Chapter
Binomial Theorem

Topic-1: Binomial Theorem for a Positive Integral Index 'X', Expansion of Binomial, General Term, Coefficient of any Power of 'X'

1 MCQs with One Correct Answer

1. For r = 0, 1, …, 10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1+x)^{10}$, **[2010]**

   $(1+x)^{20}$ and $(1+x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to

   (a) $B_{10} - C_{10}$ (b) $A_{10}(B^2_{10}C_{10}A_{10})$
   (c) 0 (d) $C_{10}-B_{10}$

2. Coefficient of t^{24} in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is **[2003S]**

   (a) ${}^{12}C_6+3$ (b) ${}^{12}C_6+1$ (c) ${}^{12}C_6$ (d) ${}^{12}C_6+2$

3. In the binomial expansion of $(a-b)^n$, $n \geq 5$, the sum of the 5^{th} and 6^{th} terms is zero. Then a/b equals **[2001S]**

   (a) $(n-5)/6$ (b) $(n-4)/5$
   (c) $5/(n-4)$ (d) $6/(n-5)$

4. The coefficient of x^4 in $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}$ is **[1983 - 1 Mark]**

   (a) $\frac{405}{256}$ (b) $\frac{504}{259}$
   (c) $\frac{450}{263}$ (d) none of these

5. Given positive integers $r > 1$, $n > 2$ and that the coefficient of $(3r)$th and $(r+2)$th terms in the binomial expansion of $(1+x)^{2n}$ are equal. Then **[1983 - 1 Mark]**

   (a) $n=2r$ (c) $n=2r+1$
   (c) $n=3r$ (d) none of these

2 Integer Value Answer/ Non-Negative Integer

6. Let a and b be two non-zero real numbers. If the coefficient of x^5 in the expansion of $\left(ax^2+\frac{70}{27bx}\right)^4$ is equal to the coefficient of x^{-5} in the expansion of $\left(ax-\frac{1}{bx^2}\right)^7$, then the value of 2b is **[Adv. 2023]**

7. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1+x)^2+(1+x)^3+\ldots+(1+x)^{49}+(1+mx)^{50}$ is $(3n+1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is **[Adv. 2016]**

8. The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n = **[Adv. 2013]**

3 Numeric/ New Stem Based Questions

9. Let $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \cdots + 10({}^{10}C_{10})^2$, where ${}^{10}C_r$, $r \in \{1, 2, \cdots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is _____. **[Adv. 2018]**

4 Fill in the Blanks

10. The sum of the rational terms in the expansion of $(\sqrt{2}+3^{1/5})^{10}$ is .... **[1997 - 2 Marks]**

11. Let n be positive integer. If the coefficients of 2nd, 3rd, and 4th terms in the expansion of $(1+x)^n$ are in A.P., then the value of n is .... **[1994 - 2 Marks]**

12. The larger of $99^{50}+100^{50}$ and 101^{50} is .... **[1982 - 2 Marks]**

6 MCQs with One or More than One Correct Answer

13. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals **[1998 - 2 Marks]**

    (a) $(n-1)a_n$ (b) na_n
    (c) $\frac{1}{2}na_n$ (d) None of the above

Topic-2: Middle Term, Greatest Term, Independent Term, Particular Term from end in Binomial Expansion, Greatest Binomial Coefficient

10 Subjective Problems

1. Prove that $\sum_{r=1}^{k}(-3)^{r-1}\ {}^{3n}C_{2r-1} = 0$, where $k = (3n)/2$ and n is an even positive integer. **[1993 - 5 Marks]**

2. Let $R = (5\sqrt{5}+11)^{2n+1}$ and $f = R - [R]$, where [ ] denotes the greatest integer function. Prove that $Rf = 4^{2n+1}$. **[1988 - 5 Marks]**

Topic-3: Properties of Binomial Coefficients, Number of Terms in the Expansion of (x + y + z)ⁿ, Binomial Theorem for any Index, Multinomial Theorem, Infinite Series

1. Coefficient of x^{11} in the expansion of $(1+x^2)^4(1+x^3)^7(1+x^4)^{12}$ is **[Adv. 2014]**
   (a) 1051 (b) 1106 (c) 1113 (d) 1120

2. The value of
$$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} ..... + \binom{30}{20}\binom{30}{30}$$ is
where $\binom{n}{r} = {}^nC_r$ **[2005S]**
   (a) $\binom{30}{10}$ (b) $\binom{30}{15}$ (c) $\binom{60}{30}$ (d) $\binom{31}{10}$

3. The sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, (where $\binom{p}{q} = 0$ if p > q) is maximum when m is **[2002S]**
   (a) 5 (b) 10 (c) 15 (d) 20

4. For $2 \le r \le n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$ **[2000S]**
   (a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$
   (c) $2\binom{n+2}{r}$ (d) $\binom{n+2}{r}$

5. If in the expansion of $(1+x)^m(1-x)^n$, the coefficients of x and x^2 are 3 and – 6 respectively, then m is **[1999 - 2 Marks]**
   (a) 6 (b) 9 (c) 12 (d) 24

6. The expression $\left[x+(x^3-1)^{\frac{1}{2}}\right]^5 + \left[x-(x^3-1)^{\frac{1}{2}}\right]^5$ is a polynomial of degree **[1992 - 2 Marks]**
   (a) 5 (b) 6 (c) 7 (d) 8

3 Numeric/ New Stem Based Questions

7. Suppose $\det\begin{bmatrix}\sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k\end{bmatrix} = 0$ holds for some positive integer n. The $\sum_{k=0}^{n}\frac{{}^nC_k}{k+1}$ equals _____ **[Adv. 2019]**

4 Fill in the Blanks

8. The sum of the coefficients of the plynomial $(1+x-3x^2)^{2163}$ is .... **[1982 - 2 Marks]**

6 MCQs with One or More than One Correct Answer

9. If C_r stands for ${}^nC_r$, then the sum of the series
$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2 - 2C_1^2 + 3C_2^2 - .... + (-1)^n(n+1)C_n^2],$$
where n is an even positive integer, is equal to **[1986 - 2 Marks]**
   (a) 0 (b) $(-1)^{n/2}(n+1)$
   (c) $(-1)^{n/2}(n+2)$ (d) $(-1)^n n$

10 Subjective Problems

10. Prove that **[2003 - 2 Marks]**
$$2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - .....(-1)^k\binom{n}{k}\binom{n-k}{0} = \binom{n}{k}.$$

11. For any positive integer m, n (with $n \geq m$), let $\binom{n}{m} = {}^nC_m$.

Prove that $\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+.....+\binom{m}{m}=\binom{n+1}{m+2}$.

