4 Applications of Derivatives

Let's Study

- Meaning of Derivatives
- Increasing and Decreasing Functions.
- Maxima and Minima
- Application of derivatives to Economics.

Introduction

Derivatives have a wide range of applications in everyday life. In this chapter, we shall discuss geometrical and physical significance of derivatives and some of their applications such as equation of tangent and normal at a point on the curve, rate measure in physical field, approximate values of functions and extreme values of a function.

Let's Learn

4.1 Meaning of Derivative:

Let $y = f(x)$ be a continuous function of x.

It represents a curve in XY-plane. (fig. 4.1).

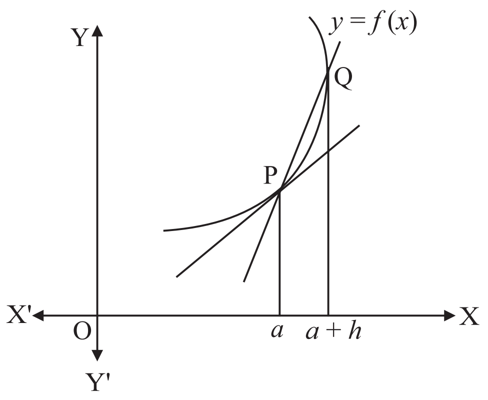

Fig. 4.1

Let P $(a, f(a))$ and Q $(a + h, f(a + h))$ be two points on the curve. Join the points P and Q.

The slope of the chord PQ $= \dfrac{f(a+h)-f(a)}{h}$

Let the point Q move along the curve such that Q$\rightarrow$P. Then the secant PQ approaches the tangent at P as h$\rightarrow$0

$\therefore \lim\limits_{Q \to P}$ (slope of secant PQ) $= \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$

Slope of tangent at P $= f'(a)$ (if limit exists)

Thus, the derivative of a function $y = f(x)$ at any point P(a,b) is the slope of the tangent at the point P(a,b) on the curve.

The slope of the tangent at any point P(a,b) is also called gradient of the curve $y = f(x)$ at point P and is denoted by $f'(a)$ or $\left(\dfrac{dy}{dx}\right)_P$.

Normal is a line perpendicular to tangent, passing through the point of tangency.

$\therefore$ Slope of the normal is the negative reciprocal of slope of tangent.

Thus, slope of normal $= \dfrac{-1}{f'(a)} = \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_P}$

Hence,

(i) The equation of tangent to the curve $y = f(x)$ at the point P(a,b) is given by $(y - b) = f'(a)(x - a)$

(ii) The equation of normal to the curve $y = f(x)$ at the point P(a,b) is given by

$$(y - b) = \frac{-1}{f'(a)}(x - a)$$

SOLVED EXAMPLES

1) Find the equation of tangent and normal to the curve $y = x^2 + 4x + 1$ at P(–1, –2).

Solution: Given equation of curve is

$y = x^2 + 4x + 1$

Differentiating with respect to x

$\therefore \quad \frac{dy}{dx} = 2x + 4$

$\therefore \quad \left(\frac{dy}{dx}\right)_{p(-1,-2)} = 2(-1) + 4$

$= 2$

$\therefore$ The slope of tangent at P(–1, –2) is 2

$\therefore$ The equation of tangent is

$y + 2 = 2(x + 1)$

$\therefore \quad y + 2 = 2x + 2$

$\therefore \quad 2x - y = 0$

Now, The slope of Normal at P(–1, –2) is $\frac{-1}{2}$

$\therefore$ The equation of normal is

$y + 2 = \frac{-1}{2}(x + 1)$

$2(y + 2) = -1(x + 1)$

$2y + 4 = -x - 1$

$x + 2y + 5 = 0$

2) Find the equation of tangent and normal to the curve $y = 6 - x^2$ where the normal is parallel to the line $x - 4y + 3 = 0$.

Solution: Let $P(x_1,y_1)$ be the point on the curve $y = 6 - x^2$ where the normal is parallel to the line $x - 4y + 3 = 0$

Consider, $y = 6 - x^2$

$\therefore \quad \frac{dy}{dx} = -2x$

$\therefore \quad \left(\frac{dy}{dx}\right)_{x=x_1} = -2x_1$

$\therefore$ The slope of the tangent as $P(x_1,y_1) = -2x_1$

$\therefore$ The slope of the normal at $P(x_1,y_1)$

$= \frac{1}{2x_1}$

Now, slope of $x - 4y + 3 = 0$ is $\frac{1}{4}$

$\therefore$ The slope of the normal $= \frac{1}{4}$ (since normal is parallel to given line)

$\therefore \quad \frac{1}{2x_1} = \frac{1}{4}$

$\therefore \quad x_1 = 2$

$P(x_1,y_1)$ lies on the curve $y = 6 - x^2$

$\therefore \quad y_1 = 6 - x_1^2$

$\therefore \quad y_1 = 6 - 4$

$\therefore \quad y_1 = 2$

$\therefore$ The point on the curve is (2, 2)

