log(\tan x) dx$

$$= \int_0^{\pi/2} \log(\tan x) dx + \int_{\pi/2}^{\pi} \log(\tan x) dx$$

$$= I_1 + I_2 \qquad \text{...(i)}$$

Now, consider $I_1 = \int_0^{\pi/2} \log(\tan x) dx \qquad \text{...(ii)}$

$$= \int_0^{\pi/2} \log[\tan(\pi/2 - x)]\, dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx \right]$$

$$= \int_0^{\pi/2} \log(\cot x)\, dx \qquad \text{...(iii)}$$

On adding Eqs. (ii) and (iii), we get

$$2I_1 = \int_0^{\pi/2} \log(\tan x)\, dx + \int_0^{\pi/2} \log(\cot x) dx$$

$$= \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)]\, dx$$

$$= \int_0^{\pi/2} \log(\tan x \cdot \cot x) dx = \int_0^{\pi/2} \log 1\, dx = 0$$

$$[\because \tan x \cdot \cot x = 1]$$

$$\Rightarrow \quad I_1 = 0 \qquad [\because \log 1 = 0]$$

and $I_2 = \int_{\pi/2}^{\pi} \log(\tan x) dx \qquad \text{...(iv)}$

$$= \int_{\pi/2}^{\pi} \log\left[\tan\left(\pi + \frac{\pi}{2} - x \right) \right] dx$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^b f(a + b - x) dx \right]$$

$$= \int_{\pi/2}^{\pi} \log\left[\tan\left(\frac{3\pi}{2} - x \right) \right] dx$$

$$= \int_{\pi/2}^{\pi} \log(\cot x)\, dx \qquad \text{...(v)}$$

On adding Eqs. (iv) and (v), we get

$$2I_2 = \int_{\pi/2}^{\pi} [\log(\tan x) + \log(\cot x)]dx$$

$$= \int_{\pi/2}^{\pi} \log|\tan x \cdot \cot x| dx = \int_{\pi/2}^{\pi} \log 1 \, dx = 0 \quad [\because \log 1 = 0]$$

$\Rightarrow I_2 = 0$

$\therefore$ From Eq. (i), we get $I = 0$

40. **Hint** Use $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$ [**Ans.** π]

41. Similar as Example 6. $\left[\textbf{Ans. } \frac{23}{2}\right]$

42. **Hint** Let $I = \int_{-1}^{2} |x^3 - x| dx$.

Again, let $f(x) = x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$

Now, break the given limit at $x = 0, 1$

[$\because$ put $f(x) = 0$, we get $x = 0, 1, -1$]

$$\therefore \quad f(x) = x^3 - x = \begin{cases} \geq 0, \ \forall \ x \in [-1, 0] \\ \leq 0, \ \forall \ x \in [0, 1] \\ \geq 0, \ \forall \ x \in [1, 2] \end{cases}$$

$$\therefore \ I = \int_{-1}^{0} (x^3 - x)dx + \int_0^1 -(x^3 - x)dx + \int_1^2 (x^3 - x)dx$$

$\left[\textbf{Ans. } \frac{11}{4}\right]$

43. Let $I = \int_0^{3/2} |x \cos \pi x| \, dx$

Here, $x \cos \pi x = 0 \Rightarrow x = 0$ or $\cos \pi x = 0$

$$\Rightarrow \qquad x = 0 \ \text{ or } \ \cos \pi x = \cos \frac{\pi}{2} = \cos \frac{3\pi}{2}$$

$$\Rightarrow \qquad x = 0 \ \text{ or } \ \pi x = \frac{\pi}{2} \ \text{ or } \ \pi x = \frac{3\pi}{2}$$

$$\Rightarrow \qquad x = 0 \ \text{ or } \ x = \frac{1}{2} \ \text{ or } \ \pi x = \frac{3}{2}$$

For $0 < x < \frac{1}{2}$; $x > 0$ and $\cos \pi x > 0$, then

$\therefore \ x \cos \pi x > 0$

For $\frac{1}{2} < x < \frac{3}{2}$; $x > 0$ and $\cos \pi x < 0$, then

$\therefore \ x \cos \pi x < 0$

$$\therefore \ |x \cos \pi x| = \begin{cases} x \cos \pi x, & \text{for } 0 < x \leq \frac{1}{2} \\ -x \cos \pi x, & \text{for } \frac{1}{2} \leq x \leq \frac{3}{2} \end{cases}$$

$$\text{Now, } \int_0^{3/2} |x \cos \pi x| \, dx = \int_0^{1/2} x \cos \pi x \, dx + \int_{1/2}^{3/2} -(x \cos \pi x)\, dx$$

$$= \left[x \frac{\sin \pi x}{\pi} - \int \frac{\sin \pi x}{\pi} dx\right]_0^{1/2} - \left[x \frac{\sin \pi x}{\pi} - \int \frac{\sin \pi x}{\pi} dx\right]_{1/2}^{3/2}$$

$$= \left[\frac{x}{\pi} \sin \pi x + \frac{1}{\pi^2} \cos \pi x\right]_0^{1/2} - \left[\frac{x}{\pi} \sin \pi x + \frac{1}{\pi^2} \cos \pi x\right]_{1/2}^{3/2}$$

$$= \left[\left(\frac{1}{2\pi} \sin \frac{\pi}{2} + \frac{1}{\pi^2} \cos \frac{\pi}{2}\right) - \left(0 + \frac{1}{\pi^2} \cos 0\right)\right]$$

$$- \left[\left(\frac{3}{2\pi} \sin \frac{3\pi}{2} + \frac{1}{\pi^2} \cos \frac{3\pi}{2}\right) - \left(\frac{1}{2\pi} \sin \frac{\pi}{2} + \frac{1}{\pi^2} \cos \frac{\pi}{2}\right)\right]$$

$$= \left[\frac{1}{2\pi} - \frac{1}{\pi^2} - \frac{3}{2\pi}(-1) + \frac{1}{2\pi}\right]$$

$$= \frac{1}{\pi} - \frac{1}{\pi^2} + \frac{3}{2\pi} = \frac{5}{\pi} - \frac{1}{\pi^2}$$

