Exercise 13.R

Answer 1CC.

A **vector function** or a vector-valued function is any function whose domain is a set of real numbers and whose range is a set of vectors.

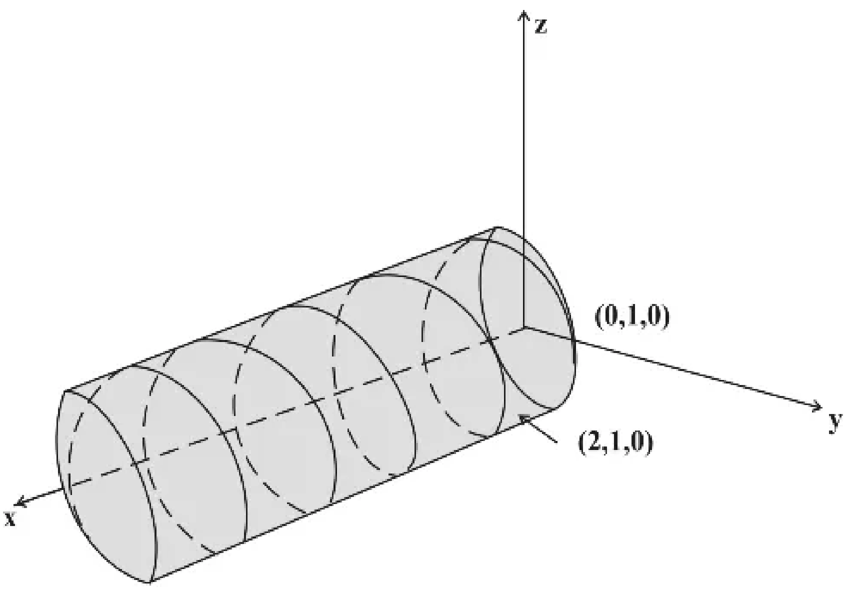
Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ be a vector valued function, where f, g, and h are differentiable functions. Then the first derivative of $\mathbf{r}(t)$ is given by $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$.

Thus, we can say that the derivative of a vector-valued function is obtained by differentiating its component functions.

The integral of a derivative function is obtained by integrating each of its component functions. Then, $\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$.

Answer 1E.

(A)



(B)

The vector function is

$$\vec{r}(t) = t\hat{i} + \cos\theta t\,\hat{j} + \sin\pi t\,\hat{k}$$

Then $\vec{r}\,'(t) = \frac{d}{dt}(t)\hat{i} + \frac{d}{dt}(\cos\pi t)\hat{j} + \sin\pi t\,\hat{k}$

i.e. $\boxed{\vec{r}\,'(t) = \hat{i} - \pi\sin\pi t\,\hat{j} + \pi\cos\pi t\,\hat{k}}$

And $\vec{r}\,''(t) = \frac{d}{dt}(t)\hat{i} - \pi\frac{d}{dt}(\sin\pi t)\hat{j} + \pi\frac{d}{dt}(\cos\pi t)\hat{k}$

i.e. $\boxed{\vec{r}\,''(t) = -\pi^2\cos\pi t\,\hat{j} - \pi^2\sin\pi t\,\hat{k}}$

Answer 1P.

(a)

The projectile reaches maximum height when

$$0 = \frac{dy}{dt} = \frac{d}{dt}\left[\left(v_0 \sin\alpha\right)t - \frac{1}{2}gt^2\right] = v_0 \sin\alpha - gt$$; that is, whenever

$$t = \frac{v_0 \sin\alpha}{g} \text{ and } y = \left(v_0 \sin\alpha\right)\left(\frac{v_0 \sin\alpha}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\alpha}{g}\right)^2 = \frac{v_0{}^2 \sin^2\alpha}{2g}$$

This is the maximum height attained when the projectile is fired with an angle of elevation α. This maximum height is largest when α = π/2. In that case, sin α = 1 and the maximum height is v02/2g.