$\therefore$ The slope of tangent at (2,2) is

$-2x_1 = -2(2) = -4$

$\therefore$ The equation of tangent is
$(y - 2) = -4\ (x - 2)$

$\therefore \quad y - 2 = -4x + 8$

$\therefore \quad 4x + y - 10 = 0$

$\therefore$ The equation of normal is

$(y - 2) = \frac{1}{4}(x - 2)$

$\therefore \quad 4(y - 2) = 1\ (x - 2)$

$\therefore \quad 4y - 8 = x - 2$

$\therefore \quad x - 4y + 6 = 0$

EXERCISE 4.1

Q.1 Find the equation of tangent and normal to the curve at the given points on it.

i) $y = 3x^2 - x + 1$ at (1,3)

ii) $2x^2 + 3y^2 = 5$ at (1,1)

iii) $x^2 + y^2 + xy = 3$ at (1,1)

Q.2 Find the equation of tangent and normal to the curve $y = x^2 + 5$ where the tangent is parallel to the line $4x - y + 1 = 0$.

Q.3 Find the equation of tangent and normal to the curve $y = 3x^2 - 3x - 5$ where the tangent is parallel to the line $3x - y + 1 = 0$.

4.2 Increasing and Decreasing Functions:

Definition : The function $y = f(x)$ is said to be an increasing function of x in the interval (a,b) if $f(x_2) > f(x_1)$, whenever $x_2 > x_1$ in the interval (a,b).

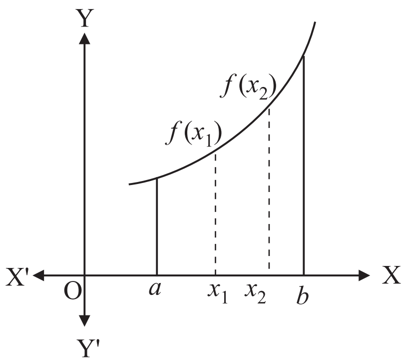

Fig. 4.2

Geometrically, as we move from left to right along the curve $y = f(x)$ in (a,b), then the curve rises. (see fig. 4.2)

$\therefore$ Slope of tangent at x: $f'(x) > 0$

$\therefore$ The slope of the tangent is positive.

If $f'(x) > 0$ for all $x \in (a,b)$ then, $y = f(x)$ is an increasing function in the interval (a,b)

Note: Sign of the Derivative can be used to find if the function $f(x)$ is increasing.

Definition: A function $y = f(x)$ is said to be a decreasing function of x in an interval (a,b). if $f(x_2) < f(x_1)$, whenever $x_2 > x_1$ for all x_1, x_2 in the interval (a, b).

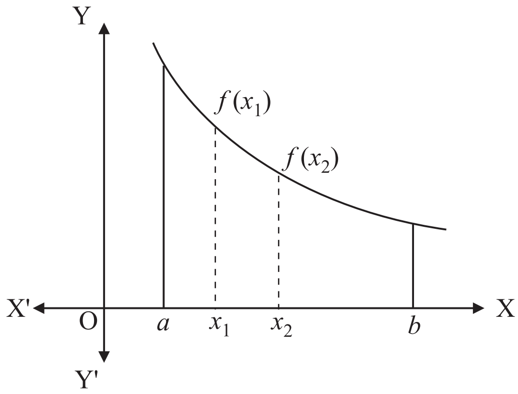

Fig. 4.3

Geometrically, as we move from left to right along the curve $y = f(x)$ in (a,b), then the curve falls. (see fig.4.3)

$\therefore$ Slope of tangent $f'(x) < 0$

$\therefore$ The slope of tangent is negative.

If $f'(x) < 0$ in (a,b) then $f(x)$ is a decreasing function in the interval (a,b).

Note: Every function may not be either increasing or decreasing.

SOLVED EXAMPLES

1) Test whether the following function is increasing or decreasing.

$f(x) = x^3 - 3x^2 + 3x - 100, x \in R$

Solution: Given $f(x) = x^3 - 3x^2 + 3x - 100, x \in R$

$\therefore$ $f'(x) = 3x^2 - 6x + 3$

$\therefore$ $f'(x) = 3(x - 1)^2$

Since $(x - 1)^2$ is always positive, $x \neq 1$

$\therefore$ $f'(x) > 0, \forall x \in R - \{1\}$

Hence, $f(x)$ is an increasing function, $\forall x \in R - \{1\}$

2) Test whether the following function is increasing or decreasing.

$f(x) = 2 - 3x + 3x^2 - x^3, \forall x \in R$

Solution: $f(x) = 2 - 3x + 3x^2 - x^3$

$\therefore \quad f'(x) = -3 + 6x - 3x^2$

$\therefore \quad f'(x) = -3(x^2 - 2x + 1)$

$\therefore \quad f'(x) = -3(x-1)^2$

Since $(x-1)^2$ is always positive, $x \neq 1$

$\therefore \quad f'(x) < 0, \forall x \in R - \{1\}$

Hence, function $f(x)$ is decreasing function $\forall x \in R - \{1\}$

3) Find the value of x, for which the function $f(x) = x^3 + 12x^2 + 36x + 6$ is increasing.

