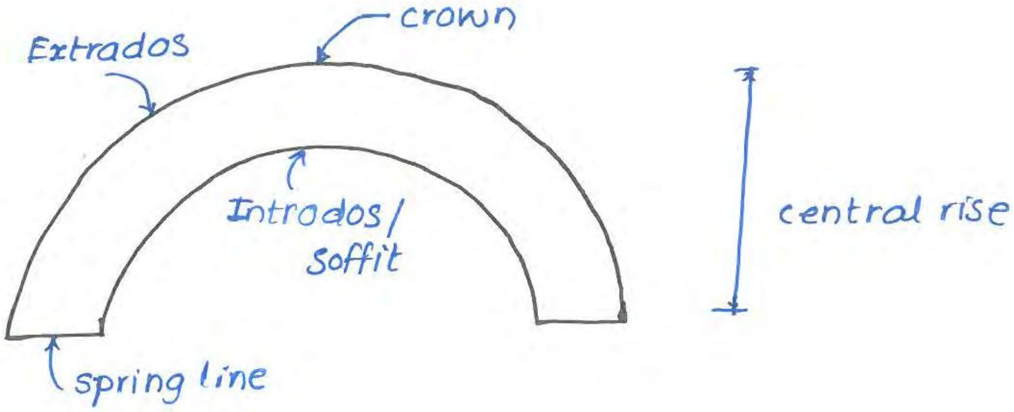
8. Arch

8.1 Introduction:

Arches are structural member which is predominantly subjected to axial compressive force and restricted for lateral movement at supports.



8.2 Reasons to construct Arch:

- Better use of materials which are strong in compression and weak in tension because arches are predominantly subjected to axial compression.
- For Large span.
- For better aesthetic.
- For cultural representation.

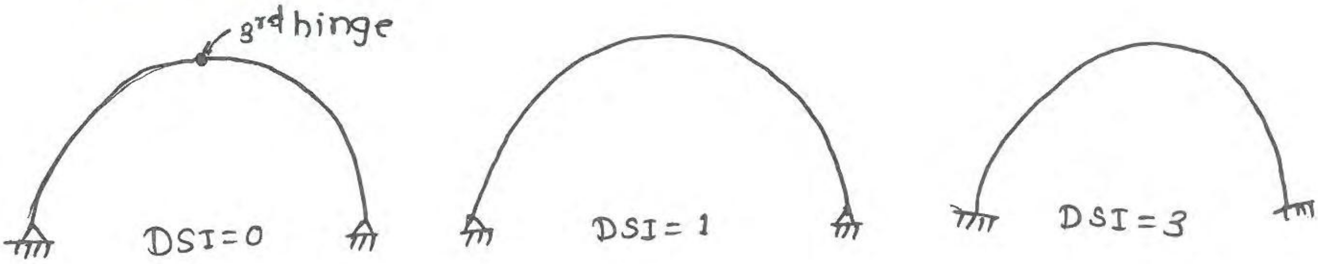
8.3 Classification of Arch:

A) Based on Architectural Point of View:-

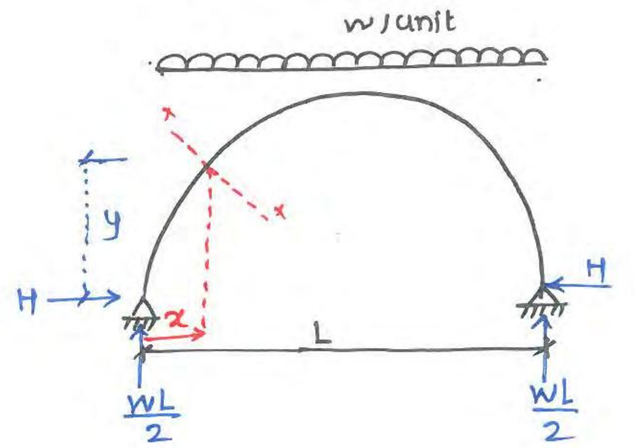
1) Parabolic
2) Circular
3) Elliptical
4) Any other shape.

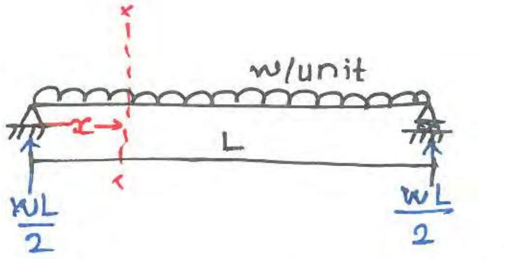
B) Based on Structural Design Point of View:-

1) 3-hinged Arch.
2) 2-hinged Arch.
3) Fixed Arch



8.4 Bending Moment in Arch:



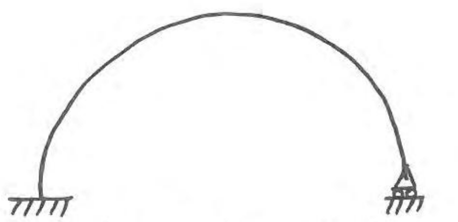


$$BM_{arch} = \underbrace{\frac{WL}{2}x - \frac{wx^2}{2}} - Hy \qquad BM_{SS} = \frac{WL}{2}\cdot x - \frac{wx^2}{2}$$

$$BM_{arch} = BM_{SS} - Hy$$

- Conclusion:-

It means lateral reactions are mandatory for being an arch because it reduces net bending moment at any section.

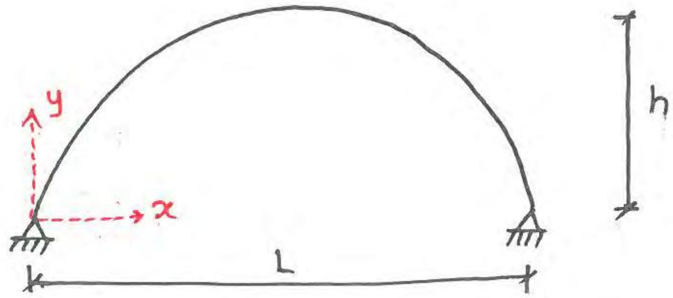


Beam curved in Elevation

Arch.

8.5 Analysis of 3-Hinged Arch:

8.5.1 3-Hinged Parabolic Arch:-

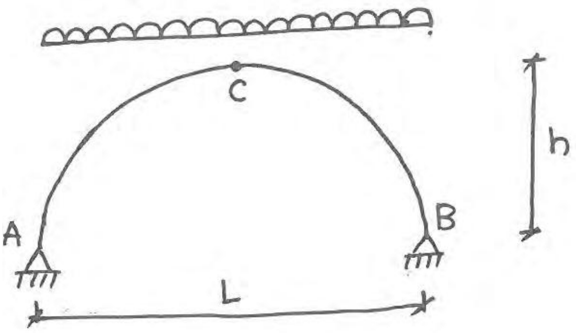


Equation of parabolic profile

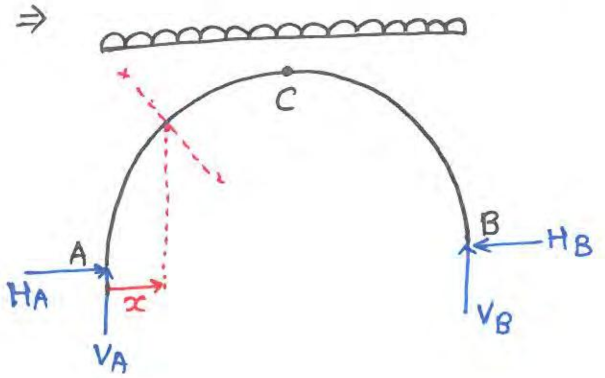
$$y = \frac{4h}{L^2}x(L-x)$$

- Origin is at left support
- Y-axis is upward.
- L is span of equal level supports.