44. Let $I = \int_0^{\pi/2} \sin 2x \tan^{-1} (\sin x) \, dx$

$$= \int_0^{\pi/2} 2 \sin x \cos x \tan^{-1} (\sin x) \, dx$$

[$\because \sin 2x = 2 \sin x \cos x$]

Now, put $\sin x = t \Rightarrow \cos x \, dx = dt$

Lower limit When $x = 0$, then $t = 0$

Upper limit When $x = \frac{\pi}{2}$, then $t = \sin \frac{\pi}{2} = 1$

$$\text{Now, } I = \int_0^1 2t \tan^{-1} t \, dt = 2\int_0^1 \underset{\text{II}}{t} \cdot \underset{\text{I}}{\tan^{-1} t} \, dt$$

$$= 2\left(\left[(\tan^{-1} t)\frac{t^2}{2}\right]_0^1 - \int_0^1 \frac{1}{1+t^2} \cdot \frac{t^2}{2} dt\right)$$

[using integration by parts]

$$= 2\left[\frac{\tan^{-1} 1}{2} - 0\right] - \int_0^1 \frac{t^2}{1+t^2} dt$$

$$= 2\left(\frac{\pi}{8}\right) - \int_0^1 \frac{(1+t^2) - 1}{1+t^2} dt$$

[adding and subtracting 1 from numerator]

$$= \frac{\pi}{4} - \int_0^1 \left(1 - \frac{1}{1+t^2}\right) dt = \frac{\pi}{4} - [t - \tan^{-1} t]_0^1$$

$$= \frac{\pi}{4} - [1 - \tan^{-1} 1 - (0 - 0)]$$

$$= \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1$$

45. **Hint** (i) Put $\cos x = t$.

(ii) Use the integral of the form $\int \frac{(px+q)}{ax^2 + bx + c} dx$

$\left[\textbf{Ans. } \log\left(\frac{9}{8}\right)\right]$

46. **Hint** Let $I = \int_{-1}^{1} \frac{x^3 + |x| + 1}{x^2 + 2|x| + 1} dx$

$$= \int_{-1}^{1} \frac{x^3}{x^2 + 2|x| + 1} dx + \int_{-1}^{1} \frac{|x| + 1}{x^2 + 2|x| + 1} dx$$

$$= 0 + 2\int_0^1 \frac{|x|+1}{x^2+2|x|+1}dx$$

$$\left[\because \int_{-1}^1 f(x)dx = \begin{cases} 2\int_0^1 f(x)dx, & \text{if } f(x) \text{ is even function} \\ 0, & \text{if } f(x) \text{ is odd function} \end{cases}\right]$$

$$= 2\int_0^1 \frac{(x+1)}{x^2+2x+1}dx \qquad [\because |x| = x, \text{ if } x \geq 0]$$

$$= 2\int_0^1 \frac{(x+1)}{(x+1)^2}dx \quad [\textbf{Ans. } 2\log 2]$$

47. Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}dx$ ...(i)

Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$, we get

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\cos x \sin x}{\cos^4 x + \sin^4 x}dx \qquad \text{...(ii)}$$

$$\left[\because \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta \text{ and } \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta\right]$$

On adding Eqs. (i) and (ii), we get

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\cos x \sin x}{\sin^4 x + \cos^4 x}dx$$

$$\Rightarrow \quad I = \frac{\pi}{4}\int_0^{\pi/2} \frac{\sin x \cos x}{(\sin^2 x)^2 + (1-\sin^2 x)^2}dx$$

$$[\because \cos^2\theta = 1 - \sin^2\theta]$$

Now, put $\sin^2 x = t \Rightarrow 2\sin x \cos x\, dx = dt$

$$\Rightarrow \quad \sin x \cos x\, dx = \frac{dt}{2}$$

Lower limit When $x = 0$, then $t = \sin^2 0 = 0$

Upper limit When $x = \frac{\pi}{2}$, then $t = \sin^2\frac{\pi}{2} = 1$

Now, $I = \frac{\pi}{4}\int_0^1 \frac{1}{t^2+(1-t)^2}\frac{dt}{2}$

$$\Rightarrow \quad I = \frac{\pi}{8}\int_0^1 \frac{1}{t^2+(1+t^2-2t)}dt$$

$$\Rightarrow \quad I = \frac{\pi}{8}\int_0^1 \frac{1}{2t^2-2t+1}dt \Rightarrow I = \frac{\pi}{16}\int_0^1 \frac{1}{t^2-t+\frac{1}{2}}dt$$

$$\Rightarrow \quad I = \frac{\pi}{16}\int_0^1 \frac{1}{t^2-t+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+\frac{1}{2}}dt$$

$$\left[\text{adding and subtracting by } \left(\frac{1}{2}\right)^2 \text{ from denominator}\right]$$

$$\Rightarrow \quad I = \frac{\pi}{16}\int_0^1 \frac{1}{\left(t-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}dt$$

$$\Rightarrow \quad I = \frac{\pi}{16}\cdot\frac{1}{(1/2)}\left[\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1 = \frac{\pi}{8}\left[\tan^{-1}2\left(t-\frac{1}{2}\right)\right]_0^1$$

$$\left[\because \int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$\Rightarrow \quad I = \frac{\pi}{8}\left[\tan^{-1}2\left(1-\frac{1}{2}\right)-\tan^{-1}2\left(0-\frac{1}{2}\right)\right]$$

$$\Rightarrow \quad I = \frac{\pi}{8}[\tan^{-1}(1)-\tan^{-1}(-1)]$$

$$\left[\because \tan^{-1}(-1) = -\tan^{-1}(1) = -\frac{\pi}{4}\right]$$

$$\Rightarrow \quad I = \frac{\pi}{8}\left[\frac{\pi}{4}+\frac{\pi}{4}\right] = \frac{\pi^2}{16}$$