Solution: Given $f(x) = x^3 + 12x^2 + 36x + 6$

$\therefore \quad f'(x) = 3x^2 + 24x + 36$

$\therefore \quad f'(x) = 3(x+2)(x+6)$

Now, $f'(x) > 0$, as $f(x)$ is increasing.

$\therefore \quad 3(x+2)(x+6) > 0$

$(ab > 0 \Leftrightarrow a > 0, b > 0$ or $a < 0, b < 0)$

Case I] $x + 2 > 0$ and $x + 6 > 0$

$\therefore \quad x > -2$ and $x > -6$

$\therefore \quad x > -2$ .... (I)

Case II] $x + 2 < 0$ and $x + 6 < 0$

$\therefore \quad x < -2$ and $x < -6$

$\therefore \quad x < -6$ .... (II)

From case I and II, $f(x)$ is increasing if $x < -6$ or $x > -2$

$\therefore \quad f(x) = x^3 + 12x^2 + 36x + 6$ is increasing if and only if $x < -6$ or $x > -2$

Hence, $x \in (-\infty, -6)$ or $x \in (-2, \infty)$.

4) Find the values of x for which the function $f(x) = 2x^3 - 9x^2 + 12x + 2$ is decreasing.

Soluction: Given $f(x) = 2x^2 - 9x^2 + 12x + 2$

$\therefore \quad f'(x) = 6x^2 - 18x + 12$

$\therefore \quad f'(x) = 6(x-1)(x-2)$

Now, $f'(x) < 0$

$\therefore \quad 6(x-1)(x-2) < 0$

(if ab < 0 either a < 0 and b > 0 or a > 0 and b < 0)

Case I] $(x-1) < 0$ and $x - 2 > 0$

$\therefore \quad x < 1$ and $x > 2$ which is contradiction

Case II] $x - 1 > 0$ and $x - 2 < 0$

$\therefore \quad x > 1$ and $x < 2$

$\therefore \quad 1 < x < 2$

$\therefore \quad f(x) = 2x^3 - 9x^2 + 12x + 2$ is decreasing function if $x \in (1,2)$.

EXERCISE 4.2

Q.1 Test whether the following fuctions are increasing or decreasing

i) $f(x) = x^3 - 6x^2 + 12x - 16, x \in R$

ii) $f(x) = x - \frac{1}{x}, x \in R, x \neq 0$

iii) $f(x) = \frac{7}{x} - 3, x \in R, x \neq 0$

Q.2 Find the values of x, such that $f(x)$ is increasing function.

i) $f(x) = 2x^3 - 15x^2 + 36x + 1$

ii) $f(x) = x^2 + 2x - 5$

iii) $f(x) = 2x^3 - 15x^2 - 144x - 7$

Q.3 Find the values of x such that $f(x)$ is decreasing function.

i) $f(x) = 2x^3 - 15x^2 - 144x - 7$

ii) $f(x) = x^4 - 2x^3 + 1$

iii) $f(x) = 2x^3 - 15x^2 - 84x - 7$

4.3 Maxima and Minima:

a) **Maximum value of $f(x)$:** A function $f(x)$ is said to have a maximum value at a point $x = c$ if $\underline{f(x) < f(c)}$ for all $x \neq c$.

The value $f(c)$ is called the maximum value of $f(x)$.

Thus, the function $f(x)$ will have a maximum at $x = c$ if $f(x)$ is increasing for $x < c$ and $f(x)$ is decreasing for $x > c$ as shown in Fig. 4.4

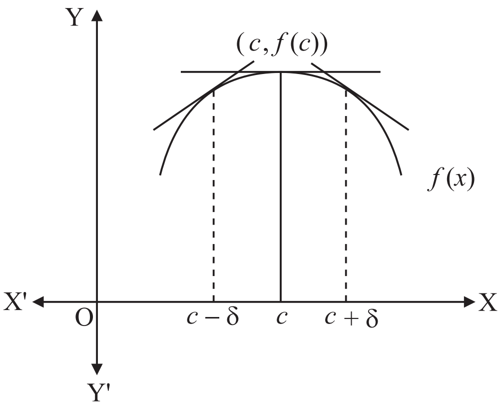

Fig. 4.4

b) Minimum value of $f(x)$: A function $f(x)$ is said to have a minimum at a point $x = c$ if $f(x) > f(c)$ for all $x \neq c$.

