Exercise 10.5

Q1E

2657-10.5-1E

AID: 9514
RID: 378

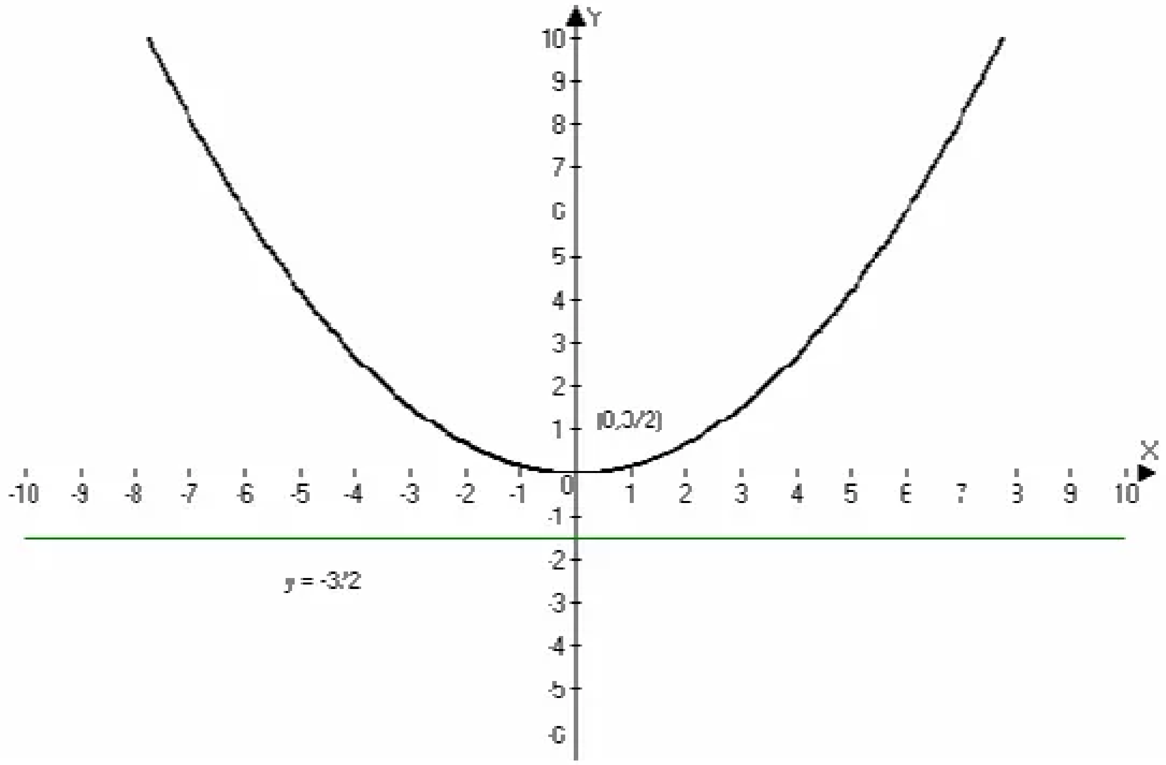
Given $x^2 = 6y$

We know $x^2 = 4py$ is an equation of the parabola with focus $(0, p)$ and directrix $y = -p$

$\Rightarrow 4p = 6 \Rightarrow p = \frac{3}{2}$

Therefore focus is $\left(0, \frac{3}{2}\right)$, directrix is $y = -\frac{3}{2}$ and vertex is $(0,0)$.

Graph



Q2E

2657-10.5-2E

AID: 9514
RID: 378

Given

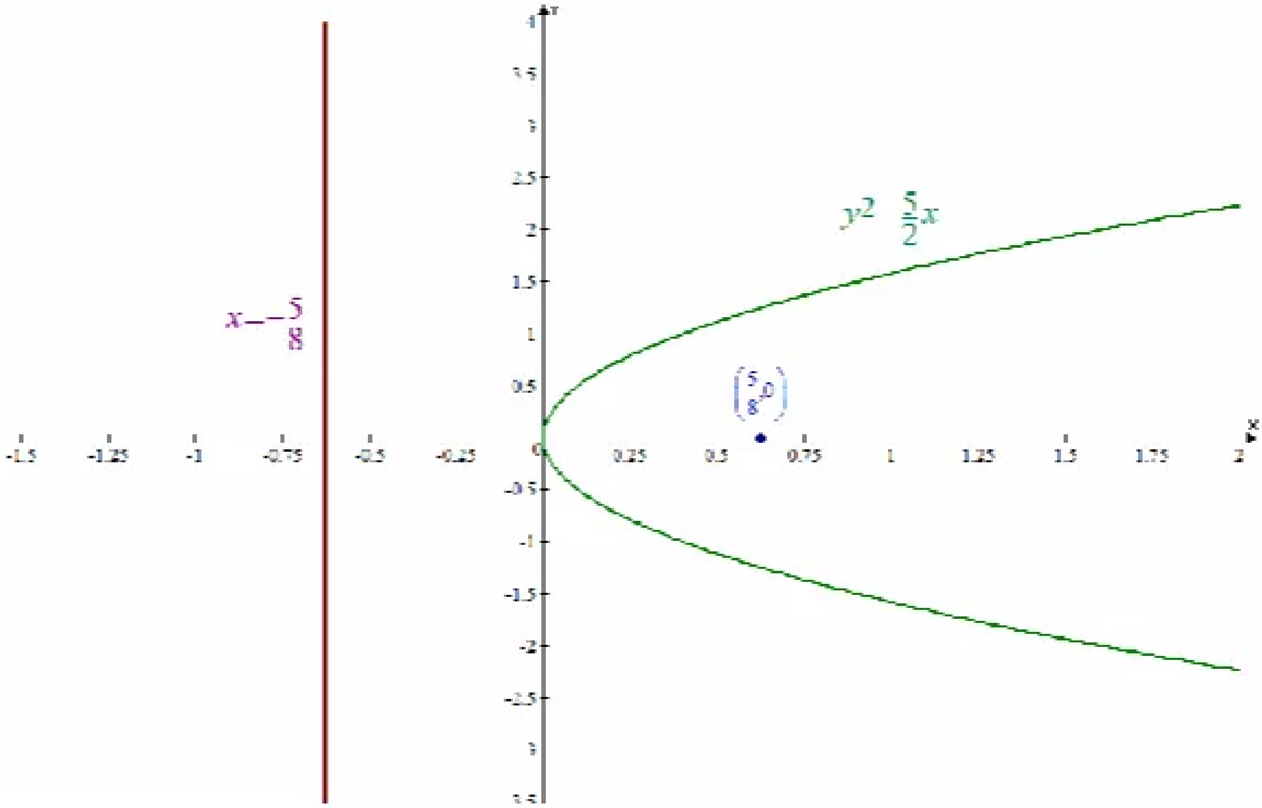
$2y^2 = 5x$

$\Rightarrow y^2 = \frac{5}{2}x$

$\Rightarrow 4p = \frac{5}{2} \Rightarrow p = \frac{5}{8}$

Therefore focus is $\left(\frac{5}{8}, 0\right)$, directrix is $x = -\frac{5}{8}$ and vertex is $(0,0)$.

Graph



Q3E

2657-10.5-3E AID: 9514
RID: 378

Given

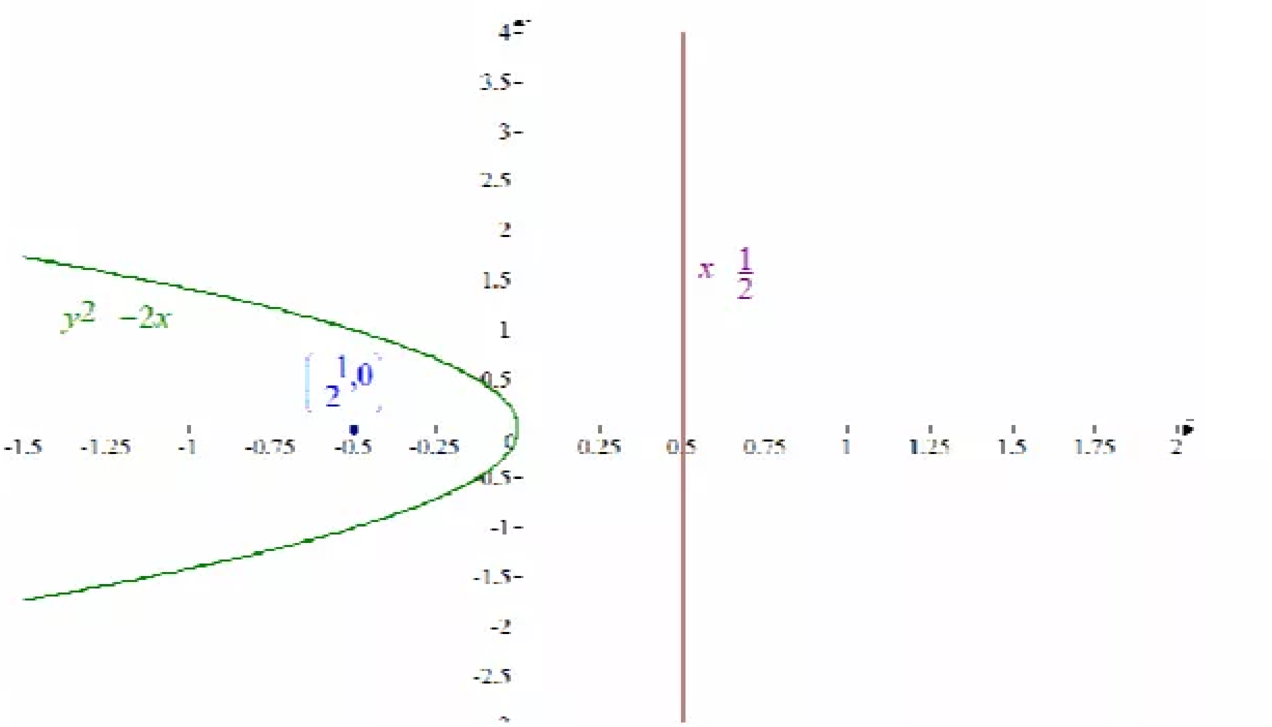
$2x = -y^2$

$\Rightarrow y^2 = -2x$

$\Rightarrow 4p = -2 \Rightarrow p = \frac{-1}{2}$

Therefore focus is $\left(\frac{-1}{2}, 0\right)$, directrix is $x = \frac{1}{2}$ and vertex is (0,0).

Graph



Q4E

Given

$3x^2 + 8y = 0$

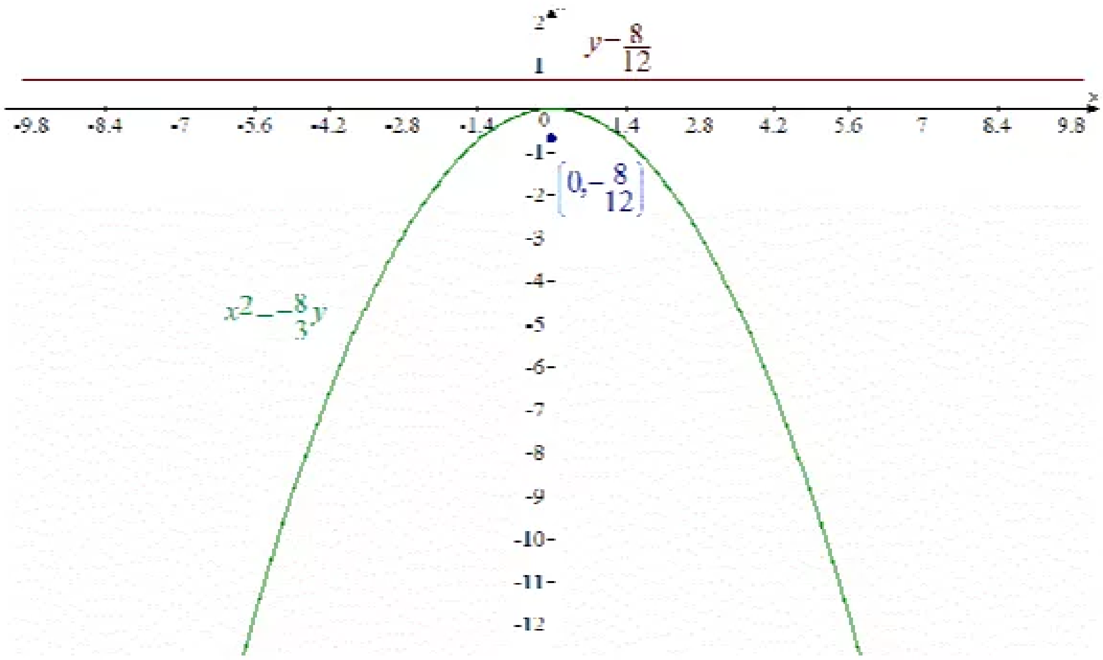
$\Rightarrow 3x^2 = -8y$

$\Rightarrow x^2 = -\frac{8}{3}y$

$\Rightarrow 4p = -\frac{8}{3} \Rightarrow p = \frac{-8}{12}$

Therefore focus is $\left(0, \frac{-8}{12}\right)$, directrix is $y = \frac{8}{12}$ and vertex is $(0,0)$.

Graph



Q5E

Given equation of the parabola is $(x+2)^2 = 8(y-3)$ ....(1)

Let $x+2 = X$ and $y-3 = Y$

Then equation becomes $X^2 = 8Y$ ....(2)

Comparing with $x^2 = 4py$

We have $4P = 8 \Rightarrow p = 2$ [for equation (2)]

So the focus is (0, P)

$\Rightarrow X = 0$ and $Y = 2$

so $x+2 = 0$ and $y-3 = 2$

$\Rightarrow x = -2$ and $y = 5$

So focus of the given parabola is $\boxed{(-2,\ 5)}$

Vertex is = (0, 0) [for equation (2)]

$\Rightarrow X = 0$ and $Y = 0$

$\Rightarrow x+2 = 0$ and $y-3 = 0$

$\Rightarrow x = -2$ and $y = 3$

Then vertex of the given parabola is $\boxed{(-2,\ 3)}$

Directrix for the equation (2) is Y = - P

$\Rightarrow y-3 = -2$

$\Rightarrow \boxed{y = 1}$

Directrix of the given parabola is $\boxed{y = 1}$

Now we sketch the curve

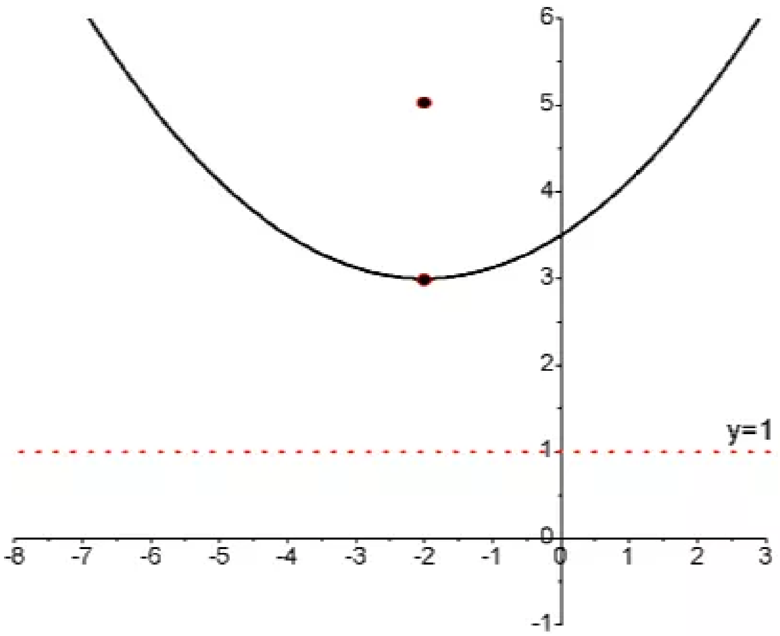


Fig.1

Q6E

Given equation of the parabola is $(x-1)=(y+5)^2$ ......(1)

Let $X = x-1$ *and* $Y = y+5$

Thus the equation becomes

$X = Y^2$ *or* $Y^2 = X$ ....(2)

Comparing with the equation becomes $y^2 = 4px$

We have 4p = 1 $\Rightarrow$ $p = 1/4$

Vertex is = (0, 0) [for equation (2)]

$\Rightarrow X = 0$ *and* $Y = 0$

$\Rightarrow x-1 = 0$ *and* $y+5 = 0$

$\Rightarrow x = 1$ *and* $y = -5$

So vertex of the given parabola is $\boxed{(1, -5)}$

Focus is =(p, 0) [for equation (2)]

$\Rightarrow X = p$ *and* $Y = 0$

$\Rightarrow x-1 = 1/4$ *and* $y+5 = 0$

$\Rightarrow x = 5/4$ *and* $y = -5$

Thus focus of the given parabola is $\boxed{(5/4, -5)}$

For equation (2) the directrix is $X=-p$

$\Rightarrow x-1=-1/4$

$\Rightarrow x=3/4$

Directrix of the given parabola is $\boxed{x=3/4}$

Now we sketch the curve

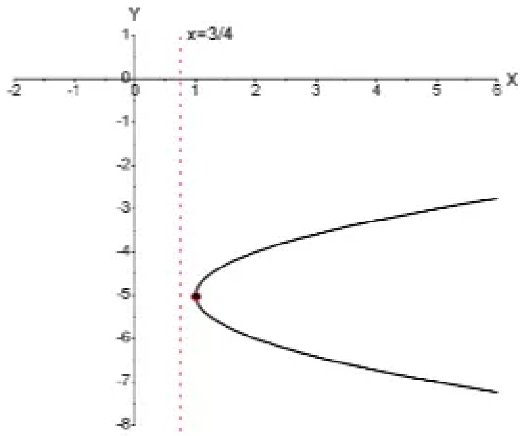


Fig.1

Q7E

Given equation of the parabola is $y^2+2y+12x+25=0$ ---(1)

Making perfect square

$\Rightarrow y^2+2y+1+12x+24=0$

$\Rightarrow (y+1)^2+12(x+2)=0$

$\Rightarrow (y+1)^2=-12(x+2)$

Let Y = y + 1 and X=x + 2

Then equation becomes

$Y^2=-12X$ --- (2)

Comparing with $Y^2=4pX$

We have

$4p=-12$ $\Rightarrow \boxed{p=-3}$

Then focus is = (p, 0) [for equation (2)]

$\Rightarrow X=p,\quad y=0$

$\Rightarrow x+2=-3 \quad and \quad y+1=0$

$\Rightarrow x=-5 \quad and \quad y=-1$

Thus focus of the given parabola is $\boxed{(-5,-1)}$

Vertex is at (0, 0) [for equation(2)]

$\Rightarrow X=0 \quad and \quad Y=0$

$\Rightarrow x+2=0 \quad and \quad y+1=0$

$\Rightarrow x=-2 \quad and \quad y=-1$

Thus vertex is $\boxed{(-2,-1)}$

Driectrix is $X = -p \quad \Rightarrow x+2=3$

Thus directrix is $\boxed{x=1}$

Now we sketch the curve

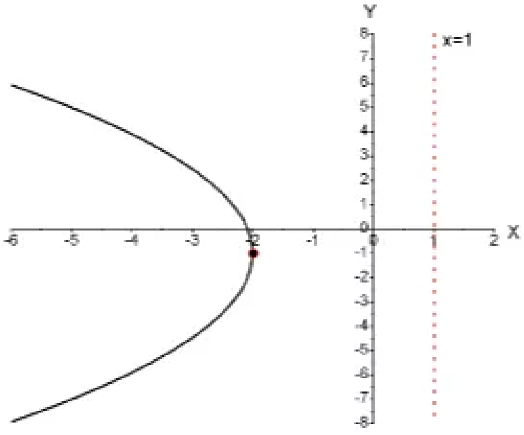


Fig.1

Q8E

Given equation of the parabola is $y+12x-2x^2=16$

$$\Rightarrow y = 16+2x^2-12x$$

$$\Rightarrow \frac{1}{2}y = x^2-6x+8$$

Making perfect square

$$\frac{1}{2}y = x^2-6x+9-1$$

$$\Rightarrow \frac{1}{2}y = (x-3)^2-1$$

$$\Rightarrow (x-3)^2 = \frac{1}{2}y+1$$

$$\Rightarrow (x-3)^2 = \frac{1}{2}(y+2)$$

Let $x-3 = X$ *and* $y+2=Y$

Then equation becomes $X^2 = \frac{1}{2}Y \qquad ---(2)$

Comparing with $X^2 = 4pY$

$\Rightarrow 4p = 1/2 \quad \Rightarrow \quad p = 1/8$

Vertex is = (0,0) [for equation(2)]

$\Rightarrow X=0$ *and* $Y=0$

$\Rightarrow x-3=0$ *and* $y+2=0$

$\Rightarrow x=3$ and $y=-2$

Thus vertex of the given parabola is $\boxed{(3,-2)}$

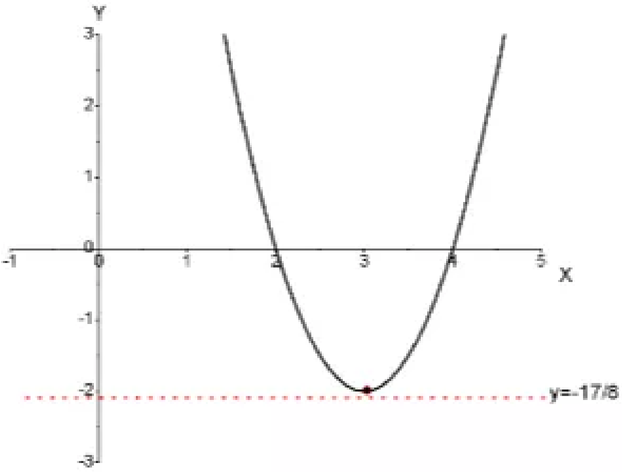
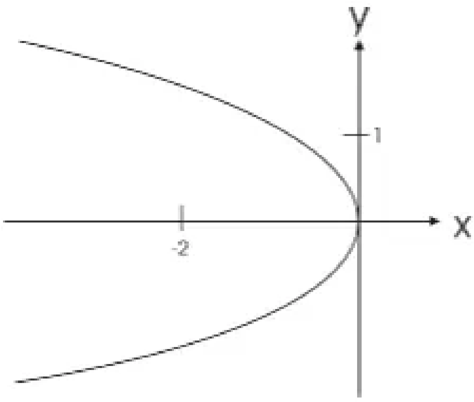