(b)

Let R = v02/g. We are asked to consider the parabola x2 + 2Ry - R2 = 0 which can be rewritten as $y = -\frac{1}{2R}x^2 + \frac{R}{2}$. The points on or inside this parabola are those for which $-R \leq x \leq R$ and $0 \leq y \leq \frac{-1}{2R}x^2 + \frac{R}{2}$. When the projectile is fired at an angle of elevation α, the points (x,y) along its path satisfy the relations x = (v0 cos α)t along with

y = (v0 sin α)t - (1/2)gt2, where $0 \leq t \leq \frac{2v_0 \sin\alpha}{g}$. Therefore, we find that

$$|x| \leq \left|v_0 \cos\alpha\left(\frac{2v_0 \sin\alpha}{g}\right)\right| = \left|\frac{v_0{}^2}{g}\sin 2\alpha\right| \leq \left|\frac{v_0{}^2}{g}\right| = |R|.$$

This result shows that $-R \leq x \leq R$.

For t in the specified range, we also have

$$y = t\left(v_0 \sin\alpha - \frac{1}{2}\text{gt}\right) = \frac{1}{2}\text{gt}\left(\frac{2v_0 \sin\alpha}{g} - t\right) \geq 0$$ and

$$y = \left(v_0 \sin\alpha\right)\frac{x}{v_0 \cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0 \cos\alpha}\right)^2 = \left(\tan\alpha\right)x - \frac{g}{2v_0^2 \cos^2\alpha}x^2 = -\frac{1}{2R\cos^2\alpha}x$$

Therefore, we now find that

$$y - \left(\frac{-1}{2R}x^2 + \frac{R}{2}\right) = \frac{-1}{2R\cos^2\alpha}x^2 + \frac{1}{2R}x^2 + \left(\tan\alpha\right)x - \frac{R}{2}$$

$$= \frac{x^2}{2R}\left(1 - \frac{1}{\cos^2\alpha}\right) + \left(\tan\alpha\right)x - \frac{R}{2} = \frac{x^2\left(1 - \sec^2\alpha\right) + 2R\left(\tan\alpha\right)x - R^2}{2R}$$

$$= \frac{-\left(\tan^2\alpha\right)x^2 + 2R\left(\tan\alpha\right)x - R^2}{2R} = \frac{-[(\tan\alpha)x - R]^2}{2R} \leq 0$$

So we have shown that every target that can be hit by the projectile lies on or inside the parabola $y = -\frac{1}{2R}x^2 + \frac{R}{2}$. Now let (a,b) be any point on or inside the parabola $y = -\frac{1}{2R}x^2 + \frac{R}{2}$. Then $-R \leq a \leq R$ and $0 \leq b \leq -\frac{1}{2R}a^2 + \frac{R}{2}$. We seek an angle α such that (a,b) lies in the path of the projectile; that is, we wish to find an angle α such that $b = -\frac{1}{2R\cos^2\alpha}a^2 + \left(\tan\alpha\right)a$ or equivalently we can also have that

$b = \frac{-1}{2R}\left(\tan^2\alpha + 1\right)a^2 + \left(\tan\alpha\right)a$. In rearranging this equation, we get

$$\frac{a^2}{2R}\tan^2\alpha - a\tan\alpha + \left(\frac{a^2}{2R} + b\right) = 0 \text{ or } a^2(\tan\alpha)^2 - 2\text{aR}\left(\tan\alpha\right) + \left(a^2 + 2\text{bR}\right) =$$

This quadratic equation for tan α has real solutions exactly when the discriminant is nonnegative. So now we have

$$B^2 - 4AC \geq 0 \Leftrightarrow (-2aR)^2 - 4a^2\left(a^2 + 2bR\right) \geq 0 \Leftrightarrow 4a^2\left(R^2 - a^2 - 2bR\right) \geq$$

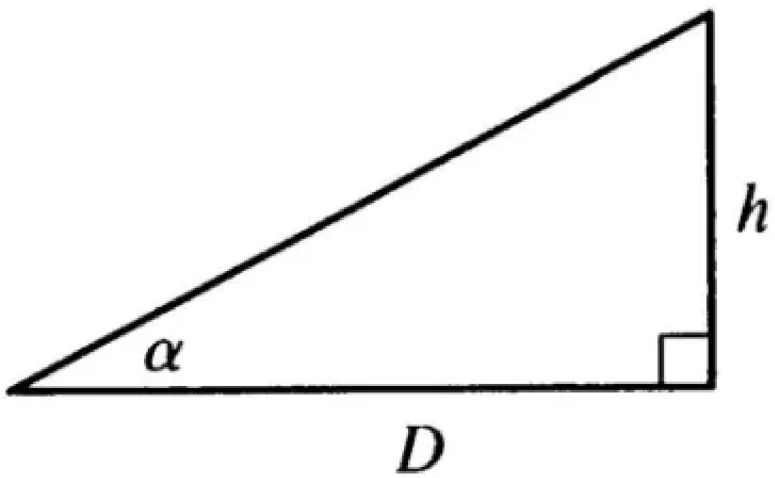
This condition is satisfied since (a,b) is on or inside the parabola $y = -\frac{1}{2R}x^2 + \frac{R}{2}$. It follows that (a,b) lies in the path of the projectile when tan α satisfies (*), which is when