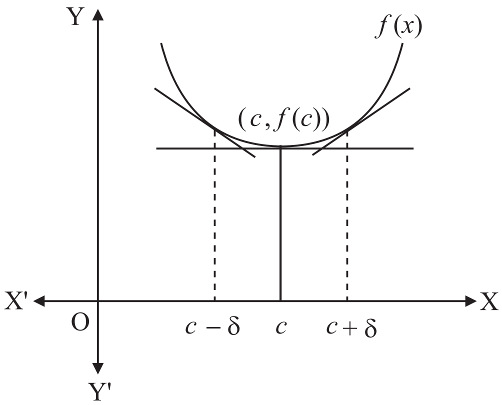

Fig. 4.5

The value of $f(c)$ is called the minimum value of $f(x)$.

The function will have a minimum at $x = c$ if $f(x)$ is decreasing for $x < c$ and $f(x)$ is increasing for $x > c$ as shown in fig. 4.5

At $x = c$ if the function is neither increasing nor decreasing, then the function is stationary at $x = c$

Note: The maximum and minimum values of a function are called its extreme values.

To find extreme values of a function, we use the following tests.

Test for maximum / minimum : For a real valued function f, defined on $[a, b]$ and differentiable on (a, b), then for $c \in (a, b)$

i) $x = c$ is a point of local maxima, if $f'(c) = 0$ and $f''(c) < 0$. The value $f(c)$ is local maximum value of f.

ii) $x = c$ is a point of local minima, if $f'(c) = 0$ and $f''(c) > 0$. In this case $f(c)$ is local minimum value of f.

Remark :

If $f'(c) = 0$ and $f'(c - h) > 0$, $f'(c + h) > 0$ or $f'(c - h) < 0$, $f'(c + h) < 0$ then $f(c)$ is neither maximum nor minimum. In this case $x = c$ is called a **point of inflection** (see.fig. 4.6)

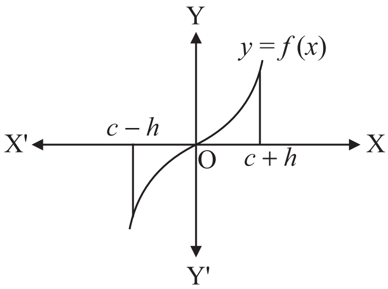

Fig. 4.6

A function may have several maxima and several minima. In such cases, the maxima are called **local maxima** and the minima are called **local minima.** (see. fig. 4.7)

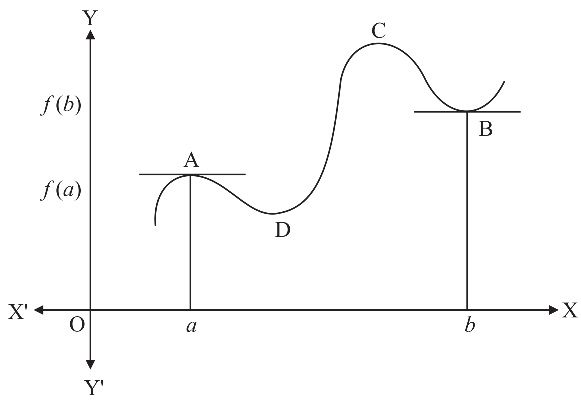

Fig. 4.7

In this figure the function has a local maximum at $x = a$ and a local minimum at $x = b$ and still $f(b) > f(a)$.

SOLVED EXAMPLES

1) Find the maximum and minimum value of the function

$f(x) = 3x^3 - 9x^2 - 27x + 15$

Solution: Given $f(x) = 3x^3 - 9x^2 - 27x + 15$

$\therefore \quad f'(x) = 9x^2 - 18x - 27$

$\therefore \quad f''(x) = 18x - 18$

For the extreme values $f'(x) = 0$

$\therefore \quad 9x^2 - 18x - 27 = 0$

$\therefore \quad 9(x^2 - 2x - 3) = 0$

$\therefore \quad (x + 1)(x - 3) = 0$

$\therefore \quad x = -1$ or $x = 3$

For $x = -1$, $f''(x) = 18x - 18$

$f''(-1) = 18(-1) - 18$

$= -18 - 18$

$= -36 < 0$

$\therefore \quad f(x)$ attains maximum at $x = -1$

Maximum value is

$f(-1) = 3(-1)^3 - 9(-1)^2 - 27(-1) + 15 = 30$

For $x = 3$, $f''(x) = 18x - 18$

$f''(3) = 18(3) - 18$

$= 54 - 18$

$= 36 > 0$

$\therefore \quad f(x)$ attains minimum at $x = 3$

Minimum value is,

$f(3) = 3(3)^3 - 9(3)^2 - 27(3) + 15 = -66$

$\therefore$ The function $f(x)$ has maximum value 30 at $x = -1$ and minimum value -66 at $x = 3$

2) Divide the number 84 into two parts such that the product of one part and square of the other is maximum.