48. Let $I = \int_0^{\pi} x \log|\sin x|\,dx$ ...(i)

$$\Rightarrow \quad I = \int_0^{\pi}(\pi - x)\log|\sin(\pi - x)|dx$$

$$= \int_0^{\pi}(\pi - x)\log|\sin x|\,dx \qquad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \pi\int_0^{\pi}\log|\sin x|\,dx \qquad \text{...(iii)}$$

$$\Rightarrow \quad 2I = 2\pi\int_0^{\pi/2}\log|\sin x|\,dx$$

$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x)\right]$$

$$\Rightarrow \quad I = \pi\int_0^{\pi/2}\log|\sin x|\,dx \qquad \text{...(iv)}$$

$$\Rightarrow \quad I = \pi\int_0^{\pi/2}\log|\sin(\pi/2 - x)|dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \pi\int_0^{\pi/2}\log|\cos x|\,dx \qquad \text{...(v)}$$

On adding Eqs. (iv) and (v), we get

$$2I = \pi\int_0^{\pi/2}(\log|\sin x| + \log|\cos x|)dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi/2}\log|\sin x \cos x|\,dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi/2}\log\left|\frac{2\sin x\cos x}{2}\right|dx$$

[multiplying by 2 from numerator and denominator]

$$\Rightarrow \quad 2I = \pi\int_0^{\pi/2}(\log|\sin 2x| - \log 2)dx$$

$$2I = \pi\int_0^{\pi/2}\log|\sin 2x|\,dx - \pi\int_0^{\pi/2}\log 2\,dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi/2}\log|\sin 2x|\,dx - \pi\log 2[x]_0^{\pi/2}$$

Now, put $2x = t \Rightarrow dx = \frac{1}{2}dt$

Lower limit When $x = 0$, then $t = 0$

Upper limit When $x = \frac{\pi}{2}$, then $t = \pi$

$\therefore \quad 2I = \frac{\pi}{2}\int_0^{\pi} \log|\sin t| dt - \frac{\pi^2}{2}\log 2$

$\Rightarrow \quad 2I = \frac{\pi}{2}\int_0^{\pi} \log|\sin x|\, dx - \frac{\pi^2}{2}\log 2$

$\Rightarrow \quad 2I = I - \frac{\pi^2}{2}\log 2$ [from Eq. (iii)]

$\therefore \quad I = -\frac{\pi^2}{2}\log 2 = \frac{\pi^2}{2}\log\left(\frac{1}{2}\right)$

49. **Hint** Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$ ...(i)

Now, put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

Lower limit When $x = 0$, then $\tan\theta = 0 \Rightarrow \theta = 0$

Upper limit When $x = 1$, then $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

Now, $I = \int_0^{\pi/4} \frac{\log|1+\tan\theta|}{(1+\tan^2\theta)} \sec^2\theta\, d\theta$

$= \int_0^{\pi/4} \frac{\log|1+\tan\theta|}{\sec^2\theta} \sec^2\theta\, d\theta$ $[\because \sec^2\theta = 1+\tan^2\theta]$

$I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$ ...(ii)

$\left[\textbf{Ans. } \frac{\pi}{8}\log 2\right]$

50. Let $I = \int_{-\pi}^{\pi} (\cos ax - \sin bx)^2\, dx$

$= 2\int_0^{\pi} (\cos ax - \sin bx)^2\, dx$

$\left[\because \int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}\right.$ and the given integrand is an even function $\Big]$

$= 2\int_0^{\pi} [\cos^2 ax + \sin^2 bx - 2\sin bx \cos ax] dx$

$= 2\int_0^{\pi} [\cos^2 ax + \sin^2 bx - \{\sin(bx+ax) + \sin(bx-ax)\}] dx$

$[\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)]$

$= 2\left[\int_0^{\pi}\cos^2 ax\, dx + \int_0^{\pi}\sin^2 bx\, dx - \int_0^{\pi}\sin(b+a)x\, dx - \int_0^{\pi}\sin(b-a)x\, dx\right]$

$= 2\left[\int_0^{\pi}\left(\frac{1+\cos 2ax}{2}\right)dx + \int_0^{\pi}\left(\frac{1-\cos 2bx}{2}\right)dx - \int_0^{\pi}\sin(b+a)x\, dx - \int_0^{\pi}\sin(b-a)x\, dx\right]$

$= 2\left[\int_0^{\pi}\left(\frac{1}{2}+\frac{\cos 2ax}{2}\right)dx + \int_0^{\pi}\left(\frac{1}{2}-\frac{\cos 2bx}{2}\right)dx - \int_0^{\pi}\sin(b+a)x\, dx - \int_0^{\pi}\sin(b-a)x\, dx\right]$

$= 2\left[\int_0^{\pi}\left(\frac{1}{2}+\frac{1}{2}\right)dx + \frac{1}{2}\int_0^{\pi}\cos 2ax\, dx - \frac{1}{2}\int_0^{\pi}\cos 2bx\, dx - \int_0^{\pi}\sin(b+a)x\, dx - \int_0^{\pi}\sin(b-a)x\, dx\right]$

$= 2\left[[x]_0^{\pi} + \frac{1}{2}\left[\frac{\sin 2ax}{2a}\right]_0^{\pi} - \frac{1}{2}\left[\frac{\sin 2bx}{2b}\right]_0^{\pi} + \left[\frac{\cos(b+a)x}{(b+a)}\right]_0^{\pi} + \left[\frac{\cos(b-a)x}{(b-a)}\right]_0^{\pi}\right]$

$= 2\left[\pi + \frac{1}{4a}[\sin 2a\pi] - \frac{1}{4b}[\sin 2b\pi] + \frac{1}{b+a}\cdot[\cos(b+a)\pi - 1] + \frac{1}{(b-a)}[\cos(b-a)\pi - 1]\right]$

$[\because \cos 0 = 1]$

$= 2\left[\pi + \frac{1}{4a}\sin 2a\pi - \frac{1}{4b}\sin 2b\pi + \frac{1}{b+a}\cos(b+a)\pi - \frac{1}{b+a} + \frac{1}{b-a}\cos(b-a)\pi - \frac{1}{b-a}\right]$