Ex. Calculate horizontal thrust at supports and BM at any section in a symmetrical 3-hinged parabolic arch subjected to udl on horizontal span



$\Rightarrow$



$\Sigma F_x = 0$

$\Rightarrow H_A - H_B = 0$ ----(i)

$\Sigma F_y = 0$

$\Rightarrow V_A + V_B - w \times L = 0$ ---(ii)

$\Sigma M_z = 0$

$\Rightarrow \Sigma M_A = 0$

$\Rightarrow w \times L \times \frac{L}{2} - V_B \times L = 0$ ---(iii)

$M_c = 0$ (LHS)

$\Rightarrow V_A \times \frac{L}{2} - w \times \frac{L}{2} \times \frac{L}{4} - Hh = 0$ ---(iv)

from eq^n (i) to (iv)

$V_A = V_B = \frac{wL}{2}$

$H_A = H_B = \frac{wL^2}{8h}$

BM at any section

$$BM_x = V_A \cdot x - \frac{wx^2}{2} - Hy$$

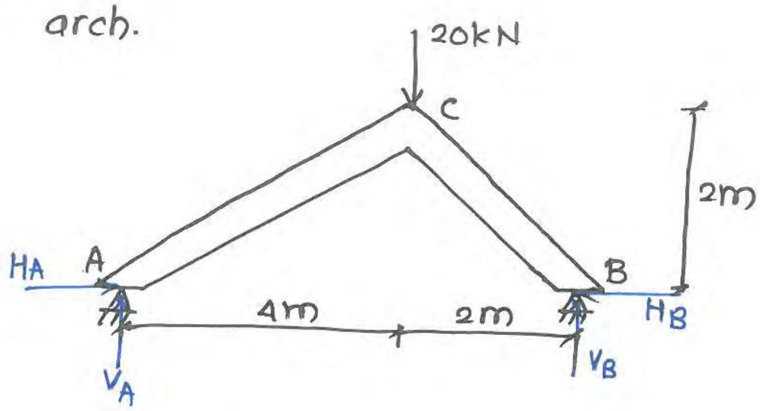
$$= \frac{wL}{2}x - \frac{wx^2}{2} - \frac{wL^2}{8h}\left\{\frac{4hx}{L^2}(L-x)\right\}$$

$$= 0$$

- Linear/ Theoretical / Funicular Arch:

If bending moment is zero throughout along the span then arch is called as linear/ funicular arch. Shape of this arch is proportional to BMD of equivalent simply supported beam.

Ex. Determine horizontal thrust at supports for given linear arch.



$\Rightarrow$ $\Sigma F_x = 0$

$\Rightarrow H_A - H_B = 0$ ------(i)

$\Sigma F_y = 0$

$\Rightarrow V_A + V_B - 20 = 0$ ----(ii)

$\Sigma M_z = 0$

$\Rightarrow \Sigma M_A = 0$

$\Rightarrow 20 \times 4 - V_B \times 6 = 0$ ---(iii)

Since arch is linear arch so BM is zero everywhere.

$M_c = 0$ (LHS)

$\Rightarrow V_A \times 4 - H_A \times 2 = 0$ --(iv)

from equation (i) to (iv)

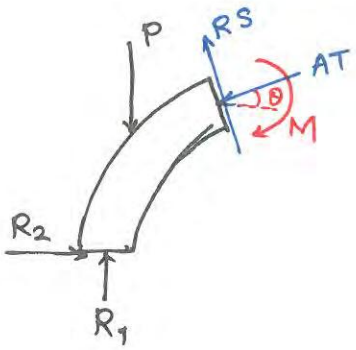
$V_A = 6.67$ kN.

$V_B = 13.33$ kN

$H_A = H_B = 13.33$ kN

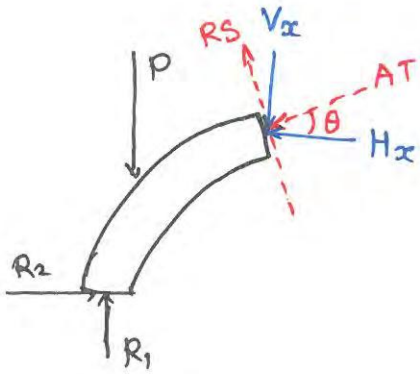
8.5.2 Normal / Axial Thrust and Radial Shear:

Method I :-



By using $\Sigma F_x = 0$ and $\Sigma F_y = 0$, AT and RS can be calculated.

Method II :



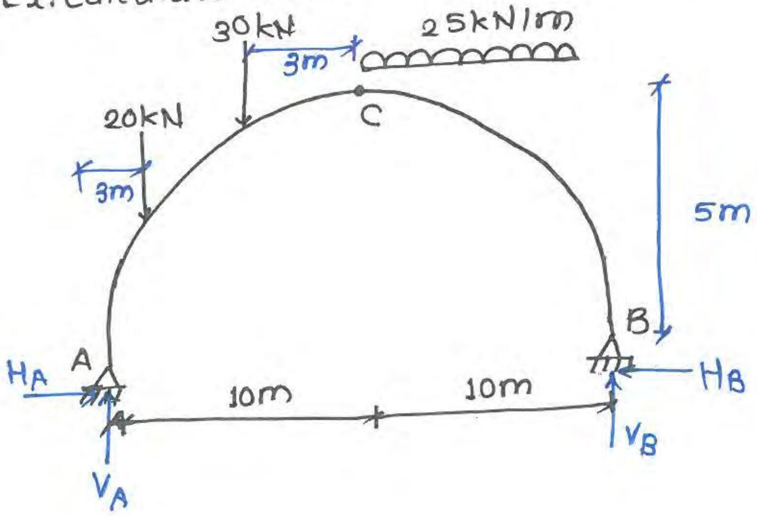
$$AT = H_x \cos\theta + V_x \sin\theta$$

$$RS = H_x \sin\theta - V_x \cos\theta$$

H_x = Net horizontal force on segment towards right

V_x = Net vertical force on segment upward.

Ex. Calculate horizontal thrust at support, Axial thrust and Radial shear at 5m from support A. Arch is parabolic.