Solution: Let one part be x then other part will be $84 - x$

$f(x) = x^2(84 - x)$

$f(x) = 84x^2 - x^3$

$f'(x) = 168(x) - 3x^2$

$f''(x) = 168 - 6x$

For extreme value $f'(x) = 0$

$\therefore \quad 168x - 3x^2 = 0$

$\therefore \quad 3x(56 - x) = 0$

$x = 0$ or $x = 56$

If $x = 0$, $f''(x) = 168 - 6x$

$f''(0) = 168 - 6(0)$

$= 168 > 0$

$\therefore \quad f(x)$ attains minimum at $x = 0$

If $x = 56$, $f''(x) = 168 - 6x$

$f''(x) = 168 - 6(56)$

$= -168 < 0$

$\therefore \quad f(x)$ attains maximum at $x = 56$

$\therefore$ Two parts of 84 are 56 and 28

3) A rod of 108 meter long is bent to form a rectangle. Find it's dimensions if the area is maximum.

Solution: Let x be the length and y be the breadth of the rectangle.

$\therefore \quad 2x + 2y = 108$

$\therefore \quad 2y = 108 - 2x$

$\therefore \quad 2y = 2(54 - x)$

$\therefore \quad y = 54 - x$ .... (1)

Now, area of the rectangle $= xy$

$= x(54 - x)$

$f(x) = 54x - x^2$

$f'(x) = 54 - 2x$

$f''(x) = -2$

For extreme value, $f'(x) = 0$

$\therefore \quad 54 - 2x = 0$

$\therefore \quad 2x = 54$

$\therefore \quad x = 27$

$f''(27) = -2 < 0$

$\therefore$ Area is maximum when $x = 27, y = 27$

$\therefore$ The dimension of rectangle are 27m × 27m.

$\therefore$ It is a square.

EXERCISE 4.3

Q.1 Determine the maximum and minimum values of the following functions.

i) $f(x) = 2x^3 - 21x^2 + 36x - 20$

ii) $f(x) = x.\log x$

iii) $f(x) = x^2 + \dfrac{16}{x}$

Q.2 Divide the number 20 in to two parts such that their product is maximum.

Q.3 A metal wire of 36cm long is bent to form a rectangle. Find it's dimensions when it's area is maximum.

Q.4 The total cost of producing x units is Rs. $(x^2 + 60x + 50)$ and the price is Rs. $(180 - x)$ per unit. For what units is the profit maximum?

4.4 Applications of derivative in Economics:

We ave discussed the following functions in XIth standard.

1. Demand Function D = f(P).

$$\text{Marginal demand} = D_m = \frac{dD}{dP}$$

2. Supply function S = g (P)

$$\text{Marginal supply} = \frac{dS}{dP}$$

3. Total cost function C = $f(x)$, where x is number of items produced,

$$\text{Marginal cost} = C_m = \frac{dC}{dx}$$

$$\text{Average cost} = C_A = \frac{C}{x}$$

4. Total Revenue R = P.D where P is price and D is demand.

$$\text{Average Revenue } R_A = \frac{R}{D} = \frac{PD}{D} = P$$

Total profit = R – C

With this knowledge, we are now in a position to discuss price elasticity of demand; which is usually referred as 'elasticity of demand' denoted by 'η'.

$$\text{Elasticity of demand} \quad \eta = \frac{-P}{D} . \frac{dD}{dP}$$

We observe the following situations in the formula for elasticity of demand.

i) Demand is a decreasing function of price.

$$\therefore \quad \frac{dD}{dP} < 0$$

Also, price P and the demand D are always positive.

$$\therefore \quad \eta = \frac{-P}{D} . \frac{dD}{dP} > 0$$

ii) If η = 0, it means the demand D is constant function of price P.

$$\therefore \quad \frac{dD}{dP} < 0$$

In this situation demand is perfectly inelastic.

iii) If $0 < \eta < 1$, the demand is relatively inelastic.

iv) If η = 1, the demand is exactly proportional to the price and demand is said to be unitary elastic.

v) If $\eta > 1$, the demand is relatively elastic.

Now let us establish the relation between marginal revenue (R_m), average revenue (R_A) and elasticity of demand (η)

As, $R_m = \frac{dR}{dD}$

But $R = P.D.$

$\therefore \quad R_m = \frac{d}{dD}(P.D)$

$= P + D\frac{dP}{dD}$

$= P\left(1 + \frac{D}{P}\frac{dP}{dD}\right) \quad \text{...... (1)}$

But $\boxed{\eta = \frac{-P}{D}.\frac{dD}{dP}}$

$\frac{-1}{\eta}. = \frac{D}{P}.\frac{dP}{dD}$

Substituting in (1) we get,

$R_m = P\left(1 - \frac{1}{\eta}\right)$

$R_m = R_A\left(1 - \frac{1}{\eta}\right) \quad (\text{as } R_A = P)$

Marginal propensity to consume: For any person with income x, his consumption expenditure (E_c) depends on x.