$$\tan\alpha = \frac{2aR \pm \sqrt{4a^2\left(R^2 - a^2 - 2bR\right)}}{2a^2} = \frac{R \pm \sqrt{R^2 - 2bR - a^2}}{a}$$

(c)

If the gun is pointed at a target with height h at a distance D downrange, then tan α = h/D. When the projectile reaches a distance D downrange, we have

$D = x = (v_0 \cos\alpha)t$, so from this we get that $t = \frac{D}{v_0 \cos\alpha}$ and $y = \left(v_0 \sin\alpha\right)t - \frac{1}{2}gt^2 = D\tan\alpha - \frac{gD^2}{2v_0^2\cos^2\alpha}$. Meanwhile, the target, whose x-coordinate is also D, has fallen from height h to height h - (1/2)gt2 = D tan α - (gD2)/(2v02 cos2α). Therefore, we know that the projectile hits the target.



Answer 1TFQ.

The given statement is **true**.

From the given vector valued function; we get the parametric equations of the curve as $x = t^3, y = 2t^3$, and $z = 3t^3$.

Obtain the corresponding Cartesian equation of the curve. On adding $y = 2t^3$ and $z = 3t^3$, we get $y + z = 5t^3$.

Replace t^3 with x in $y + z = 5t^3$.
$y + z = 5x$

We get the Cartesian equation as $y + z = 5x$, which represents a straight line.

Therefore, the given vector represents a **straight line**.

Answer 2CC.

A vector function or a vector-valued function is any function whose domain is a set of real numbers and whose range is a set of vectors. This means for every t in the domain of **r** there is a unique vector in 3-D space denoted by $\mathbf{r}(t)$. If , g, and h are differentiable functions, then et $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a vector valued function. Now, there is a close connection between continuous vector functions and space curves. Suppose that f, g, and h are continuous real-valued functions on an interval I. Then, the set C of all points (x, y, z) space, where $x = f(t), y = g(t)$, and $z = h(t)$ and t varies throughout the interval I, is called a space curve.

Answer 2E.

$$\vec{r}(t) = < \sqrt{2-t}, \frac{(e^t - 1)}{t}, \ln(t+1) >$$

(A)

In the domain of $\vec{r}$ each function

$$f(t) = \sqrt{2-t}$$

$$g(t) = {(e^t - 1)}/{t}$$

$$h(t) = \ln(t+1)$$

Must be defined

Now f(t) is defined for $t \le 2$
g(t) is defined for $t \ne 0$
And h(t) is defined for $t + 1 \ge 0$
Hence the domain of r is $(-1, 0) \cup (0, 2)$

(B)

$$\lim_{t\to 0}\vec{r}(t) = <\lim_{t\to 0}\sqrt{2-t}, \lim_{t\to 0}\left(\frac{e^t-1}{t}\right), \lim_{t\to 0}\ln(t+1)>$$
$$= <\sqrt{2}, 1, 0>$$

(C)

$$\vec{r}\,'(t) = <\frac{d}{dt}\sqrt{2-t}, \frac{d}{dt}\left(\frac{e^t-1}{t}\right), \frac{d}{dt}\ln(t+1)>$$
$$= <\frac{-1}{2\sqrt{2-t}}, \frac{e^t(t-1)+1}{t^2}, \frac{1}{t+1}>$$

Answer 2P.