51. Let $I = \int_0^{\pi/2} (2\log|\sin x| - \log|\sin 2x|)\, dx$

$= \int_0^{\pi/2} [\log(\sin^2 x) - \log(\sin 2x)] dx$

$[\because m\log n = \log n^m]$

$= \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{\sin 2x}\right) dx$ $\left[\because \log m - \log n = \log\frac{m}{n}\right]$

$= \int_0^{\pi/2} \log\left(\frac{\sin^2 x}{2\sin x\cos x}\right) dx$ $[\because \sin^2 x = 2\sin x\cos x]$

$= \int_0^{\pi/2} \log\left(\frac{\tan x}{2}\right) dx = \int_0^{\pi/2} [\log(\tan x) - \log 2]\, dx$

$= \int_0^{\pi/2} \log(\tan x)\, dx - \int_0^{\pi/2} \log 2\, dx$

$= \int_0^{\pi/2} \log(\tan x) dx - \log 2\int_0^{\pi/2} 1\, dx$

$\Rightarrow I = I_1 - (\log 2)[x]_0^{\pi/2} = I_1 - \left(\frac{\pi}{2} - 0\right)\log 2$ ...(i)

where, $I_1 = \int_0^{\pi/2} \log(\tan x)\, dx$ ...(ii)

Now, $I_1 = \int_0^{\pi/2} \log\left[\tan\left(\frac{\pi}{2} - x\right)\right] dx$

$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$

$\Rightarrow \quad I_1 = \int_0^{\pi/2} \log(\cot x)\, dx$ ...(iii)

On adding Eqs. (ii) and (iii), we get

$2I_1 = \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)] dx$

$= \int_0^{\pi/2} \log(\tan x \cot x)\, dx$

$[\because \log m + \log n = \log(mn)]$

$= \int_0^{\pi/2} \log 1\, dx = 0$ $\left[\because \tan x = \frac{1}{\cot x}\right]$

$\Rightarrow \quad I_1 = 0$

From Eq. (i), we get $I = 0 - \frac{\pi}{2}\log 2$

$\Rightarrow \quad I = -\frac{\pi}{2}\log 2$

SUMMARY

- **Indefinite Integral** Let $F(x)$ and $f(x)$ be two functions connected together such that $\frac{d}{dx}F(x) = f(x)$, then $F(x)$ is called **integral of $f(x)$** or **indefinite integral** or **anti-derivative.**
- **Properties of Indefinite Integrals**
  (i) $\frac{d}{dx}\int f(x)dx = f(x)$ and $\int f'(x)\,dx = f(x) + C$
  (ii) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.
  (iii) $\int \{f(x) \pm g(x)\}\,dx = \int f(x)dx \pm \int g(x)dx$
  (iv) $\int k \cdot f(x)dx = k \cdot \int f(x)dx$, where k is any non-zero real number.
  (v) $\int [k_1f_1(x) + k_2f_2(x) + \ldots + k_nf_n(x)]dx$
  $$= k_n\int f_1(x)dx + k_2\int f_2(x)dx + \ldots + k_n\int f_n(x)dx$$
- **Methods of Integration**
  (i) **Integration by Substitutions** The method of reducing a given integral into one of the standard integrals by a proper substitution is called method of substitution.
  To evaluate an integral of the type $\int f\{g(x)\}\,g'(x)dx$, we substitute $g(x) = t$, so that $g'(x)dx = dt$.
  (ii) **Integration by Partial Fractions** Suppose the given integral is of the form $\int \frac{P(x)}{Q(x)}dx$, where $P(x)$ and $Q(x)$ are polynomials in x, $Q(x) \neq 0$ and degree of $P(x)$ is less than degree of $Q(x)$.
  Then, to solve such integrals, firstly take the given integrand $\frac{P(x)}{Q(x)}$ and decompose it into suitable partial fraction form and then integrate.
  (iii) **Integration by Parts** Let u and v be two differentiable functions of a single variable x, then the integral of the product of two functions is given by
  $$\int uv\,dx = u\int v\,dx - \int\left(\frac{d}{dx}(u)\int v\,dx\right)dx.$$
- **Some Special Types of Integrals**
  (i) Integral of the types $\int \frac{dx}{ax^2 + bx + c}$ or $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ can be transformed into standard form by expressing
  $$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$
  $$= a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$
  (ii) Integral of the types $\int \frac{(px+q)}{ax^2 + bx + c}dx$
  or $\int \frac{(px+q)\,dx}{\sqrt{ax^2 + bx + c}}$
  transformed into standard form by expressing
  $px + q = A\frac{d}{dx}(ax^2 + bx + c) + B = A(2ax + b) + B$, where A and B are determined by comparing coefficients on both sides.
  (iii) Integral of the type $\int e^x\,\{f(x) + f'(x)\}\,dx$ can be evaluated by using the formula
  $$\int e^x\,\{f(x) + f'(x)\}\,dx = e^x f(x) + C.$$
- **Definite Integral** An integral of the form $\int_a^b f(x)dx$ is known as definite integral and is given by $\int_a^b f(x)dx = g(b) - g(a)$, where $f(x)$ is derivative of $g(x)$ or $g(x)$ is an anti-derivative of $f(x)$. Here, a and b are called **lower** and **upper limits** of definite integral.
- **Properties of Definite Integrals**
  (i) $\int_a^b f(x)dx = \int_a^b f(t)dt$
  (ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$
  (iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
  (iv) $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$
  (v) $\int_0^a f(x)dx = \int_0^a f(a - x)dx$ [particular case of property (iv)]
  (vi) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a - x)dx$
  (vii) $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$
  (viii) $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f \text{ is an even function, i.e. } f(-x) = f(x) \\ 0, & \text{if } f \text{ is an odd function, i.e. } f(-x) = -f(x) \end{cases}$

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1 If $\frac{d}{dx}\{f(x)\} = \frac{1}{1+x^2}$, then $\frac{d}{dx}\{f(x^3)\}$ is equal to