Fig.1

Q9E



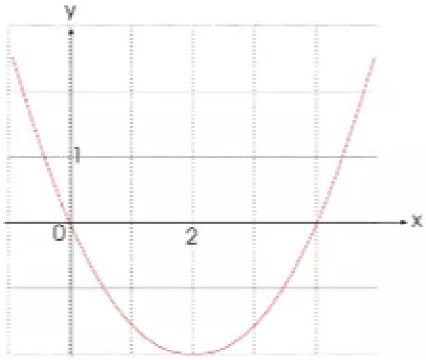
From figure we see that vertex of parabola is at the origin
Since directrix of the parabola will be parallel to Y axis.
So equation of the parabola is $y^2 = 4px, \quad p < 0$
This parabola passes through $(-1,1)$ *and* $(-1,-1)$

So these points will satisfy the equation of parabola thus

$$(-1)^2 = 4p(-1)$$
$$\Rightarrow 1 = -4p$$
$$\Rightarrow p = -1/4$$
$$\Rightarrow 4p = -1$$

Then equation of the parabola is $\boxed{y^2 = -x}$ or $\boxed{x = -y^2}$
This is the equation of parabola.
Focus of the parabola is $(p,0) = \boxed{(-1/4,\ 0)}$
And directrix is $x = -p \Rightarrow \boxed{x = 1/4}$

Q10E



From the graph we see that vertex of the parabola is at (2,-2)

Then equation of parabola will be $(x-2)^2 = 4p(y+2)$

Since this parabola passes through the points (4, 0) and (0, 0)

So these points will satisfy the equation of parabola

$$(0-2)^2 = 4p(0+2)$$
$$4 = 4p \times 2$$
$$\Rightarrow p = 1/2$$

Then equation of parabola becomes $\boxed{(y+2) = \frac{1}{2}(x-2)^2}$

Focus of the parabola is $(0, p) = \boxed{(2, -3/2)}$

Directrix is $\boxed{y = -5/2}$

Q11E

2657-10.5-11E

AID: 9514
RID: 378

Given

$$\frac{x^2}{2} + \frac{y^2}{4} = 1, 2 > \sqrt{2} > 0$$

$$\Rightarrow a = 2, b = \sqrt{2}$$

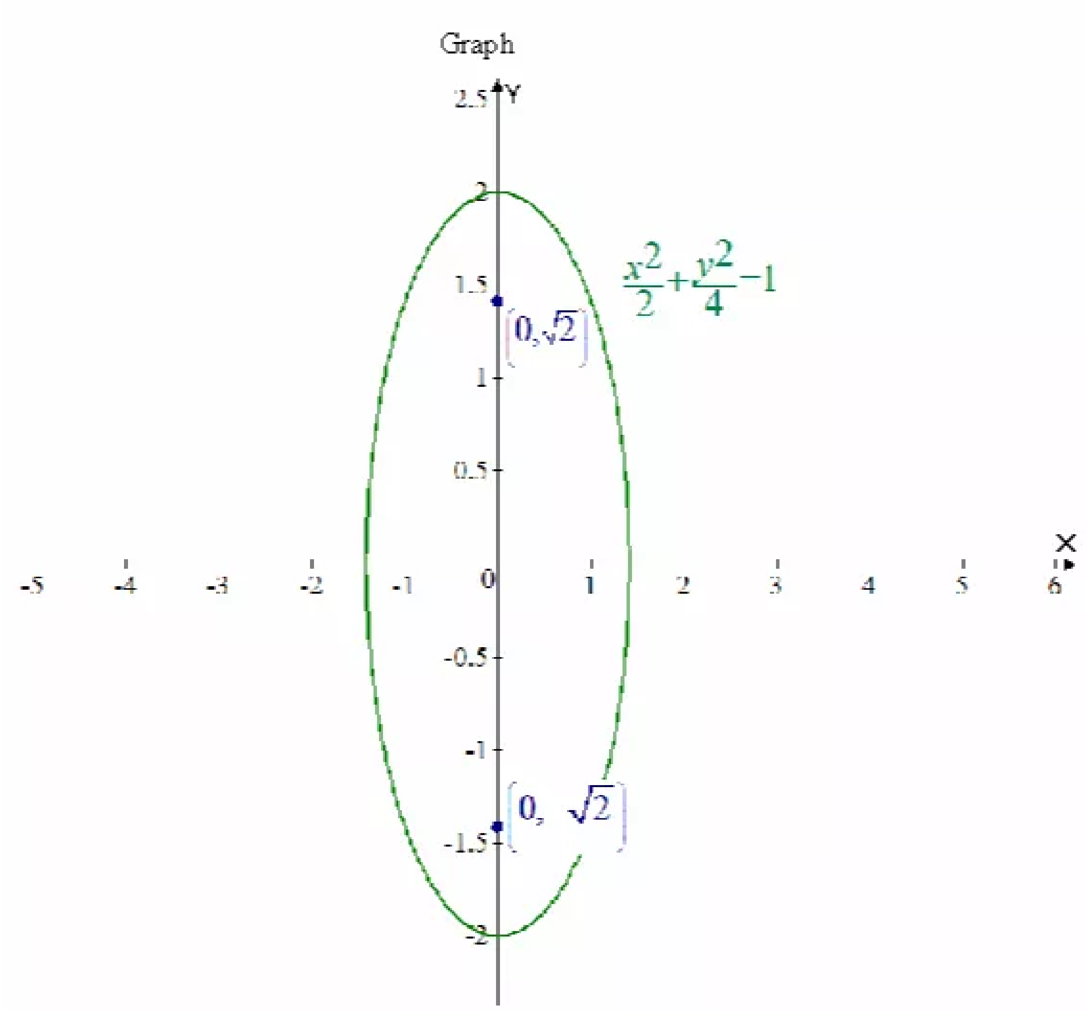
Compare this ellipse with $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

Therefore foci is $(0, \pm c)$

where $c^2 = a^2 - b^2$

$$\Rightarrow c^2 = 4 - 2$$
$$= 2$$
$$\Rightarrow c = \pm\sqrt{2}$$

Vertices are $(0, \pm a) = (0, \pm 2)$



Q12E

Given

$$\frac{x^2}{36}+\frac{y^2}{8}=1, 2>\sqrt{2}>0$$

$$\Rightarrow a=6, b=\sqrt{8}$$

Compare this ellipse with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Therefore foci is $(\pm c,0)$

where $c^2=a^2-b^2$

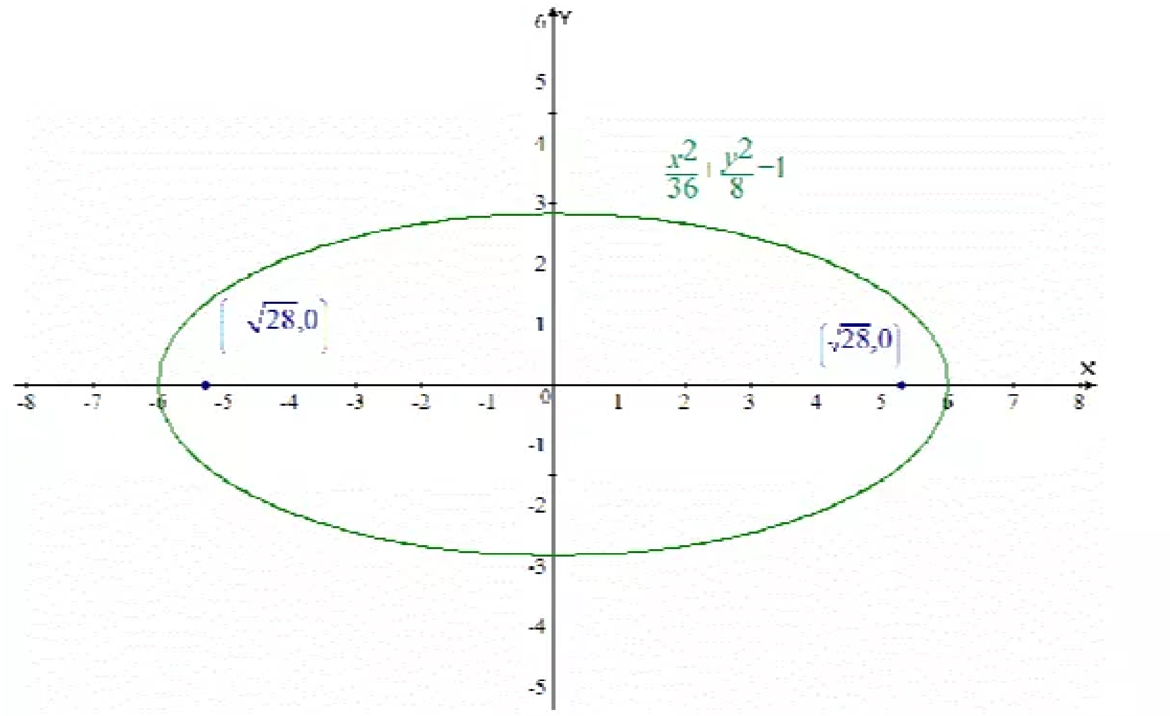
$$\Rightarrow c^2=36-8$$

$$=28$$

$$\Rightarrow c=\pm\sqrt{28}$$

Vertices are $(\pm a,0)=(\pm 6,0)$

Graph



Q13E

Given

$x^2+9y^2=9$

$\Rightarrow \frac{x^2}{9}+\frac{y^2}{1}=1$

$\Rightarrow a=3, b=1$

Compare this ellipse with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Therefore foci is $(\pm c, 0)$

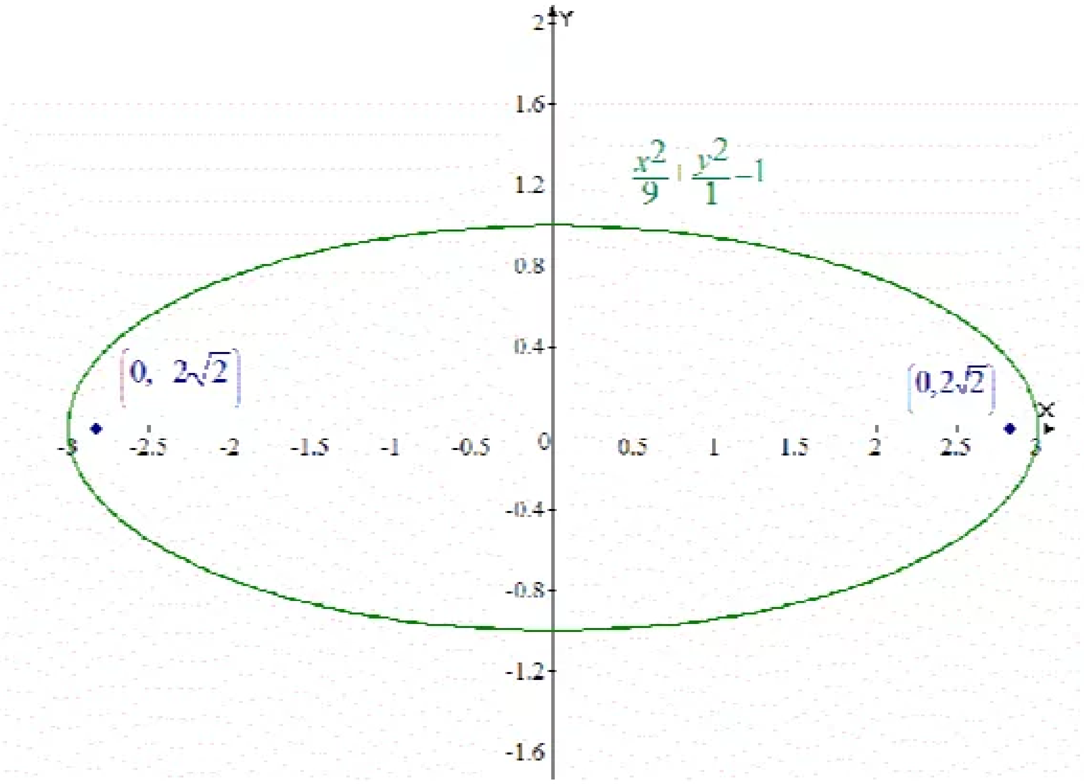
where $c^2=a^2-b^2$

$\Rightarrow c^2=8$

$\Rightarrow c=\pm 2\sqrt{2}$

Vertices are $(\pm a, 0)=(\pm 3, 0)$

Graph



Q14E

Consider the equation

$$100x^2+36y^2=225$$
$$\frac{100x^2+36y^2}{225}=\frac{225}{225}$$
$$\frac{x^2}{225/100}+\frac{y^2}{225/36}=1 \quad \cdots\cdots(1)$$

Standard form of the ellipse: The ellipse

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, \quad a\geq b>0 \quad \cdots\cdots(2)$$

has foci $(0,\pm c)$, where $c^2=a^2-b^2$, and vertices $(0,\pm a)$.

Compare equation (1) with equation (2), obtain that

$$a^2=\frac{225}{36}, b^2=\frac{225}{100}$$
$$a=\frac{15}{6}, \quad b=\frac{15}{10}$$

So, the *x*-intercepts are $\pm\frac{15}{10}$ and the *y*-intercepts are $\pm\frac{15}{6}$.

Therefore

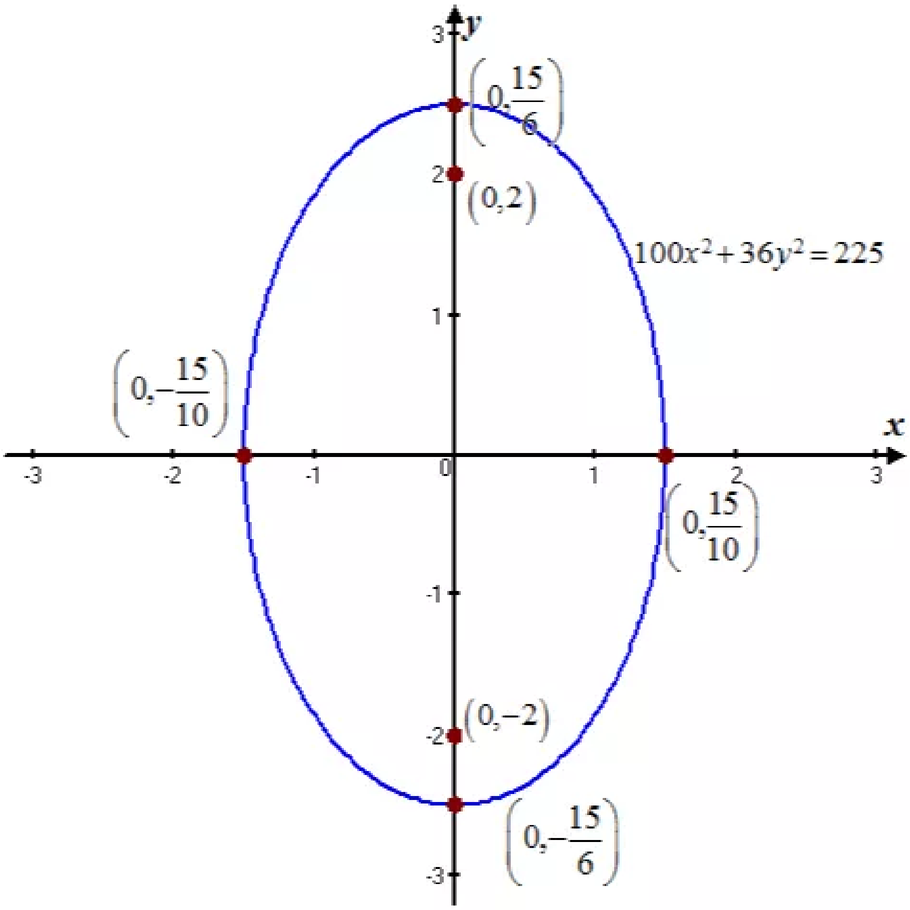
$$
\begin{aligned}
c^2 &= a^2 - b^2 \\
&= \frac{225}{36} - \frac{225}{100} \\
&= 225\left(\frac{64}{36 \times 100}\right) \\
&= 4
\end{aligned}
$$

$$c = \pm 2$$

Hence, Foci are $(0, \pm c) = (0, \pm 2)$

And Vertices are $(0, \pm a) = \left(0, \pm \frac{15}{6}\right)$

Sketch the graph of the ellipse is shown below:



Q15E

Given equation of the ellipse is $9x^2 - 18x + 4y^2 = 27$

Making perfect square.

$$9x^2 - 2\times(3x)\times 3 + 4y^2 + 9 = 27 + 9$$

$$\Rightarrow 9x^2 + 9 - 2\times 3\times 3x + 4y^2 = 36$$

$$\Rightarrow (3x-3)^2 + 4y^2 = 36$$

$$\Rightarrow 9(x-1)^2 + 4y^2 = 36$$

$$\Rightarrow \frac{(x-1)^2}{4} + \frac{y^2}{9} = 1$$ (Equation of ellipse shifted 1 unit to the right)

Comparing with $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

We have $a^2 = 9$, $b^2 = 4$

then $c^2 = a^2 - b^2 = 9 - 4 = 5$ $\Rightarrow c = \pm\sqrt{5}$

Then Foci are $\boxed{(1,\ \pm\sqrt{5})}$ and vertices are $\boxed{(1,\ \pm 3)}$

Now we sketch the graph

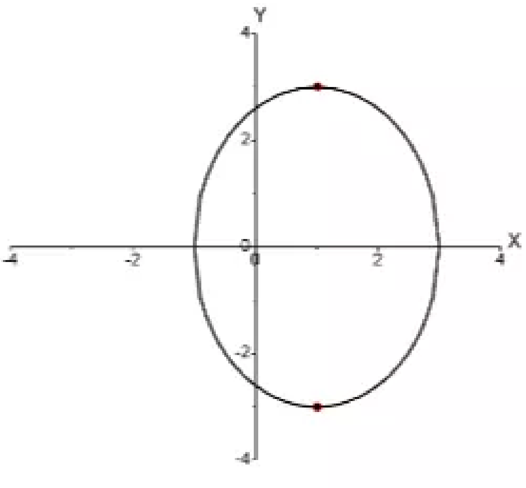


Fig.1

Q16E

Consider the equation of the ellipse,

$$x^2 + 3y^2 + 2x - 12y + 10 = 0$$

Recollect the standard form of the equation of the ellipse which has foci $(h \pm c, k)$,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,\quad a \ge b > 0$$

Where $c^2 = a^2 - b^2$, and vertices $(h \pm a, k)$.

To change the equation to the standard form of the hyperbola, rewrite the equation to complete square form:

$$x^2+2x+3y^2-12y+10=0$$
$$(x^2+2x)+3(y^2-4y)=-10$$
$$(x^2+2x+1)+3(y^2-4y+4)=-10+1+12 \qquad \text{Add 1 and 12.}$$
$$(x+1)^2+3(y-2)^2=3$$
$$\frac{(x-(-1))^2}{3}+\frac{(y-2)^2}{1}=1$$

This is in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ where $h=-1$ and $k=2$.

Here

$$a^2=3 \Rightarrow a=\sqrt{3},$$
$$b^2=1 \Rightarrow b=1$$

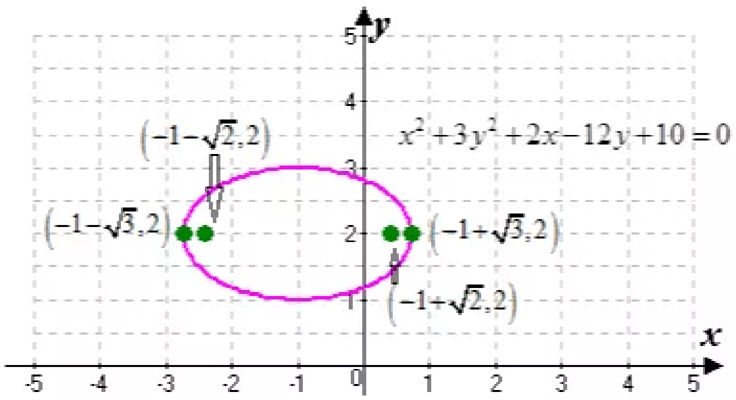
Then

$$c^2=a^2-b^2$$
$$=3-1$$
$$=2$$
$$c=\sqrt{2}$$

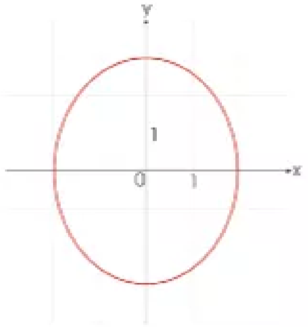
Therefore, the foci are $(h \pm c, k)=\boxed{(-1 \pm \sqrt{2}, 2)}$

And the vertices are $(h \pm a, k)=\boxed{(-1 \pm \sqrt{3}, 2)}$.