⇒ $\Sigma F_x = 0$

$H_A - H_B = 0$ ·····(i)

$\Sigma F_y = 0$

$V_A + V_B - 20 - 30 - 25 \times 10 = 0$ ···(ii)

$\Sigma M_z = 0$

$\Rightarrow \quad \Sigma M_A = 0$

$\Rightarrow \quad 20\times3 + 30\times7 + 25\times10\times15 - V_B\times20 = 0 \quad \cdots(iii)$

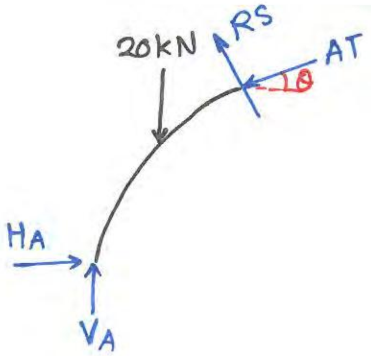
$M_c = 0 \quad$ RHS

$\Rightarrow \quad -V_B\times10 + H_B\times5 + 25\times10\times5 = 0 \quad \cdots\cdots(iv)$

from equation (i) to (iv)

$$\boxed{\begin{aligned} H_A &= H_B = 152\,kN \\ V_A &= 99\,kN \\ V_B &= 201\,kN. \end{aligned}}$$

For RS and AT:-



$$y = \frac{4h}{L^2}x(L-x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{4h}{L^2}(L-2x)$$

at $x=5$

$$\Rightarrow \tan\theta = \frac{4\times5}{20^2}(20-2\times5)$$

$$= 0.5$$

$\sin\theta = 0.45$

$\cos\theta = 0.89$

Now, $\Sigma F_x = 0$

$H_A - AT\cos\theta - RS\sin\theta = 0$

$\Rightarrow 152 - AT\times0.89 - RS\times0.45 = 0 \quad \cdots(v)$

$\Sigma F_y = 0$

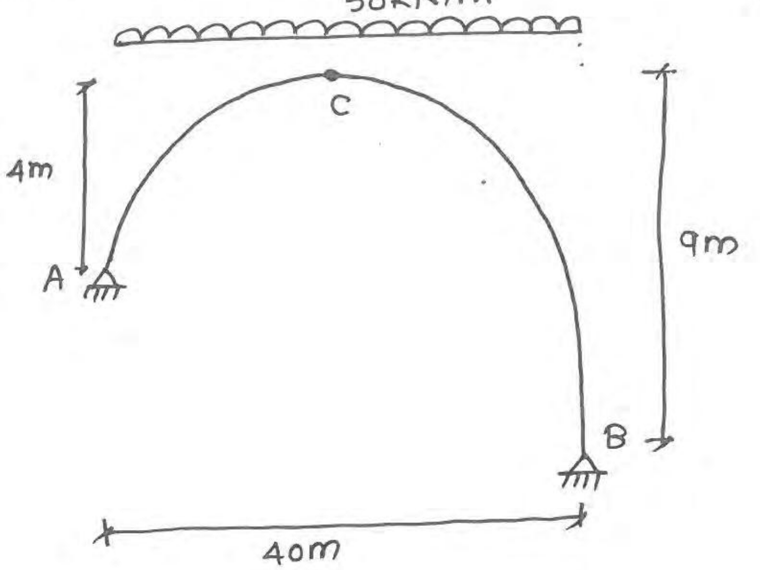
$V_A - 20 - AT\sin\theta + RS\cos\theta = 0$

$\Rightarrow 99 - 20 - AT\times0.45 + RS\times0.89 = 0 \quad \cdots(vi)$

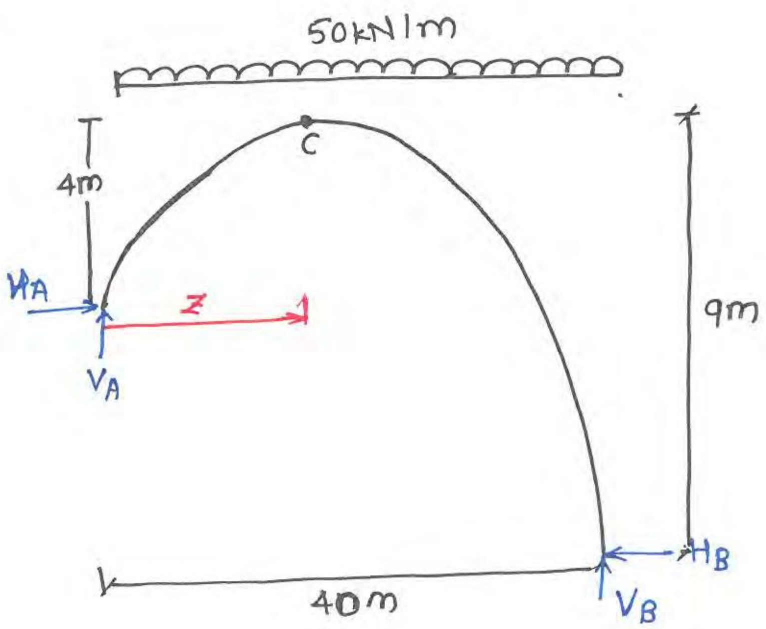
from eq^n (v) and (vi)

$$\boxed{\begin{aligned} AT &= 171.98\,kN \\ RS &= -2.68\,kN. \end{aligned}}$$

Ex Calculate horizontal thrust for parabolic arch given below. Hinge is at crown.



$\Rightarrow$



$$\Sigma F_x = 0$$

$$\Rightarrow H_A - H_B = 0 \quad \text{--(i)}$$

$$\Sigma F_y = 0$$

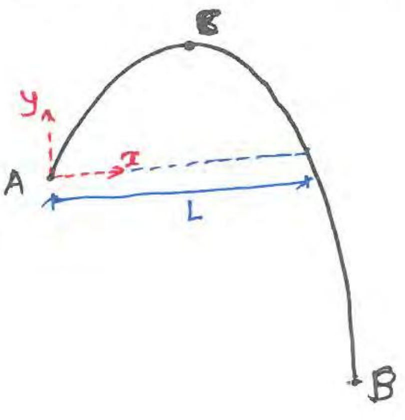
$$\Rightarrow V_A + V_B - 50 \times 40 = 0 \quad \text{---(ii)}$$

$$\Sigma M_z = 0$$

$$\Rightarrow \quad \Sigma M_A = 0$$

$$\Rightarrow$$

In above 4 equations, 5 unknowns (H_A, V_A, H_B, V_B and z) are present so one more equation is required. 5th equation can be formulated using equation of parabola



$$y = \frac{4h}{L^2} x(L-x)$$

$$\Rightarrow y = \frac{4 \times 4}{L^2} x(L-x)$$

At $x = 0$, $y = 0$

$$\Rightarrow 0 = \frac{4 \times 4}{L^2} \times 0 \times (L-0)$$

$$\Rightarrow 0 = 0 \quad \text{(useless)}$$

At $x = \frac{L}{2}$, $y = 4m$

$$\Rightarrow 4 = \frac{4 \times 4}{L^2} \frac{L}{2} \left(L - \frac{L}{2} \right)$$

$$\Rightarrow 4 = 4 \qquad \text{(useless)}$$

At $x = 40$, $y = -5m$

$$\Rightarrow -5 = \frac{4 \times 4}{L^2} \times 40 (L - 40)$$

$$\Rightarrow L = 32m$$

$$\text{So } z = \frac{L}{2} = \frac{32}{2} = 16m \quad \cdots (v)$$

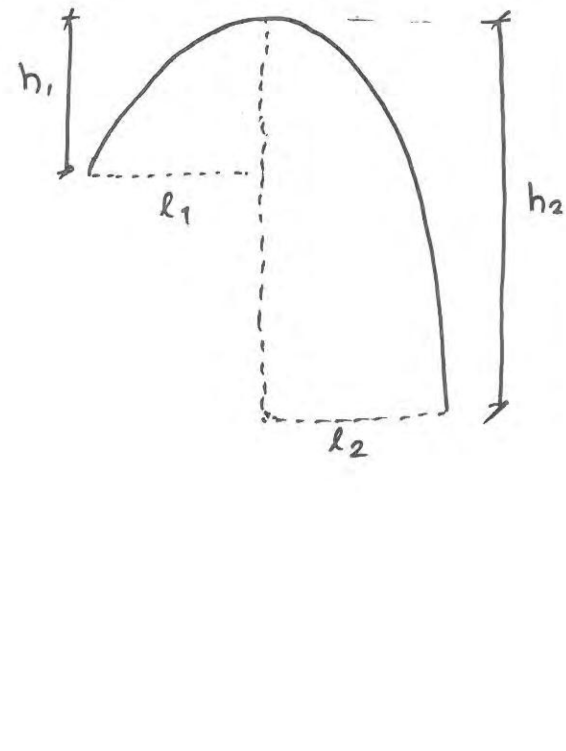
from (i) to (v)

$H_A = H_B = 1600$ kN

$V_A = 800$ kN

$V_B = 1200$ kN.