(a)

As in Problem 3, $r(t) = (v_0 \cos\alpha)t\,i + \left[(v_0 \sin\alpha)t - \frac{1}{2}gt^2\right]j$, and so from this we have $x = (v_0 \cos\alpha)t$ and $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$. The difference here is that the projectile travels until it reaches a point where x > 0 and y = -(tan θ)x. From the parametric equations, we obtain

$$t = \frac{x}{v_0 \cos\alpha} \text{ and } y = \frac{(v_0 \sin\alpha)x}{v_0 \cos\alpha} - \frac{gx^2}{2v_0^2\cos^2\alpha} = (\tan\alpha)x - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

Therefore from this we can determine that the projectile hits the inclined plane at the point where $(\tan\alpha)x - \frac{gx^2}{2v_0^2\cos^2\alpha} = -(\tan\theta)x$.

Now, since $\frac{gx^2}{2v_0^2\cos^2\alpha} = (\tan\alpha + \tan\theta)x$ and x > 0, then we know that we must have that $\frac{gx}{2v_0^2\cos^2\alpha} = \tan\alpha + \tan\theta$. It follows that $x = \frac{2v_0^2\cos^2\alpha}{g}(\tan\alpha + \tan\theta)$ and that $t = \frac{x}{v_0\cos\alpha} = \frac{2v_0\cos\alpha}{g}(\tan\alpha + \tan\theta)$. This means that the parametric equations are defined for t in the interval $\left[0, \frac{2v_0\cos\alpha}{g}(\tan\alpha + \tan\theta)\right]$.

(b)

The downhill range is R(α) = x sec θ, where x is the coordinate of the landing point calculated in part (a). Therefore, we now find that

$$R\left(\alpha\right) = \frac{2v_0^2\cos^2\alpha}{g}\left(\tan\alpha + \tan\theta\right)\sec\theta = \frac{2v_0^2}{g}\left(\frac{\sin\alpha\cos\alpha}{\cos\theta} + \frac{\cos^2\alpha\sin\theta}{\cos^2\theta}\right)$$

$$= \frac{2v_0^2\cos\alpha}{g\cos^2\theta}\left(\sin\alpha\cos\theta + \cos\alpha\sin\theta\right) = \frac{2v_0^2\cos\alpha\sin\left(\alpha+\theta\right)}{g\cos^2\theta}$$

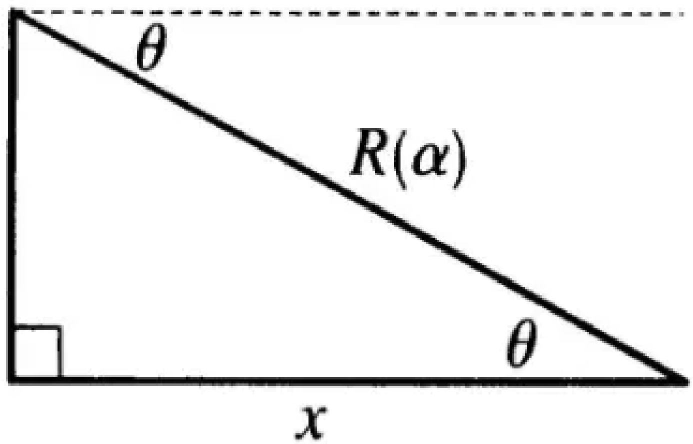
R(α) is maximized when

$$0 = R'\left(\alpha\right) = \frac{2v_0^2}{g\cos^2\theta}\left[-\sin\alpha\sin(\alpha+\theta) + \cos\alpha\cos(\alpha+\theta)\right]$$

$$= \frac{2v_0^2}{g\cos^2\theta}\cos\left[(\alpha+\theta)+\alpha\right] = \frac{2v_0^2\cos\left(2\alpha+\theta\right)}{g\cos^2\theta}$$

So we determine that this condition implies that

$$\cos\left(2\alpha+\theta\right) = 0 \Rightarrow 2\alpha+\theta = \frac{\pi}{2} \Rightarrow \alpha = \frac{1}{2}\left(\frac{\pi}{2} - \theta\right)$$



(c)

The solution is similar to the solutions to parts (a) and (b). This time the projectile travels until it reaches a point where x > 0 and y = (tan θ)x. Since tan θ = -tan(-θ), we obtain the solution from the previous one by replacing θ with -θ. The desired angle is α = 1/2(π/2 + θ).