(a) $\frac{3x}{1+x^3}$ (b) $\frac{3x^2}{1+x^6}$

(c) $\frac{-6x^5}{(1+x^6)^2}$ (d) $\frac{-6x^5}{1+x^6}$

2 $\int x^2 e^{x^3} dx$ is equal to **[NCERT]**

(a) $\frac{1}{3}e^{x^3} + C$ (b) $\frac{1}{3}e^{x^2} + C$

(c) $\frac{1}{2}e^{x^3} + C$ (d) $\frac{1}{2}e^{x^2} + C$

3 $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$ is equal to **[NCERT]**

(a) $-\cot(ex^x) + C$ (b) $\tan(xe^x) + C$

(c) $\tan(e^x) + C$ (d) $\cot(e^x) + C$

4 Let $f(x) = \frac{\sin^2 \pi x}{1+\pi^x}$. Then, $\int [f(x) + f(-x)]\, dx$ is equal to

(a) 0 (b) $x + C$

(c) $\frac{x}{2} - \frac{\sin 2\pi x}{4\pi} + C$ (d) $\frac{x}{2} - \frac{\cos \pi x}{2\pi} + C$

5 $\int \frac{1}{x^2 + 2x + 2} dx$ is equal to **[NCERT]**

(a) $x \tan^{-1}(x+1) + C$ (b) $\tan^{-1}(x+1) + C$

(c) $(x+1)\tan^{-1} x + C$ (d) $\tan^{-1} x + C$

6 $\int \frac{1}{\sqrt{9x - 4x^2}} dx$ is equal to **[NCERT]**

(a) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$ (b) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right) + C$

(c) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$ (d) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right) + C$

20 Given, $\int e^x(\tan x + 1)\sec x\, dx = e^x f(x) + C$. Write $f(x)$ satisfying above.

21 Evaluate $\int \frac{2}{1+\cos 2x} dx$.

22 Evaluate $\int \frac{x + \cos 6x}{3x^2 + \sin 6x} dx$.

[**Hint** Put $3x^2 + \sin 6x = t$]

23 Evaluate $\int_0^2 \sqrt{4 - x^2}\, dx$.

24 Evaluate $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$.

7 If $\int_0^1 \frac{e^t}{1+t} dt = a$, then $\int_0^1 \frac{e^t}{(1+t)^2} dt$ is equal to **[NCERT Exemplar]**

(a) $a - 1 + \frac{e}{2}$ (b) $a + 1 - \frac{e}{2}$

(c) $a - 1 - \frac{e}{2}$ (d) $a + 1 + \frac{e}{2}$

8 The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ is **[NCERT]**

(a) 1 (b) zero

(c) -1 (d) $\frac{\pi}{4}$

9 $\int_0^{\pi/2} \sqrt{1 - \sin 2x}\, dx$ is equal to **[NCERT Exemplar]**

(a) $2\sqrt{2}$ (b) $2(\sqrt{2} + 1)$

(c) 2 (d) $2(\sqrt{2} - 1)$

VERY SHORT ANSWER Type Questions

10 Evaluate $\int (4e^{3x} - 2) dx$.

11 Evaluate $\int \frac{\sec^2(\log x)}{x} dx$.

12 Evaluate $\int (\text{cosec}^2 x - \cot x) e^x\, dx$.

13 Evaluate $\int \frac{\log(\sin x)}{\tan x} dx$.

14 Evaluate $\int \sec^2(7 - 4x) dx$. **[All India 2010]**

15 Evaluate $\int \sec x^\circ \tan x^\circ\, dx$. $\left[\textbf{Hint} \text{ Write } x^\circ = \frac{\pi x}{180}\right]$

16 Evaluate $\int \frac{x^3 - 1}{x^2} dx$. **[Delhi 2010C]**

17 Evaluate $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$.

18 Evaluate $\int \frac{dx}{x^2 + 16}$. **[Delhi 2011]**

19 Evaluate $\int (1 - x)\sqrt{x}\, dx$. **[Delhi 2012]**

36 Evaluate $\int_0^\pi \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2}\right) dx$.

$\left[\textbf{Hint} \cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right]$ **[NCERT]**

37 Evaluate $\int_0^2 x\sqrt{x+2}\, dx$.

[**Hint** Put $x + 2 = t^2$] **[NCERT]**

SHORT ANSWER Type II Questions

38 Evaluate $\int \frac{\sin(x-a)}{\sin(x+a)} dx$.

[**Hint** Put $x + a = t$] **[Delhi 2013]**

25 Evaluate $\int_0^1 \frac{dx}{4-x^2}$. [All India 2014]

26 Evaluate $\int_{-1}^{1} \sin^5 x \cos^4 x \, dx$. [NCERT]

27 Evaluate $\int \frac{\sin^6 x}{\cos^8 x} dx$. [All India 2014C]

28 Evaluate $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$. [Delhi 2012C, 2011]

SHORT ANSWER Type I Questions

29 Evaluate $\int e^{3\log x} (x^4+1)^{-1} dx$.

$\left[\textbf{Hint } I = \int \frac{x^3}{(x^4+1)} dx\right]$

30 Evaluate $\int \frac{x}{x^4-1} dx$. [NCERT Exemplar]

$\left[\textbf{Hint } \int \frac{x}{x^4-1} dx = \int \frac{x}{(x^2)^2-1} dx, \text{ put } x^2 = t\right]$

31 Evaluate $\int x\sqrt{x^4-1}\, dx$.

32 Evaluate $\int \sqrt{1+\sin x}\, dx$. [NCERT Exemplar]

$\left[\textbf{Hint } 1+\sin x = \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2\right]$

33 Evaluate $\int \sqrt{\tan x}(1+\tan^2 x) dx$.

34 Evaluate $\int \frac{\sin^4 x + \cos^4 x}{\sin^3 x + \cos^3 x} dx$.

35 Evaluate $\int_2^8 |x-5| dx$.

39 Evaluate $\int \frac{\sin 2x}{(a+b\cos x)^2} dx$.

[**Hint** $\sin 2x = 2\sin x \cos x$ and put $a + b\cos x = t$]

40 Evaluate $\int \sin^4 x \, dx$. [All India 2011]

$\left[\textbf{Hint} \sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2 \text{ and } \cos^2 2x = \frac{1+\cos 4x}{2}\right]$