Alternatively:-



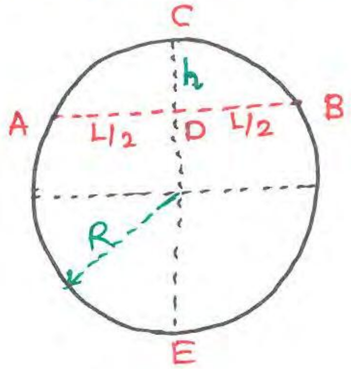
$$\boxed{\sqrt{\frac{h_1}{h_2}} = \frac{l_1}{l_2}} \quad **$$

$$\Rightarrow \sqrt{\frac{4}{9}} = \frac{z}{40 - z}$$

$$\Rightarrow z = 16m$$

8.5.3 Circular 3-Hinged Arch:

Property of circle:



$$AD \times DB = CD \times DE$$

$$\Rightarrow L/2 \times L/2 = h \times (2R - h)$$

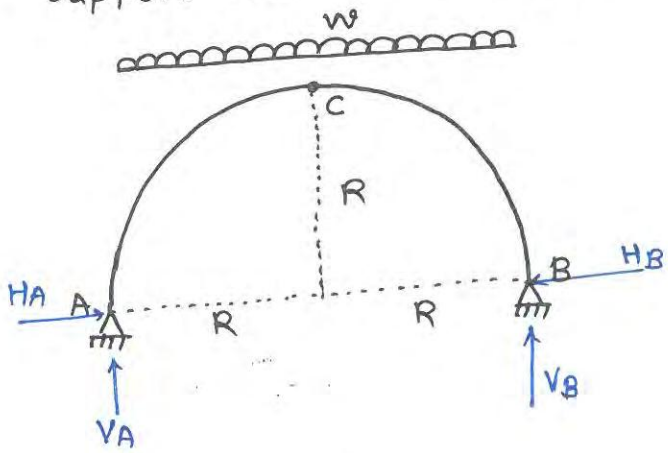
Ex. What is the central rise of a segmental circular arch of radius 250 m and span 80m.

$$\frac{L}{2} \times \frac{L}{2} = h \times (2R - h)$$

$$\frac{80}{2} \times \frac{80}{2} = h \times (2 \times 250 - h)$$

$$h = 3.22 m$$

Ex. A 3-hinged symmetrical semicircular arch is subjected to udl 'w' on horizontal span. calculate horizontal thrust at support and BM at any section.



$$\Sigma F_x = 0$$

$$\Rightarrow H_A - H_B = 0 \quad \text{---(i)}$$

$$\Sigma F_y = 0$$

$$V_A + V_B - w(2R) = 0 \quad \text{---(ii)}$$

$$\Sigma M_z = 0$$

$$\Rightarrow \Sigma M_A = 0$$

$$\Rightarrow w \times (2R) \times R - V_B \times (2R) = 0 \quad \text{---(iii)}$$

$M_c = 0$ (LHS)

$$\Rightarrow V_A \times R - H_A \times R - \frac{wR^2}{2} = 0 \quad \text{---(iv)}$$

From eq^n (i) to (iv)

$$V_A = V_B = wR$$

$$H_A = H_B = \frac{wR}{2}$$

• **Note:**

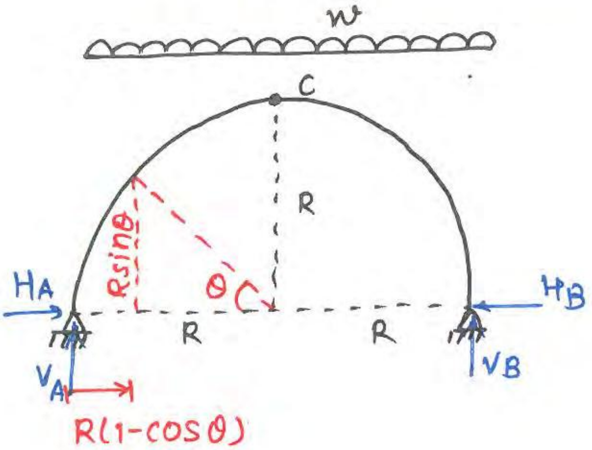
Expression of H of 3-Hinged parabolic arch subjected to UDL.

$$H = \frac{wL^2}{8h}$$

putting $L = 2R$ & $h = R$

$$H = \frac{wR}{2}$$

It means horizontal thrust of 3-hinged arch does not depend on shape of arch.



BM at any section:-

$$BM_\theta = V_A \times R(1-\cos\theta) - H_A \cdot R\sin\theta - \frac{w[R(1-\cos\theta)]^2}{2}$$

$$= wR \cdot R(1-\cos\theta) - \frac{wR}{2} \times R\sin\theta - \frac{wR^2}{2}(1 - 2\cos\theta + \cos^2\theta)$$

$$= wR^2 - \cancel{wR^2\cos\theta} - \frac{wR^2}{2}\sin\theta - \frac{wR^2}{2} + \cancel{wR^2\cos\theta} - \frac{wR^2}{2}\cos^2\theta$$

$$\Rightarrow BM_\theta = \frac{wR^2}{2} - \frac{wR^2}{2}\cos^2\theta - \frac{wR^2}{2}\sin\theta$$

$$= \frac{wR^2}{2}(1 - \cos^2\theta - \sin\theta)$$

$$= \frac{wR^2}{2}(\sin^2\theta - \sin\theta)$$

for maximum BM :-

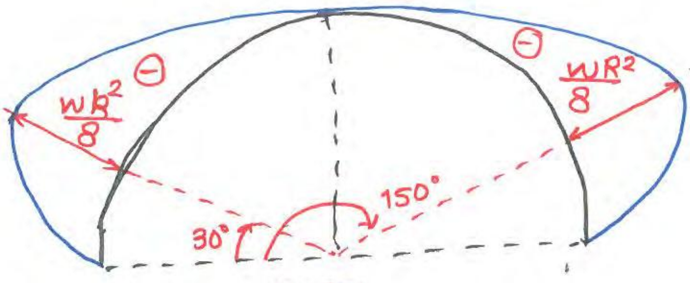
$$\frac{d(BM_\theta)}{d\theta} = 0$$

$$\Rightarrow \frac{wR^2}{2}(2\sin\theta\cos\theta - \cos\theta) = 0$$

$$\Rightarrow \cos\theta(2\sin\theta - 1) = 0$$

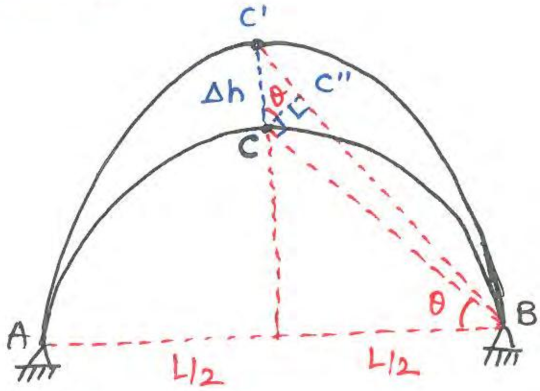
Now, $2\sin\theta - 1 = 0 \Rightarrow \sin\theta = 1/2 \Rightarrow \theta = 30°$ & $150°$

At $\theta = 30^\circ$ and 150° $BM = \frac{-wR^2}{8}$



BMD.