Hence or otherwise, prove that

$$\binom{n}{m}+2\binom{n-1}{m}+3\binom{n-2}{m}+.....+(n-m+1)\binom{m}{m}=\binom{n+2}{m+2}.$$

[2000 - 6 Marks]

12. Let n be a positive integer and

$(1+x+x^2)^n = a_0 + a_1 x + ...+ a_{2n} x^{2n}$

Show that $a_0^2 - a_1^2 + a_2^2 ....+ a_{2n}^2 = a_n$

[1994 - 5 Marks]

13. If $\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$ and $a_k = 1$ for all $k \geq n$, then show that $b_n = {}^{2n+1}C_{n+1}$ **[1992 - 6 Marks]**

14. Prove that

$$C_0 - 2^2 C_1 + 3^2 C_2 .... + (-1)^n (n+1)^2 C_n = 0,$$

$n > 2$, where $C_r = {}^nC_r$. **[1989 - 5 Marks]**

15. Given $s_n = 1 + q + q^2 + ... + q^n$;

$S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n, q \neq 1$ Prove that ${}^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + ... + {}^{n+1}C_{n+1} s_n = 2^n S_n$

[1984 - 4 Marks]

16. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$ then show that the sum of the products of the C_i's taken two at a time, represented by $\sum\sum_{0 \leq i < j \leq n} C_i C_j$ is equal to

$$2^{2n-1} - \frac{(2n)!}{2(n!)^2}$$

[1983 - 3 Marks]

17. Given that

$$C_1 + 2C_2 x + 3C_3 x^2 + ... + 2n\, C_{2n} x^{2n-1} = 2n(1+x)^{2n-1}$$

where $C_r = \frac{(2n)!}{r!(2n-r)!}$ $r = 0, 1, 2, ..., 2n$

Prove that

$$C_1^2 - 2C_2^2 + 3C_3^2 - .... - 2nC_{2n}^2 = (-1)^n n\, C_n.$$

[1979]

Answer Key

Topic-1 : Binomial Theorem for a Positive Integral Index 'x', Expansion of Binomial, General Term, Coefficient of any Power of 'x'

1. (d) **2.** (d) **3.** (b) **4.** (a) **5.** (a) **6.** (3) **7.** (5) **8.** (6) **9.** (646) **10.** (41)
11. (7 or 2) **12.** $(101)^{50}$ **13.** (c)

Topic-3 : Properties of Binomial Coefficients, Number of Terms in the Expansion of (x+y+z)ⁿ, Binomial Theorem for any Index, Multinomial Theorem, Infinite Series

1. (c) **2.** (a) **3.** (c) **4.** (d) **5.** (c) **6.** (c) **7.** (6.20) **8.** (–1) **9.** (c)

Hints & Solutions

Topic-1: Binomial Theorem for a Positive Integral Index 'x', Expansion of Binomial, General Term, Coefficient of any Power of 'x'

1. **(d)** Clearly $A_r = {}^{10}C_r, B_r = {}^{20}C_r, C_r = {}^{30}C_r$

Now $\sum_{r=1}^{10} A_r (B_{10}B_r - C_{10}A_r)$

$= \sum_{r=1}^{10} {}^{10}C_r \left({}^{20}C_{10}\, {}^{20}C_r - {}^{30}C_{10}\, {}^{10}C_r\right)$

$= {}^{20}C_{10} \sum_{r=1}^{10} {}^{10}C_r\, {}^{20}C_r - {}^{30}C_{10} \sum_{r=1}^{10} {}^{10}C_r \times {}^{10}C_r$

$= {}^{20}C_{10}\left({}^{10}C_1\, {}^{20}C_1 + {}^{10}C_2\, {}^{20}C_2 + ... + {}^{10}C_{10}\, {}^{20}C_{10}\right)$

$- {}^{30}C_{10}\left({}^{10}C_1 \times {}^{10}C_1 + {}^{10}C_2 \times {}^{10}C_2 + ... + {}^{10}C_{10}\, {}^{10}C_{10}\right)$ ....(i)

Now on expanding $(1 + x)^{10}$ and $(1 + x)^{20}$ and comparing the coefficients of x^{20} in their product on both sides, we get

${}^{10}C_0\, {}^{20}C_0 + {}^{10}C_1\, {}^{20}C_1 + {}^{10}C_2\, {}^{20}C_2 + ... + {}^{10}C_{10}\, {}^{20}C_{10}$

$=$ Coeff. of x^{20} in $(1+x)^{30} = {}^{30}C_{20} = {}^{30}C_{10}$

$\therefore\ {}^{10}C_1\, {}^{20}C_1 + {}^{10}C_2\, {}^{20}C_2 + ... + {}^{10}C_{10}\, {}^{20}C_{10} = {}^{30}C_{10} - 1$ ....(ii)

Again on expanding $(1 + x)^{10}$ and $(x + 1)^{10}$ and comparing the coefficients of x^{10} in their product on both sides, we get

$\therefore\ ({}^{10}C_0)^2\, ({}^{10}C_1)^2 + ({}^{10}C_2)^2 + ... + ({}^{10}C_{10})^2$

$=$ Coeff. of x^{10} in $(1 + x)^{20} = {}^{20}C_{10}$

$\therefore\ ({}^{10}C_1)^2 + ({}^{10}C_2)^2 + ... + ({}^{10}C_{10})^2 = {}^{20}C_{10} - 1$ ...(iii)