41 Evaluate $\int \sin 4x \cdot \cos 3x \, dx$. [All India 2011]

42 Evaluate $\int \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^3} dx$. [Delhi 2014C]

$\left[\textbf{Hint } \text{Use } \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \text{ and simplify it, then put } \cos\frac{x}{2} + \sin\frac{x}{2} = t\right]$

43 Evaluate $\int \frac{5x-2}{1+2x+3x^2} dx$. [Delhi 2013]

$\left[\textbf{Hint } \text{Let } 5x-2 = A\frac{d}{dx}(1+2x+3x^2) + B\right]$

44 Evaluate $\int \frac{2x^2+1}{x^2(x^2+4)} dx$. [Delhi 2013]

[**Hint** Put $x^2 = t$ and then use partial fractions]

45 Evaluate $\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$. [All India 2013]

$\left[\textbf{Hint } \text{Write given integral as } \int \frac{(x+1)}{\sqrt{x^2+2x+3}} dx + \int \frac{1}{\sqrt{x^2+2x+3}} dx\right]$

46 Evaluate $\int \frac{x+2}{\sqrt{4x-x^2}} dx$. [All India 2013]

$\left[\textbf{Hint } \text{Let } x+2 = A\frac{d}{dx}(4x-x^2) + B\right]$

47 Evaluate $\int x \tan^{-1} x \, dx$. [Delhi 2011]

48 Evaluate $\int \sec^3 x \, dx$.

[**Hint** $\int \sec^3 x \, dx = \int \sec^2 x \cdot \sec x \, dx$]

49 Evaluate $\int \frac{xe^{2x}}{(1+2x)^2} dx$.

50 Evaluate $\int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$. [NCERT Exemplar]

51 Evaluate $\int \frac{dx}{x(x^5+3)}$. [All India 2013]

[**Hint** Multiply numerator and denominator by x^4]

52 Evaluate $\int \left(\frac{1+\sin x}{1+\cos x}\right) e^x dx$. [All India 2012C]

$\left[\textbf{Hint } 1+\sin x = 1 + 2\sin\frac{x}{2}\cos\frac{x}{2} \text{ and } 1+\cos x = 2\cos^2\frac{x}{2}\right]$

61 Evaluate $\int \frac{(3x+1)}{(x+1)^2(x+3)} dx$. [Delhi 2013C]

62 Evaluate $\int \frac{x^2+x+1}{(x+1)^2(x+2)} dx$. [All India 2017C; Delhi 2014C]

63 Evaluate $\int \frac{x^2}{x^4-x^2+12} dx$. [NCERT Exemplar]

$\left[\textbf{Hint } \int \frac{x^2}{x^4-x^2+12} dx = \int \frac{x^2}{(x^2-4)(x^2-3)} dx\right]$

64 Evaluate $\int [\sin(\log x) + \cos(\log x)] dx$.

[**Hint** Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t \, dt$

$\therefore I = \int e^t (\sin t + \cos t) dt$]

65 Evaluate $\int \frac{\log x}{(1+\log x)^2} dx$.

53 Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2+4}}$.

54 Evaluate $\int \frac{1}{x^4+1}\,dx$

$\left[\text{Hint } \int \frac{dx}{x^4+1} = \frac{1}{2}\int \frac{2dx}{x^4+1} = \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}dx\right]$

55 Evaluate $\int \frac{x}{x^4+x^2+1}\,dx$. [**Hint** Put $x^2 = t$]

56 Evaluate $\int \sqrt{\frac{x}{a^3-x^3}}\,dx$. **[Delhi 2016]**

$\left[\text{Hint } \int \sqrt{\frac{x}{a^3-x^3}}\,dx = \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2-(x^{3/2})^2}}\,dx, \text{ put } x^{3/2}=t\right]$

57. Evaluate $\sqrt{\frac{1+x}{x}}\,dx$.

$\left[\text{Hint } \int \sqrt{\frac{1+x}{x}}\,dx = \int \sqrt{\frac{1+x}{x}}\cdot\sqrt{\frac{1+x}{1+x}}\,dx = \int \frac{1+x}{\sqrt{x^2+x}}dx\right]$

58 Evaluate $\int \frac{dx}{\sin x + \sqrt{3}\cos x}\,dx$.

59 Evaluate $\int \frac{5x}{(x+1)(x^2+9)}\,dx$. **[NCERT]**

60 Evaluate $\int \frac{x^2+1}{(x^2+4)(x^2+25)}\,dx$. **[Delhi 2013]**

75 Evaluate $\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$. **[Foreign 2014]**

76 Evaluate $\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$. **[NCERT; All India 2011]**

77 Evaluate $\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}\,dx$. **[All India 2011]**

78 Evaluate $\int_0^1 \frac{x^4+1}{x^2+1}\,dx$. **[All India 2011C]**

$\left[\text{Hint } \int_0^1 \frac{x^4+1}{x^2+1}dx = \int_0^1 \frac{(x^4-1)+2}{x^2+1}dx = \int_0^1 \frac{(x^2-1)(x^2+1)+2}{(x^2+1)}dx\right]$

79 Evaluate $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right)e^{2x}dx$. **[NCERT, CBSE 2020]**

80 Evaluate $\int_{1/3}^{1} \frac{(x-x^3)^{1/3}}{x^4}dx$. **[NCERT]**

$\left[\text{Hint Put } \log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt, \text{ so } I = \int \frac{e^t t\,dt}{(1+t)^2}\right]$

66 Evaluate $\int \sin(\log x)\,dx$.

[**Hint** Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$\therefore\ I = \int e^t \sin t\,dt$, apply integration by parts]

67 Evaluate $\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\,d\theta$.

68 Evaluate $\int \frac{\sin x}{\sin 4x}\,dx$.

69 Evaluate $\int \tan^8 x \sec^4 x\,dx$. **[NCERT Exemplar]**

70 Evaluate $\int x\sin^{-1} x\,dx$. **[Delhi 2016C]**

71 Evaluate $\int \frac{\sqrt{x}}{x+1}\,dx$. $\left[\text{Hint } \int \frac{\sqrt{x}}{x+1}dx = \int \frac{x\,dx}{\sqrt{x}(x+1)}\right]$

72. Evaluate $\int_0^{\pi} x\sin x\cos^2 x\,dx$.

[**Hint** Use

$\int_0^{\pi} x\sin x\cos^2 x\,dx = (\pi - x)\sin(\pi - x)\cos(\pi - x)\,dx$]

73 Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$.