8.5.4 Effect of Temperature of 3-Hinged Arch:-



$C'C'' = BC \cdot \alpha \cdot T$

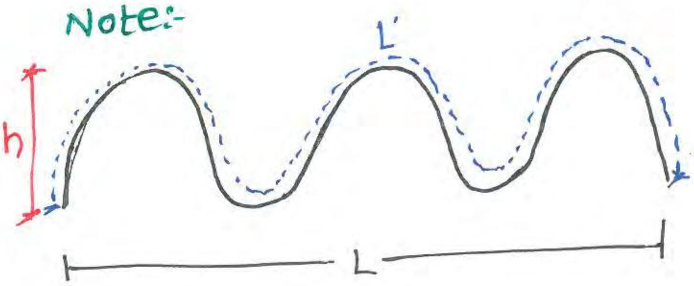
$= \sqrt{h^2 + (L/2)^2} \cdot \alpha \cdot T$

$\Delta h = CC' = \frac{C'C''}{\sin\theta}$

$= \frac{\sqrt{h^2+(L/2)^2} \cdot \alpha \cdot T}{h / \sqrt{h^2+(L/2)^2}}$

$$\boxed{\Delta h = \frac{(4h^2+L^2)\,\alpha T}{4h}}$$

Note:-



horizontal expansion $= L\alpha T$

Vertical expansion $= h\alpha T$

Ex. A 3-hinged arch of span 20 m and central rise 4 m is subjected to udl 25 kN/m. This arch is also subjected to rise in temperature of 40°C. What is the change in horizontal thrust due to increase of temperature? $\alpha = 12 \times 10^{-6}/^\circ C$.

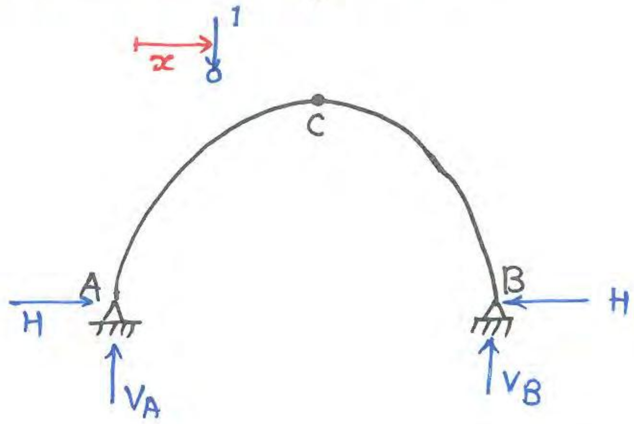
$\Rightarrow$ $H = \frac{wL^2}{8h}$

$\frac{dH}{dh} = -\frac{wL^2}{8h^2}$ $\Rightarrow$ $dH = -\frac{wL^2}{8h^2}\, dh = -\frac{wL^2}{8h^2}\left(\frac{(4h^2+L^2)\cdot \alpha T}{4h}\right)$

$= -1.08\,kN.$

–ve sign indicates that 'H' is decreasing with increase in 'h'

8.5.5 ILD of 3-Hinged Arch:-



For V_A:-

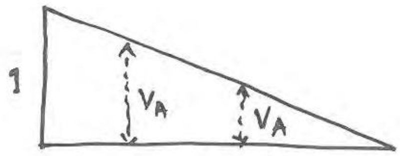
$\Sigma M_B = 0$

$\Rightarrow V_A \times L - 1 \times (L - x) = 0$

$\Rightarrow V_A = \dfrac{L-x}{L}$

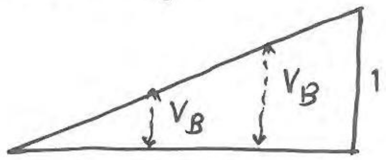
At $x = 0$, $V_A = 1$

$x = L$, $V_A = 0$



ILD of V_A

Similarly,



ILD of V_B.

For H:-

Case I:- $0 \le x \le \frac{L}{2}$

$M_c = 0$ (RHS)

$\Rightarrow V_A \times \frac{L}{2} - H \times h - 1 \times (\frac{L}{2} - x) = 0$

$\Rightarrow \left(\frac{L-x}{L}\right) \frac{L}{2} - H \times h - \left(\frac{L}{2} - x\right) = 0$

$\Rightarrow H = \dfrac{x}{2h}$

At $x = 0$, $H = 0$

$x = \frac{L}{2}$, $H = \frac{L}{4h}$

ILD of H

Case II:- $\frac{L}{2} \le x \le L$

$M_c = 0$ (LHS)

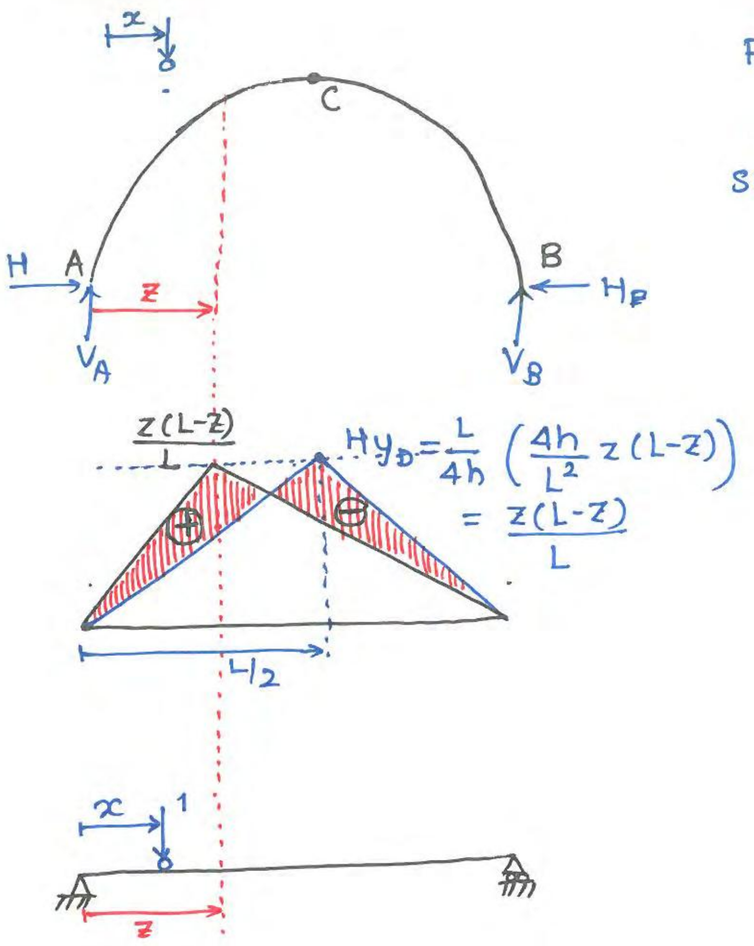
$\Rightarrow V_A \times \frac{L}{2} - H \times h = 0$

$\Rightarrow \left(\frac{L-x}{L}\right) \times \frac{L}{2} - H \times h = 0$

$\Rightarrow H = \dfrac{L-x}{2h}$

At $x = \frac{L}{2}$, $H = \frac{L}{4h}$

$x = L$, $H = 0$



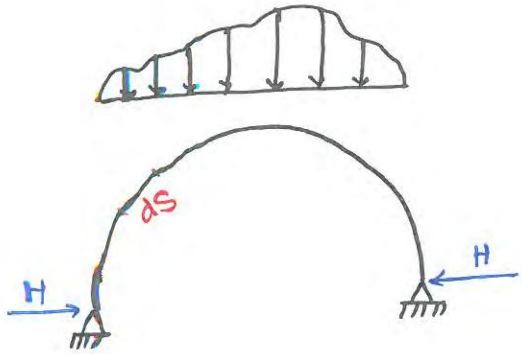
For M_D:-

It is plotted by method of superposition.

$$BM_{arch} = BM_{ss.} - Hy$$

8.6 Two-Hinged Arch:-

8.6.1 Expression of Horizontal Thrust:-



DSI = 1

So strain energy method is being used to analyze.