Now, from equations (i), (ii) and (iii), we get

Required value $= {}^{20}C_{10}\left({}^{30}C_{10} - 1\right) - {}^{30}C_{10}\left({}^{20}C_{10} - 1\right)$

$= {}^{30}C_{10} - {}^{20}C_{10} = C_{10} - B_{10}$

2. **(d)** $(1 + t^2)^{12}\ (1 + t^{12})\ (1 + t^{24})$

$= (1 + t^{12} + t^{24} + t^{36})\ (1 + t^2)^{12}$

$\therefore$ Coeff. of $t^{24} = 1\times$ Coeff. of t^{24} in $(1+ t^2)^{12} + 1 \times$ Coeff. of t^{12} in $(1 + t^2)^{12} + 1 \times$ constant term in $(1 + t^2)^{12}$

$= {}^{12}C_{12} + {}^{12}C_6 + {}^{12}C_0 = 1 + {}^{12}C_6 + 1 = {}^{12}C_6 + 2$

3. **(b)** In binomial expansion $(a - b)^n, n \geq 5$;

$T_5 + T_6 = 0$

$\Rightarrow {}^nC_4\, a^{n-4}\, b^4 - {}^nC_5\, a^{n-5}\, b^5 = 0$

$\Rightarrow \frac{{}^nC_4}{{}^nC_5} \cdot \frac{a}{b} = 1 \Rightarrow \frac{5}{n-4} \cdot \frac{a}{b} = 1 \Rightarrow \frac{a}{b} = \frac{n-4}{5}$

4. **(a)** General term in the expansion $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is

$T_{r+1} = {}^{10}C_r \left(\frac{x}{2}\right)^{10-r} \left(\frac{-3}{x^2}\right)^r = {}^{10}C_r x^{10-3r} \frac{(-1)^r 3^r}{2^{10-r}}$

To find coeff of x^4, put $10 - 3r = 4 \Rightarrow r = 2$

$\therefore$ Coeff of $x^4 = {}^{10}C_2 \frac{(-1)^2 3^2}{2^8} = \frac{405}{256}$

5. **(a)** Given : r and n are positive integers such that $r > 1$, $n > 2$

Also, in the expansion of $(1+ x)^{2n}$

Coeff. of $(3r)^{th}$ term = Coeff. of $(r + 2)^{th}$ term

$\Rightarrow {}^{2n}C_{3r-1} = {}^{2n}C_{r+1} \Rightarrow 3r - 1 = r + 1$ or $3r - 1 + r + 1 = 2n$

$\left[\because \text{If } {}^nC_p = {}^nC_q, \text{ then } p = q \text{ or } p + q = n\right]$

$\Rightarrow r = 1$ or $2r = n$

But $r > 1 \quad \therefore \quad n = 2r$

6. **(3)** Given expansion $\left(ax^2 + \frac{70}{27bx}\right)^4$

$T_{r+1} = {}^4C_r \left(ax^2\right)^{4-r} \left(\frac{70}{27bx}\right)^r$

$= {}^4C_r\, a^{4-r} \left(\frac{70}{27b}\right)^r .x^{8-3r}$

Here, $8 - 3r = 5 \Rightarrow r = 1$

So, coefficient of $x^5 = {}^4C_1 a^3 . \frac{70}{27b}$

For expansion $\left(ax - \frac{1}{bx^2}\right)^7$

$T_{r+1} = {}^7C_r\, (ax)^{7-r} \left(\frac{-1}{bx^2}\right)^r = {}^7C_r\, a^{7-r} \left(\frac{-1}{b}\right)^r x^{7-3r}$

Here, $7 - 3r = -5 \Rightarrow r = 4$

So, coefficient of $x^{-5} = {}^7C_4\, a^3 \left(\frac{-1}{b}\right)^3$

A.T.Q, ${}^4C_1\, a^3 \frac{70}{27b} = {}^7C_4\, a^3 . \frac{-1}{b^3}$

$\Rightarrow b = \frac{3}{2} \Rightarrow 2b = 3.$

7. **(5)** $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$

$= (1 + x)^2 \left[\frac{(1+x)^{48} - 1}{(1+x) - 1}\right] + (1 + mx)^{50}$

$= \frac{1}{x}\left[(1+x)^{50} - (1+x)^2\right] + (1 + mx)^{50}$

Coeff. of x^2 in the above expansion

= Coeff. of x^3 in $(1 + x)^{50}$ + Coeff. of x^2 in $(1 + mx)^{50}$

$= {}^{50}C_3 + {}^{50}C_2\, m^2$

$\therefore\ (3n + 1)\ {}^{51}C_3 = {}^{50}C_3 + {}^{50}C_2\, m^2$

$\Rightarrow (3n+1) = \frac{^{50}C_3}{^{51}C_3} + \frac{^{50}C_2}{^{51}C_3} m^2$

$\Rightarrow 3n+1 = \frac{16}{17} + \frac{1}{17}m^2 \Rightarrow n = \frac{m^2-1}{51}$

$\therefore$ Least positive integer m for which n is an integer is m = 16 and then n = 5

8. **(6)** Let the coefficients of three consecutive terms of $(1+x)^{n+5}$ be $^{n+5}C_{r-1}$, $^{n+5}C_r$, $^{n+5}C_{r+1}$, then we have $^{n+5}C_{r-1} : {}^{n+5}C_r : {}^{n+5}C_{r+1} = 5:10:14$

$\frac{^{n+5}C_{r-1}}{^{n+5}C_r} = \frac{5}{10} \Rightarrow \frac{r}{n+6-r} = \frac{1}{2}$

$\Rightarrow n - 3r + 6 = 0$ ...(i)

Also $\frac{^{n+5}C_r}{^{n+5}C_{r+1}} = \frac{10}{14} \Rightarrow \frac{r+1}{n-r+5} = \frac{5}{7}$

$\Rightarrow 5n - 12r + 18 = 0$ (ii)

Solving (i) and (ii), we get $n = 6$.