Sketch of the graph of the ellipse $x^2+3y^2+2x-12y+10=0$ as shown below:



Q16E



The equation of the ellipse is $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

Where y intercepts are $\pm a$

And x intercepts are $\pm b$

From figure we see that y intercepts are ± 3

And x intercepts are ± 2

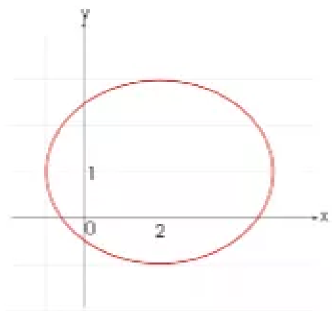
So equation of the ellipse becomes $\boxed{\frac{x^2}{4}+\frac{y^2}{9}=1}$

We have $c^2=a^2-b^2 \quad \Rightarrow c^2=9-4=5$

$\Rightarrow c=\pm\sqrt{5}$

Then foci are $\boxed{\left(0,\pm\sqrt{5}\right)}$

Q18E



Equation of the ellipse whish is shifted h units to the right and k units upward is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

From figure we see that h = 2 and k = 1

Then equation becomes as $\frac{(x-2)}{a^2}+\frac{(y-1)^2}{b^2}=1$

Since this ellipse passes through the points $(2,3)$

Then $\frac{(2-2)^2}{a^2}+\frac{(3-1)^2}{b^2}=1$

$\Rightarrow \frac{4}{b^2}=1 \Rightarrow b^2=4$

">

And since ellipse passes through the point $(-1, 1)$

Then $\frac{(-1-2)^2}{a^2}+0=1$

$\Rightarrow \frac{9}{a^2}=1 \qquad \Rightarrow \boxed{a^2=9}$

i”>

Thus equation of the ellipse becomes $\boxed{\frac{(x-2)^2}{9}+\frac{(y-1)^2}{4}=1}$

Since $c^2=a^2-b^2$

$\Rightarrow c^2=9-4=5$

$\Rightarrow c=\pm\sqrt{5}$

Then foci are $\boxed{\left(2\pm\sqrt{5},\ 1\right)}$

Q19E

Given

$\frac{y^2}{25}-\frac{x^2}{9}=1$

Here $a=5, b=3$

$\Rightarrow c^2=a^2+b^2$

$=25+9$

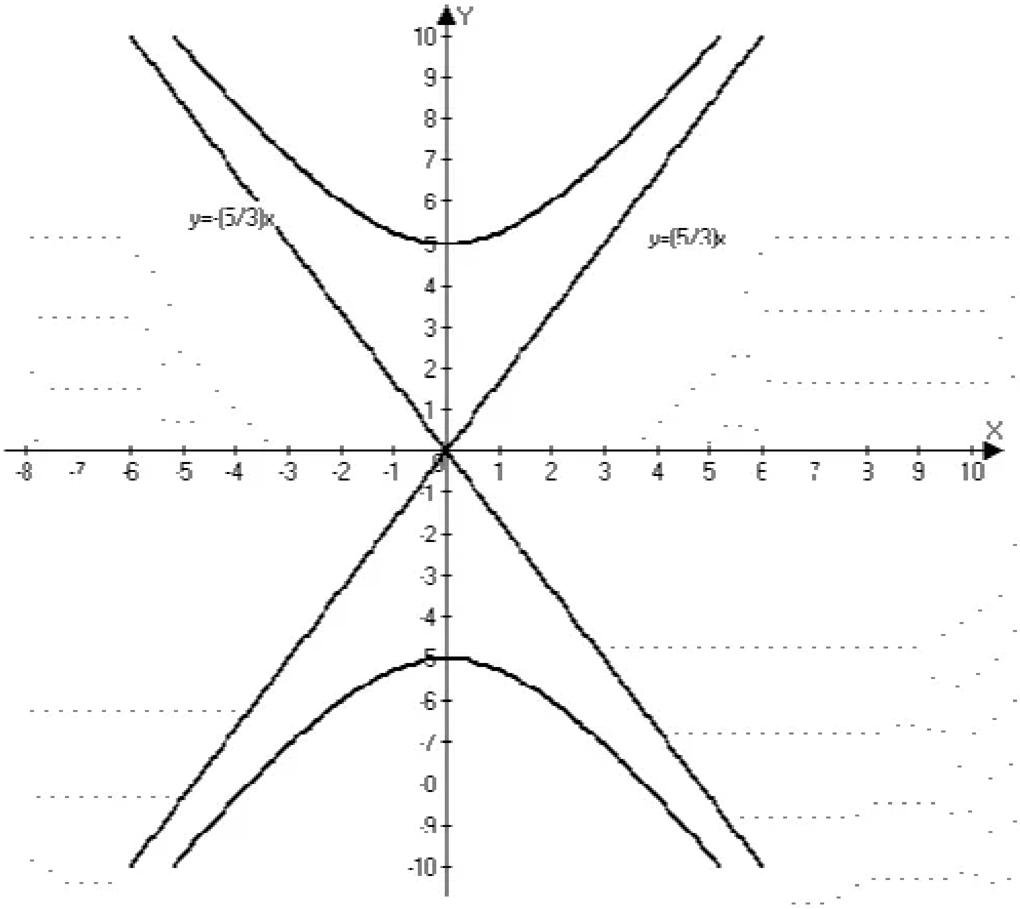
$=34$

Foci is $(0,\pm c)=(0,\pm\sqrt{34})$

Vertices are $(0,\pm a)=(0,\pm 5)$

Asymptotes are $y=\pm\frac{a}{b}x=\pm\frac{5}{3}x$

Graph



Q20E

Given

$$\frac{x^2}{36}-\frac{y^2}{64}=1$$

Here $a=6, b=8$

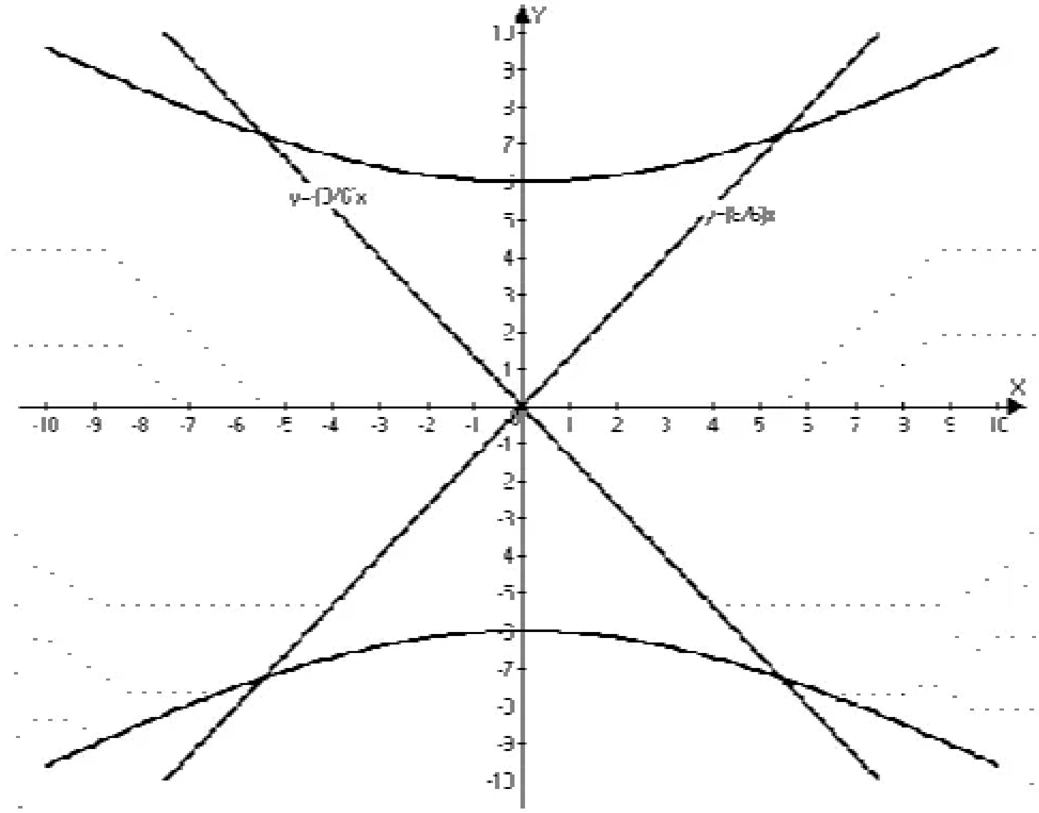
$$\Rightarrow c^2=a^2+b^2$$
$$=36+64$$
$$=100$$

Foci is $(\pm c,0)=(\pm 10,0)$

Vertices are $(\pm a,0)=(\pm 6,0)$

Asymptotes are $y=\pm\frac{a}{b}x=\pm\frac{8}{6}x$

Graph



Q21E

2657-10.5-21E

AID: 9514
RID: 378

Given

$$\frac{x^2}{100}-\frac{y^2}{100}=1$$

Here $a=10, b=10$

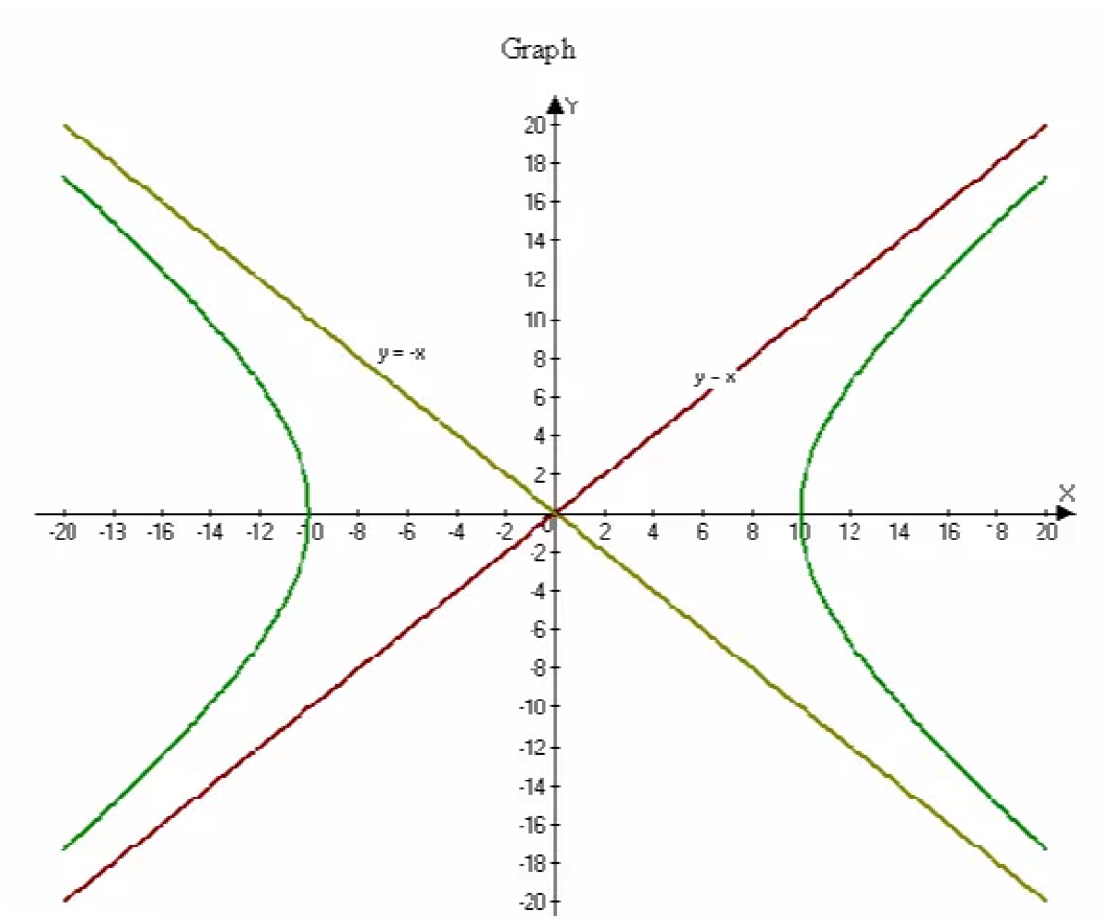
$\Rightarrow c^2=a^2+b^2$

$\quad = 200$

Foci is $(\pm c,0)=(\pm\sqrt{200},0)$

Vertices are $(\pm a,0)=(\pm 10,0)$

Asymptotes are $y=\pm\frac{a}{b}x=\pm x$



Q22E

Given

$$\frac{y^2}{16} - \frac{x^2}{1} = 1$$

Here $a = 4, b = 1$

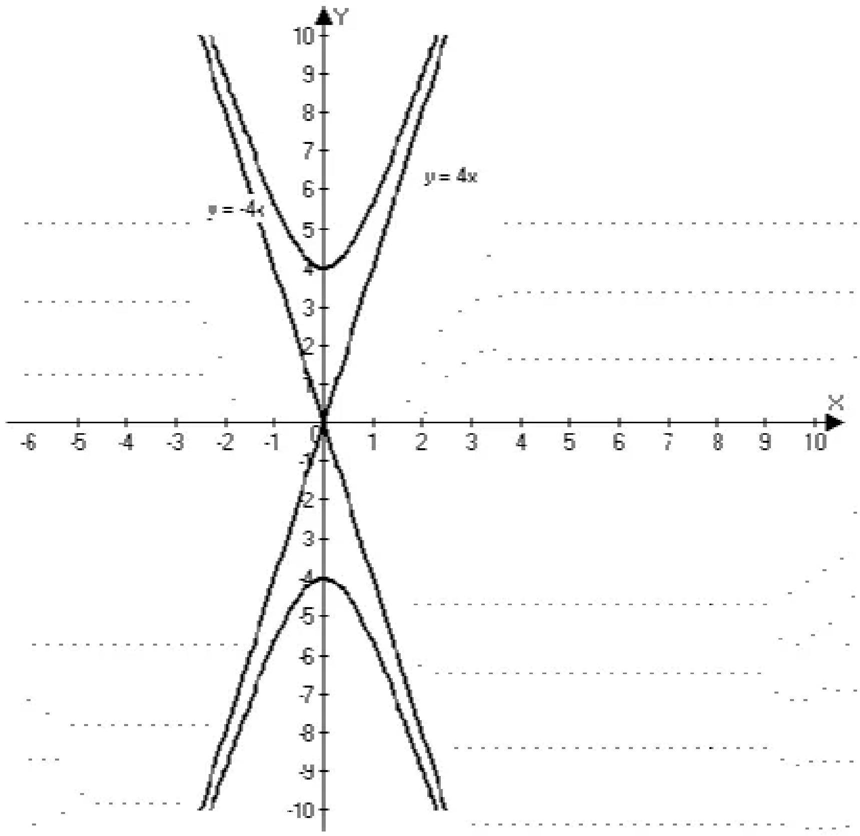
$$\Rightarrow c^2 = a^2 + b^2$$
$$= 16 + 1$$
$$= 17$$

Foci is $(0, \pm c) = (0, \pm\sqrt{17})$

Vertices are $(0, \pm a) = (0, \pm 4)$

Asymptotes are $y = \pm\frac{a}{b}x = \pm 4x$

Graph



Q23E

Consider the equation of the hyperbola,

$$4x^2 - y^2 - 24x - 4y + 28 = 0$$

Recollect the standard form of the equation of the hyperbola which has foci $(h \pm c, k)$,

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$, vertices $(h \pm a, k)$ and asymptotes $y - k = \pm\frac{b}{a}(x-h)$.

To change the equation to the standard form of the hyperbola, rewrite the equation to complete square form:

$$4x^2 - 24x - y^2 - 4y + 28 = 0$$

$$4(x^2 - 6x) - (y^2 + 4y) = -28$$

$$4(x^2 - 6x + 9) - (y^2 + 4y + 4) = -28 + 36 - 4 \quad \text{Add 36 and } -4.$$

$$4(x-3)^2 - (y+2)^2 = 4$$

$$\frac{(x-3)^2}{1} - \frac{(y-(-2))^2}{4} = 1$$

This is in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ where $h = 3$ and $k = -2$.

So, the hyperbola is shifted three units to the right and two units downward.

Here

$a^2 = 1 \Rightarrow a = 1,$
$b^2 = 4 \Rightarrow b = 2$

Then

$$c^2 = a^2 + b^2$$
$$= 1 + 4$$
$$= 5$$
$$c = \sqrt{5}$$

Therefore, the foci are $(h \pm c, k) = \boxed{(3 \pm \sqrt{5}, -2)}$,

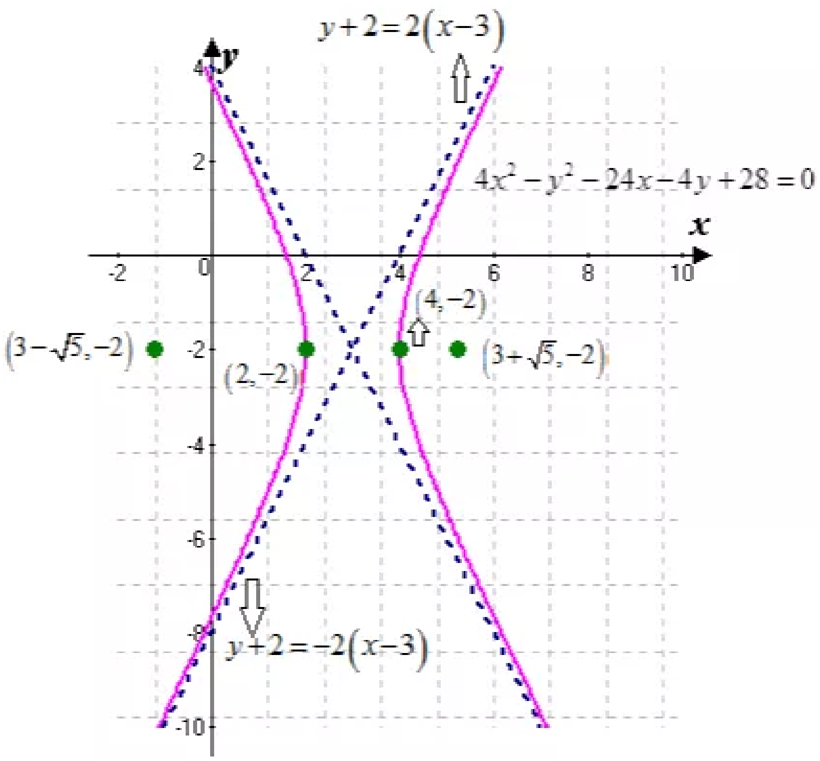
And the vertices are $(h \pm a, k) = (3 \pm 1, -2)$

$= \boxed{(4,-2),(2,-2)}$.

The asymptotes are

$$y - k = \pm \frac{b}{a}(x - h)$$
$$y - (-2) = \pm \frac{2}{1}(x - 3)$$
$$\boxed{y + 2 = \pm 2(x - 3)}.$$

Sketch of the graph of the hyperbola $4x^2 - y^2 - 24x - 4y + 28 = 0$ as shown below:



Q24E

Consider

$y^2-4x^2-2y+16x=31$

Transform the equation in to standard form.

The equation for a hyperbola that is not centred at the origin is

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ or $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$

Where the Centre is (h,k)

The Vertices are $(h\pm a,k)$ or $(h,k\pm a)$

The Foci are $(h\pm c,k)$ or $(h,k\pm c)$ $,c^2=a^2+b^2$

The Asymptote are $y-k=\pm\frac{b}{a}(x-h)$ or $y-k=\pm\frac{a}{b}(x-h)$

First adjust the equation above into standard form.

$y^2-4x^2-2y+16x=31$

$y^2-4x^2-2y+16x-31=0$

$y^2-2y-4x^2+16x-31=0$

$$(y-1)^2-\left(4x^2-2\cdot 2x\cdot 4+16-16\right)-32=0$$
$$(y-1)^2-\left((2x-4)^2-16\right)-32=0$$
$$(y-1)^2-4(x-2)^2-16=0$$

$$(y-1)^2-4(x-2)^2=16$$
$$\frac{(y-1)^2}{16}-\frac{(x-2)^2}{4}=1$$
$$\frac{(y-1)^2}{4^2}-\frac{(x-2)^2}{2^2}=1$$

This is an equation of the form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$

Determine the vertices, foci and asymptotes of the hyperbola.

Determine the value of the centre.

Since $h=2$ and $k=1$, conclude that the centre of the hyperbola is $(h,k)=(2,1)$.

Determine the vertices of the hyperbola

The vertices are $(h,k\pm a)$

$(h,k+a)=(2,1+4)$ $(h,k-a)=(2,1-4)$

$=(2,5)$ $=(2,-3)$

Next solve for c.

$$c^2 = a^2 + b^2$$

$$c^2 = 16 + 4$$

$$c^2 = 20$$

$$c = \sqrt{20}$$

Now determine the foci of the hyperbola.

The foci are $(h, k \pm c)$

$$(h, k + c) = \left(2, 1 + \sqrt{20}\right) \quad (h, k - a) = \left(2, 1 - \sqrt{20}\right)$$

Finally determine the asymptotes of the hyperbola.

$$y - k = \pm\frac{a}{b}(x - h)$$

$$y - 1 = \pm\frac{4}{2}(x - 2)$$

Now, distributive the sign and find the equations of the asymptotes separately.

$$y - 1 = \frac{4}{2}(x - 2)$$

$$y - 1 = 2(x - 2)$$

$$y - 1 = 2x - 4$$

$$y = 2x - 3$$

$$y - 1 = -\frac{4}{2}(x - 2)$$

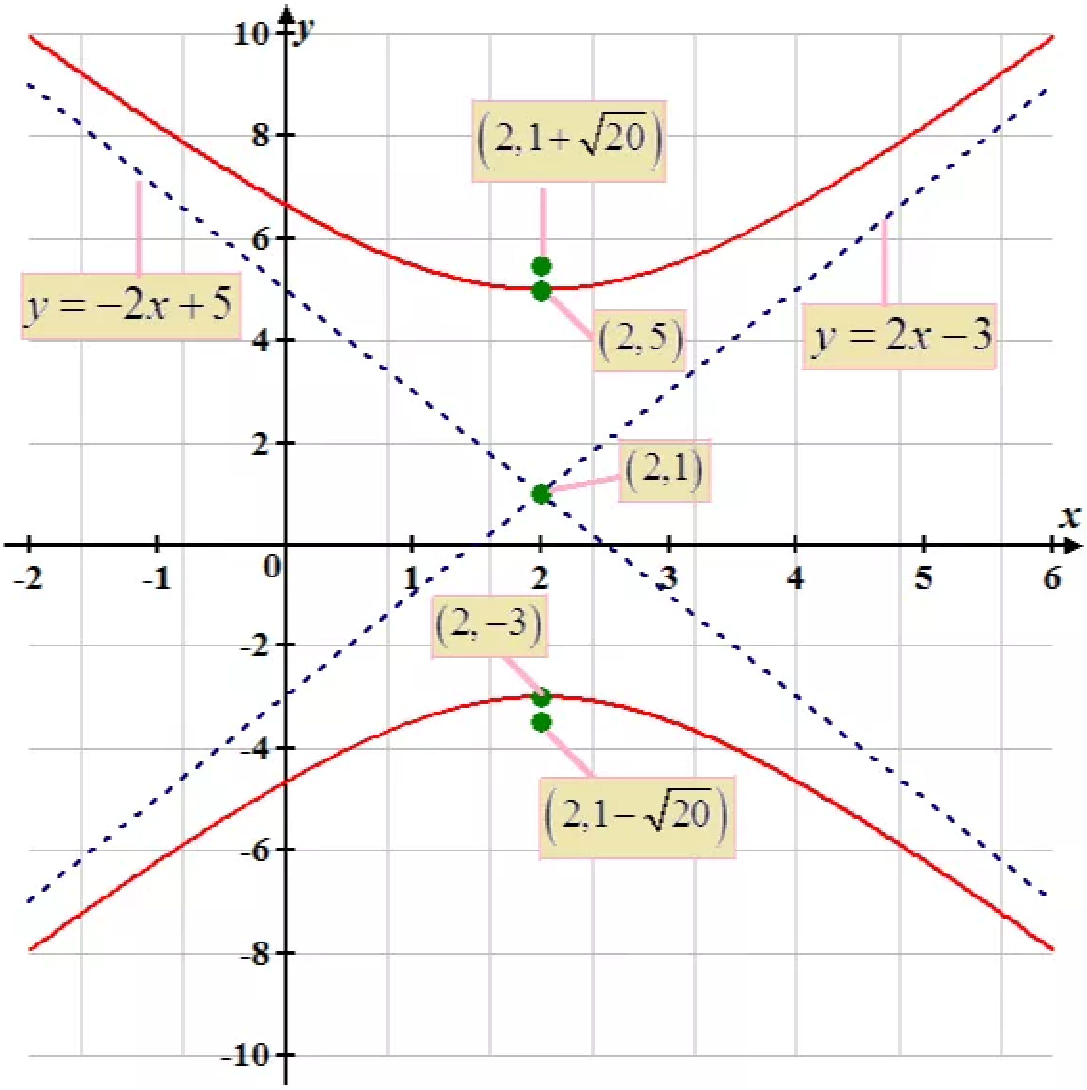
$$y - 1 = -2(x - 2)$$

$$y - 1 = -2x + 4$$

$$y = -2x + 5$$

Hence, the centre, vertices, foci and asymptotes of the hyperbola are $\boxed{(2,1)}$ $\boxed{(2,5),(2,-3)}$ $\boxed{\left(2,1+\sqrt{20}\right),\left(2,1-\sqrt{20}\right)}$ $\boxed{y = 2x - 3, y = -2x + 5}$ respectively.