Assuming horizontal thrust as redundant.

$$\frac{\partial U}{\partial H} = 0$$

$$\Rightarrow \frac{\partial \int \frac{M^2 ds}{2EI}}{\partial H} = 0$$

$$\Rightarrow \int \frac{M \frac{\partial M}{\partial H} ds}{EI} = 0$$

As we know that,

$$M_{arch} = M_{s.s.} - Hy$$

$$\Rightarrow M = M_s - Hy$$

$$\Rightarrow \frac{\partial M}{\partial H} = -y$$

Now,

$$\int \frac{(M_s - Hy) \cdot (-y) ds}{EI} = 0$$

$$\Rightarrow \boxed{H = \frac{\int \frac{M_s y \, ds}{EI}}{\int y^2 ds / EI}}$$

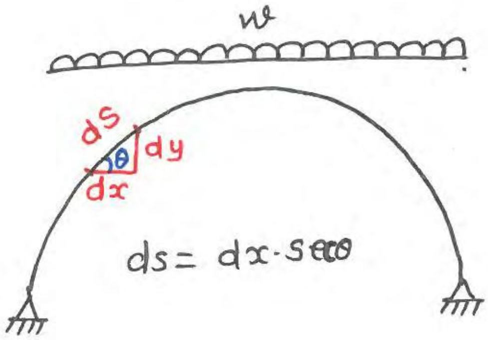
In case of support settlement and temperature change:-

$$\frac{\partial U}{\partial H} = L \cdot \alpha \cdot T + \Delta$$

$$\boxed{H = \frac{\int \frac{M_s \cdot y \cdot ds}{EI} + L\alpha T + \Delta}{\int \frac{y^2 ds}{EI}}}$$

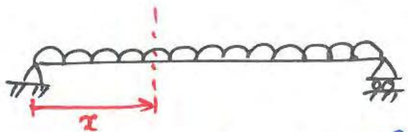
8.6.2 Two-Hinged Parabolic Arch:-

Ex. A symmetrical parabolic Two-hinged arch is subjected to udl on horizontal span. Calculate horizontal thrust.



$$H = \frac{\int \frac{M_s \cdot y \, ds}{EI}}{\int \frac{y^2 ds}{EI}}$$

For M_s:-



$$M_s = \frac{wL}{2} \cdot x - \frac{wx^2}{2}.$$

For y:-

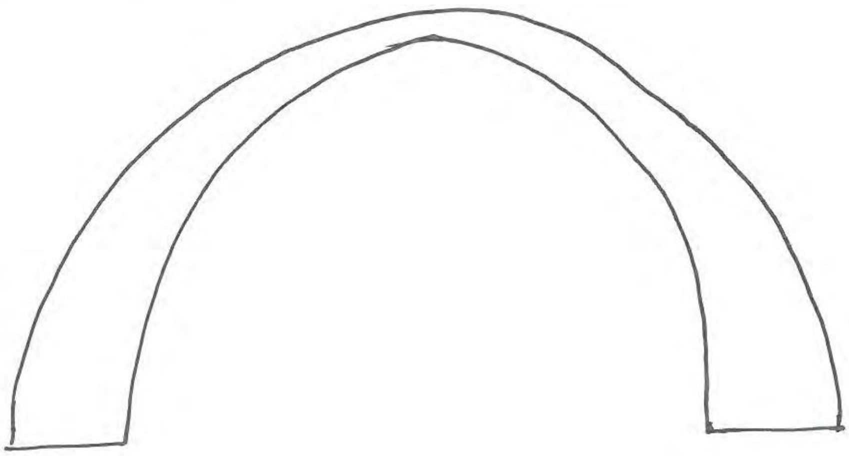
$$y = \frac{4h}{L^2} x (L-x)$$

For ds:-

$$ds = dx \cdot \sec\theta.$$

$$H = \frac{\int_0^L \frac{\left(\frac{wL}{2} \cdot x - \frac{wx^2}{2}\right) \times \left(\frac{4h}{L^2} \cdot x \cdot (L-x)\right) \cdot dx \sec\theta}{EI}}{\int_0^L \frac{\left(\frac{4h}{L^2} x (L-x)\right)^2 \cdot dx \sec\theta}{EI}}$$

In above integration, $\sec\theta$ also depends on x which makes above integration very difficult. To simplify this, considering variation of MI of parabolic arch in above calculation.



$I = I_0 \sec\theta$

where

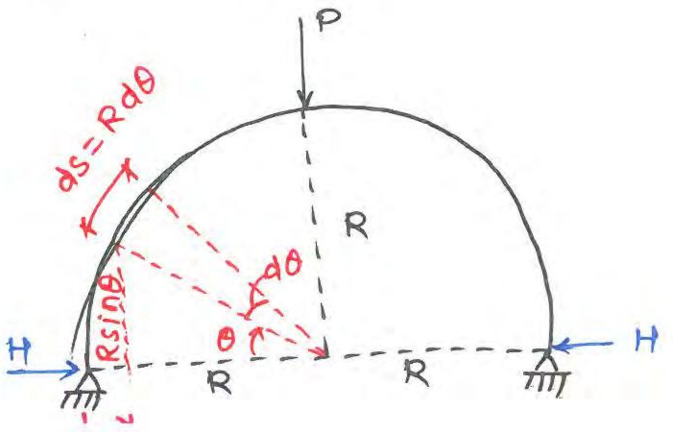
I_0 = MI at crown

$$H = \frac{\displaystyle\int_0^L \frac{\left(\frac{wL}{2}x - \frac{wx^2}{2}\right)\left(\frac{4h}{L^2}\cdot x(L-x)\right) dx\, \cancel{\sec\theta}}{\cancel{EI_0}\, \cancel{\sec\theta}}}{\displaystyle\int_0^L \frac{\left(\frac{4h}{L^2}x(L-x)\right)^2 dx\, \cancel{\sec\theta}}{\cancel{EI_0}\, \cancel{\sec\theta}}}$$

$$\Rightarrow \boxed{H = \frac{wL^2}{8h}}$$

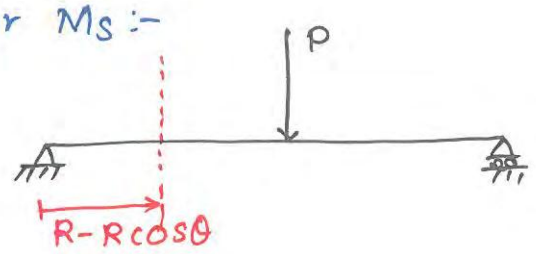
8.6.3 Two-Hinged Circular Arch:-

Ex. A symmetrical two-hinged circular arch is subjected to point load at crown. Calculate horizontal thrust.