9. **(646)** $\sum_{r=0}^{n} r({}^nC_r)^2 = n\sum_{r=0}^{n} {}^nC_r {}^{n-1}C_{r-1}$

$= n\sum_{r=1}^{n} {}^nC_{n-r} {}^{n-1}C_{r-1} = n\,{}^{2n-1}C_{n-1}$

Now, $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + ... + 10({}^{10}C_{10})^2$

$= \sum_{n=0}^{10} r({}^{10}C_r)^2 = 10\,{}^{19}C_9 \therefore \frac{X}{1430} = \frac{1}{143}\,{}^{19}C_9 = 646$

10. Given expression : $(\sqrt{2} + 3^{1/5})^{10}$

$\therefore T_{r+1} = {}^{10}C_r (\sqrt{2})^{10-r}.(3^{1/5})^r. (0 \le r \le 10)$

$= \frac{10!}{r!(10-r)!}.2^{5-r/2}.3^{r/5}$

T_{r+1} will be rational if $2^{5-r/2}$ and $3^{r/5}$ are rational numbers.

$\Rightarrow 5 - \frac{r}{2}$ and $\frac{r}{5}$ are integers.

$\Rightarrow r = 0$ and $r = 10 \Rightarrow T_1$ and T_{11} are rational terms.

Now, $T_1 + T_{11} = {}^{10}C_0 2^{5-0}.3^0 + {}^{10}C_{10} 2^{5-5}.3^2$

$= 1.32.1 + 1.1.9 = 32 + 9 = 41$

11. We know that for a positive integer n

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ...+ {}^nC_n x^n$

Since coefficients of 2^{nd}, 3^{rd} and 4^{th} terms are in A.P.

$\therefore {}^nC_1, {}^nC_2, {}^nC_3$ are in A.P.

$\Rightarrow 2.{}^nC_2 = {}^nC_1 + {}^nC_3$

$\Rightarrow 2 \times \frac{n(n-1)}{2} = n + \frac{n(n-1)(n-2)}{3!}$

$\Rightarrow n - 1 = 1 + \frac{n^2 - 3n + 2}{6} \Rightarrow n^2 - 9n + 14 = 0$

$\Rightarrow (n-7)(n-2) = 0 \Rightarrow n = 7$ or 2

But for the existance of 4^{th} term, n = 7.

12. $(101)^{50} - \{(99)^{50} + (100)^{50}\}$

$= (100+1)^{50} - (100-1)^{50} - (100)^{50}$

$= (100)^{50}[(1+0.0\,1)^{50} - (1-0.01)^{50} - 1]$

$= (100)^{50}[2({}^{50}C_1(0.01) + {}^{50}C_3(0.01)^3 + ...) - 1]$

$= (100)^{50}\left[2 \times 50 \times \frac{1}{100} + 2\left({}^{50}C_3(0.01)^3 + ...\right) - 1\right]$

$= (100)^{50}[2({}^{50}C_3(0.01)^3 + ...)] > 0$

$\therefore (101)^{50} > (99)^{50} + (100)^{50} \therefore (101)^{50}$ is greater.

13. **(c)** Let $b = \sum_{r=0}^{n} \frac{r}{^nC_r} = \sum_{r=0}^{n} \frac{n-(n-r)}{^nC_r}$

$= na_n \sum_{r=0}^{n} \frac{n-r}{^nC_{n-r}} = na_n - b \quad [\because {}^nC_r = {}^nC_{n-r}]$

$\Rightarrow 2b = na_n \Rightarrow b = \frac{n}{2}a_n$

Topic-2: Middle Term, Greatest Term, Independent Term, Particular Term from end in Binomial Expansion, Greatest Binomial Coefficients

1. Given $k = \frac{3n}{2}$, where n is an even positive integer. Now let $n = 2m$ so that $k = 3m$

Since $\sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1} = \sum_{r=1}^{3m}(-3)^{r-1}\,{}^{6m}C_{2r-1}$

$= {}^{6m}C_1 - 3\,{}^{6m}C_3 + 3^2\,{}^{6m}C_5 - 3^3\,{}^{6m}C_7$ .....

Consider $(\cos\theta - i\sin\theta)^{6m}$

$= \cos^{6m}\theta - {}^{6m}C_1\cos^{6m-1}\theta\, i\sin\theta$

$+ {}^{6m}C_2\cos^{6m-2}\theta\, i^2\sin^2\theta - {}^{6m}C_3\cos^{6m-3}\theta\, i^3\sin^3\theta....$

L.H.S. $= \cos 6m\theta - i\sin 6m\theta$

Equating imaginary parts we get $\sin 6m\theta$

$= {}^{6m}C_1\cos^{6m-1}\theta\sin\theta - {}^{6m}C_3\cos^{6m-3}\theta\sin^3\theta + ...$

Now put $\theta = \frac{\pi}{3} \therefore 6\theta = 2\pi$

or $\sin 6m\theta = \sin 2m\pi = 0$

$\therefore 0 = {}^{6m}C_1\frac{\sqrt{3}}{2}\left(\frac{1}{2}\right)^{6m-1} ...... - {}^{6m}C_3\left(\frac{\sqrt{3}}{2}\right)^3\left(\frac{1}{2}\right)^{6m-2} + ...$

or $0 = \frac{\sqrt{3}}{2(6m)}\left[{}^{6m}C_1 - 3\,{}^{6m}C_3 + 3^2\,{}^{6m}C_5 + ...\right]$

$\therefore {}^{6m}C_1 - 3\,{}^{6m}C_3 + 3^2\,{}^{6m}C_5 ..... = 0$

$\Rightarrow \sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1} = 0.$

2. $5\sqrt{5} - 11 = \frac{4}{5\sqrt{5}+11} < 1 \therefore 0 < 5\sqrt{5} - 11 < 1$

$\Rightarrow 0 < (5\sqrt{5}-11)^{2n+1} < 1$ for every positive integer n.