(d)

As observed in part (c), firing the projectile up an inclined plane with angle of inclination θ involves the same equations as in parts (a) and (b) but with θ replaced by -θ. So if R is the distance up an inclined plane, then we know from part (b) that

$$R = \frac{2{v_0}^2 \cos\alpha \sin(\alpha-\theta)}{g\cos^2(-\theta)} \Rightarrow {v_0}^2 = \frac{Rg\cos^2\theta}{2\cos\alpha\sin(\alpha-\theta)}$$

${v_0}^2$ is minimized with respect to α when

$$0 = \frac{d}{d\alpha}\left({v_0}^2\right) = \frac{Rg\cos^2\theta}{2} \cdot \frac{-(\cos\alpha\cos(\alpha-\theta) - \sin\alpha\sin(\alpha-\theta))}{[\cos\alpha\sin(\alpha-\theta)]^2}$$

$$= \frac{-Rg\cos^2\theta}{2} \cdot \frac{\cos[\alpha+(\alpha-\theta)]}{[\cos\alpha\sin(\alpha-\theta)]^2} = \frac{-Rg\cos^2\theta}{2} \cdot \frac{\cos(2\alpha-\theta)}{[\cos\alpha\sin(\alpha-\theta)]^2}$$

Since 0 < α < π/2, this implies cos(2α - θ) = 0 ↔ 2α - θ = π/2 → α = 1/2(π/2 + θ). Therefore, the initial speed, and thus the energy required, is minimized for α = 1/2(π/2 + θ).

Answer 2TFQ.

The given statement is **true**.

From the given vector valued function; we get the parametric equations of the curve as $x = 0, y = t^2$, and $z = 4t$.

From the equation for z, we get $t = \frac{z}{4}$.

Replace t with $\frac{z}{4}$ in $y = t^2$.

$$y = \left(\frac{z}{4}\right)^2$$
$$16y = z^2$$

We get the Cartesian equation as $z^2 = 16y$, which represents a parabola.

Therefore, the given vector represents a **parabola**.

Answer 3CC.

Let $\mathbf{r}(t)$ a vector function, then the tangent vector to the curve defined by $\mathbf{r}$ at the point P is given by $\mathbf{r}'(t)$. The tangent vector is defined only if $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq 0$. Now, the tangent line to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The unit tangent $\mathbf{T}(t)$ vector at P is given by $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

Answer 3E.

Let C be the curve of intersection of given cylinder $x^2 + y^2 = 16$ and plane $x + z = 5$

The projection of C onto xy − plane is a circle $x^2 + y^2 = 16,\ z = 0$

Then we can write

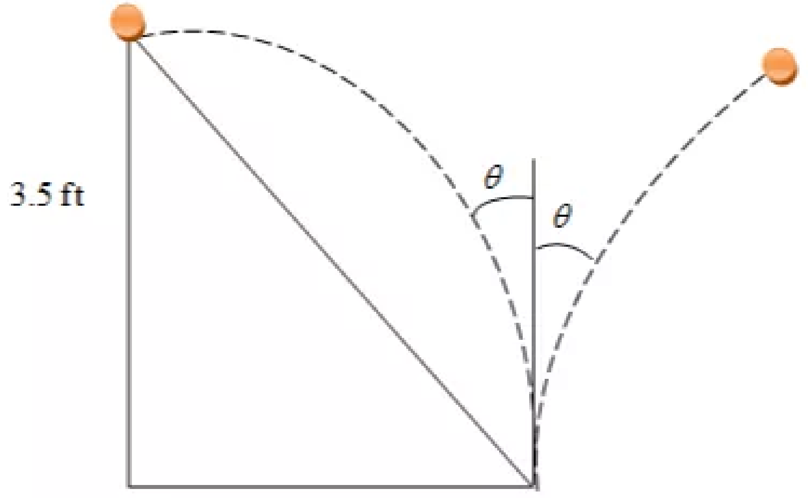
$$x = 4\cos t,\ y = 4\sin t,\ 0 \le t \le 2\pi$$

From the equation of plane

$$z = 5 - x = 5 - 4\cos t$$

Answer 3P.

Observe the following diagram.



Horizontal velocity of the ball is 2 feet per second.

Height of the table is 3.5 feet.