[All India 2017C, 2013, 2012; Delhi 2017, 2011C]

[**Hint** $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$]

74 Evaluate $\int_0^4 (|x|+|x-2|+|x-4|)\,dx$. **[Delhi 2013]**

87 Evaluate $\int_0^{3/2} |x\sin \pi x|\,dx$. **[Delhi 2017]**

LONG ANSWER Type Questions

88 Evaluate $\int x(\log x)^2 dx$. **[NCERT]**

[**Hint** Take $(\log x)^2$ as Ist function and x as IInd function]

89 Evaluate $\int \frac{3x+5}{x^3-x^2-x+1}dx$. **[NCERT]**

90 Evaluate $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$. **[Delhi 2012]**

91 Evaluate $\int_{-\pi/4}^{\pi/4} \log(\cos x + \sin x)dx$. **[NCERT Exemplar]**

[**Hint** Use $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$]

92 Prove that $\int_0^{\pi} \frac{x}{1+\cos\alpha\sin x}\,dx = \frac{\pi\alpha}{\sin\alpha}$.

79 Evaluate $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx.$ **[NCERT, CBSE 2020]**

80 Evaluate $\int_{1/3}^{1} \frac{(x - x^3)^{1/3}}{x^4} dx.$ **[NCERT]**

$\left[\textbf{Hint } \frac{(x - x^3)^{1/3}}{x^4} = \frac{\left[x^3\left(\frac{1}{x^2} - 1\right)\right]^{1/3}}{x^4} = \frac{(x^3)^{1/3}\left(\frac{1}{x^2} - 1\right)^{1/3}}{x^4} = \frac{\left(\frac{1}{x^2} - 1\right)^{1/3}}{x^3}. \text{ Now, put } \frac{1}{x^2} - 1 = t\right]$

81 Evaluate $\int_0^{\pi/2} \log \sin x \, dx.$ **[NCERT]**

82 Evaluate $\int_2^4 \frac{(x^2 + x)}{\sqrt{2x + 1}} dx.$

[**Hint** Integrate by parts by taking $(x^2 + x)$ as Ist function and $\frac{1}{\sqrt{2x + 1}}$ as IInd function]

83 Evaluate $\int_0^{\pi/2} \frac{x}{(\sin x + \cos x)} dx.$

84 Evaluate $\int_{\pi/5}^{3\pi/10} \frac{\sin x}{(\sin x + \cos x)} dx.$

85 Evaluate $\int_0^{\pi/2} |\cos x - \sin x| dx.$

86 Let $f(x) = x - [x]$, for every real number x, where $[x]$ is the integral part of x, then evaluate $\int_{-1}^{1} f(x) dx.$

$\left[\textbf{Hint } [x] = \begin{cases} -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \end{cases}\right]$

(v) $\int_{-\pi/2}^{\pi/2} \sin^7 x \, dx$ is equal to

(a) 0 (b) 1
(c) 2 (d) π

94. If $f(x)$ is a continuous function defined on $[0, a]$, then $\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx.$

[CBSE Question Bank]

Based on the above information, answer the following questions.

(i) $\int_0^a \frac{f(x) \, dx}{f(x) + f(a - x)}$ is equal to

(a) a (b) $\frac{a}{2}$ (c) $\frac{-a}{2}$ (d) $2a$

(ii) If $f(x) = \frac{\sin x - \cos x}{1 + \sin x \cos x}$, then $f\left(\frac{\pi}{2} - x\right)$ is

(a) $f(x)$ (b) $-f(x)$ (c) $\frac{1}{f(x)}$ (d) $2f(x)$

[**Hint** Use $\int_a^b f(x) \, dx = \int_a^b f(a + b - x) \, dx$]

92 Prove that $\int_0^{\pi} \frac{x}{1 + \cos\alpha \sin x} dx = \frac{\pi\alpha}{\sin\alpha}.$

CASE BASED Questions

93. For a function $f(x)$, if $f(-x) = f(x)$, then $f(x)$ is an even function and $f(-x) = -f(x)$, then $f(x)$ is a odd function. Again, we have

$$\int_{-a}^{a} f(x) \, dx = \begin{cases} 2\int_0^a f(x) dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$$

[CBSE Question Bank]

Based on the above information, answer the following questions.

(i) $f(x) = x^2 \sin x$ is a

(a) even
(b) odd
(c) neither even nor odd
(d) None of the above

(ii) $\int_{-\pi}^{\pi} f(x) \, dx$ is equal to

(a) $\frac{\pi}{4}$ (b) 2π (c) $\frac{\pi}{2}$ (d) 0

(iii) If $g(x) = x \sin x$, then $\int_{-\pi}^{\pi} x \sin x \, dx$ is

(a) π (b) 2π
(c) 3π (d) 4π

(iv) $\int_{-\pi/2}^{\pi/2} |\sin x| \, dx$ is equal to

(a) 0 (b) 1
(c) 2 (d) 3

(iii) $\int_0^{\frac{\pi}{2}} f(x) \, dx$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) 0 (d) None of these

(iv) If $g(x) = \log(1 + \tan x)$, then $g\left(\frac{\pi}{4} - x\right)$ is

(a) $\log 2 - g(x)$ (b) $g(x) - \frac{1}{2} \log 2$
(c) $g(x) - \log 2$ (d) $\frac{1}{2} \log 2 + g(x)$

(v) $\int_0^{\frac{\pi}{4}} g(x) \, dx$ is equal to

(a) $\frac{\pi}{4} \log 2$ (b) $\frac{\pi}{8} \log 2$
(c) $-\frac{\pi}{4} \log 2$ (d) $-\frac{\pi}{8} \log 2$