Also $(5\sqrt{5}+11)^{2n+1} - (5\sqrt{5}-11)^{2n+1}$

$= 2[{}^{2n+1}C_1(5\sqrt{5})^{2n}.11 + {}^{2n+1}C_3(5\sqrt{5})^{2n-2}.11^3 +$

$....+ {}^{2n+1}C_{2n+1}11^{2n+1}]$

$= 2[{}^{2n+1}C_1(125)^n.11 + {}^{2n+1}C_3(125)^{n-1}.11^3 + \ldots + {}^{2n+1}C_{2n+1}11^{2n+1}]$

$= 2k$ ....(i)

where k is some positive integer.

Let $F = (5\sqrt{5} - 11)^{2n+1}$

Then equation (i) becomes

$R - F = 2k$ $\left[\because R = (5\sqrt{5}+11)^{2n+1}\right]$

$\Rightarrow [R] + R - [R] - F = 2k \Rightarrow [R] + f - F = 2k$

$\Rightarrow f - F = 2k - [R] \Rightarrow f - F$ is an integer.

But $0 \le f < 1$ and $0 < F < 1$, $\therefore -1 < f - F < 1$

$\because f - F$ is an integer, we must have $f - F = 0$

$\Rightarrow f = F$.

Now, $Rf = RF = (5\sqrt{5}+11)^{2n+1}(5\sqrt{5}-11)^{2n+1}$

$= [(5\sqrt{5})^2 - 12]^{2n+1} = 4^{2n+1}$

Topic-3: Properties of Binomial Coefficients, Number of Terms in the Expansion of $(x+y+z)^n$ Binomial Theorem for any Index, Multinomial Theorem, Infinite Series

1. (c) Coeff. of x^{11} in exp. of $\left(1+x^2\right)^4\left(1+x^3\right)^7\left(1+x^4\right)^{12}$

= [Coeff. of x^a in $(1 + x^2)^4$] × [Coeff. of x^b in $(1 + x^3)^7$] × [Coeff. of x^c in $(1 + x)^4$]

Such that $a + b + c = 11$

Here $a = 2m$, $b = 3n$, $c = 4p$

$\therefore 2m + 3n + 4p = 11$

Case I : $m = 0, n = 1, p = 2$

Case II : $m = 1, n = 3, p = 0$

Case III : $m = 2, n = 1, p = 1$

Case IV : $m = 4, n = 1, p = 0$

$\therefore$ Required coefficient.

$= {}^4C_0 \times {}^7C_1 \times {}^{12}C_2 + {}^4C_1 \times {}^7C_3 \times {}^{12}C_0 + {}^4C_2 \times {}^7C_1 \times {}^{12}C_1 + {}^4C_4 \times {}^7C_1 \times {}^{12}C_0$

$= 462 + 140 + 504 + 7 = 1113$

2. (a) To find

${}^{30}C_0{}^{30}C_{10} - {}^{30}C_1{}^{30}C_{11} + {}^{30}C_2{}^{30}C_{12} - \ldots + {}^{30}C_{20}{}^{30}C_{30}$

$\because (1+x)^{30} = {}^{30}C_0 + {}^{30}C_1x + {}^{30}C_2x^2 + \ldots + {}^{30}C_{20}x^{20} + \ldots {}^{30}C_{30}x^{30}$ ....(i)

and $(x-1)^{30} = {}^{30}C_0x^{30} - {}^{30}C_1x^{29} + \ldots + {}^{30}C_{10}x^{20} - {}^{30}C_{11}x^{19} + {}^{30}C_{12}x^{18} + \ldots {}^{30}C_{30}x^0$ ....(ii)

On multiplying equations (i) and (ii), we get

$(x^2 - 1)^{30} = (\ \ ) \times (\ \ )$

Equating the coefficients of x^{20} on both sides, we get

${}^{30}C_{10} = {}^{30}C_0{}^{30}C_{10} - {}^{30}C_1{}^{30}C_{11} + {}^{30}C_2{}^{30}C_{12} - \ldots + {}^{30}C_{20}{}^{30}C_{30}$

$\therefore$ Required value $= {}^{30}C_{10}$

3. (c) $\sum_{i=0}^{m} {}^{10}C_i{}^{20}C_{m-i} = {}^{10}C_0{}^{20}C_m + {}^{10}C_1{}^{20}C_{m-1} + {}^{10}C_2{}^{20}C_{m-2} + \ldots + {}^{10}C_m{}^{20}C_0$

= Coeff. of x^m in the expansion of product $(1+x)^{10}(1+x)^{20}$

= Coeff. of x^m in the expansion of $(1+x)^{30} = {}^{30}C_m$

$\sum_{i=0}^{n} {}^{10}C_i{}^{20}C_{m-1}$ will be maximum, if ${}^{30}C_m$ will be maximum.

Clearly, ${}^{30}C_m$ will be maximum when $m = \frac{30}{2} = 15$

$$\left[\because \text{Max}\cdot({}^nC_r) = \begin{cases} {}^nC_{n/2} \text{ if } n \text{ is even} \\ {}^nC_{\frac{n+1}{2}} \text{ if } n \text{ is odd} \end{cases}\right]$$

4. (d) $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$

$= \left[\binom{n}{r} + \binom{n}{r-1}\right] + \left[\binom{n}{r-1} + \binom{n}{r-2}\right]$

$= \left[\text{Here } \binom{n}{r}, \binom{n}{r-1} \text{ and } \binom{n}{r-2} \text{ represent } {}^nC_r, {}^nC_{r-1} \text{ and } {}^nC_{r-2}\right]$

$= \binom{n+1}{r} + \binom{n+1}{r-1} = \binom{n+2}{r}$ $[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

5. (c) $(1 + x)^m (1 - x)^n$

$= \left[1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots\right]\left[1 - nx + \frac{n(n-1)}{2!}x^2 - \ldots\right]$

$= 1 + (m-n)x + \left[\frac{m(m-1)}{2} + \frac{n(n-1)}{2} - mn\right]x^2 + \ldots$

Given, $m - n = 3$ ....(i)

and $\frac{1}{2}m(m-1) + \frac{1}{2}n(n-1) - mn = -6$

$\Rightarrow m^2 + n^2 - 2mn - (m+n) = -12$

$\Rightarrow (m-n)^2 - (m+n) = -12$

$\Rightarrow m + n = 9 + 12 = 21$ ....(ii)

From (i) and (ii), we get $m = 12$

6. (c) Given expression :

$(x + \sqrt{x^3 - 1})^5 + (x - \sqrt{x^3 - 1})^5$

We know that using binomial theorem,

$(x + a)^n + (x - a)^n = 2\,[{}^nC_0x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \ldots]$

$\therefore$ The given expression

$= 2\,[{}^5C_0x^5 + {}^5C_2x^3(x^3 - 1) + {}^5C_4x(x^3 - 1)^2]$

Since maximum power of x involved in the expansion is 7. Also only +ve integral powers of x are involved in the expansion, therefore given expression is a polynomial of degree 7.