$$H = \frac{\int \frac{M_s y\, ds}{EI}}{\int \frac{y^2 ds}{EI}}$$

For M_s :-



$$M_s = \frac{P}{2}\left(R - R\cos\theta\right)$$

For y :-

$$y = R\sin\theta$$

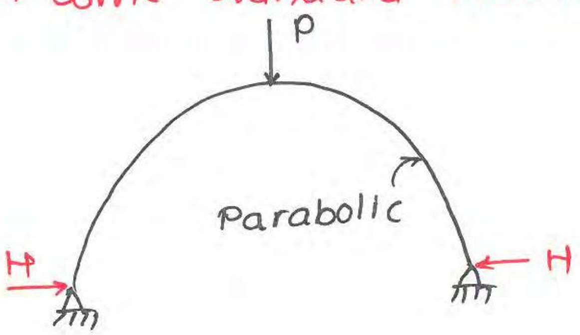
For ds :-

$$ds = R\,d\theta$$

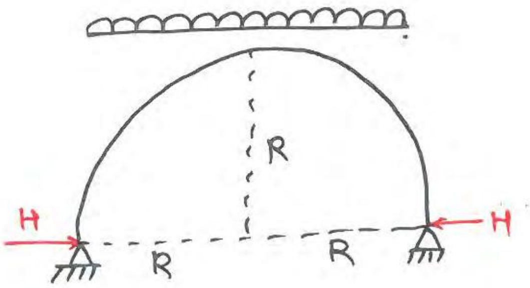
$$H = \frac{2\times\displaystyle\int_0^{\pi/2} \frac{\left[\frac{P}{2}\left(R-R\cos\theta\right)\right]\left[R\sin\theta\right]\cdot R\,d\theta}{\cancel{EI}}}{2\times\displaystyle\int_0^{\pi/2} \frac{(R\sin\theta)^2\cdot R\,d\theta}{\cancel{EI}}}$$

$$\boxed{H = \frac{P}{\pi}}$$

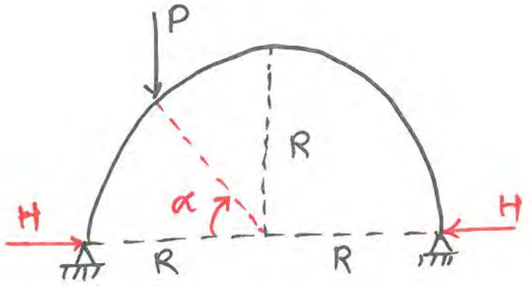
8.6.4 Some Standard Results :-



** $\boxed{H = \frac{25}{128}\,\frac{PL}{h}}$



** $\boxed{H = \frac{4}{3}\,\frac{wR}{\pi}}$



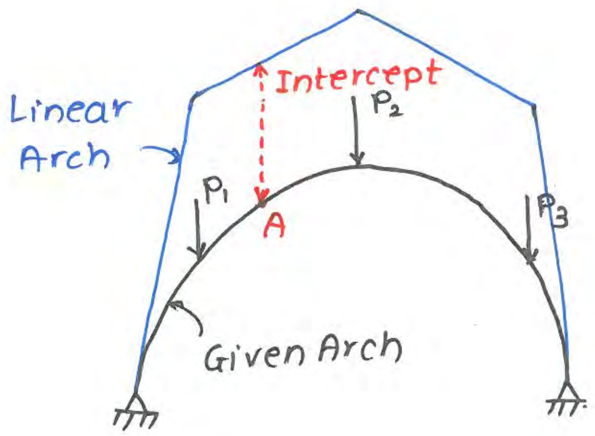
$$\boxed{H = \frac{P}{\pi} \cdot \sin^2\alpha}$$

For $\alpha = \pi/2$

$$H = \frac{P}{\pi}$$

8.7 Eddy's Theorem:-

If a linear arch is superimposed over the given arch then BM at any section in given arch is proportional to the intercept of given arch and linear arch.

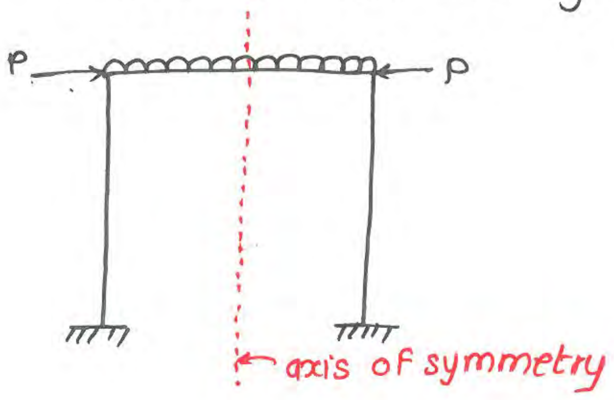


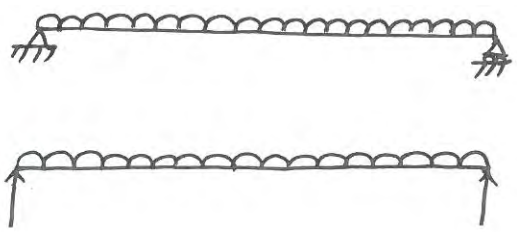
$BM_A \propto$ Intercept.

8.8 Concept of Symmetry and Antisymmetry:-

Symmetrical Structure:-

If structure and loading both are symmetrical.

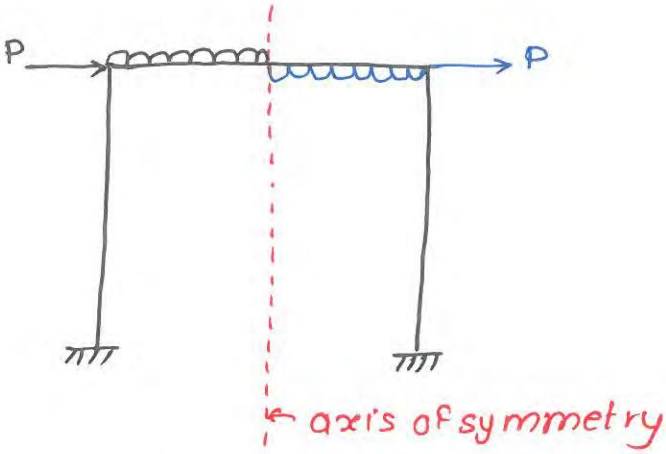




Visually unsymmetrical but behaves as symmetrical.

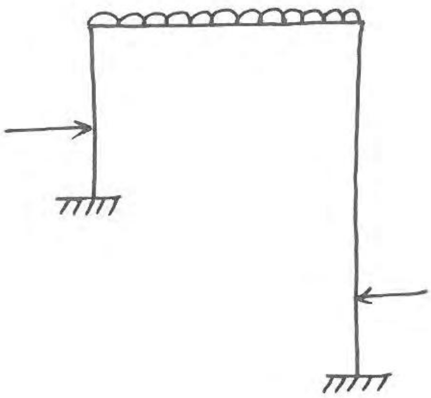
Anti-symmetrical Structure:-

If a structure is symmetrical and loading one one half is just opposit to mirror image of other half.