$\therefore E_c = f(x)$

Marginal propensity to consume

$(MPC) = \frac{dE_c}{dx}$

Average propensity to consume

$(APC) = \frac{E_c}{x}$

Marginal propensity to save (MPS): If S is a saving of a person with income x then

$MPS = \frac{dS}{dx}$

Average propensity to save (APS) $= \frac{S}{x}$

Note here that $x = E_c + S$

Differentiating both sides w.r.t.x

$$1 = \frac{dE_c}{dx} + \frac{dS}{dx}$$

$\therefore \quad MPC + MPS = 1$

Also as $x = E_C + S$,

$\therefore \quad 1 = \frac{E_c}{x} + \frac{S}{x}$

$\therefore \quad 1 = APC + APS$

SOLVED EXAMPLES

1) The revenue function is given by $R = D^2 - 40D$, where D is demand of the commodity. For what values of D, the revenue is increasing?

Solution: Given $R = D^2 - 40D$

Differentiating w.r.t.D

$$\frac{dR}{dD} = 2D - 40$$

As revenue is increasing

$\therefore \quad \frac{dR}{dD} > 0$

$\therefore \quad 2D - 40 > 0$

$\therefore \quad D > 20$

Revenue is increasing for $D > 20$

2) The cost C of producing x articles is given as $C = x^3 - 16x^2 + 47x$. For what values of x the average cost is decreasing?

Solution: Given $C = x^3 - 16x^2 + 47x$

Average cost $\quad C_A = \frac{C}{x}$

$C_A = x^2 - 16x + 47$

Differentiating w.r.t.x

$$\frac{dC_A}{dx} = 2x - 16$$

Now C_A is decreasing if $\frac{dC_A}{dx} < 0$

that is $2x - 16 < 0$

$\therefore \quad x < 8$

Average cost is decreasing for $x < 8$

3) In a factory, for production of Q articles, standing charges are 500/-, labour charges are 700/- and processing charges are 50Q. The price of an article is 1700 - 3Q. For what values of Q, the profit is increasing?

Solution: Cost of poduction of Q aricles

C = standing charges + labour charges + processing charges

$\therefore \quad C = 500 + 700 + 50Q$

$\therefore \quad C = 1200 + 50Q$

Revenue R = P.Q.

$= (1700 - 3Q)\,Q$

$= 1700Q - 3Q^2$

Proit $\pi = R - C$

$= 1700Q - 3Q^2 - (1200 + 50Q)$

$\therefore \pi = 1650Q - 3Q^2 - 1200$

Differentiating w.r.t.Q,

$$\frac{d\pi}{dQ} = 1650 - 6Q$$

If profit is increasing, then $\frac{d\pi}{dQ} > 0$

$\therefore \quad 1650 - 6Q > 0$

That is $1650 > 6Q$

$\therefore \quad Q < 275$

$\therefore$ Profit is increasing for $Q < 275$

4) Demand function x, for a certain commodity is given as $x = 200 - 4p$, where p is the unit price. Find

i) elasticity of demand as a function of p.

ii) elasticity of demand when $p = 10$; $p = 30$. Interpret your results.

iii) the price p for which elasticity of demand is equal to one.

Solution: (i) Elasticity of demand

$$\eta = \frac{-p}{x} \cdot \frac{dx}{dp}$$

For $x = 200 - 4p$,

$$\frac{dx}{dp} = -4$$

$$\therefore \quad \eta = \frac{-p}{x} \cdot \frac{dx}{dp}$$

$$= \frac{-p}{(200-4p)}(-4) \quad (\text{For } p < 50)$$

$$\therefore \quad \eta = \frac{p}{(50-p)} \qquad (\text{For } p < 50)$$

(ii) When $P = 10$

$$\eta = \frac{10}{(50-10)}$$

$$= \frac{10}{40}$$

$= 0.25 < 1$

$\therefore$ Demand is inelastic for $p = 10$

When $p = 30$

$$\eta = \frac{30}{(50-30)}$$

$$\eta = \frac{30}{20}$$

$= 1.5 > 1$

$\therefore$ Demand is elastic when $p = 30$

(iii) To find the price when $\eta = 1$

As $\eta = 1$,

$$\therefore \quad \frac{p}{50-p} = 1$$

$\therefore \quad p = 50 - p$

$\therefore \quad 2p = 50$

$\therefore \quad p = 25$

$\therefore$ For elasticity equal to 1 then price is 25/unit.

5) If the average revenue R_A is 50 and elasticity of demand η is 5, find marginal revenue R_m.

Solution: Given $R_A = 50$ and $\eta = 5$,

$$R_m = R_A\left(1-\frac{1}{\eta}\right)$$

$$= 50\left(1-\frac{1}{5}\right)$$

$$= 50\left(\frac{4}{5}\right)$$

$$R_m = 40$$

6) The consumption expenditure E_C of a person with income x, is given by

$E_C = 0.0006x^2 + 0.003x$. Find average propensity to consume, marginal propensity to consume when his income is Rs. 200/- Also find his marginal propensity to save.