7. (6.20) Here $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$

$\sum_{k=0}^{n} {}^nC_k k^2 = \sum_{k=1}^{n} \frac{n}{k}.{}^{n-1}C_{k-1}.k^2 = \sum_{k=1}^{n} n.{}^{n-1}C_{k-1}.k$

$= n\sum_{k=1}^{n} {}^{n-1}C_{k-1}(k-1+1)$

$$= n\left[\sum_{k=2}^{n}\frac{n-1}{k-1}\,{}^{n-2}C_{k-2}(k-1)+\sum_{k=1}^{n}{}^{n-1}C_{k-1}\right]$$

$$= n(n-1)2^{n-2}+n\times 2^{n-1}$$

$$\sum_{k=0}^{n}{}^{n}C_k\times k=\sum_{k=1}^{n}\frac{n}{k}\,{}^{n-1}C_{k-1}\times k=n\times 2^{n-1}$$

and $\sum_{k=0}^{n}{}^{n}C_k 3^k=4^n$

$$\therefore \det\begin{bmatrix}\sum_{k=0}^{n}k & \sum_{k=0}^{n}{}^{n}C_k k^2\\ \sum_{k=0}^{n}{}^{n}C_k k & \sum_{k=0}^{n}{}^{n}C_k 3^k\end{bmatrix}=0$$

$$\Rightarrow \begin{vmatrix}\frac{n(n+1)}{2} & n(n-1)2^{n-2}+n\times 2^{n-1}\\ n\times 2^{n-1} & 4^n\end{vmatrix}=0$$

$$\Rightarrow n(n+1)\times 2^{2n-1}-n^2\left[(n-1)2^{2n-3}+2^{2n-2}\right]=0$$

$$\Rightarrow 2^{2n-3}\times n\left[4(n+1)-n[n-1+2]\right]=0$$

$$\Rightarrow 2^{2n-3}\times n\left[4n+4-n^2-n\right]=0$$

$$\Rightarrow n^2-3n-4=0\Rightarrow n=4$$

$$\therefore \sum_{k=0}^{n}\frac{{}^{n}C_k}{k+1}=\sum_{k=0}^{4}\frac{{}^{4}C_k}{k+1}={}^4C_0+\frac{{}^4C_1}{2}+\frac{{}^4C_2}{3}+\frac{{}^4C_3}{4}+\frac{{}^4C_4}{5}$$

$$=1+2+2+1+\frac{1}{5}=6.20$$

8. On putting $x=1$ in the expansion of $(1+x-3x^2)^{2163}$ $=A_0+A_1x+A_2x^2+\ldots$, we will get the sum of coefficients of given polynomial, which clearly comes to be -1.

9. (c) $\because$ n is even, let $n=2m$ then

$$\text{LHL}=S=\frac{2.m!m!}{(2m)!}[C_0^2-2C_1^2+3C_2^2\ldots\ldots$$
$$+(-1)^{2m}(2m+1)C_{2m}^2\ldots]\quad\ldots\text{(i)}$$

$$=\frac{2.m!.m!}{(2m)!}\Big[C_{2m}^2-2C_{2m-1}^2+3C_{2m-2}^2-\ldots\ldots$$
$$+(-1)^{2m}(2m+1)C_0^2\ldots\Big]\quad[\text{using } C_r=C_{n-r}]$$

$$\Rightarrow S=\frac{2.m!m!}{(2m)!}[(2m+1)\,C_0^2-2mC_1^2$$
$$+(2m-1)\,C_2^2\ldots\ldots-2C_{2m-1}^2+C_{2m}^2]\quad\ldots\text{(ii)}$$

On adding (i) and (ii),

$$2S=2\frac{m!m!}{(2m)!}[2m+2]\,[C_0^2-C_1^2+C_2^2+\ldots..+C_{2m}^2]$$

Now keeping in mind that if n is even, then

$$C_0^2-C_1^2+C_2^2-\ldots\ldots+C_n^2=(-1)^{n/2}\,{}^{n}C_{n/2}$$

$\therefore$ we get

$$S=\frac{m!m!}{(2m)!}(2m+2)\,[(-1)^m\,{}^{2m}C_m]$$

$$=\frac{m!m!}{(2m)!}\left(2.\frac{n}{2}+2\right)(-1)^m\frac{(2m)!}{m!m!}$$

$$=(n+2)(-1)^{n/2}=(-1)^{n/2}(n+2)$$

10. To show that

$$2^k.{}^nC_0.{}^nC_k-2^{k-1}.{}^nC_1.{}^{n-1}C_{k-1}+2^{k-2}.{}^nC_2.{}^{n-2}C_{k-2}$$
$$-\ldots..+(-1)^k\,{}^nC_k\,{}^{n-k}C_0={}^nC_k$$