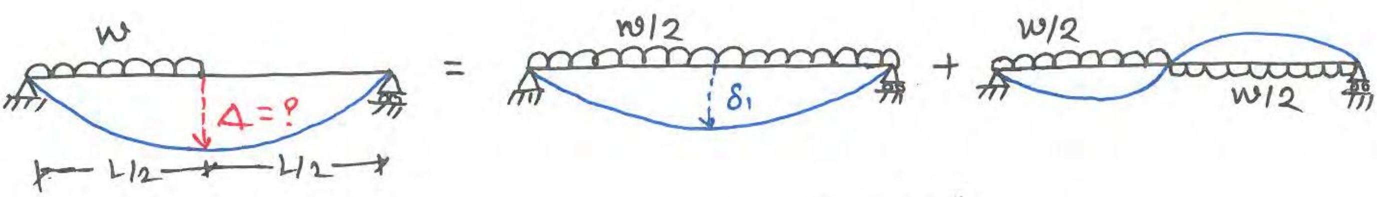
Unsymmetrical Structure:-

If loading or structure or both are unsymmetrical



- **Note:-**

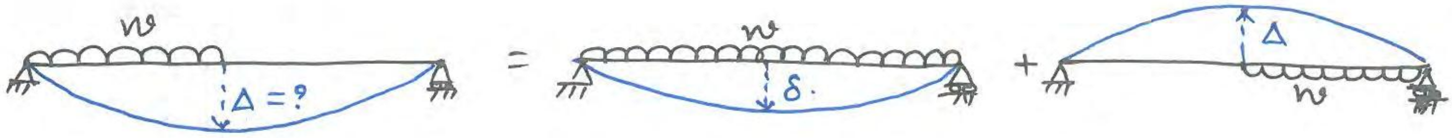
A symmetrical structure with any loading can be representd as sum of symmetrical and antisymmetrical structure.



$$\Delta = \delta_1 + \delta_2 = \frac{5}{384}\frac{(w/2)L^4}{EI} + 0$$

$$\Delta = \frac{5\,wL^4}{768\,EI}$$

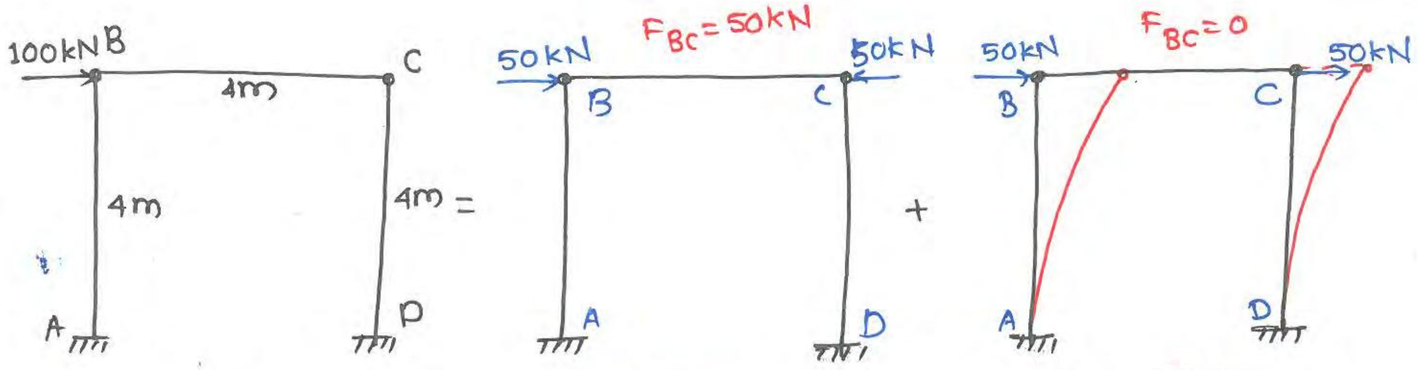
Alternatively:-



$$\Delta = \delta - \Delta$$

$$\Rightarrow \Delta = \frac{\delta}{2} = \frac{5}{768}\,\frac{wL^4}{EI}.$$

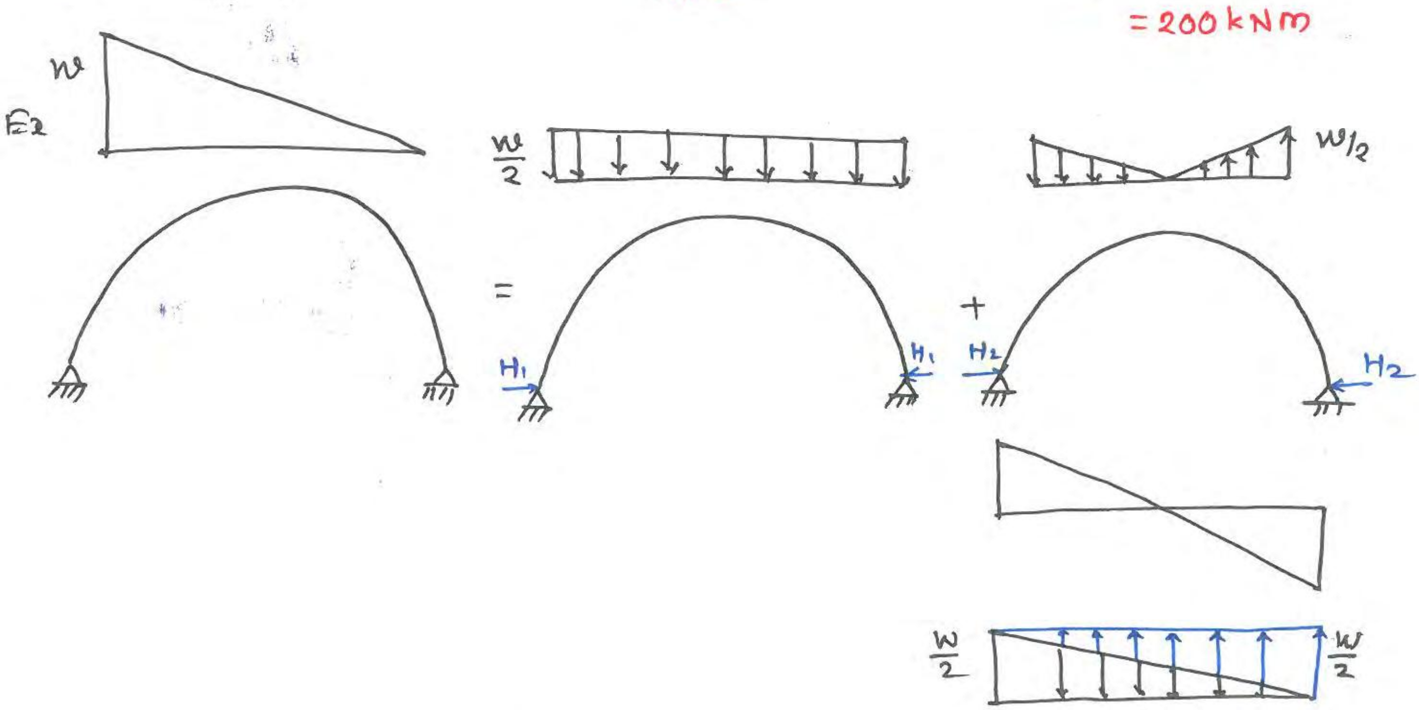
Ex.



$M_A = ?$ $\qquad$ $M_A = 0$ $\qquad$ $M_A = 50 \times 4$

$= 200\ \text{kNm}$

Ex



$$H = H_1 + H_2$$

$$= \frac{(w/2)\,L^2}{8h} + 0$$

$$= \frac{wL^2}{16h}$$