Derive a formula to calculate the time at which ball hits the ground.

$$y = v_y t + \frac{1}{2} g t^2$$

where y vertical distance at which the object is located from the ground v_y vertical velocity of the object and it is zero for this problem cause the object has only horizontal velocity.

So put $v_y = 0$ into the equation $y = v_y t + \frac{1}{2} g t^2$ solve for t.

$$t = \sqrt{\frac{2y}{g}}$$

Hence, the formula of time taken the by ball to hit ground is as follows:

$$t = \sqrt{\frac{2 \times \text{Height of the table}}{g}}$$

Since $\text{Height of the table} = 3.5\,\text{ft}$ and $g = 32\ \text{ft/sec}^2$

$$\begin{aligned} t &= \sqrt{\frac{2 \times 3.5}{32}} \\ &= \sqrt{\frac{7}{32}} \\ &= 0.4678\,\text{seconds} \end{aligned}$$

Hence ball hits the ground after $0.4678\,\text{seconds}$.

Now the distance travelled by the ball is given by the following equation.

$$\begin{aligned} d &= \text{speed} \times \text{time} \\ &= 2 \times 0.4678 \\ &= 0.9356\ \text{ft.} \end{aligned}$$

Hence the ball hits the ground 0.9356 feet away from the bottom of the table.

Use the formula $v = u + at$ to find the speed of the ball when it hits the ground.

When the ball hits the ground its final velocity will be zero that is $v = 0$.

Now the equation becomes $0 = u - gt \Rightarrow u = gt$

This follows that $u = gt$

Since $t = 0.4678\,\text{seconds}$.

$$u = 32 \times 0.4678 \Rightarrow u = 14.9696 \approx \boxed{15\text{ft/sec}}$$

Hence the velocity of ball when it hits ground is 15ft/sec.

Use the following formula to find the angle θ.

$$\tan\theta = \frac{v_x^{\ 2}}{Rg}$$

Where v_x -horizontal velocity R -range (distance from the bottom of the table to the position where the ball hit the ground)

Since the horizontal velocity of the ball is 2 feet per second and R is 0.9356 feet.

$$\begin{aligned} \tan\theta &= \frac{2^2}{0.9356 \cdot 32} \\ &= 0.1334 \\ &\Rightarrow \theta = \arctan(0.1334) \approx \boxed{7.6^\circ}. \end{aligned}$$

Hence the required angle is 7.6 degrees.

The speed of the ball when it hits ground is 15 ft/sec and it is reduced up to 20 percent.

Which means the speed of ball when it takes off from the ground will be

$$(15 - 20\% \text{ of } 15) \text{ ft/sec} = 12 \text{ ft/sec.}$$

In the second jump, the ball acts like a projectile hence distance at which the ball hits ground after the second jump will obtained by the following formula.

$$\begin{aligned} R &= \frac{v_0^2}{g}\sin 2\theta \\ &= \frac{12^2}{32}\sin 2(7.6) \\ &= \frac{144}{32}\times 0.2621891 \\ &= 1.19 \end{aligned}$$

Hence the ball lands at distance of 1.19 feet.

This follows that it hits ground after second jump at distance from the bottom of the table is

$$0.94 \text{ feet} + 1.19 \text{ feet} = \boxed{2.13\text{feet}}.$$

Answer 3TFQ.

The given statement is **false**.

From the given vector valued function; we get the parametric equations of the curve as $x = 2t, y = 3 - t$, and $z = 0$.

From the equation for y, we get $t = 3 - y$.

Replace t with $3 - y$ in $x = 2t$.

$$\begin{aligned} x &= 2(3 - y) \\ x &= 6 - 2y \\ x + 2y &= 6 \end{aligned}$$

We get the Cartesian equation as $x + 2y = 6$, which represents a straight line with x-*intercept* 6 and y-intercept 3.

Therefore, the given vector does not represent a straight line passing through the origin.

Answer 4CC.

(a) The derivative of the sum of two vector functions can be obtained by differentiating each of the vector functions and then adding the differentials.

$$\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \mathbf{u}'(t)+\mathbf{v}'(t)$$

(b) The derivative of constant times a function is obtained by multiplying the derivative of the function by the constant.