Sketch the graph of the hyperbola along with the asymptotes as shown below.



Q25E

Given equation is $x^2 = y+1$

Let $X = x$ *and* $Y = y+1$

Then $X^2 = Y$, which represent a parabola.

Comparing with standard equation of parabola $X^2 = 4pY$

We have 4p=1 $\Rightarrow p = 1/4$

Vertex of the parabola is (0, 0)

Here $X=0$ and $Y=0$

$\Rightarrow x = 0$ *and* $y+1=0$

$\Rightarrow x = 0$ *and* $y=-1$

So vertex of the parabola is $(0,-1)$

Focus of the parabola is (0, p)

here $X=0$ *and* $Y=p$

$\Rightarrow x = 0$ *and* $y+1=1/4$

$\Rightarrow x = 0$ *and* $y=-3/4$

So focus of the parabola is $(0,-3/4)$

The given equation represents a parabola with focus (0,-3/4) and vertex (0,-1)

Q26E

Given equation is $x^2 = y^2$

$\Rightarrow \frac{x^2}{1^2} - \frac{y^2}{1^2} = 1$ This represents a hyperbola.

Comparing with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We have $a^2 = 1$ and $b^2 = 1$

Then $c^2 = 2$ $(c^2 = a^2 + b^2)$

Foci are $(\pm c, 0) = (\pm\sqrt{2},\ 0)$

Vertices are $(\pm a, 0) = (\pm 1,\ 0)$

Asymptotes are $y = \pm(b/a)x$

$\Rightarrow y = \pm x$

So given equation represents

A hyperbola with foci $(\pm\sqrt{2},\ 0)$, vertices $(\pm 1, 0)$ and asymptotes $y = \pm x$.

Q27E

Given equation is $x^2 = 4y - 2y^2$

$\Rightarrow x^2 + 2y^2 - 4y = 0$

Making perfect square

$x^2 + 2y^2 - 4y = 0$

$\Rightarrow x^2 + 2(y^2 - 2y + 1 - 1) = 0$

$\Rightarrow x^2 + 2(y-1)^2 - 2 = 0$

$\Rightarrow x^2 + 2(y-1)^2 = 2$

$\Rightarrow \frac{x^2}{2} + \frac{(y-1)^2}{1} = 1$

This represents an ellipse

Comparing with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

We have $h = 0,\ \ k = 1,\ \ a^2 = 2$ *and* $b^2 = 1$

So $c^2 = a^2 - b^2 = 2 - 1 = 1$ $\Rightarrow c = \pm 1$

Foci are $(h \pm c, k) = (\pm 1,\ 1)$

And vertices are $(h \pm a, k) = (\pm\sqrt{2},\ 1)$

Thus given equation represents **an ellipse with foci $(\pm 1,\ 1)$ and vertices $(\pm\sqrt{2},\ 1)$**.

Q28E

Step 1 of 2

Given equation is $y^2 - 8y = 6x - 16$

$\Rightarrow y^2 - 8y + 16 = 6x$

$\Rightarrow (y-4)^2 = 6x$

This represents a parabola.

Comparing with $(y-k)^2 = 4p(x-h)$

We have $\quad 4p = 6, k = 4 \quad and \quad h = 0$

$\Rightarrow \boxed{p = 3/2}$

Thus focus of the parabola is $(h+p, k) = (3/2,\ 4)$

Vertex of the parabola is (h, k) $= (0,\ 4)$

Thus given equation represents a $\boxed{\text{parabola with focus } (3/2,\ 4) \text{ and vertex } (0, 4)}$

Q29E

Given equation is $y^2 + 2y = 4x^2 + 3$

$\Rightarrow y^2 + 2y + 1 = 4x^2 + 3 + 1$

$\Rightarrow (y+1)^2 = 4x^2 + 4$

$\Rightarrow (y+1)^2 - 4x^2 = 4$

$\Rightarrow \frac{(y+1)^2}{4} - \frac{x^2}{1} = 1$, represents a hyperbola

Comparing with $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

We have $h = 0, \quad k = -1, a^2 = 4 \quad$ and $\quad b^2 = 1$

Then $\quad c^2 = a^2 + b^2 = 5, \Rightarrow c = \pm\sqrt{5}$

Then foci are $(h, k \pm c) = (0, -1 \pm \sqrt{5})$

Vertices are $(h, k \pm a) = (0, -1 \pm 2) = (0, -3)$ and $(0,\ 1)$

So given equation represents

$\boxed{\text{A hyperbola with foci } (0, -1 \pm \sqrt{5}) \text{ and vertices } (0, -3), (0,\ 1)}$

Q30E

Given equation is $4x^2 + 4x + y^2 = 0$

$\Rightarrow 4(x^2 + x) + y^2 = 0$

$\Rightarrow 4\left(x^2 + x + \frac{1}{4} - \frac{1}{4}\right) + y^2 = 0$

$\Rightarrow 4\left(x^2 + x + \frac{1}{4}\right) + y^2 = 1$

$\Rightarrow 4\left(x + \frac{1}{2}\right)^2 + y^2 = 1$

$\Rightarrow \frac{(x + 1/2)^2}{(1/4)} + \frac{y^2}{1} = 1$

This represents an ellipse

Comparing with the equation of ellipse shifted at (h, k)

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

We have $h=-1/2,\ k=0\ \ and\ \ a^2=1,\ b^2=1/4$

Then $c^2=a^2-b^2=1-1/4=3/4 \quad \Rightarrow c=\pm\sqrt{3}/2$

Foci are $(h,k\pm c)=\left(-1/2,\ \pm\sqrt{3}/2\right)$

Vertices are $(h,k\pm a)=(-1/2,\ \pm 1)$

Thus given equation represents

$\boxed{\text{An ellipse with foci } \left(-1/2,\ \pm\sqrt{3}/2\right) \text{ and vertices } (-1/2,\ \pm 1)}$

Q31E

Given parabola with vertex (0, 0) and focus (1, 0)

The equation of the parabola with focus (p, 0) is $y^2=4px$

Since here p=1 the equation for the parabola is $\boxed{y^2=4x}$

Q32E

Given parabola with focus $(0,0)$ and directrix $y=6$

An equation of the parabola with focus $(0,p)$ and directrix $y=-p$ is $x^2=4py$

$\boxed{\Rightarrow x^2=0\ or\ x^2=-24y}$

Q33E

Let $P(x,y)$ be any point on the parabola

Then the distance from P to the focus equal to the distance from P to the directrix x = 2

That is, $\sqrt{(x+4)^2+(y-0)^2}=|x-2|$

Squiring, $(x+4)^2+y^2=(x-2)^2$

$$\Rightarrow x^2+8x+16+y^2=x^2-4x+4$$

$$\Rightarrow 12x+12=-y^2$$

$$\Rightarrow 12(x+1)=-y^2$$

$\Rightarrow \boxed{y^2=-12(x+1)}$ is the equation of the parabola

Q34E

Focus (3, 6), and vertex (3, 2)

Let Z (a, b) be the foot of perpendicular from the focus on the directrix. Then the point (3, 2) is the mid point of the segment [FZ]

That is $3=\dfrac{3+a}{2}\Rightarrow a=3$ and $2=\dfrac{b+6}{2}\Rightarrow b=-2$

Since the axis of parabola is $x=3$, therefore the equation of directrix is $y=-2$

That is $y+2=0$

That is $\sqrt{(x-3)^2+(y-6)^2}=|y+2|$

$$\Rightarrow (x-3)^2+(y-6)^2=(y+2)^2$$
$$\Rightarrow x^2-6x+9+y^2-12y+36=y^2+4y+4$$
$$\Rightarrow \boxed{x^2-6x-16y+41=0}$$

Q35E

Consider the data

The axis of the parabola is vertical axis

Vertex of the parabola is $(2,3)$

And the point through which it passes is $(1,5)$

Determine the equation of the parabola for the given data.

Recall that,

The equation of a parabola with a vertical axis and origin shifted to point (h,k) is

$(x-h)^2=4p(y-k)$ ...... (1)

Where vertex is (h,k)

Here, $(h,k)=(2,3)$

Then, from (1) the equation of the parabola with given vertex is

$(x-2)^2=4p(y-3)$ Substitute $h=2, k=3$

As the parabola passes through the point $(1,5)$, substitute 1 for x and 5 for y in the above equation.

$(1-2)^2=4p(5-3)$ Substitute $x=1, y=5$

$(-1)^2=4p(2)$

$1=8p$ Simplify

$p=\frac{1}{8}$

Finally, substitute the value of p in $(x-2)^2=4p(y-3)$.

$(x-2)^2=4\cdot\frac{1}{8}(y-3)$

$(x-2)^2=\frac{1}{2}(y-3)$

Therefore, the equation of the parabola along the vertical axis is $\boxed{(x-2)^2=\frac{1}{2}(y-3)}$.

Q36E

Consider the data

The axis of the parabola is horizontal axis

And the parabola passes through the points $(-1,0),(1,-1)$ and $(3,1)$

Determine the equation of the parabola for the given data.

Recall that,

The equation of a parabola with a horizontal axis and origin shifted to point (h,k) in the quadratic form is

$x = ay^2 + by + c$ ...... (1)

Substitute the point $(-1,0)$ in (1).

$0 = a(-1)^2 + b(-1) + c$ Plug in – 1 for *x* and 0 for *y*

$0 = a - b + c$ ...... (2)

Substitute the point $(1,-1)$ in (1).

$-1 = a(1)^2 + b(1) + c$ Plug in – 1 for *x* and 0 for *y*

$-1 = a + b + c$ ...... (3)

Substitute the point $(3,1)$ in (1).

$1 = a(3)^2 + b(3) + c$ Plug in – 1 for *x* and 0 for *y*

$1 = 9a + 3b + c$ ...... (4)

Eliminate *b* from the equations (2), (3) and (4).

Add the equation (2) and (3). This implies

$-1 + 0 = (a - b + c) + (a + b + c)$ Add

$-1 = a - b + c + a + b + c$

$-1 = 2a + 2c$ ...... (5)

Multiply (2) by 3 and add it to (4).

$0 + 1 = (3a - 3b + 3c) + (9a + 3b + c)$ Add

$1 = 3a - 3b + 3c + 9a + 3b + c$

$1 = 12a + 4c$ ...... (6)

Eliminate c from (5) and (6) and find a.

Multiply (5) by 2 and then subtract it from (6).

$$1-(-2)=(12a+4c)-(4a+4c) \text{ Subtract}$$

$$1+2=12a+4c-4a-4c$$

$$3=8a$$

$$a=\frac{3}{8}$$

Substitute a value in (5) to get c.

$$-1=2a+2c$$

$$-1=2\left(\frac{3}{8}\right)+2c$$

$$-1=\frac{3}{4}+2c$$

$$-1-\frac{3}{4}=2c$$

$$-\frac{7}{4}=2c$$

$$c=-\frac{7}{8}$$

Plug in the values of a, c in (2) to get b.

$$0=a-b+c$$

$$0=\frac{3}{8}-b-\frac{7}{8}$$

$$b=-\frac{4}{8}$$

$$b=-\frac{1}{2}$$

Substitute a, b and c values in (1) to get required parabola.

$$x=ay^2+by+c$$

$$x=\left(\frac{3}{8}\right)y^2+\left(-\frac{1}{2}\right)y+\left(-\frac{7}{8}\right)$$

$$x=\frac{1}{8}\left(3y^2-4y-7\right)$$

$$8x=3y^2-4y-7$$

$$y^2-4y-8x-7=0$$

Therefore, the equation of the parabola along the horizontal axis is $\boxed{y^2-4y-8x-7=0}$.

Q37E

Foci of the ellipse are $(\pm 2,0)$ *ie.* $c=2$

And vertices are $(\pm 5,0)$ *ie.* $a=5$

Since $c^2=a^2-b^2$, thus $b^2=a^2-c^2$

$$=5^2-2^2$$
$$=25-4$$
$$b^2=21$$

The equation of the ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$$\Rightarrow \frac{x^2}{5^2}+\frac{y^2}{21}=1$$

$$\Rightarrow \boxed{\frac{x^2}{25}+\frac{y^2}{21}=1}$$

Q38E

Foci of the ellipse are $(0,\pm 5)$ *ie.* $c=5$

And vertices $(0,\pm 13)$ *ie.* $a=13$

Since $c^2=a^2-b^2$

Then $b^2=a^2-c^2$

$$=13^2-5^2$$
$$=169-25=144$$

The equation of the ellipse is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

$$\Rightarrow \boxed{\frac{x^2}{144}+\frac{y^2}{169}=1}$$

Q39E

Foci of the ellipse are $(0,2)$, $(0,6)$

And vertices are $(0,0)$, $(0,8)$

The major axis is the line segment that joins the vertices (0, 0) and (0, 8)

It has vertical length 8, so $a=4$

The distance between foci is 4, so $c=2$

Thus $b^2=a^2-c^2 \quad =16-4=12$

Since the centre of the ellipse is $(0,4)$

Therefore, an equation of the ellipse is $\boxed{\dfrac{x^2}{12}+\dfrac{(y-4)^2}{16}=1}$

Q40E

Foci of the ellipse are $(0,-1),(8,-1)$

The horizontal distance between foci is 8, so $c=4$

And center becomes $(4,-1)$

Now one vertex is (9, -1) and let second vertex be (L, M)
Using mid point formula, we have

$$\frac{L+9}{2}=4 \quad \text{and} \quad \frac{M+(-1)}{2}=-1$$

$$\Rightarrow L=-1 \quad \text{and} \quad M=-1$$

Thus the second vertex is (-1, -1)

The major axis has length 10, thus $a=5$.

So $\quad b^2=a^2-c^2$
$\quad\quad =25-16$
$\quad\quad =9$

Therefore, an equation of the ellipse is $\boxed{\frac{(x-4)^2}{25}+\frac{(y+1)^2}{9}=1}$

The major axis has length 10, thus $a=5$.

So $\quad b^2=a^2-c^2$
$\quad\quad =25-16$
$\quad\quad =9$

Therefore, an equation of the ellipse is $\boxed{\frac{(x-4)^2}{25}+\frac{(y+1)^2}{9}=1}$

Q41E

Consider the data

Centre of the ellipse is $(-1,4)$

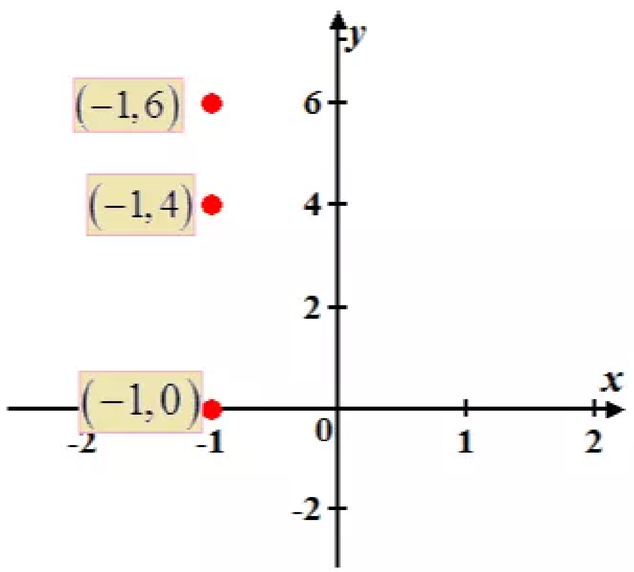
Vertex of the ellipse is $(-1,0)$

Focus of the ellipse is $(-1,6)$

Determine the equation of the ellipse from the above data.

The centre of the ellipse is (h,k). So, $h=-1, k=4$.

Plot the points centre, focus, vertex on the same line.



From the graph it is clear that, all the points lie on the same line $x=-1$. So, the major axis is parallel to vertical axis.

Recall that,

The equation of the ellipse with major axis parallel to vertical axis is

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \quad \ldots\ldots (1)$$

And the distance between the centre and focus is *c* and the distance from the centre to vertex is *a*.

Find the distances from the centre to focus and from centre to vertex.

$c=\sqrt{(-1-1)^2+(6-4)^2}$ Distance between $(-1,4)$ and $(-1,6)$

$=\sqrt{(-2)^2+(2)^2}$

$=\sqrt{8}$ Simplify

$=2\sqrt{2}$

$a=\sqrt{(-1-1)^2+(0-4)^2}$ Distance between $(-1,4)$ and $(-1,0)$

$=\sqrt{(-2)^2+(-4)^2}$

$=\sqrt{4+16}$ Simplify

$=\sqrt{20}$

$=2\sqrt{5}$

Find the value of *b* by using $c^2=a^2-b^2$

$$\begin{aligned}
\left(2\sqrt{2}\right)^2 &= \left(2\sqrt{5}\right)^2-b^2 \\
4(2) &= 4(5)-b^2 \quad \text{Simplify} \\
8 &= 20-b^2 \\
b^2 &= 20-8 \\
b^2 &= 12 \\
b &= \sqrt{12} \\
&= 2\sqrt{3}
\end{aligned}$$

Substitute a,b,h and k values in (1).

$$\frac{(x-(-1))^2}{\left(2\sqrt{3}\right)^2}+\frac{(y-4)^2}{\left(2\sqrt{5}\right)^2}=1$$

$$\frac{(x+1)^2}{12}+\frac{(y-4)^2}{20}=1$$

Thus, the equation of the conic ellipse is $\boxed{\frac{(x+1)^2}{12}+\frac{(y-4)^2}{20}=1}$.

Q42E

We must find an equation of the conic that satisfies the following conditions:
Ellipse, foci(±4, 0), passing through (-4, 1.8)

Since the foci are (±4, 0), the major and minor axes of the ellipse are the x and y axes respectively.

The ellipse equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The focus is:

$4 = f = \sqrt{a^2 - b^2}$

$\Rightarrow a^2 - b^2 = 16$

Ellipse goes through (-4, 1.8)

$\frac{16}{a^2}+\frac{81}{25b^2}=1$

$\Rightarrow \frac{16}{16+b^2}+\frac{81}{25b^2}=1 \qquad \left[\text{Since } a^2 = 16+b^2\right]$

$\Rightarrow 16+\frac{81(16+b^2)}{25b^2}=16+b^2$

$\Rightarrow \frac{81(16+b^2)}{25b^2}=b^2$

$\Rightarrow 25b^4 = 1296+81b^2$

$\Rightarrow -25b^4 + 81b^2 + 1296 = 0$

Let $z = b^2 > 0$

$25b^4 - 81b^2 - 1296 = 0$

$\Rightarrow 25z^2 - 81z - 1296 = 0$

$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow z = \frac{81 \pm \sqrt{(-81)^2 - 4(25)(-1296)}}{2(25)} = 9$

$\Rightarrow z = b^2 = 9$

$a^2 - b^2 = 16$

$\Rightarrow a^2 - 9 = 16$

$a^2 = 25$

Therefore, Equation of the ellipse is $\boxed{\frac{x^2}{25}+\frac{y^2}{9}=1}$

Q43E

Consider the data

Vertices of the hyperbola are $(\pm 3, 0)$

Foci of the hyperbola are $(\pm 5, 0)$

Recall that,

The equation of the hyperbola with vertices $(\pm a, 0)$ and $(\pm c, 0)$ is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

With the condition $c^2 = a^2 + b^2$

The vertices are $(\pm a, 0)$, so $a = 3$ and foci are $(\pm c, 0)$, so $c = 5$.

Find the value of b by substitution of a, c in $c^2 = a^2 + b^2$.

$$c^2 = a^2 + b^2$$

$(5)^2 = (3)^2 + b^2$ Substitute $a = 3, c = 5$

$$25 = 9 + b^2$$

$$b^2 = 25 - 9$$ Simplify

$$b = \sqrt{16}$$

$$b = 4$$

Substitute a, b in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$$\frac{x^2}{3^2} - \frac{y^2}{4^2} = 1$$

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Thus, the equation of the conic hyperbola is $\boxed{\frac{x^2}{9} - \frac{y^2}{16} = 1}$.

Q44E

Consider the data

Vertices of the hyperbola are $(0, \pm 2)$

Foci of the hyperbola are $(0, \pm 5)$

Recall that,

The equation of the hyperbola with vertices $(0, \pm a)$ and $(0, \pm c)$ is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

With the condition $c^2 = a^2 + b^2$

The vertices are $(0, \pm a)$, so $a = 2$ and foci are $(0, \pm c)$, so $c = 5$.

Find the value of b by substitution of a, c in $c^2 = a^2 + b^2$.

$$c^2 = a^2 + b^2$$

$$(5)^2 = (2)^2 + b^2 \quad \text{Substitute } a = 2, c = 5$$

$$25 = 4 + b^2$$

$$b^2 = 25 - 4 \quad \text{Simplify}$$

$$b = \sqrt{21}$$

Substitute a, b in $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

$$\frac{y^2}{2^2} - \frac{x^2}{\left(\sqrt{21}\right)^2} = 1$$

$$\frac{y^2}{4} - \frac{x^2}{21} = 1$$

Thus, the equation of the conic hyperbola is $\boxed{\frac{y^2}{4} - \frac{x^2}{21} = 1}$.