Taking LHS

$$2^k.{}^nC_0.{}^nC_k-2^{k-1}.{}^nC_1.{}^{n-1}C_{k-1}+\ldots+(-1)^k.{}^nC_k.{}^{n-k}C_0$$

$$=\sum_{r=0}^{k}(-1)^r.2^{k-r}.{}^nC_r.{}^{n-r}C_{k-r}$$

$$=\sum_{r=0}^{k}(-1)^r2^{k-r}.\frac{n!}{r!(n-r)!}.\frac{(n-r)!}{(k-r)!(n-k)!}$$

$$=\sum_{r=0}^{k}(-1)^r.2^{k-r}.\frac{n!}{(n-k)!k!}.\frac{k!}{r!(k-r)!}$$

$$=\sum_{r=0}^{k}(-1)^r.2^{k-r}\,{}^nC_k.{}^kC_r=2^k.{}^nC_k\left\{\sum_{r=0}^{k}(-1)^r.\frac{1}{2^r}.{}^kC_r\right\}$$

$$=2^k.{}^nC_k\left(1-\frac{1}{2}\right)^k={}^nC_k=\text{R.H.S.}$$

11. Given that for positive integers m and n such that $n\ge m$, then to prove that

$${}^nC_m+{}^{n-1}C_m+{}^{n-2}C_m+\ldots.+{}^mC_m={}^{n+1}C_{m+1}$$

L.H.S. ${}^mC_m+{}^{m+1}C_m+{}^{m+2}C_m+\ldots.+{}^{n-1}C_m+{}^nC_m$

[writing L.H.S. in reverse order]

$=({}^{m+1}C_{m+1}+{}^{m+1}C_m)+{}^{m+2}C_m+\ldots.+{}^{n-1}C_m+{}^nC_m$

$[\because\ {}^mC_m={}^{m+1}C_{m+1}]$

$=({}^{m+2}C_{m+1}+{}^{m+2}C_m)+{}^{m+3}C_m+\ldots.+{}^nC_m$

$[\because\ {}^nC_{r+1}+{}^nC_r={}^{n+1}C_{r+1}]$

$={}^{m+3}C_{m+1}+{}^{m+3}C_m+\ldots.+{}^nC_m$

Combining in the same way we get

$={}^nC_{m+1}+{}^nC_m={}^{n+1}C_{m+1}=$ R.H.S.

Again we have to prove

$${}^nC_m+2\,{}^{n-1}C_m+3\,{}^{n-2}C_m+\ldots.+(n-m+1)\,{}^mC_m={}^{n+2}C_{m+2}$$

$=[{}^nC_m+{}^{n-1}C_m+{}^{n-2}C_m+\ldots.+{}^mC_m]+[{}^{n-1}C_m$
$+{}^{n-2}C_m+\ldots.+{}^mC_m]+[{}^{n-2}C_m+\ldots.+{}^mC_m]+\ldots.+[{}^mC_m]$

[$n-m+1$ bracketed terms]

$={}^{n+1}C_{m+1}+{}^nC_{m+1}\ {}^{n-1}C_{m+1}\ldots.+{}^{m+1}C_{m+1}$

[using previous result.]

$={}^{n+2}C_{m+2}$

[Replacing n by $n+1$ and m by $m+1$ in the previous result.]

= R.H.S.

12. Given : $(1+x+x^2)^n=a_0+a_1x+\ldots.+a_{2n}x^{2n}$ ....(i)
where n is a +ve integer.

On replacing x by $-\frac{1}{x}$ in equation(i), we get

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n=a_0-\frac{a_1}{x}+\frac{a_2}{x^2}-\frac{a_3}{x^3}+\ldots.+\frac{a_{2n}}{x^{2n}}\quad\ldots.\text{(ii)}$$

Multiplying equation (i) and (ii) :

$$\frac{(1+x+x^2)^n\,(x^2-x+1)^n}{x^{2n}}$$

$$=(a_0+a_1x+\ldots.+a_{2n}x^{2n})\left(a_0-\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots+\frac{a_{2n}}{x^{2n}}\right)$$

Equating the constant terms on both sides we get

$a_0^2-a_1^2+a_2^2-a_3^2+\ldots.+a_{2n}^2=$ constant term in the expansion of $\dfrac{[(1+x+x^2)(1-x+x^2)]^n}{x^{2n}}$

= Coeff. of x^{2n} in the expansion of $(1 + x^2 + x^4)^n$

But replacing x by x^2 in equation (i), we have

$(1 + x^2 + x^4)^n = a_0 + a_1x^2 + ... + a_{2n}(x^2)^{2n}$

$\therefore$ Coeff. of $x^{2n} = a_n$

$\therefore \quad a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2 = a_n$

13. Given : $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ ....(i)

and $a_k = 1, \forall k \geq n$

To prove : $b_n = {}^{2n+1}C_{n+1}$

In the given equation (i) let us put $x - 3 = y$

$\Rightarrow x - 2 = y + 1$

$\therefore \quad \sum_{r=0}^{2n} a_r(1+y)^r = \sum_{r=0}^{2n} b_r(y)^r$ [From (i)]

$\Rightarrow a_0 + a_1(1+y) + ... + a_{n-1}(1+y)^{n-1} + (1+y)^n + (1+y)^{n+1} + ... + (1+y)^{2n}$

$= \sum_{r=0}^{2n} b_r y^r$ [Using $a_k = 1, \forall k \geq n$]

Equating the coefficients of y^n on both sides we get

$\Rightarrow {}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + ... + {}^{2n}C_n = b_n$

$\Rightarrow ({}^{n+1}C_{n+1} + {}^{n+1}C_n) + {}^{n+2}C_n + ... + {}^{2n}C_n = b_n$

[$\because {}^nC_n = {}^{n+1}C_{n+1} = 1$]

$\Rightarrow b_n = {}^{n+2}C_{n+1} + {}^{n+2}C_n + ... + {}^{2n}C_n$

[$\because {}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r$]

Combining the terms in similar way, we get

$\Rightarrow b_n = {}^{2n}C_{n+1} + {}^{2n}C_n \Rightarrow \quad b_n = {}^{2n+1}C_{n+1}$