ANSWERS

1. (b) 2. (a) 3. (b) 4. (c) 5. (b)
6. (b) 7. (b) 8. (b) 9. (d)

10. $\frac{4}{3}e^{3x} - 2x + C$
11. $\tan(\log|x|) + C$
12. $-e^x \cot x + C$
13. $\frac{(\log|\sin x|)^2}{2} + C$
14. $-\frac{\tan(7-4x)}{4} + C$
15. $\frac{180}{\pi}\sec x^\circ + C$
16. $\frac{x^2}{2} + \frac{1}{x} + C$
17. $2\sin\sqrt{x} + C$
18. $\frac{1}{4}\tan^{-1}\frac{x}{4} + C$
19. $\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C$
20. $f(x) = \sec x$
21. $\tan x + C$
22. $\frac{1}{6}\log\left|3x^2 + \sin 6x\right| + C$
23. π
24. $\frac{\pi}{12}$
25. $\frac{1}{4}\log 3$
26. 0
27. $\frac{\tan^7 x}{7} + C$
28. $\tan x - x + C$
29. $\frac{1}{4}\log(x^4+1) + C$
30. $\frac{1}{4}\log\left|\frac{x^2-1}{x^2+1}\right| + C$
31. $\frac{1}{2}\left[\frac{x^2\sqrt{x^4-1}}{2} - \frac{1}{2}\log\left|x^2 + \sqrt{x^4-1}\right|\right] + C$
32. $-2\cos\frac{x}{2} + 2\sin\frac{x}{2} + C$
33. $\frac{2}{3}(\tan x)^{3/2} + C$
34. $\log|\sec x + \tan x| + \log|\operatorname{cosec} x - \cot x| + C$
35. 9
36. 0
37. $\frac{16\sqrt{2}}{15}(\sqrt{2}+1)$
38. $(x+a)\cos 2a - \sin 2a \log|\sin(x+a)| + C$
39. $-\frac{2}{b^2}\left[\log|a + b\cos x| + \frac{a}{a + b\cos x}\right] + C$
40. $\frac{3}{8}x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$
41. $-\frac{\cos 7x}{14} - \frac{\cos x}{2} + C$
42. $-\frac{2}{\cos\frac{x}{2} + \sin\frac{x}{2}} + C$
43. $\frac{5}{6}\log|1 + 2x + 3x^2| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$
44. $-\frac{1}{4x} + \frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right) + C$
45. $\sqrt{x^2+2x+3} + \log\left|(x+1) + \sqrt{x^2+2x+3}\right| + C$
46. $-\sqrt{4x - x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$
47. $\frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + C$
48. $\frac{1}{2}[\sec x \tan x + \log|\sec x + \tan x|] + C$
49. $\frac{1}{4}\cdot\frac{e^{2x}}{1+2x} + C$
50. $\frac{1}{a^2-b^2}\left[a\tan^{-1}\frac{x}{a} - b\tan^{-1}\frac{x}{b}\right] + C$
51. $\frac{1}{15}\log\left|\frac{x^5}{x^5+3}\right| + C$
52. $e^x\tan\frac{x}{2} + C$
53. $-\frac{1}{\sqrt{5}}\log\left|\frac{1}{x-1} + \frac{1}{5} + \sqrt{\frac{x^2+4}{5(x-1)^2}}\right| + C$
54. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2 - \sqrt{2}x + 1}{x^2 + x\sqrt{2} + 1}\right| + C$
55. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C$
56. $\frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$
57. $\sqrt{x^2+x} + \frac{1}{2}\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2+x}\right| + C$
58. $\frac{1}{2}\log\left|\tan\left(\frac{x}{2} + \frac{\pi}{6}\right)\right| + C$
59. $-\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + C$
60. $-\frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right) + C$
61. $2\log\left|\frac{x+1}{x+3}\right| + \frac{1}{(x+1)} + C$
62. $-2\log|x+1| - \frac{1}{(x+1)} + 3\log|x+2| + C$
63. $\frac{1}{7}\log\left|\frac{x-2}{x+2}\right| + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}} + C$
64. $x\sin(\log x) + C$
65. $\frac{x}{(\log x + 1)} + C$
66. $\frac{x}{2}[\sin(\log x) - \cos(\log x)] + C$
67. $-\frac{1}{3}\log|1 + \tan\theta| + \frac{1}{6}\log|\tan^2\theta - \tan\theta + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C$
68. $-\frac{1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + C$
69. $\frac{\tan^{11}x}{11} + \frac{\tan^9 x}{9} + C$
70. $\frac{\sin^{-1}x}{4}(2x^2-1) + \frac{x}{4}\sqrt{1-x^2} + C$
71. $2[\sqrt{x} - \tan^{-1}\sqrt{x}] + C$
72. $\frac{\pi}{3}$
73. $\frac{\pi^2}{4}$
74. 20
75. $\frac{\pi^2}{2ab}$
76. $5 - \frac{45}{2}\log\frac{5}{4} + \frac{5}{2}\log\frac{3}{2}$
77. $\frac{\pi}{2}$
78. $\frac{3\pi - 4}{6}$
79. $\frac{e^2(e^2-2)}{4}$
80. 6
81. $-\frac{\pi}{2}\log 2$
82. $\sqrt{\frac{57 - 5\sqrt{5}}{5}}$
83. $\frac{\pi}{4\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$
84. $\frac{\pi}{20}$
85. $2(\sqrt{2}-1)$
86. 1
87. $\frac{2}{\pi} + \frac{1}{\pi^2}$
88. $\frac{x^2}{2}(\log|x|)^2 - \frac{x^2}{2}\log|x| + \frac{x^2}{4} + C$
89. $\frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{(x-1)} + C$
90. $\frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + C$
91. $\frac{\pi}{8}\log\left(\frac{1}{2}\right)$
93. (i) → (b), (ii) → (d), (iii) → (b), (iv) → (c), (v) → (a)
94. (i) → (b), (ii) → (b), (iii) → (c), (iv) → (a), (v) → (a)