Solution: Given $E_C = 0.006x^2 + 0.003x$

$$\therefore \quad APC = \frac{E_c}{x}$$

$$= 0.0006x + 0.003$$

At $x = 200$,

$$APC = 0.0006 \times 200 + 0.003$$

$$= 0.12 + 0.003$$

$$= 0.123$$

$$MPC = \frac{dE_c}{dx}$$

$$= \frac{d}{dx}(0.0006x^2 + 0.003x)$$

$$= 0.0006\,(2x) + 0.003$$

At $x = 200$,

$$MPC = 0.0006 \times 400 + 0.003$$

$$= 0.24 + 0.003$$

$$= 0.243$$

As $MPC + MPS = 1$

$$\therefore \quad MPS = 1 - MPC$$

$$= 1 - 0.243$$

$$= 0.757$$

EXERCISE 4.4

1) The demand function of a commodity at price is given as, $D = 40 - \frac{5P}{8}$. Check whether it is increasing or decreasing function.

2) The price P for demand D is given as $P = 183 + 120D - 3D^2$; find D for which price is increasing.

3) The total cost function for production of articles is given as $C = 100 + 600x - 3x^2$. Find the values of x for which total cost is decreasing.

4) The manufacturing company produces x items at the total cost of Rs. $180 + 4x$. The demand function for this product is $P = (240 - x)$. Find x for which (i) revenue is increasing, (ii) profit is increasing.

5) For manufacturing x units, labour cost is $150 - 54x$ and processing cost is x^2. Price of each unit is $p = 10800 - 4x^2$. Find the values of x for which.

   i) Total cost is decreasing

   ii) Revenue is increasing

6) The total cost of manufacturing x articles $C = 47x + 300x^2 - x^4$. Find x, for which average cost is (i) increasing (ii) decreasing.

7) i) Find the marginal revenue, if the average revenue is 45 and elasticity of demand is 5.

   ii) Find the price, if the marginal revenue is 28 and elasticity of demand is 3.

   iii) Find the elasticity of demand, if the marginal revenue is 50 and price is Rs. 75/-.

8) If the demand function is $D = \left(\frac{p+6}{p-3}\right)$, find the elasticity of demand at $p = 4$.

9) Find the price for the demand function $D = \frac{2p+3}{3p-1}$, when elasticity of demand is $\frac{11}{14}$.

10) If the demand function is $D = 50 - 3p - p^2$. Find the elasticity of demand at (i) $p = 5$ (ii) $p = 2$. Comment on the result.

11) For the demand function $D = 100 - \frac{p^2}{2}$.
Find the elasticity of demand at (i) $p = 10$ (ii) $p = 6$ and comment on the results.

12) A manufacturing company produces x items at a total cost of Rs. $40 + 2x$. Their price is given as $p = 120 - x$. Find the value of x for which (i) revenue is increasing. (ii) profit is increasing. (iii) Also find elasticity of demand for price 80.

13) Find MPC, MPS, APC and APS, if the expenditure E_c of a person with income I is given as

$E_C = (0.0003)I^2 + (0.075)I$

when I = 1000.

Let's Remember

- A function f is said to be increasing at a point c if $f'(c) > 0$.
- A function f is said to be decreasing at a point c if $f'(c) < 0$.
- Elasticity of demand $\eta = \frac{-P}{D} \cdot \frac{dD}{dP}$
- $R_m = P\left(1 - \frac{1}{\eta}\right) = R_A\left(1 - \frac{1}{\eta}\right)$
- For a person with income x, consumption or expenditure E_c and saving S,

  (i) $x = E_c + S$

  (ii) MPC + MPS = 1

  (iii) APC + APS = 1
- A function $y = f(x)$ is said to have local maximum at $x = c$, if $f'(c) = 0$ and $f''(c) < 0$.
- A function $y = f(x)$ is said to have local minimum at $x = c$, if $f'(c) = 0$ and $f''(c) > 0$.

MISCELLANEOUS EXERCISE - 4

I) Choose the correct alternative.

1) The equation of tangent to the curve $y = x^2 + 4x + 1$ at $(-1, -2)$ is

(a) $2x - y = 0$ (b) $2x + y - 5 = 0$

(c) $2x - y - 1 = 0$ (d) $x + y - 1 = 0$

2) The equation of tangent to the curve $x^2 + y^2 = 5$ where the tangent is parallel to the line $2x - y + 1 = 0$ are

(a) $2x - y + 5 = 0$; $2x - y - 5 = 0$

(b) $2x + y + 5 = 0$; $2x + y - 5 = 0$

(c) $x - 2y + 5 = 0$; $x - 2y - 5 = 0$

(d) $x + 2y + 5$; $x + 2y - 5 = 0$

3) If elasticity of demand $\eta = 1$ then demand is

(a) constant (b) in elastic

(c) unitary elastic (d) elastic

4) If $0 < \eta < 1$, then the demand is

(a) constant (b) in elastic

(c) unitary elastic (d) elastic

5) The function $f(x) = x^3 - 3x^2 + 3x - 100$, $x \in R$ is

(a) Increasing for all $x \in R$, $x \neq 1$

(b) decreasing

(c) Neither, increasing nor decreasing

(d) Decreasing for all $x \in R$, $x \neq 1$

6) If $f(x) = 3x^3 - 9x^2 - 27x + 15$ then

(a) f has maximum value 66

(b) f has minimum value 30

(c) f has maxima at $x = -1$

(d) f has minima at $x = -1$

II) Fill in the blanks:

1) The slope of tangent at any point (a,b) is called as ....