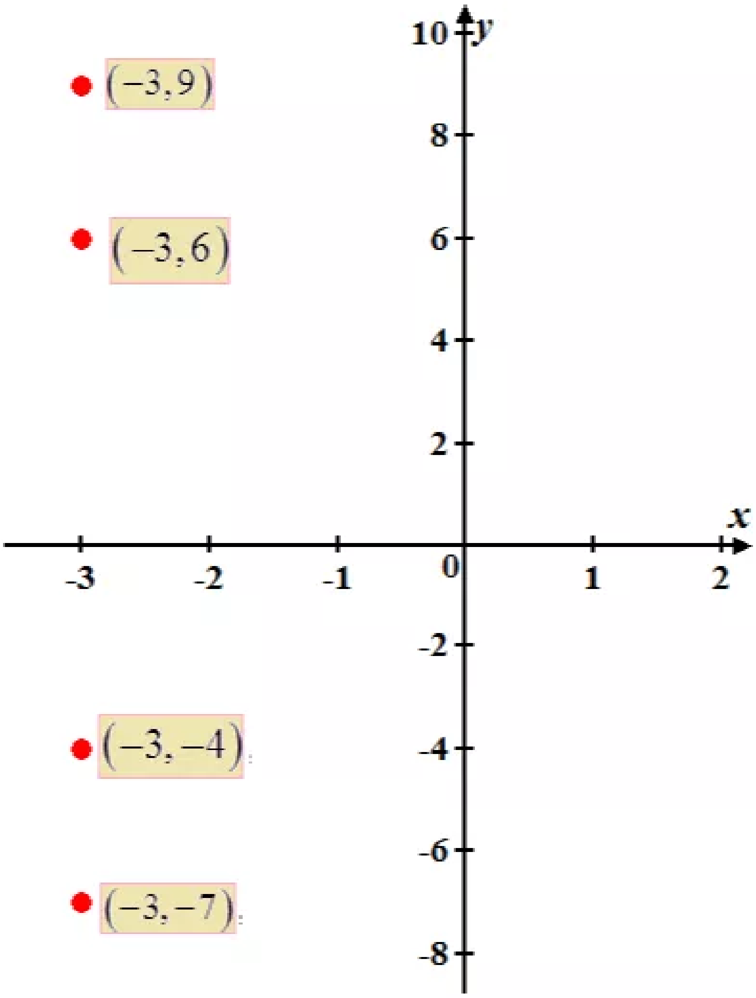
Q45E

Consider the data

Vertices of the hyperbola are $(-3,-4),(-3,6)$

Foci of the hyperbola are $(-3,-7),(-3,9)$

Plot the points on the coordinate axes as follows.



From the graph it is clear that, all the points lie on the same line $x=-3$. So, the axis of the hyperbola is parallel to vertical axis.

Recall that,

The equation of the hyperbola is

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1, c^2=a^2+b^2$$

with foci $(h,k\pm c)$, vertices $(h,k\pm a)$, centre (h,k) and asymptotes $y-k=\pm\frac{a}{b}(x-h)$.

And recollect the fact that the centre is the midpoint of the vertices of the hyperbola.

The midpoint of the vertices $(-3,-4),(-3,6)$ is

$$\left(\frac{-3-3}{2},\frac{-4+6}{2}\right)=\left(\frac{-6}{2},\frac{2}{2}\right)$$
$$=(-3,1)$$

So, the centre of the hyperbola is $(h,k)=(-3,1)$.

We know that,

The distance between the centre to either of the focus is *c* and the distance from the centre to either of the vertex is *a*.

Find the distances from the centre to focus and from centre to vertex.

$c=\sqrt{(-3+3)^2+(-7-1)^2}$ Distance between $(-3,1)$ and $(-3,-7)$

$=\sqrt{0^2+8^2}$

$=\sqrt{64}$ Simplify

$=8$

$a=\sqrt{(-3+3)^2+(-4-1)^2}$ Distance between $(-3,1)$ and $(-3,-4)$

$=\sqrt{(0)^2+(-5)^2}$

$=\sqrt{0+25}$ Simplify

$=5$

Find the value of *b* by using $c^2=a^2+b^2$

$(8)^2=(5)^2+b^2$

$64=25+b^2$ Simplify

$b^2=64-25$

$b^2=39$

$b=\sqrt{39}$

Substitute a, b, h and k values in (1).

$$\frac{(y-1)^2}{(5)^2}-\frac{(x-(-3))^2}{(\sqrt{39})^2}=1$$

$$\frac{(y-1)^2}{25}+\frac{(x+3)^2}{39}=1$$

Thus, the equation of the conic hyperbola is $\boxed{\frac{(y-1)^2}{25}+\frac{(x+3)^2}{39}=1}$.

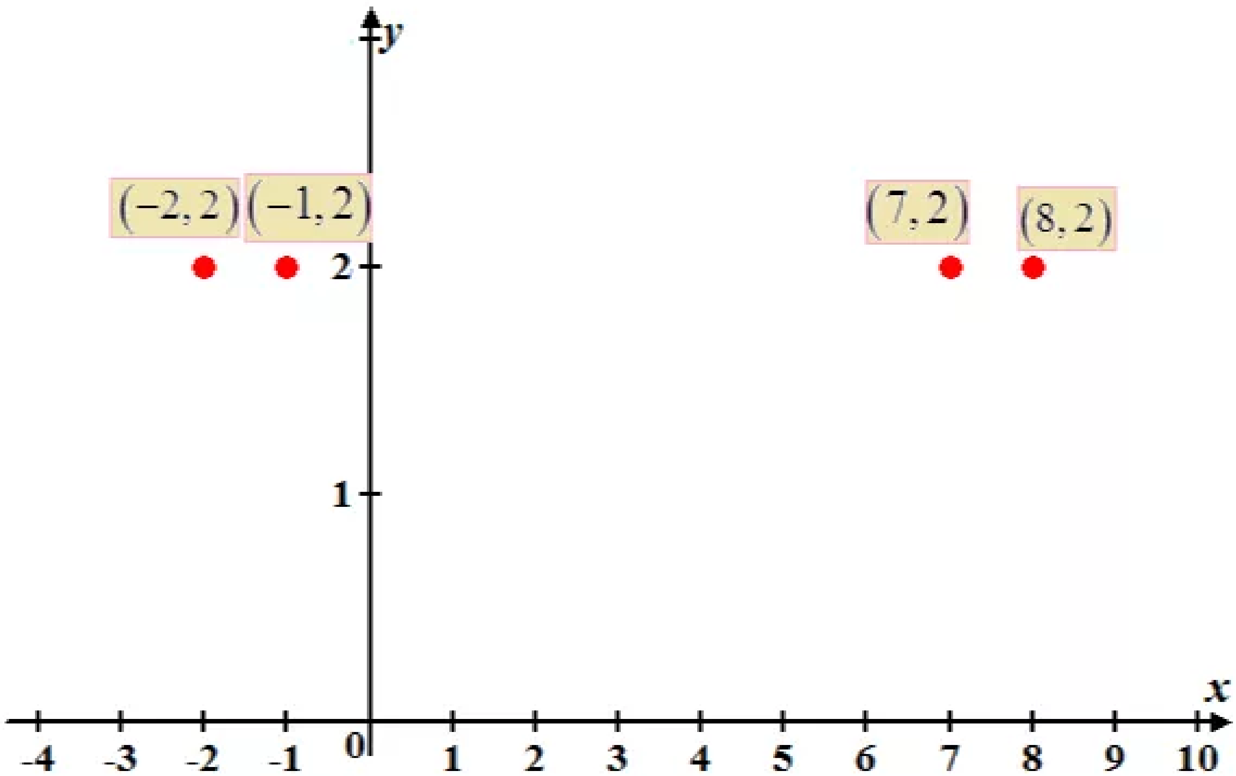
Q46E

Consider the data

Vertices of the hyperbola are $(-1,2),(7,2)$

Foci of the hyperbola are $(-2,2),(8,2)$

Plot the points on the coordinate axes as follows.



From the graph it is clear that, all the points lie on the same line $x=-3$. So, the axis of the hyperbola is parallel to vertical axis.

Recall that,

The equation of the hyperbola is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1, c^2=a^2+b^2$$

with foci $(h\pm c,k)$, vertices $(h\pm a,k)$, centre (h,k) and asymptotes $y-k=\pm\frac{b}{a}(x-h)$.

And recollect the fact that the centre is the midpoint of the vertices of the hyperbola.

The midpoint of the vertices $(-1,2),(7,2)$ is

$$\left(\frac{-1+7}{2},\frac{2+2}{2}\right)=\left(\frac{6}{2},\frac{4}{2}\right)$$
$$=(3,2)$$

So, the centre of the hyperbola is $(h,k)=(3,2)$.

We know that,

The distance between the centre to either of the focus is *c* and the distance from the centre to either of the vertex is *a*.

Find the distances from the centre to focus and from centre to vertex.

$c=\sqrt{(-2-3)^2+(2-2)^2}$ Distance between $(-2,2)$ and $(2,2)$

$=\sqrt{5^2+0^2}$

$=\sqrt{25}$ Simplify

$=5$

$a=\sqrt{(-1-3)^2+(2-2)^2}$ Distance between $(-1,2)$ and $(2,2)$

$=\sqrt{4^2+0^2}$

$=\sqrt{16}$ Simplify

$=4$

Find the value of *b* by using $c^2=a^2+b^2$

$(5)^2=(4)^2+b^2$

$25=16+b^2$ Simplify

$b^2=25-16$

$b^2=9$

$b=3$

Substitute a, b, h and k values in (1).

$$\frac{(x-3)^2}{(4)^2}-\frac{(y-2)^2}{(3)^2}=1$$

$$\frac{(x-3)^2}{16}+\frac{(y-2)^2}{9}=1$$

Thus, the equation of the conic hyperbola is $\boxed{\frac{(x-3)^2}{16}+\frac{(y-2)^2}{9}=1}$.

"

Q47E

Consider the data

Foci of the hyperbola are $(\pm 3,0)$

Asymptotes of the hyperbola are $y=\pm 2x$

Find the hyperbola that have the given foci and asymptotes.

The foci are of the form $(\pm c,0)$. So, form this it is clear that the transverse axis is along *x*-axis.

Recall the equation for the hyperbola with transverse axis along *y*-axis,

The equation of the hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, c^2=a^2+b^2$$

with foci $(\pm c,0)$, vertices $(\pm a,0)$, centre $(0,0)$ and asymptotes $y=\pm\frac{b}{a}x$.

And the distance from centre to either of the focus is *c*.

Compare the equations of the given asymptotes with $y=\pm\frac{b}{a}x$.

From this,

$$\frac{b}{a}=2$$

$$b=2a$$

Find the distances from the centre to focus.

$c=\sqrt{(3-0)^2+(0-0)^2}$ Distance between $(2,0)$ and $(0,0)$

$=\sqrt{3^2+0^2}$

$=\sqrt{9}$ Simplify

$=3$

So, $c=3$

Substitute $c=3, b=2a$ in $c^2=a^2+b^2$.

$$3^2=a^2+(2a)^2$$

$$9=a^2+4a^2$$

$$9=5a^2$$

$$b^2=\frac{9}{5}$$

$$b=\sqrt{\frac{9}{5}}$$

$$=\frac{3}{\sqrt{5}}$$

Plug the value of a in $b=2a$.

$$b=2\left(\frac{3}{\sqrt{5}}\right)$$

$$=\frac{6}{\sqrt{5}}$$

Substitute h,k,a and b values in $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

$$\frac{x^2}{\left(\frac{3}{\sqrt{5}}\right)^2}-\frac{y^2}{\left(\frac{6}{\sqrt{5}}\right)^2}=1$$

$$\frac{x^2}{\frac{9}{5}}-\frac{y^2}{\frac{36}{5}}=1$$

$$\frac{5x^2}{9}-\frac{5y^2}{36}=1$$

Therefore, the equation of the hyperbola is $\boxed{\frac{5x^2}{9}-\frac{5y^2}{36}=1}$.

Q48E

Consider the foci $(2,0)$ and $(2,8)$

Asymptotes are $y=3+\frac{1}{2}x$ and $y=5-\frac{1}{2}x$

Find the hyperbola that have the given foci and asymptotes.

The foci are of the form $(h,k\pm c)$. So, form this it is clear that the transverse axis is along *y*-axis.

Recall the equation for the hyperbola with transverse axis along *y*-axis,

The equation of the hyperbola is

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1, c^2=a^2+b^2$$

with foci $(h,k\pm c)$, vertices $(h,k\pm a)$, centre (h,k) and asymptotes $y-k=\pm\frac{a}{b}(x-h)$.

And recollect the fact the intersection of the asymptotes of the hyperbola is the centre.

And the distance from centre to either of the focus is *c*

Find the intersections of the asymptotes of the hyperbola. To do this, equate the two equations.

$$3+\frac{1}{2}x=5-\frac{1}{2}x$$

$\frac{1}{2}x+\frac{1}{2}x=5-3$ Add $\frac{1}{2}x$ and -3 on each side

$x=2$ Simplify

Substitute $x=2$ in one the asymptotes to find *y*.

$$\begin{aligned} y &= 3+\frac{1}{2}x \\ &= 3+\frac{1}{2}(2) \\ &= 3+1 \\ &= 4 \end{aligned}$$

Therefore, the centre of the hyperbola is $(h,k)=(2,4)$

Compare the equations of the given asymptotes with $y-k=\pm\frac{a}{b}(x-h)$.

From this,

$$\frac{a}{b}=\frac{1}{2}$$

$$b=2a$$

Find the distances from the centre to focus.

$c=\sqrt{(2-2)^2+(0-4)^2}$ Distance between $(2,0)$ and $(2,4)$

$=\sqrt{0^2+4^2}$

$=\sqrt{16}$ Simplify

$=4$

So, $c=4$

Substitute $c=4, b=2a$ in $c^2=a^2+b^2$.

$$4^2=a^2+(2a)^2$$

$$16=a^2+4a^2$$

$$16=5a^2$$

$$a^2=\frac{16}{5}$$

$$a=\sqrt{\frac{16}{5}}$$

$$=\frac{4}{\sqrt{5}}$$

Plug the value of a in $b=2a$.

$$b=2\left(\frac{4}{\sqrt{5}}\right)$$

$$=\frac{8}{\sqrt{5}}$$

Substitute h, k, a and b values in $\frac{(y-k)^2}{a^2} - \frac{(x-k)^2}{b^2} = 1$.

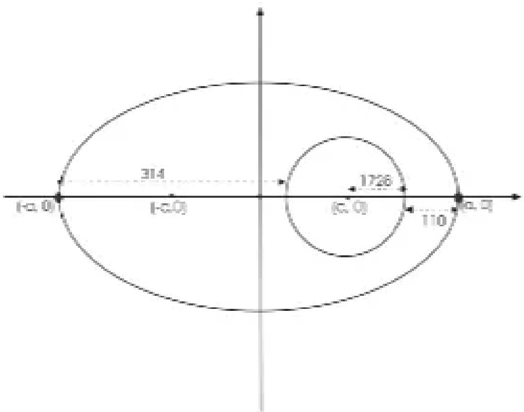
$$\frac{(y-4)^2}{\left(\frac{4}{\sqrt{5}}\right)^2} - \frac{(x-2)^2}{\left(\frac{8}{\sqrt{5}}\right)^2} = 1$$

$$\Rightarrow \frac{(y-4)^2}{\frac{16}{5}} - \frac{(x-2)^2}{\frac{64}{5}} = 1$$

$$\frac{5(y-4)^2}{16} - \frac{5(x-2)^2}{64} = 1$$

Therefore, the equation of the hyperbola is $\boxed{\frac{5(y-4)^2}{16} - \frac{5(x-2)^2}{64} = 1}$.

Q49E



From figure we see that the point on the ellipse closest to a focus (c, o), is the vertex (a, o)

Distance from the focus = a-c

And the farthest point on the ellipse from this focus is the vertex (-a, o),

Distance from the focus = a + c

Thus for the orbit

$$(a-c)+(a+c) = 2a$$

$$\Rightarrow 2a = (1728+110)+(1728+314)$$

$$\Rightarrow 2a = 3880 \qquad \Rightarrow \boxed{a = 1940}$$

And $(a+c)-(a-c) = 2c$

$$\Rightarrow 314-110 = 2c \qquad \Rightarrow \boxed{c = 102}$$

Since $c^2 = a^2 - b^2$ *then* $b^2 = a^2 - c^2 = (1940)^2 - (102)^2$

$$b^2 = 3753196$$

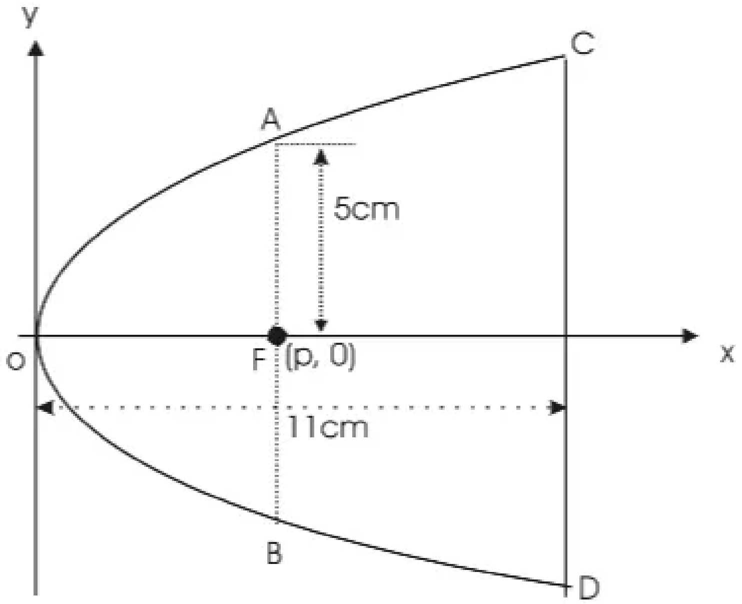
Equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\Rightarrow \frac{x^2}{(1940)^2}+\frac{y^2}{3753196}=1$$

$$\Rightarrow \boxed{\frac{x^2}{3763600}+\frac{y^2}{3753196}=1}$$

Q50E

(A)



We place the vertex of the parabola at the origin. The equation of this parabola by definition we have $y^2=4px \quad where\ p>0$

Co-ordinates of the focus F are (p, 0)

Since the line $|AB|$ is perpendicular to the X-axis at the point F, so co-ordinates of the point A are (p, 5) which will satisfy the equation of parabola so ,

$$(5)^2=4p(p)$$

$$\Rightarrow \frac{(5)^2}{4}=p^2 \qquad \Rightarrow \boxed{p=5/2} \quad [p>0]$$

Then equation of parabola becomes

$$y^2=4\times\frac{5}{2}x \quad \Rightarrow y^2=10x$$

$$\Rightarrow \boxed{x=\frac{1}{10}y^2}$$ This is the equation of parabola.

(B) Given that opening $|CD|$ is at the distance 11 cm from the origin

So x-coordinate of the point C is 11

Then from the equation of parabola $x = \frac{1}{10}y^2$

We have $11 = \frac{1}{10}y^2$

$\Rightarrow y^2 = 110$

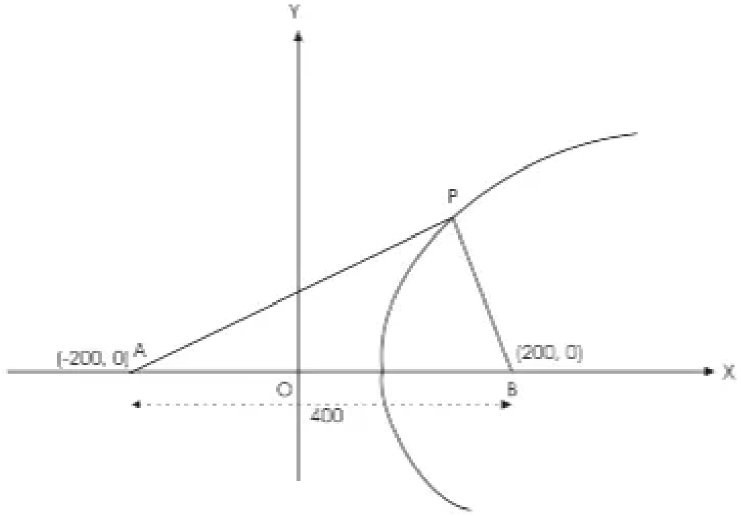
$\Rightarrow y^2 = \sqrt{110}$ (c is in the first quadrant)

Then diameter of the opening $|CD|$ is $= 2 \times \sqrt{110}$

Or $\boxed{|CD| = 2\sqrt{110}}$ cm

Q51E

(A)



We place the points A and B on the x-axis and origin at the mid point of $|AB|$.

Co-ordinates of the points A and B are (-200, 0) and (200, 0) Let A and B are the foci of the hyperbola.