14. We know

$(1-x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + ... + (-1)^n C_nx^n$

On multiplying both sides by x,

$x(1-x)^n = C_0x - C_1x^2 + C_2x^3 - C_3x^4 + ... + (-1)^n C_nx^{n+1}$

On differentiating both sides w.r. to x,

$(1-x)^n - nx(1-x)^{n-1}$

$= C_0 - 2C_1x + 3C_2x^2 - 4C_3x^3 + ... + (-1)^n(n+1)C_nx^n$

Again on multiplying both sides by x,

$x(1-x)^n - nx^2(1-x)^{n-1}$

$= C_0x - 2C_1x^2 + 3C_2x^3 - 4C_3x^4 + ... + (-1)^n(n+1)C_nx^{n+1}$

On differentiating both sides with respect to x,

$(1-x)^n - nx(1-x)^{n-1} - 2nx(1-x)^{n-1} + nx^2(n-1)(1-x)^{n-2}$

$= C_0 - 2^2C_1x + 3^2C_2x^2 - 4^2C_3x^3 + ... + (-1)^n(n+1)^2C_nx^n$

Putting $x = 1$, in above, we get

$0 = C_0 - 2^2C_1 + 3^2C_2 - 4^2C_3 + ... + (-1)^n(n+1)^2C_n$

15. ${}^{n+1}C_1 + {}^{n+1}C_2s_1 + {}^{n+1}C_3s_2 + ... + {}^{n+1}C_{n+1}s_n$

$= \sum_{r=1}^{n+1} {}^{n+1}C_r s_{r-1},$

where $S_n = 1 + q + q^2 + ... + q^n = \dfrac{1-q^{n+1}}{1-q}$

$\therefore \sum_{r=1}^{n+1} {}^{n+1}C_r\left(\dfrac{1-q^r}{1-q}\right) = \dfrac{1}{1-q}\left(\sum_{r=1}^{n+1} {}^{n+1}C_r - \sum_{r=1}^{n+1} {}^{n+1}C_rq^r\right)$

$= \dfrac{1}{1-q}[(1+1)^{n+1} - (1+q)^{n+1}]$

$= \dfrac{1}{1-q}\left[2^{n+1} - (1+q)^{n+1}\right]$ .... (i)

Also, $S_n = 1 + \left(\dfrac{q+1}{2}\right) + \left(\dfrac{q+1}{2}\right)^2 + ... + \left(\dfrac{q+1}{2}\right)^n$

$= \dfrac{1-\left(\dfrac{q+1}{2}\right)^{n+1}}{1-\left(\dfrac{q+1}{2}\right)} = \dfrac{2^{n+1} - (q+1)^{n+1}}{2^n(1-q)}$ ....(ii)

From equations (i) and (ii),

${}^{n+1}C_1 + {}^{n+1}C_2s_1 + {}^{n+1}C_3s_2 + ... + {}^{n+1}C_{n+1}s_n = 2^n S_n$

16. $S = \sum \sum C_iC_j$

$0 \leq i < j \leq n$

$\Rightarrow S = C_0(C_1 + C_2 + C_3 + ... + C_n) + C_1(C_2 + C_3 + ... + C_n) + C_2(C_3 + C_4 + C_5 + ... C_n) + ... C_{n-1}(C_n)$

$\Rightarrow S = C_0(2^n - C_0) + C_1(2^n - C_0 - C_1) + C_2(2^n - C_0 - C_1 - C_2) + ... + C_{n-1}(2^n - C_0 - C_1 .... C_{n-1}) + C_n(2^n - C_0 - C_1 .... C_n)$

$\Rightarrow S = 2^n(C_0 + C_1 + C_2 + ... + C_{n-1} + C_n) - (C_0^2 + C_1^2 + C_2^2 + ... + C_n^2) - S$

$\Rightarrow 2S = 2^n.2^n - \dfrac{2n!}{(n!)^2} = 2^{2n} - \dfrac{2n!}{(n!)^2}$

$\Rightarrow S = 2^{2n-1} - \dfrac{2n!}{2(n!)^2}$

17. Given :

$C_1 + 2C_2x + 3C_3x^2 + ... + 2nC_{2n}x^{2n-1} = 2n(1+x)^{2n-1}$ ....(i)

where $C_r = \dfrac{2n!}{r!(2n-r)!}$

Integrating both sides with respect to x, under the limits 0 to x, we get

$[C_1x + C_2x^2 + C_3x^3 + ... + C_{2n}x^{2n}]_0^x = [(1+x)^{2n}]_0^x$

$\Rightarrow C_1x + C_2x^2 + C_3x^3 + ... + C_{2n}x^{2n} = (1+x)^{2n} - 1$

$\Rightarrow C_0 + C_1x + C_2x^2 + C_3x^3 + ... + C_{2n}x^{2n} = (1+x)^{2n}$ ....(ii)

Changing x by $-\dfrac{1}{x}$, we get

$\Rightarrow C_0 - \dfrac{C_1}{x} + \dfrac{C_2}{x^2} - \dfrac{C_3}{x^3} + ... + (-1)^{2n}\dfrac{C_{2n}}{x^{2n}} = \left(1 - \dfrac{1}{x}\right)^{2n}$

$\Rightarrow C_0x^{2n} - C_1x^{2n-1} + C_2x^{2n-2} - C_3x^{2n-3} + ... + C_{2n} = (x-1)^{2n}$ ....(iii)

Multiplying eqn. (i) and (iii) and equating the coefficients of x^{2n-1} on both sides, we get

$-C_1^2 + 2C_2^2 - 3C_3^2 + ... + 2nC_{2n}^2$

$=$ coeff.of x^{2n-1} in $2n(x-1)(x^2-1)^{2n-1}$

$= 2n$ [coeff. of x^{2n-2} in $(x^2-1)^{2n-1}$ − coeff. of x^{2n-1} in $(x^2-1)^{2n-1}$]

$= 2n[{}^{2n-1}C_{n-1}(-1)^{n-1} - 0]$

$= (-1)^{n-1}.2n{}^{2n-1}C_{n-1}$

$\Rightarrow C_1^2 - 2C_2^2 + 3C_3^2 + ... + 2nC_{2n}^2$

$= (-1)^n.2n{}^{2n-1}C_{n-1} = (-1)^n n.\left(\dfrac{2n}{n}.{}^{2n-1}C_{n-1}\right)$

$= (-1)^n n.{}^{2n}C_n = (-1)^n n.C_n.$ $(\because {}^{2n}C_n = C_n)$