2) If $f(x) = x^3 - 3x^2 + 3x - 100$, $x \in R$ then $f''(x)$ is ....

3) If $f(x) = \frac{7}{x} - 3$, $x \in R$, $x \neq 0$ then $f''(x)$ is ....

4) A rod of 108m length is bent to form a rectangle. If area at the rectangle is maximum then its dimension are ....

5) If $f(x) = x.\log.x$ then its maximum value is ....

III) State whether each of the following is True or false:

1) The equation of tangent to the curve $y = 4xe^x$ at $(-1, -4/e)$ is $y.e. + 4 = 0$.

2) $x + 10y + 21 = 0$ is the equation of normal to the curve $y = 3x^2 + 4x - 5$ at (1,2).

3) An absolute maximum must occur at a critical point or at an end point.

IV) Solve the following.

1) Find the equation of tangent and normal to the following curves

   i) $xy = c^2$ at $(ct, \frac{c}{t})$ where t is parameter

   ii) $y = x^2 + 4x$ at the point whose ordinate is -3

   iii) $x = \frac{1}{t}$, $y = t - \frac{1}{t}$, at $t = 2$

   iv) $y = x^3 - x^2 - 1$ at the point whose abscissa is -2.

2) Find the equation of normal to the curve $y = \sqrt{x-3}$ which is perpendicular to the line $6x + 3y - 4 = 0$

3) Show that the function $f(x) = \frac{x-2}{x+1}$, $x \neq -1$ is increasing

4) Show that the function $f(x) = \frac{3}{x} + 10$, $x \neq 0$ is decreasing

5) If $x + y = 3$ show that the maximum value of x^2y is 4.

6) Examine the function for maxima and minima $f(x) = x^3 - 9x^2 + 24x$

Activities

(1) Find the equation of tangent to the curve $\sqrt{x} - \sqrt{y} = 1$ at P(9,4).

Solution : Given equation of curve is

$$\sqrt{x} - \sqrt{y} = 1$$

Diff. w.r.to x

$$\therefore \frac{1}{2\sqrt{x}} - \frac{1}{2}\frac{dy}{\square x} = 0$$

$$\therefore \frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{1}{2}\square$$

$$\therefore \frac{1}{\sqrt{y}}\frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{y}}{\sqrt{x}}$$

$$\therefore \left(\frac{dy}{dx}\right)_{p=(9,4)} = \frac{\sqrt{9}}{\square} = \frac{3}{2}$$

$\therefore$ slope of tangent is $\frac{3}{2}$

$\therefore$ Eqation of the tangent at P(9.4) is

$y - 4 = \square(x - 9)$

$\therefore 2(y - 4) = 3(x - 9)$

$\therefore 2y - \square = \square + 27$

$\therefore 3x - 2y + 8 + \square = 0$

$\therefore 3x - 2y + 35 = 0$

(2): A rod of 108 meters long is bent to form rectangle. Find its dimensions if the area of rectangle is maximum.

Solution: Let x be the length and y be breadth of the rectangle.

$\therefore \quad 2x + 2y = 108$

$\therefore \quad x + y = \square$

$\therefore \quad y = 54 - \square$

Now area of the rectangle = $x\,y$

$= x \; \square$

$\therefore \quad f(x) = 54x - \square$

$\therefore \quad f'(x) = \square - 2x$

$\therefore \quad f'(x) = \square$

For extreme values, $f'(x) = 0$

$\therefore \quad 54 - 2x = 0$

$\therefore \quad -2x = \square$

$\therefore \quad x = \dfrac{-54}{-2}$

$\therefore \quad x = \square$

$\therefore \quad f''(27) = -2 < 0$

$\therefore$ area is maximum when $x = 27$, $y = 27$

$\therefore$ The dimensions of rectangles are 27m × 27m

(3): Find the value of x for which the function $f(x) = 2x^3 - 9x^2 + 12x + 2$ is decreasing.

Solution: Given $f(x) = 2x^3 - 9x^2 + 12x + 2$

$\therefore \quad f'(x) = \square x^2 - \square + \square$

$\therefore \quad f'(x) = 6(x - 1)(\square)$

Now $f'(x) < 0$

$\therefore \quad 6(x - 1)(x - 2) < 0$

since ab < 0 ⇔ a < 0 & b > 0 or a > 0 & b < 0

Case I] $(x - 1) < 0$ and $x - 2 > 0$

$\therefore \quad x < \square$ and $x > \square$

Which is contradiction

Case II] $x - 1 > 0$ and $x - 2 < 0$

$\therefore \quad x > \square$ and $x < \square$

$1 < \square < 2$

$f(x)$ is decreasing if and only if $x \in (1,2)$.

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