Given time difference = 1200 μs

Then according to the problem

$$|PA| - |PB| = \text{time difference} \times \text{speed of the signal}$$
$$= 1200\,\mu s \times 980 \text{ ft}/\mu s$$
$$= 1176000\, ft$$

Since 1 mile = 5280 ft so

$$|PA| - |PB| = \frac{1176000}{5280}\text{miles} = \frac{2450}{11}\, miles$$

By the definition of hyperbola we have $|PA| - |PB| = 2a$

$$\Rightarrow 2a = \frac{2450}{11} \Rightarrow \boxed{a = \frac{1225}{11}}$$

And from figure we have c = 200

$$\text{Since } c^2 = a^2 + b^2$$
$$\text{then } b^2 = c^2 - a^2$$
$$\Rightarrow b^2 = (200)^2 - \left(\frac{1225}{11}\right)^2 = 40000 - \frac{1500625}{121}$$
$$\Rightarrow b^2 = \frac{3339375}{121}$$

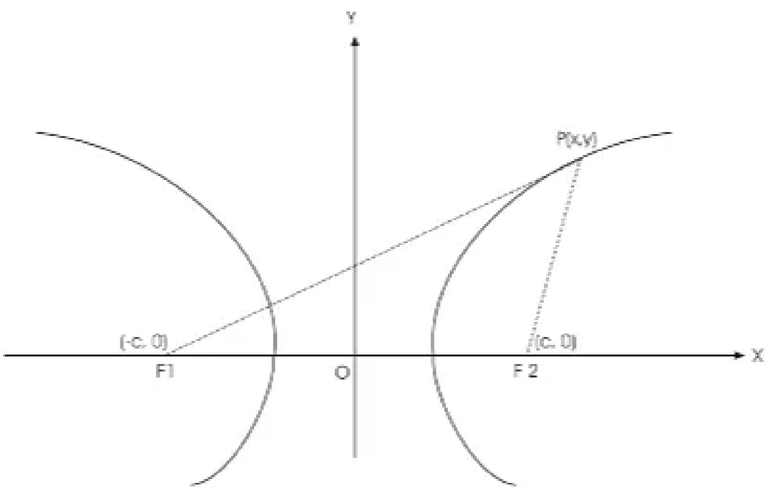
Then equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\Rightarrow \boxed{\frac{121x^2}{1560625} - \frac{121y^2}{3339375} = 1} \qquad ---(1)$$

(B) According to the problem ship is due north of B, so x-coordinate of the point P is $x = 200$. Then from (1)

$$\frac{121(200)^2}{1500625} - \frac{121y^2}{3339375} = 1$$
$$\Rightarrow \frac{4840000}{1500625} - \frac{121y^2}{3339375} = 1$$
$$\Rightarrow \frac{121y^2}{3339375} = \frac{4840000}{1500625} - 1 = \frac{3339375}{1500625}$$
$$\Rightarrow y^2 = \frac{(3339375)^2}{(1500625)\times(121)} \Rightarrow y = \frac{3339375}{1225\times 11}$$
$$\Rightarrow y = \frac{133575}{539} \approx 248\,\text{miles}$$
$$\Rightarrow \text{Distance from the coastline is} \approx \boxed{248\,\text{miles}}$$

Q52E



Let the coordinates of the point P, be (x,y) which is lying on the hyperbole. F_1 and F_2 are two fixed point having coordinates (-c, 0) and (c, 0) respectively

By the definition of hyperbola we have

$$|PF_1| - |PF_2| = \pm 2a$$

By the distance formula

$$\Rightarrow \sqrt{(x+c)^2+(y-0)}-\sqrt{(x-c)^2(y-0)^2}=2a$$

$$\Rightarrow \sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a$$

$$\Rightarrow \sqrt{(x+c)^2+y^2}=\sqrt{(x+c)^2+y^2}+2a$$

Squaring both sides

$$(x+c)^2+y^2=(x-c)^2+y^2+4a^2+4a\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow x^2+c^2+2xc=x^2+c^2-2xc+4a^2+4a\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow 4xc=4a^2+4a\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow xc-a^2=a\sqrt{(x-c)^2\,y^2}$$

Squaring again

$$x^2c^2+a^4-2a^2xc=a^2(x-c)^2+a^2y^2$$

$$\Rightarrow x^2c^2+a^4-2a^2xc=a^2\left(x^2+c^2-2xc\right)+a^2y^2$$

$$\Rightarrow x^2c^2+a^4-2a^2xc=a^2x^2+a^2c^2-2a^2xc+a^2y^2$$

$$\Rightarrow x^2c^2-a^2x^2-a^2y^2=a^2c^2-a^4$$

$$\Rightarrow x^2\left(c^2-a^2\right)-a^2y^2=a^2\left(c^2-a^2\right)$$

Let $c^2-a^2=b^2$

Then $b^2x^2-a^2y^2=a^2b^2$

Dividing both sides by a^2b^2

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$$

Which is the equation of hyperbola with foci $(\pm c,0)$ and vertices $(\pm a,0)$

Where $c^2=a^2+b^2$

Q53E

Equation of hyperbole is $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

Solving for y $\Rightarrow y^2=a^2\left(1+\frac{x^2}{b^2}\right)$

Thus equation of upper branch of the hyperbola is $y=a\sqrt{1+\frac{x^2}{b^2}}$

$$\Rightarrow y=\frac{a}{b}\sqrt{\left(b^2+x^2\right)}$$

Differentiating with respect to x

$$y'=\frac{a}{b}\frac{1}{2\sqrt{b^2+x^2}}\frac{2x}{}$$

$$\Rightarrow y'=\frac{a}{b}\frac{x}{\sqrt{b^2+x^2}}$$

Again differentiating

$$y'' = \frac{a}{b}\left[\frac{\sqrt{b^2+x^2} - \frac{1}{2\sqrt{b^2+x^2}}\frac{2x}{}x}{b^2+x^2}\right]$$

$$\Rightarrow y'' = \frac{a}{b}\left[\frac{b^2+x^2-x^2}{\left(b^2+x^2\right)^{3/2}}\right]$$

$$\Rightarrow y'' = ab\left(b^2+x^2\right)^{-3/2} > 0 \qquad \textit{for all } x \text{ and a, b} > 0$$

And so y is concave upward.

Q54E

Consider the data

Foci of the ellipse $(1,1)$ and $(-1,-1)$

Length of major axis of ellipse is 4

With the given data find the equation of the ellipse.

Suppose that $P(x,y)$ be any point on the ellipse.

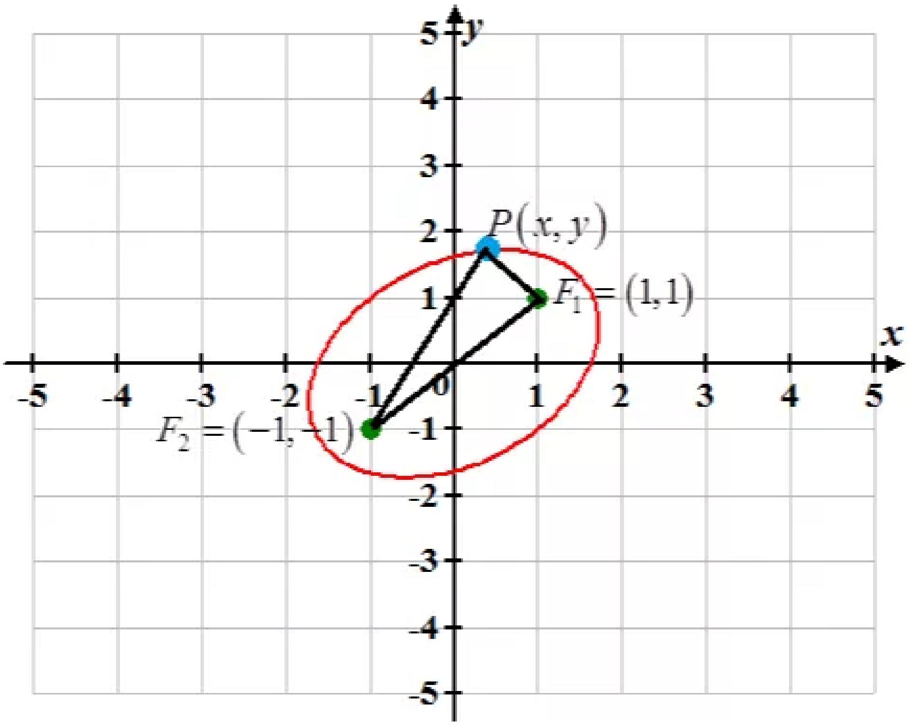
Recall that,

For an ellipse, the sum of the distances from the foci F_1 and F_2 from a point $P(x,y)$ on the ellipse is constant.

Here, $F_1 = (1,1), F_2 = (-1,-1)$

Since the length of the major axis is 4, the constant value is 4.

Make a rough sketch of the ellipse with the given foci.



Solve the equation $|PF_1|+|PF_2|=4$ to get the equation of the ellipse through the point P.

Now,

$$|PF_1|+|PF_2|=4$$

$$\sqrt{(x-1)^2+(y-1)^2}+\sqrt{(x+1)^2+(y+1)^2}=4$$

$$\left(\sqrt{(x-1)^2+(y-1)^2}\right)^2=\left(4-\sqrt{(x+1)^2+(y+1)^2}\right)^2$$

$$(x-1)^2+(y-1)^2=4^2-2\sqrt{(x+1)^2+(y+1)^2}\cdot 4+(x+1)^2+(y+1)^2$$

$$(x-1)^2+(y-1)^2=16-8\sqrt{(x+1)^2+(y+1)^2}+(x+1)^2+(y+1)^2$$

$$8\sqrt{(x+1)^2+(y+1)^2}=16+(x+1)^2+(y+1)^2-(x-1)^2-(y-1)^2$$

$$8\sqrt{(x+1)^2+(y+1)^2}=16+\left((x+1)^2-(x-1)^2\right)+\left((y+1)^2-(y-1)^2\right)$$

$$8\sqrt{(x+1)^2+(y+1)^2}=16+(2\cdot 2x)+(2\cdot 2y)$$

$$8\sqrt{(x+1)^2+(y+1)^2}=16+4x+4y$$

$$2\sqrt{(x+1)^2+(y+1)^2}=4+x+y$$

$$\left(2\sqrt{(x+1)^2+(y+1)^2}\right)^2=(4+x+y)^2$$

$$4\left((x+1)^2+(y+1)^2\right)=\left(4+(x+y)\right)^2$$

$$4\left(x^2+2x+1+y^2+2y+1\right)=4^2+2\cdot 4\cdot(x+y)+(x+y)^2$$

$$4x^2+4\cdot 2x+4\cdot 1+4\cdot y^2+4\cdot 2y+4\cdot 1=16+8x+8y+x^2+2xy+y^2$$

$$4x^2+8x+4y^2+8y+8=x^2+2xy+8x+y^2+8y+16$$

$$3x^2+3y^2=2xy+16-8$$

$$3x^2+3y^2=2xy+8$$

Therefore, the equation of the ellipse $\boxed{3x^2+3y^2=2xy+8}$.

Q55E

(A) Equation of the conic is $\frac{x^2}{k}+\frac{y^2}{k-16}=1$

Case: - when $k>16$

Then $k>k-16>0$

So this equation represents an ellipse

(B) When $0<k<16$

So k-16<0

And then we can write the equation

$$\frac{x^2}{k}+\frac{y^2}{-(16-k)}=1$$

$$\Rightarrow \frac{x^2}{k}-\frac{y^2}{16-k}=1$$

This represents a hyperbola

(C) When $k < 0$

So $k - 16 < 0$

Then comparing with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Where $a \geq b > 0$

But here $k = a^2 < 0$ and $k-16=b^2<0$

So there is a contradiction

So for $k < 0$, this equation does not represent any curve

(D) First we consider the case: $k > 16$,

Equation of the curve is $\frac{x^2}{k}+\frac{y^2}{k-16}=1$

Here $a^2=k,\ b^2=k-16$

And $c^2=a^2-b^2$

$\Rightarrow c^2 = k-(k-16)$

$=16$

$\Rightarrow c=\pm 4$

So foci are $(\pm 4, 0)$

Now for the case: $0 < k < 16$

The equation becomes $\frac{x^2}{k}-\frac{y^2}{16-k}=1$ which is a hyperbola

Here $a^2=k$ and $b^2=16-k$

$\Rightarrow c^2 = a^2+b^2=(k)+(16-k)=16$

$\Rightarrow c=\pm 4$

$\Rightarrow$ foci are $(\pm 4, 0)$

So we get the same foci for different values of k.

Q56E

Consider the parabola

$y^2 = 4px$ ...... (1)

And the parabola has tangent at the point (x_0, y_0)

It is required to show that the equation of the tangent is $yy_0 = 2px + 2px_0$.

First find the slope of the tangent line to (1). The slope of the tangent line is the first derivative of the equation (1) at the point (x_0, y_0).

Differentiate (1) with respect to *x*.

$$2yy' = 4p$$

$$y' = \frac{4p}{2y}$$

Substitute the point (x_0, y_0) in y'. Then

$$y' = \frac{4p}{2y_0}$$

So, the slope of tangent line is $\frac{4p}{2y_0}$.

State the point slope formula

The equation of the line with slope m and passing through the point (x_0, y_0) is

$$y - y_0 = m(x - x_0)$$

Therefore, the equation of the tangent line with slope $\frac{4p}{2y_0}$ and passing through the point (x_0, y_0) is

$$y - y_0 = \frac{4p}{2y_0}(x - x_0)$$

$$y - y_0 = \frac{2p}{y_0}(x - x_0)$$ This is the equation of the tangent line

$$y_0(y - y_0) = 2p(x - x_0)$$

$$yy_0 - y_0^2 = 2px - 2px_0$$ ...... (2)

Plug in the point (x_0, y_0) in the parabola given by (1).

$$y_0^2 = 4px_0$$

Substitute the value of y_0^2 in (2).

$$yy_0 - y_0^2 = 2px - 2px_0$$

$$yy_0 - 4px_0 = 2px - 2px_0$$

$$yy_0 = 2px - 2px_0 + 4px_0$$

$$yy_0 = 2px + 2px_0$$

$$yy_0 = 2p(x + x_0)$$

Hence, it is proved that the equation of the tangent line can be written as $\boxed{yy_0 = 2p(x + x_0)}$.

Plug in the point (x_0, y_0) in the parabola given by (1).

$$y_0^2 = 4px_0$$

Substitute the value of y_0^2 in (2).

$$yy_0 - y_0^2 = 2px - 2px_0$$

$$yy_0 - 4px_0 = 2px - 2px_0$$

$$yy_0 = 2px - 2px_0 + 4px_0$$

$$yy_0 = 2px + 2px_0$$

$$yy_0 = 2p(x + x_0)$$

Hence, it is proved that the equation of the tangent line can be written as $\boxed{yy_0 = 2p(x + x_0)}$.

(b)

To find the *x*-intercept of the tangent line set $y=0$ in $y-y_0=\frac{2p}{y_0}(x-x_0)$.

$yy_0=2p(x+x_0)$ Tangent line

$(0)y_0=2p(x+x_0)$

$0=2p(x+x_0)$

$0=(x+x_0)$

$x=-x_0$

Therefore, the *x*-intercept of the tangent line is $\boxed{(-x_0,0)}$.

The slope of the tangent line is $\frac{2p}{y_0}$

We know that slope is $\frac{\text{Rise}}{\text{Run}}$

Rise is difference between the *y*-coordinates and run is the difference between the *x*-coordinates.

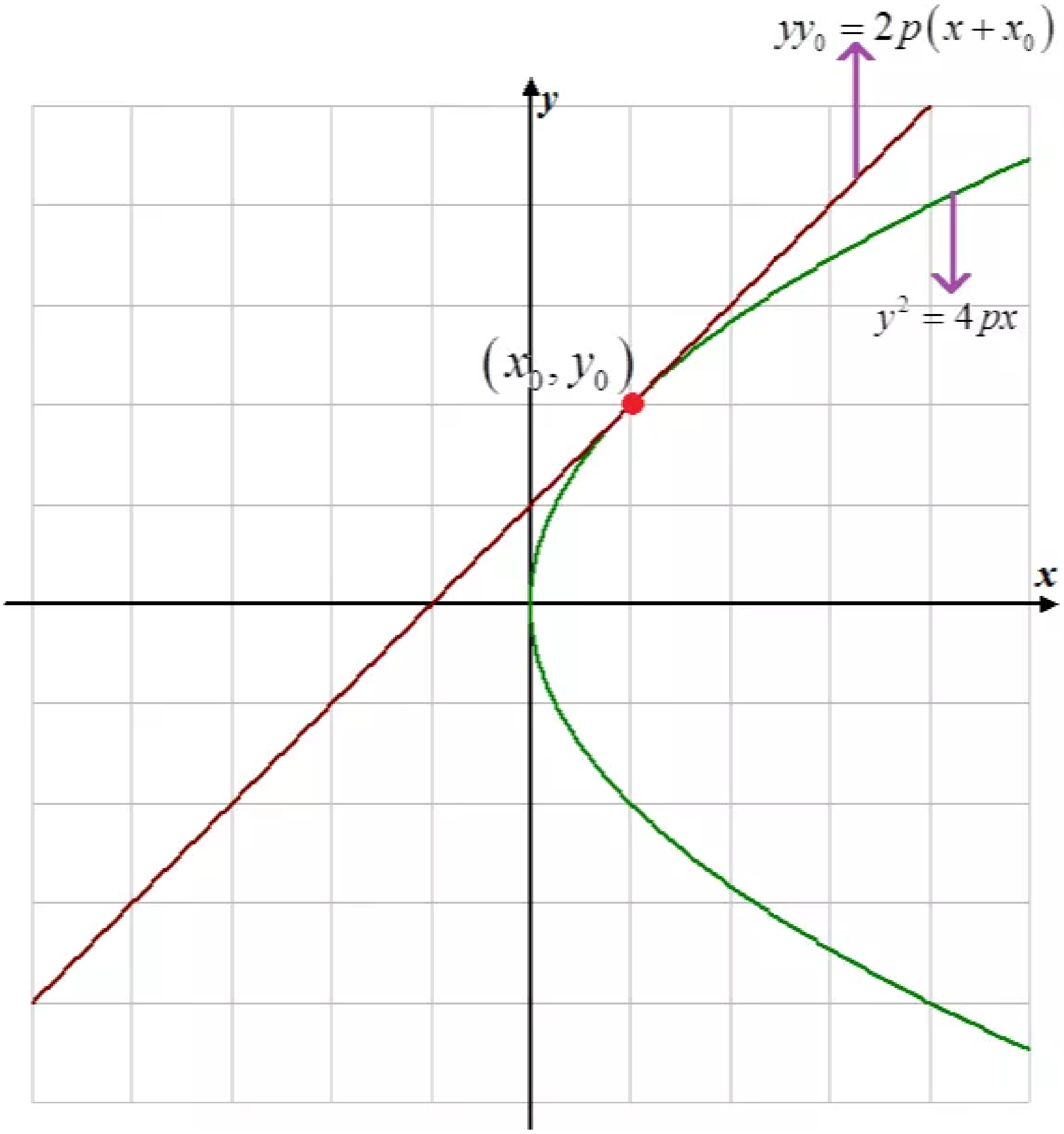
So, the coordinate next to *x*-intercept of the tangent line is obtained by adding y_0 to *x*-coordinate of *x*-intercept and by adding $2p$ to *y*-coordinate of *x*-intercept.

The next coordinate is $(-x_0+y_0,2p)$.

But depending on the value of *p*, the parabola opens to the left of right.

Assume that $p>0$, and then draw the tangent to the parabola that opens to the right is as follows.

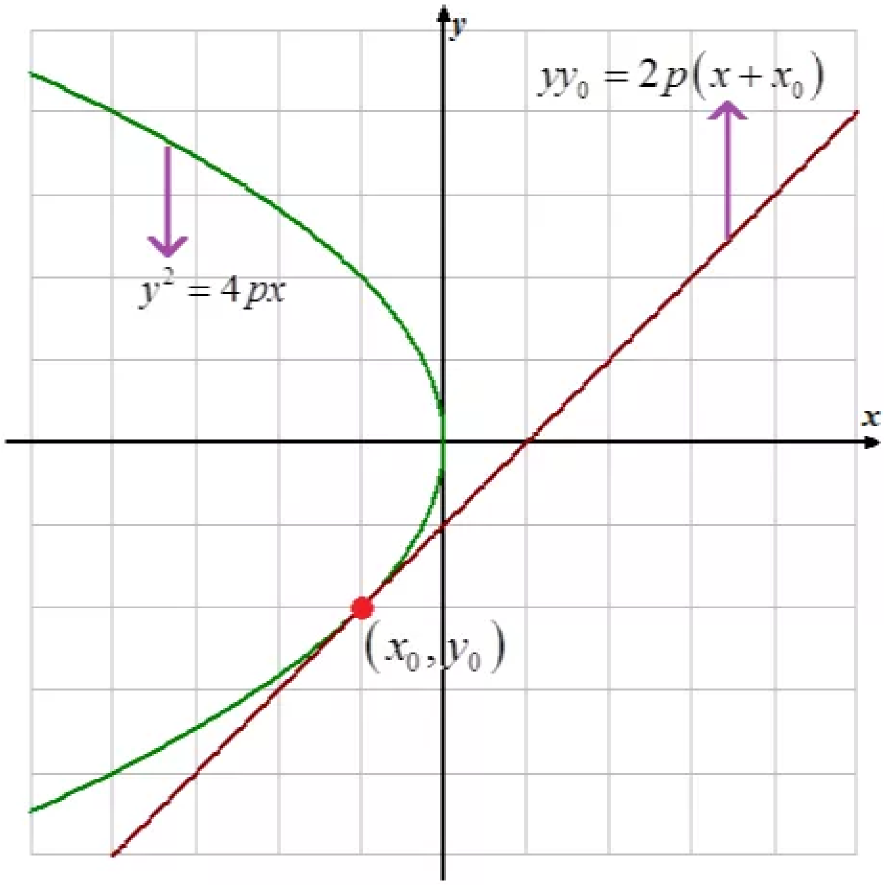
Use the two points $(-x_0,0)$ and $(-x_0+y_0,2p)$ to draw the tangent line. The tangent to the parabola $y^2=4px, p>0$ is as follows.



$yy_0 = 2p(x + x_0)$
y
$y^2 = 4px$
(x_0, y_0)
x

Assume that $p<0$, and then draw the tangent to the parabola that opens to the left is as follows.

Use the two points $(-x_0,0)$ and $(-x_0+y_0,2p)$ to draw the tangent line. The tangent to the parabola $y^2=4px, p<0$ is as follows.



Q57E

Consider the equation of the parabola

$x^2=4py$

Recollect that the equation of the tangent lines to the parabola, $x^2=4py$ at point (x_0,y_0) is given by

$xx_0=2p(y+y_0)$

Again the equation of the directrix of parabola $x^2=4py$ is $y=-p$

It is given that the tangent

Here, the tangent is on the directrix; therefore, the end point to the directrix is $(x_0,-p)$

Now, the slope of the tangent passing through (x_0,y_0) and $(x_0,-p)$ is

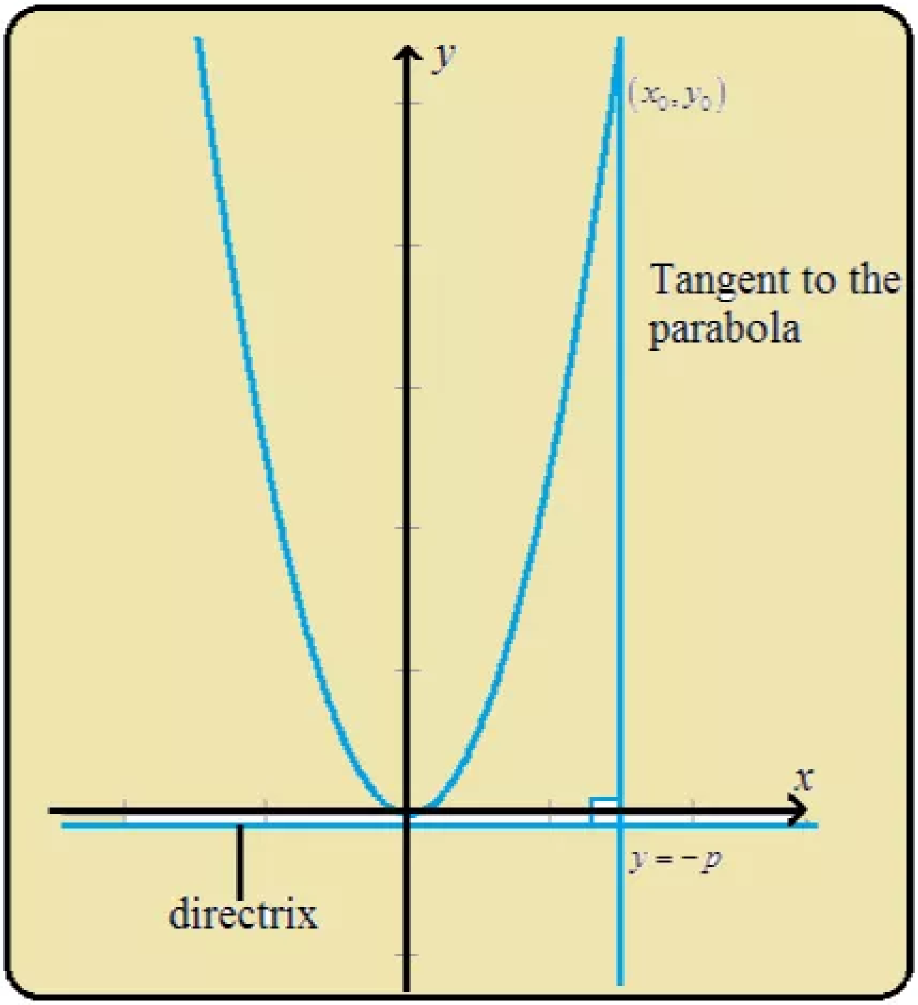
$$m=\frac{y_2-y_1}{x_2-x_1}$$
$$=\frac{-p-y_0}{x_0-x_0}$$

$$\tan\theta=\frac{-p-y_0}{0} \quad [\theta \text{ is the angle between tangent and directrix}]$$

$$\tan\theta=\infty$$

$$\theta=\frac{\pi}{2}$$

Sketch the tangent line at point (x_0, y_0) on the parabola $x^2 = 4py$.



Therefore, the tangent lines to the parabola $x^2 = 4py$ drawn from any point on the directrix are perpendicular.

Q58E

Consider the standard ellipse equation, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the standard hyperbola equation, $\frac{x^2}{a_1^2}-\frac{y^2}{b_1^2}=1$.

Let the ellipse and hyperbola have the same foci $(\pm c, 0)$, then $c=a^2-b^2=a_1^2+b_1^2$.

Find the points of intersection of ellipse and hyperbola.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x^2}{a_1^2}-\frac{y^2}{b_1^2}$$

$$\frac{y^2}{b_1^2}+\frac{y^2}{b^2}=\frac{x^2}{a_1^2}-\frac{x^2}{a^2}$$

$$y^2\left(\frac{1}{b_1^2}+\frac{1}{b^2}\right)=x^2\left(\frac{1}{a_1^2}-\frac{1}{a^2}\right)$$

$$y^2\left(\frac{b^2+b_1^2}{b_1^2b^2}\right)=x^2\left(\frac{a^2-a_1^2}{a^2a_1^2}\right)$$

$$\frac{1}{b_1^2b^2}y^2=x^2\frac{1}{a^2a_1^2}$$ Since $a^2-b^2=a_1^2+b_1^2$

$$y^2=x^2\frac{b_1^2b^2}{a^2a_1^2}$$

Substitute $y^2=x^2\frac{b_1^2b^2}{a^2a_1^2}$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$\frac{x^2}{a^2}+\frac{1}{b^2}\left(\frac{b_1^2b^2}{a^2a_1^2}x^2\right)=1$$

$$\frac{x^2}{a^2}+\frac{b_1^2}{a^2a_1^2}x^2=1$$

$$x^2\left(\frac{a_1^2+b_1^2}{a^2a_1^2}\right)=1$$

$$x^2=\frac{a^2a_1^2}{a_1^2+b_1^2}$$

$$x=\pm\frac{aa_1}{\sqrt{a_1^2+b_1^2}}$$

Substitute $x^2 = \frac{a^2 a_1^2}{a_1^2 + b_1^2}$ in $y^2 = x^2 \frac{b_1^2 b^2}{a^2 a_1^2}$.

$$y^2 = \frac{b_1^2 b^2}{a_1^2 + b_1^2}$$

$$y = \pm \frac{b_1 b}{\sqrt{a_1^2 + b_1^2}}$$

Therefore, the points of intersection of an ellipse and hyperbola are,

$$\boxed{\left(\pm \frac{a_1 a}{\sqrt{a_1^2 + b_1^2}}, \pm \frac{b_1 b}{\sqrt{a_1^2 + b_1^2}} \right)}.$$

To find the slope of the tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, differentiate $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to x.

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

$$\frac{x}{a^2} = -\frac{yy'}{b^2}$$

$$y' = -\frac{b^2 x}{a^2 y}$$

Therefore, the slope of the tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x, y) is $\boxed{m = -\frac{b^2 x}{a^2 y}}$.

To find the slope of the tangent to the curve $\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1$, differentiate $\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1$ with respect to x.

$$\frac{2x}{a_1^2} - \frac{2yy'}{b_1^2} = 0$$

$$\frac{x}{a_1^2} = \frac{yy'}{b_1^2}$$

$$y' = \frac{b_1^2 x}{a_1^2 y}$$

Therefore, the slope of the tangent to the curve $\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1$ at (x, y) is $\boxed{m_1 = \frac{b_1^2 x}{a_1^2 y}}$.

Find the slope of the tangents to the curves at each point of intersection.

Slope of the tangent to the curve at the point $(x,y)=\left(\frac{a_1a}{\sqrt{a_1^2+b_1^2}},\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ is as follows:

Substitute $(x,y)=\left(\frac{a_1a}{\sqrt{a_1^2+b_1^2}},\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ in $m=-\frac{b^2x}{a^2y}$.

$$m=-\frac{b^2}{a^2}\left(\frac{a_1a}{\sqrt{a_1^2+b_1^2}}\cdot\frac{\sqrt{a_1^2+b_1^2}}{b_1b}\right)$$

$$m=-\frac{b^2}{a^2}\left(\frac{a_1a}{b_1b}\right)$$

$$m=-\frac{ba_1}{ab_1}$$

Substitute $(x,y)=\left(\frac{a_1a}{\sqrt{a_1^2+b_1^2}},\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ in $m_1=\frac{b_1^2x}{a_1^2y}$.

$$m_1=\frac{b_1^2}{a_1^2}\left(\frac{a_1a}{\sqrt{a_1^2+b_1^2}}\cdot\frac{\sqrt{a_1^2+b_1^2}}{b_1b}\right)$$

$$m_1=\frac{b_1^2}{a_1^2}\left(\frac{a_1a}{b_1b}\right)$$

$$m_1=\frac{b_1a}{ba_1}$$

Multiply the slopes $m=-\frac{ba_1}{ab_1}$ and $m_1=\frac{b_1a}{ba_1}$

$$\begin{aligned}m_1m&=-\frac{ba_1}{ab_1}\cdot\frac{b_1a}{ba_1}\\&=-1\end{aligned}$$

As the product of the slopes of their tangents is -1, so the tangent lines to the respective curves at the point $(x,y)=\left(\frac{a_1a}{\sqrt{a_1^2+b_1^2}},\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ is perpendicular.

Slope of the tangent to the curves at the point $(x,y)=\left(\frac{a_1a}{\sqrt{a_1^2+b_1^2}},-\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ is as follows:

Substitute $(x,y)=\left(\frac{a_1a}{\sqrt{a_1^2+b_1^2}},-\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ in $m=-\frac{b^2x}{a^2y}$.

$$m=-\frac{b^2}{a^2}\left(\frac{a_1a}{\sqrt{a_1^2+b_1^2}}\cdot\left(-\frac{\sqrt{a_1^2+b_1^2}}{b_1b}\right)\right)$$
$$m=\frac{b^2}{a^2}\left(\frac{a_1a}{b_1b}\right)$$
$$m=\frac{ba_1}{ab_1}$$

Substitute $(x,y)=\left(\frac{a_1a}{\sqrt{a_1^2+b_1^2}},-\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ in $m_1=\frac{b_1^2x}{a_1^2y}$.

$$m_1=\frac{b_1^2}{a_1^2}\left(\frac{a_1a}{\sqrt{a_1^2+b_1^2}}\cdot\left(-\frac{\sqrt{a_1^2+b_1^2}}{b_1b}\right)\right)$$
$$m_1=-\frac{b_1^2}{a_1^2}\left(\frac{a_1a}{b_1b}\right)$$
$$m_1=-\frac{b_1a}{ba_1}$$

Multiply the slopes $m=\frac{ba_1}{ab_1}$ and $m_1=-\frac{b_1a}{ba_1}$

$$m_1m=\frac{ba_1}{ab_1}\cdot\left(-\frac{b_1a}{ba_1}\right)$$
$$=-1$$

As the product of the slopes of their tangents is -1, so the tangent lines to the respective curves at the point $(x,y)=\left(\frac{a_1a}{\sqrt{a_1^2+b_1^2}},-\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ is perpendicular.

Slope of the tangent to the curves at the point $(x,y)=\left(-\frac{a_1a}{\sqrt{a_1^2+b_1^2}},\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ is as follows:

Substitute $(x,y)=\left(-\frac{a_1a}{\sqrt{a_1^2+b_1^2}},\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ in $m=-\frac{b^2x}{a^2y}$.

$$m=-\frac{b^2}{a^2}\left(-\frac{a_1a}{\sqrt{a_1^2+b_1^2}}\cdot\frac{\sqrt{a_1^2+b_1^2}}{b_1b}\right)$$

$$m=\frac{b^2}{a^2}\left(\frac{a_1a}{b_1b}\right)$$

$$m=\frac{ba_1}{ab_1}$$

Substitute $(x,y)=\left(-\frac{a_1a}{\sqrt{a_1^2+b_1^2}},\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ in $m_1=\frac{b_1^2x}{a_1^2y}$.

$$m_1=\frac{b_1^2}{a_1^2}\left(-\frac{a_1a}{\sqrt{a_1^2+b_1^2}}\cdot\frac{\sqrt{a_1^2+b_1^2}}{b_1b}\right)$$

$$m_1=-\frac{b_1^2}{a_1^2}\left(\frac{a_1a}{b_1b}\right)$$

$$m_1=-\frac{b_1a}{ba_1}$$

Multiply the slopes $m=\frac{ba_1}{ab_1}$ and $m_1=-\frac{b_1a}{ba_1}$

$$m_1m=\frac{ba_1}{ab_1}\cdot\left(-\frac{b_1a}{ba_1}\right)$$
$$=-1$$

As the product of the slopes of their tangents is -1, so the tangent lines to the respective curves at the point $(x,y)=\left(-\frac{a_1a}{\sqrt{a_1^2+b_1^2}},\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ is perpendicular.

Slope of the tangent to the curves at the point $(x,y)=\left(-\frac{a_1a}{\sqrt{a_1^2+b_1^2}},-\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ is as follows:

Substitute $(x,y)=\left(-\frac{a_1a}{\sqrt{a_1^2+b_1^2}},-\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ in $m=-\frac{b^2x}{a^2y}$.

$$m=-\frac{b^2}{a^2}\left(-\frac{a_1a}{\sqrt{a_1^2+b_1^2}}\cdot\left(-\frac{\sqrt{a_1^2+b_1^2}}{b_1b}\right)\right)$$

$$m=-\frac{b^2}{a^2}\left(\frac{a_1a}{b_1b}\right)$$

$$m=-\frac{ba_1}{ab_1}$$

Substitute $(x,y)=\left(-\frac{a_1a}{\sqrt{a_1^2+b_1^2}},-\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ in $m_1=\frac{b_1^2x}{a_1^2y}$.

$$m_1=\frac{b_1^2}{a_1^2}\left(-\frac{a_1a}{\sqrt{a_1^2+b_1^2}}\cdot\left(-\frac{\sqrt{a_1^2+b_1^2}}{b_1b}\right)\right)$$

$$m_1=\frac{b_1^2}{a_1^2}\left(\frac{a_1a}{b_1b}\right)$$

$$m_1=\frac{b_1a}{ba_1}$$

Multiply the slopes $m=-\frac{ba_1}{ab_1}$ and $m_1=\frac{b_1a}{ba_1}$

$$m_1m=-\frac{ba_1}{ab_1}\cdot\frac{b_1a}{ba_1}$$
$$=-1$$

As the product of the slopes of their tangents is -1, so the tangent lines to the respective curves at the point $(x,y)=\left(-\frac{a_1a}{\sqrt{a_1^2+b_1^2}},-\frac{b_1b}{\sqrt{a_1^2+b_1^2}}\right)$ is perpendicular.

Hence, if an ellipse and a hyperbola have the same foci, then their tangent lines at each point of intersection are perpendicular.

Q60E

Equation the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Parametric equations for this ellipse are $x=a\cos\theta$ and $y=b\sin\theta$

We have length of major axis $=1.18\times10^{10}\text{km}=2a$

$\Rightarrow a=0.59\times10^{10}$

And length of the minor axis $=1.14\times10^{10}\text{km}=2b$

$\Rightarrow b=0.57\times10^{10}$

Then parametric equation becomes

$x = 0.59 \times 10^{10} \cos\theta$ and $y = 0.57 \times 10^{10} \sin\theta$ $\quad 0 \le \theta \le 2\pi$

Then $\dfrac{dx}{d\theta} = -0.59 \times 10^{10} \sin\theta$ and $\dfrac{dy}{d\theta} = 0.57 \times 10^{10} \cos\theta$

If center of the ellipse is at the origin than it has the same length in each quadrant

So length of ellipse

$$L = 4\int_0^{\pi/2} \sqrt{\left(dx/d\theta\right)^2 + \left(dy/d\theta\right)^2}\ d\theta$$

$$= 4\int_0^{\pi/2} \sqrt{\left(0.3481 \times 10^{20}\right)\cos^2\theta + \left(0.3249 \times 10^{20}\right)\sin^2\theta}\ d\theta$$

Let $f(\theta) = \sqrt{\left(0.3481 \times 10^{20} \cos^2\theta\right) + \left(0.32\ 49 \times 10^{20} \sin^2\theta\right)}$

Taking $n = 10,$ $\quad \Delta\theta = \pi/20$

Then subintervals are $[0, \pi/20], [\pi/20, \pi/10], \ldots [9\pi/20, \pi/2]$

By Simpson's rule

$$L \approx 4 \times \frac{\Delta\theta}{3}\left[f(0) + 4f(\pi/20) + 2f(\pi/10) + \ldots + 4f(9\pi/20) + f(\pi/2)\right]$$

$$= \frac{4}{3}(\pi/20)\left[f(0) + 4f(\pi/20) + 2f(\pi/10) + \ldots + 4f(9\pi/20) + f(\pi/2)\right]$$

$L \approx 3.645 \times 10^{10}$ km

$\Rightarrow \boxed{L \approx 3.645 \times 10^{10}\ \text{km}}$

Q61E

Consider the equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

This represents a hyperbola through the origin with vertices $(\pm a,0)$ and foci $(\pm c,0)$ with $c=\sqrt{a^2+b^2}$.

The vertical lines through focus are $x=c$ and $x=-c$

It is required to find the area of the region bounded by hyperbola and the vertical lines through focus.

Express the hyperbola as a function of *y*.

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ Given equation

$\frac{y^2}{b^2}=\frac{x^2}{a^2}-1$

$y^2=b^2\left(\frac{x^2}{a^2}-1\right)$

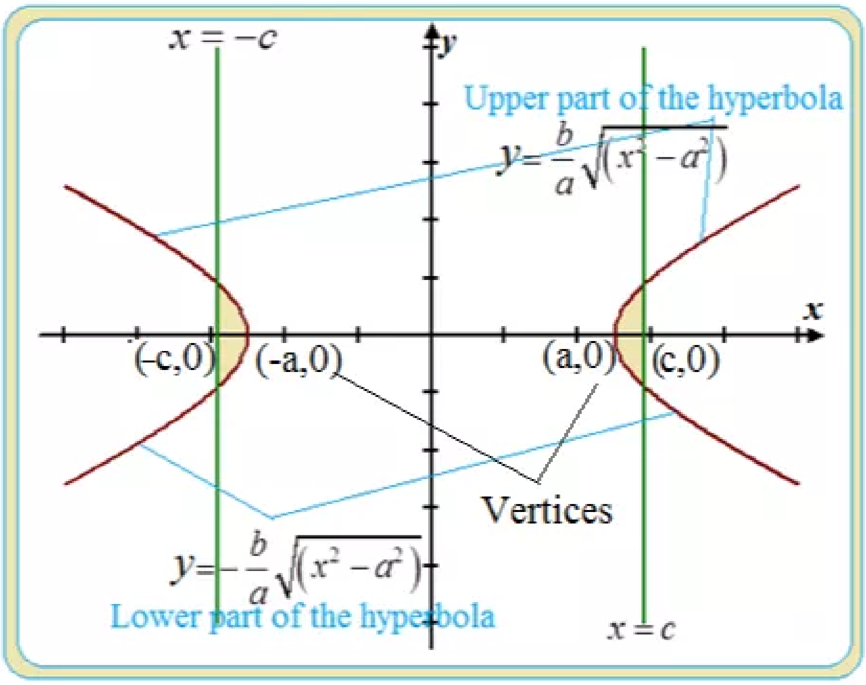
$y=\pm\sqrt{b^2\left(\frac{x^2}{a^2}-1\right)}$

$=\pm\sqrt{\frac{b^2}{a^2}\left(x^2-a^2\right)}$

$=\pm\frac{b}{a}\sqrt{\left(x^2-a^2\right)}$

To evaluate the area using integration method, need to find the bounds of the integration.

Sketch the graph of the hyperbola and vertical lines $x=c$ and $x=-c$.



From the graph left and right parts are symmetric with respect to y-axis. So, find the area for the right curve is same as that of finding the area of left curve.

Since the vertex of the hyperbola for the right part is $(a,0)$ and the focus is $(c,0)$, so the limits of integration are from a to c.

From the graph, the upper part of the graph is $y=\frac{b}{a}\sqrt{(x^2-a^2)}$

From the graph, the lower part of the graph is $y=-\frac{b}{a}\sqrt{(x^2-a^2)}$

Now, integrating the difference of upper part and lower part between a and c gives the area of the region.

$$\text{Area}=\int_a^c\left(\frac{b}{a}\sqrt{x^2-a^2}-\left(-\frac{b}{a}\sqrt{x^2-a^2}\right)\right)dx$$

$$=\int_a^c\left(\frac{b}{a}\sqrt{x^2-a^2}+\frac{b}{a}\sqrt{x^2-a^2}\right)dx$$

$$=\int_a^c\left(2\frac{b}{a}\sqrt{x^2-a^2}\right)dx$$

$$=\frac{2b}{a}\int_a^c\left(\sqrt{x^2-a^2}\right)dx$$

$$=\frac{2b}{a}\left[\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|\right]_a^c$$

Use the formula $\int\sqrt{u^2-a^2}\,du=\frac{u}{2}\sqrt{u^2-a^2}-\frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right|$

Continue the above steps.

$$\text{Area}=\frac{2b}{a}\left[\frac{c}{2}\sqrt{c^2-a^2}-\frac{a^2}{2}\ln\left|c+\sqrt{c^2-a^2}\right|-\frac{a}{2}\sqrt{a^2-a^2}-\frac{a^2}{2}\ln\left|a+\sqrt{a^2-a^2}\right|\right]$$

Apply the limits

$$=\frac{2b}{a}\left[\frac{c}{2}\sqrt{c^2-a^2}-\frac{a^2}{2}\ln\left|c+\sqrt{c^2-a^2}\right|-\frac{a^2}{2}\ln|a|\right]$$

$$=\frac{2b}{a}\left[\frac{\sqrt{a^2+b^2}}{2}\sqrt{a^2+b^2-a^2}-\frac{a^2}{2}\ln\left|\sqrt{a^2+b^2}+\sqrt{a^2+b^2-a^2}\right|+\frac{a^2}{2}\ln|a|\right]$$

$$=\frac{2b}{a}\left[\frac{\sqrt{a^2+b^2}}{2}\sqrt{b^2}-\frac{a^2}{2}\ln\left|\sqrt{a^2+b^2}+\sqrt{b^2}\right|+\frac{a^2}{2}\ln|a|\right]$$

$$=\frac{2b}{a}\left[\frac{b\sqrt{a^2+b^2}}{2}-\frac{a^2}{2}\ln\left|\sqrt{a^2+b^2}+b\right|+\frac{a^2}{2}\ln|a|\right]$$

$$=\frac{b^2\sqrt{a^2+b^2}}{a}-ab\ln\left|\sqrt{a^2+b^2}+b\right|+ab\ln|a|$$

Thus, the area of the region bounded by hyperbola and lines through foci is

$$\boxed{\frac{b^2\sqrt{a^2+b^2}}{a}-ab\ln\left|\sqrt{a^2+b^2}+b\right|+ab\ln|a|}.$$

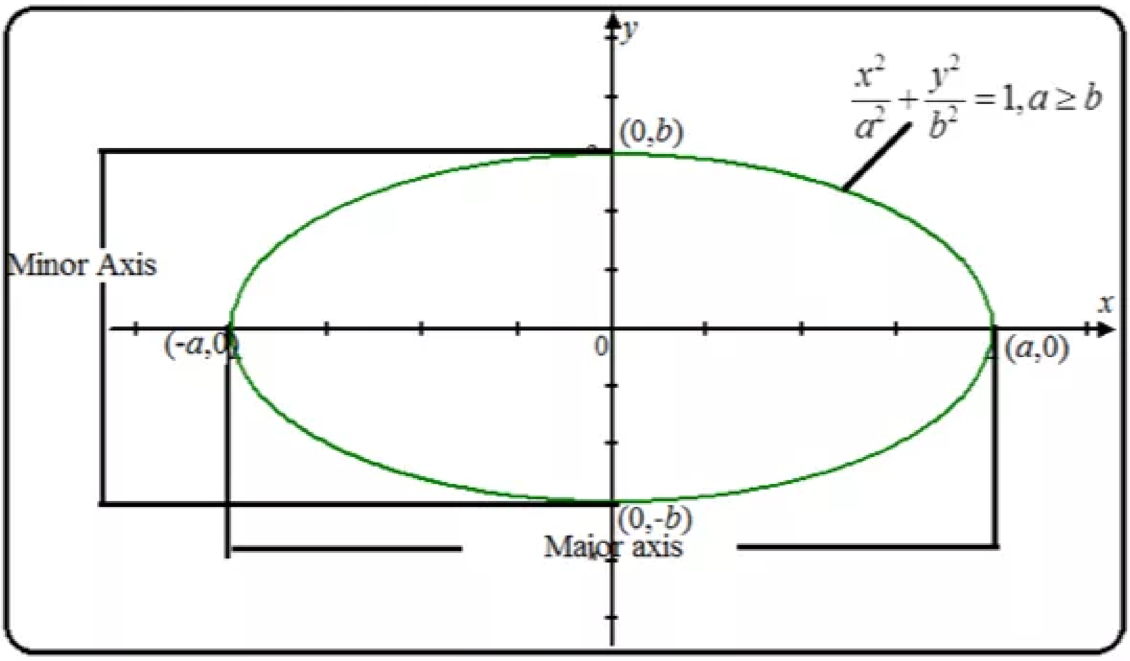
Q62E

Consider the equation of the ellipse through the origin

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

For this ellipse, the major axis is x –axis and the minor axis is y –axis.

Take a rough sketch of the ellipse.



(a)

To find the volume rotated about the major axis, first solve the equation for y.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$=\frac{a^2-x^2}{a^2}$$

$$y^2=b^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$y=\pm\sqrt{b^2\left(\frac{a^2-x^2}{a^2}\right)}$$

$$=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

From the figure, the limits of the integration, to find the volume are from $-a$ to a. But the ellipse is symmetric with respect to the axis so, finding the volume from 0 to a is same as that of the volume from $-a$ to 0.

Therefore, the volume of the solid ellipse rotated about the major axis is

$$V=\pi\int_0^a\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^2dx+\pi\int_0^a\left(-\frac{b}{a}\sqrt{a^2-x^2}\right)^2dx$$

$$=2\pi\int_0^a\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^2dx$$

$$=2\pi\int_0^a\left(\frac{b^2}{a^2}\left(a^2-x^2\right)\right)dx$$

$=2\pi\frac{b^2}{a^2}\int_0^a\left(a^2-x^2\right)dx$ Use constant multiple rule of integration

$=\frac{2\pi b^2}{a^2}\left[\int_0^a a^2dx-\int_0^a x^2dx\right]$ Use sum rule of integration

$=\frac{2\pi b^2}{a^2}\left[a^2\int_0^a dx-\int_0^a x^2dx\right]$ Again use constant multiple rule of integration

$=\frac{2\pi b^2}{a^2}\left[a^2\left[x\right]_0^a-\left[\frac{x^3}{3}\right]_0^a\right]$ Use power rule

$=\frac{2\pi b^2}{a^2}\left[a^2\left[a-0\right]-\left[\frac{a^3}{3}-\frac{0^3}{3}\right]\right]$ Apply limits

$$=\frac{2\pi b^2}{a^2}\left[a^2\left(a\right)-\left[\frac{a^3}{3}\right]\right]$$

$$=\frac{2\pi b^2}{a^2}\left[a^3-\frac{a^3}{3}\right]$$

$$=\frac{2\pi b^2}{a^2}\left[\frac{3a^3-a^3}{3}\right]$$

$$=\frac{2\pi b^2}{a^2}\left[\frac{2a^3}{3}\right]$$

$$=\frac{4\pi}{3}ab^2$$

Thus, the volume of the solid generated by rotation of the ellipse along the major axis is $\boxed{\frac{4\pi}{3}ab^2}$.

(b)

To find the volume rotated about the major axis, first solve the equation for *y*.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}$$

$$=\frac{b^2-y^2}{b^2}$$

$$x^2=a^2\left(\frac{b^2-y^2}{b^2}\right)$$

$$x=\pm\sqrt{a^2\left(\frac{b^2-x^2}{b^2}\right)}$$

$$=\pm\frac{a}{b}\sqrt{b^2-y^2}$$

From the figure, the limits of the integration, to find the volume are from $-b$ to b. But the ellipse is symmetric with respect to the axis so, finding the volume from 0 to b is same as that of the volume from $-b$ to 0.

Therefore, the volume of the solid ellipse rotated about the major axis is

$$V=\pi\int_0^b\left(\frac{a}{b}\sqrt{b^2-y^2}\right)^2dy+\pi\int_0^b\left(-\frac{a}{b}\sqrt{b^2-y^2}\right)^2dy$$

$$=2\pi\int_0^b\left(\frac{a}{b}\sqrt{b^2-y^2}\right)^2dy$$

$$=2\pi\int_0^b\left(\frac{a^2}{b^2}\left(b^2-y^2\right)\right)dy$$

$= 2\pi \frac{a^2}{b^2}\int_0^b (b^2 - y^2)dy$ Use constant multiple rule of integration

$= \frac{2\pi a^2}{b^2}\left[\int_0^b b^2 dx - \int_0^b y^2 dy\right]$ Use sum rule of integration

$= \frac{2\pi a^2}{b^2}\left[b^2\int_0^b dx - \int_0^b y^2 dy\right]$ Again use constant multiple rule of integration

$= \frac{2\pi a^2}{b^2}\left[b^2 [y]_0^b - \left[\frac{y^3}{3}\right]_0^b\right]$ Use power rule

$= \frac{2\pi a^2}{b^2}\left[b^2 [b-0] - \left[\frac{b^3}{3} - \frac{0^3}{3}\right]\right]$ Apply limits

$$= \frac{2\pi a^2}{b^2}\left[b^2 (b) - \left[\frac{b^3}{3}\right]\right]$$

$$= \frac{2\pi b^2}{a^2}\left[b^3 - \frac{b^3}{3}\right]$$

$$= \frac{2\pi a^2}{b^2}\left[\frac{3b^3 - b^3}{3}\right]$$

$$= \frac{2\pi a^2}{b^2}\left[\frac{2b^3}{3}\right]$$

$$= \frac{4\pi}{3}a^2 b$$

Thus, the volume of the solid generated by rotation of the ellipse along the major axis is

$\boxed{\frac{4\pi}{3}a^2 b}$.

Q63E

We must find the centroid of the region enclosed by the x-axis and the top half of the ellipse:

$$9x^2 + 4y^2 = 36$$

The coordinates of the centroid for a plane figure are:

$$C_x = \frac{\int x S_y(x)\,dx}{A}$$

$$C_y = \frac{\int y S_x(y)\,dy}{A}$$

The region above the x-axis and inside of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ has:

$$S_y(x)=y$$
$$=\frac{\sqrt{36-9x^2}}{2}$$
$$S_x(y)=x$$
$$=\frac{\sqrt{36-4y^2}}{3}$$

The area of an ellipse is πab where $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
For this ellipse $a=2, b=3$, and the area is 6π.

The area of the considered region is half the area of the ellipse.
For this case, $A=3\pi$.

Therefore,

$$C_x=\frac{\frac{3}{2}\int_{-2}^{2} x\sqrt{4-x^2}dx}{3\pi}$$
$$=\frac{\frac{3}{2}\left[-\frac{1}{3}\left(4-x^2\right)^{\frac{3}{2}}\right]_{-2}^{2}}{3\pi}$$
$$=\frac{0}{3\pi}$$
$$=0$$

$$C_x=0$$

And

$$C_y=\frac{2\int_{0}^{3} y\sqrt{4-\frac{4}{9}y^2}dy}{3\pi}$$
$$=\frac{2\int_{0}^{3} y\frac{1}{3}\sqrt{36-4y^2}dy}{3\pi}$$
$$=\frac{\frac{2}{3}\left[-\frac{2}{3}\left(9-y^2\right)^{\frac{3}{2}}\right]_{0}^{3}}{3\pi}$$

$$=\frac{4}{27\pi}\left[-0+\left(9^{\frac{3}{2}}\right)\right]$$
$$=\frac{4}{\pi}$$
$$C_y=\frac{4}{\pi}$$

The centroid of the region is $\boxed{\left(0, \frac{4}{\pi}\right)}$

Q64E

(a) We must calculate the surface area of the ellipsoid that is generated by rotating an ellipse about its major axis.

The equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
The x axis is the major axis if $a>b$.
Let $a>b$.
The rotation of the function

$$y=f(x)$$
$$=b\sqrt{1-\frac{x^2}{a^2}}$$

Defined on $[-a,a]$ gives rise to an ellipsoid with surface area

$$S=\int_{-a}^{a}2\pi f(x)\sqrt{1+[f'(x)]^2}dx$$

$$f'(x)=\frac{1}{2}b\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}\left(\frac{-2x}{a^2}\right)$$

$$S=\int_{-a}^{a}2\pi f(x)\sqrt{1+[f'(x)]^2}dx$$

Simplify:

$$S=\int_{-a}^{a}2\pi b\sqrt{1-\frac{x^2}{a^2}}\sqrt{1+\left[\frac{1}{2}b\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}\left(\frac{-2x}{a^2}\right)\right]^2}\,dx$$

$$S=\int_{-a}^{a}2\pi b\sqrt{1-\frac{x^2}{a^2}+\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x^2}{a^2}\right)^{-1}\left(\frac{-bx}{a^2}\right)^2}\,dx$$

$$S=\int_{-a}^{a}2\pi b\sqrt{1-\frac{x^2}{a^2}+\frac{b^2x^2}{a^4}}dx$$

$$S=\int_{-a}^{a}2\pi b\sqrt{1+x^2\left(-\frac{1}{a^2}+\frac{b^2}{a^4}\right)}dx$$

$$S=\int_{-a}^{a}2\pi b\sqrt{1+x^2\left(\frac{b^2-a^2}{a^4}\right)}dx$$

Integrate:

$$S=\int_{-a}^{a}2\pi b\sqrt{1+x^2\left(\frac{b^2-a^2}{a^4}\right)}dx$$

$$S=2\pi b\left[\frac{1}{2}\left(x\sqrt{\left(\frac{b^2-a^2}{a^4}\right)x^2+1}+\frac{\sinh^{-1}\left(\sqrt{\frac{b^2-a^2}{a^4}}x\right)}{\sqrt{\frac{b^2-a^2}{a^4}}}\right)\right]_{-a}^{a}+C$$

Where C is some constant.

$$S=2\pi b\left[\frac{x\sqrt{\left(\frac{b^2-a^2}{a^4}\right)x^2+1}}{2}+\frac{\sinh^{-1}\left(\sqrt{\frac{b^2-a^2}{a^4}}x\right)}{2\sqrt{\frac{b^2-a^2}{a^4}}}\right]_{-a}^{a}+C$$

$$S=2\pi b\left[\left[\frac{a\sqrt{\left(\frac{b^2-a^2}{a^2}\right)+1}}{2}+\frac{\sinh^{-1}\left(\sqrt{\frac{b^2-a^2}{a^4}}a\right)}{2\sqrt{\frac{b^2-a^2}{a^4}}}\right]-\left[\frac{-a\sqrt{\left(\frac{b^2-a^2}{a^2}\right)+1}}{2}+\frac{\sinh^{-1}\left(-\sqrt{\frac{b^2-a^2}{a^4}}a\right)}{2\sqrt{\frac{b^2-a^2}{a^4}}}\right]\right]+C$$

$$S=2\pi b\left[\left[\frac{b}{2}+\frac{\sinh^{-1}\left(\frac{\sqrt{b^2-a^2}}{a}\right)}{2\frac{\sqrt{b^2-a^2}}{a^2}}\right]-\left[\frac{-b}{2}+\frac{\sinh^{-1}\left(-\frac{\sqrt{b^2-a^2}}{a}\right)}{2\frac{\sqrt{b^2-a^2}}{a^2}}\right]\right]+C$$

(b) If the ellipsoid is rotated about its minor axis:
The rotation of the function

$$x=f(y)$$
$$=a\sqrt{1-\frac{y^2}{b^2}}$$

Defined on $[-b,b]$ gives rise to an ellipsoid with surface area

$$S=\int_{-b}^{b}2\pi f(y)\sqrt{1+\left[f'(y)\right]^2}dy$$

$$f'(y)=\frac{1}{2}a\left(1-\frac{y^2}{b^2}\right)^{-\frac{1}{2}}\left(\frac{-2y}{b^2}\right)$$

$$S=\int_{-b}^{b} 2\pi f(y)\sqrt{1+\left[f'(y)\right]^2}\,dy$$

Simplify:

$$S=\int_{-b}^{b} 2\pi a\sqrt{1-\frac{y^2}{b^2}}\sqrt{1+\left[\frac{1}{2}a\left(1-\frac{y^2}{b^2}\right)^{-\frac{1}{2}}\left(\frac{-2y}{b^2}\right)\right]^2}\,dy$$

$$S=\int_{-b}^{b} 2\pi a\sqrt{1-\frac{y^2}{b^2}+\left(1-\frac{y^2}{b^2}\right)\left(1-\frac{y^2}{b^2}\right)^{-1}\left(\frac{-ay}{b^2}\right)^2}\,dy$$

$$S=\int_{-b}^{b} 2\pi a\sqrt{1-\frac{y^2}{b^2}+\frac{a^2y^2}{b^4}}\,dy$$

$$S=\int_{-b}^{b} 2\pi a\sqrt{1+y^2\left(-\frac{1}{b^2}+\frac{a^2}{b^4}\right)}\,dy$$

$$S=\int_{-b}^{b} 2\pi a\sqrt{1+y^2\left(\frac{a^2-b^2}{b^4}\right)}\,dy$$

Integrate:

$$S=\int_{-b}^{b} 2\pi a\sqrt{1+y^2\left(\frac{a^2-b^2}{b^4}\right)}\,dy$$

$$S=2\pi a\left[\frac{1}{2}\left(y\sqrt{\left(\frac{a^2-b^2}{b^4}\right)y^2+1}+\frac{\sinh^{-1}\left(\sqrt{\frac{a^2-b^2}{b^4}}x\right)}{\sqrt{\frac{a^2-b^2}{b^4}}}\right)\right]_{-b}^{b}+C$$

Where C is some constant.

$$S=2\pi a\left[\frac{y\sqrt{\left(\frac{a^2-b^2}{b^4}\right)y^2+1}}{2}+\frac{\sinh^{-1}\left(\sqrt{\frac{a^2-b^2}{b^4}}y\right)}{2\sqrt{\frac{a^2-b^2}{b^4}}}\right]_{-b}^{b}+C$$

$$S=2\pi a\left[\left[\frac{b\sqrt{\left(\frac{a^2-b^2}{b^2}\right)+1}}{2}+\frac{\sinh^{-1}\left(\sqrt{\frac{a^2-b^2}{b^4}}b\right)}{2\sqrt{\frac{a^2-b^2}{b^4}}}\right]-\left[\frac{-b\sqrt{\left(\frac{a^2-b^2}{b^2}\right)+1}}{2}+\frac{\sinh^{-1}\left(-\sqrt{\frac{a^2-b^2}{b^4}}b\right)}{2\sqrt{\frac{a^2-b^2}{b^4}}}\right]\right]+C$$

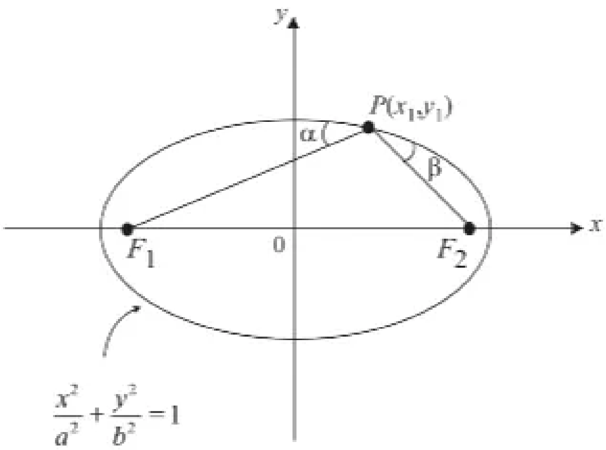
$$S = 2\pi a\left[\left[\frac{a}{2}+\frac{\sinh^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)}{2\frac{\sqrt{a^2-b^2}}{b^2}}\right]-\left[\frac{-a}{2}+\frac{\sinh^{-1}\left(-\frac{\sqrt{a^2-b^2}}{b}\right)}{2\frac{\sqrt{a^2-b^2}}{b^2}}\right]\right]+C$$

$$S = 2\pi a\left[a+\frac{\sinh^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)}{2\frac{\sqrt{a^2-b^2}}{b^2}}-\frac{\sinh^{-1}\left(-\frac{\sqrt{a^2-b^2}}{b}\right)}{2\frac{\sqrt{a^2-b^2}}{b^2}}\right]+C$$

$$S = 2\pi a^2+\pi b^2 a\left[\frac{\sinh^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)-\sinh^{-1}\left(-\frac{\sqrt{a^2-b^2}}{b}\right)}{\sqrt{a^2-b^2}}\right]+C$$

$$\boxed{S = 2\pi a^2+2\pi b^2 a\left[\frac{\sinh^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)}{\sqrt{a^2-b^2}}\right]+C}$$

Q65E



We need to prove that $\alpha = \beta$

We need to use implicit differentiation:

Differentiate both sides of the equation

$$\frac{d}{dx}\left[\frac{x^2}{a^2}\right]+\frac{d}{dx}\left[\frac{y^2}{b^2}\right]=\frac{d}{dx}[1]$$

Remembering that y is a function of x and using the Chain Rule:

$$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0$$

Solving for y'

$$y' = -\frac{b^2 x}{a^2 y}$$

The slope of the tangent line at $P(x_1, y_1)$ is $m = -\dfrac{b^2 x_1}{a^2 y_1}$

The slope of F_2P where $F_2 = (c, 0)$

$$m = \frac{y_1}{x_1 - c} \text{ and } \tan\alpha = \frac{\dfrac{y_1}{x_1 + c} + \dfrac{h^2 x_1}{a^2 y_1}}{1 - \dfrac{h^2 x_1 y_1}{a^2 y_1 (x_1 + c)}}$$

$$\tan\alpha = \frac{a^2 y_1^2 + b^2 x_1 (x_1 + c)}{a^2 y_1 (x_1 + c) - b^2 x_1 y_1}$$

We know that,

1) $c^2 = a^2 - b^2$

2) $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1$ so $\dfrac{b^2 x_1^2 + a^2 y_1^2}{a^2 b^2} = 1$ therefore $b^2 x_1^2 + a^2 y_1^2 = a^2 b^2$

So,

$$\tan\alpha = \frac{a^2 b^2 + b^2 c x_1}{c^2 x_1 y_1 + a^2 c y_1}$$

$$\tan\alpha = \frac{b^2 (c x_1 + a^2)}{c y_1 (c x_1 + a^2)}$$

$$\tan\alpha = \frac{b^2}{c y_1}$$

$$\tan\beta = \frac{-\dfrac{y_1}{x_1 - c} - \dfrac{h^2 x_1}{a^2 y_1}}{1 - \dfrac{h^2 x_1 y_1}{a^2 y_1 (x_1 - c)}}$$

$$\tan\beta = \frac{-a^2 y_1^2 - b^2 x_1 (x_1 - c)}{}$$

Using (1) and (2)

$$\tan\beta = \frac{-a^2b^2 + b^2cx_1}{c^2x_1y_1 - a^2cy_1}$$

$$\tan\beta = \frac{b^2\left(cx_1 - a^2\right)}{cy_1\left(cx_1 - a^2\right)}$$

$$\tan\beta = \frac{b^2}{cy_1}$$

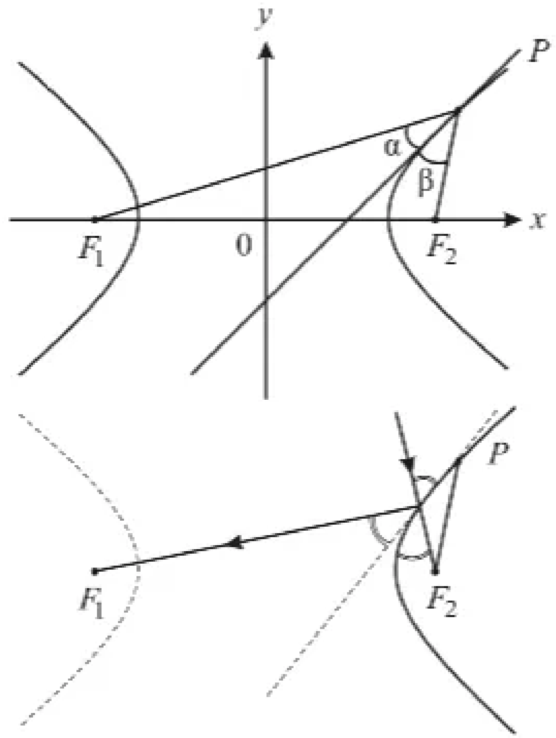
So we have that ,

$$\tan\alpha = \frac{b^2}{cy_1}$$

$$\tan\beta = \frac{b^2}{cy_1}$$

$\tan\alpha = \tan\beta$ and therefore $\alpha=\beta$

Q66E



We need to prove that $\alpha = \beta$

We need to use implicit differentiation:

Differentiate both sides of the equation

$$\frac{d}{dx}\left[\frac{x^2}{a^2}\right] - \frac{d}{dx}\left[\frac{y^2}{b^2}\right] = \frac{d}{dx}[1]$$

Remembering that y is a function of x and using the Chain Rule:

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$$

Solving for y'

$$y' = \frac{b^2 x}{a^2 y}$$

The slope of the tangent line at $P(x_1, y_1)$ is

$$m = \frac{b^2 x_1}{a^2 y_1} \quad m =$$

The slope of F_2P where $F_2 = (-c, 0)$

$$m = \frac{y_1}{x_1 - c}$$

The slope of F_1P where $F_1 = (-c, 0)$

$$m = \frac{y_1}{x_1 + c}$$

And

$$\tan\alpha = \frac{\frac{h^2 x_1}{a^2 y_1} - \frac{y_1}{x_1 + c}}{1 + \frac{h^2 x(y)}{a^2 y_1 (x_1 + c)}}$$

For my convenience I rewrite as:

$$\tan\alpha = \frac{\frac{-y_1}{x_1 + c} + \frac{h^2 x_1}{a^2 y_1}}{1 + \frac{h^2 x[y]}{a^2 y_1 (x_1 + c)}}$$

$$\tan\alpha = \frac{-a^2 y^2 + b^2 x_1 (x_1 + c)}{a^2 y_1 (x_1 + c) + b^2 x_1 y_1}$$

We know that

1) $c^2 = a^2 + b^2$

2) $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ So $\frac{b^2 x_1^2 - a^2 y_1^2}{a^2 b^2} = 1$ therefore $b^2 x_1^2 - a^2 y_1^2 = a^2 b^2$

So $$\tan\alpha = \frac{a^2 b^2 + b^2 c x_1}{c^2 x_1 y_1 + a^2 c y_1}$$

$$\tan\alpha = \frac{b^2 (c x_1 + a^2)}{c y_1 (c x_1 + a^2)}$$

$$\tan\alpha = \frac{b^2}{c y_1}$$

$$\tan\beta = \frac{-\frac{h^2 x_1}{a^2 y_1} + \frac{y_1}{x_1 - c}}{1 + \frac{h^2 x[y]}{a^2 y_1 (x_1 - c)}}$$

Again for my convenience I rewrite as :

$$\tan\beta = \frac{\dfrac{y_1}{x_1 - c} - \dfrac{h^2 x_1}{a^2 y_1}}{1 + \dfrac{h^2 x[y]}{a^2 y_1 (x_1 - c)}}$$

$$\tan\beta = \frac{a^2 y_2^1 - b^2 x_1 (x_1 - c)}{a^2 y_1 (x_1 - c) + b^2 x_1 y_1}$$

Using (1) and (2)

$$\tan\beta = \frac{-a^2 b^2 + b^2 c x_1}{c^2 x_1 y_1 - a^2 c y_1}$$

$$\tan\beta = \frac{b^2 \left(c x_1 - a^2\right)}{c y_1 \left(c x_1 - a^2\right)}$$

$$\tan\beta = \frac{b^2}{c y_1}$$

So we have that $\tan\alpha = \dfrac{b^2}{c y_1} = \tan\beta = \dfrac{b^2}{c y_1}$

$\tan\alpha = \tan\beta$

Therefore $\alpha = \beta$