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t).$$

(c) The derivative of the product of two vector functions is obtained by adding the product of the derivative of the first function and the second function to the product of the derivative of the second function and the first function.

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$$

(d) The derivative of the dot product of two vector functions is obtained by adding the dot product of the derivative of the first function and the second function to the dot product of the derivative of the second function and the first function.

$$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\mathbf{v}(t)+\mathbf{u}(t)\mathbf{v}'(t)$$

(e) The derivative of the dot product of two vector functions is obtained by adding the cross product of the derivative of the first function and the second function to the cross product of the derivative of the second function and the first function.

$$\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$$

(f) The derivative of $\mathbf{u}(f(t))$ can be obtained by applying the chain rule of differentiation.

$$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$$

Answer 4E.

(a)

Consider the vector functions.

$$\mathbf{u}(t)+\mathbf{v}(t)$$

The differentiation rule of the given vector function is the **sum rule**.

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] &= \frac{d}{dt}\mathbf{u}(t)+\frac{d}{dt}\mathbf{v}(t)\\ &= \mathbf{u}'(t)+\mathbf{v}'(t)\end{aligned}$$

(b)

Consider the vector functions.

$$c\mathbf{u}(t)$$

The differentiation rule of the given vector function is the **scalar multiple rule**.

$$\frac{d}{dt}\left[c\mathbf{u}(t)\right] = c\mathbf{u}'(t)$$

(c)

Consider the vector functions.

$$f(t)\mathbf{u}(t)$$

The differentiation rule of the given vector function is the **product rule**.

$$\frac{d}{dt}\left[f(t)\mathbf{u}(t)\right] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

(d)

Consider the vector functions.

$$\mathbf{u}(t)\cdot\mathbf{v}(t)$$

The differentiation rule of the given vector function is the **dot product rule**.

$$\frac{d}{dt}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$$

(e)

Consider the vector functions.

$$\mathbf{u}(t)\times\mathbf{v}(t)$$

The differentiation rule of the given vector function is the **cross product rule**.

$$\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$$

(f)

Consider the vector functions.

$$\mathbf{u}(f(t))$$

The differentiation rule of the given vector function is the **chain rule**.

$$\frac{d}{dt}\left[\mathbf{u}(f(t))\right] = f'(t)\mathbf{u}'(f(t))$$

Answer 4P.

By the Fundamental Theorem of Calculus,

$$r'(t) = \left\langle \sin\left(\frac{\pi t^2}{2}\right), \cos\left(\frac{\pi t^2}{2}\right)\right\rangle, \ |r'(t)| = 1 \text{ so } T(t) = r'(t)$$

Therefore,

$$T'(t) = \pi t\left\langle \sin\left(\frac{\pi t^2}{2}\right), \cos\left(\frac{\pi t^2}{2}\right)\right\rangle$$ and the curvature is given by

$$k = |T'(t)| = \sqrt{(\pi t)^2(1)} = \pi|t|$$

Answer 4TFQ.

The given statement is **true**.

We know that the derivative of any vector valued function of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is given by $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$.

Therefore, the derivative of a vector function is obtained by differentiating each component function.

Answer 5CC.

If C is a smooth curve given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, on an interval $[a, b]$, then the arc length of C on the interval is $s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt = \int_a^b |\mathbf{r}'(t)|\, dt$.

Answer 5E.

Now the integral $\int_0^1 \mathbf{r}(t)\,dt$ can be calculated as follows:

$$\int_0^1 \mathbf{r}(t)\,dt = \left[\left(\frac{t^3}{3}\right)\mathbf{i} + \left[t\left(\frac{\sin \pi t}{\pi}\right) + \frac{\cos \pi t}{\pi^2}\right]\mathbf{j} - \left(\frac{\cos \pi t}{\pi}\right)\mathbf{k}\right]_0^1$$

$$= \left(\left(\frac{1^3}{3}\right)\mathbf{i} + \left[(1)\left(\frac{\sin \pi(1)}{\pi}\right) + \frac{\cos \pi(1)}{\pi^2}\right]\mathbf{j} - \left(\frac{\cos \pi(1)}{\pi}\right)\mathbf{k}\right)$$

$$- \left(\left(\frac{0^3}{3}\right)\mathbf{i} + \left[(0)\left(\frac{\sin \pi(0)}{\pi}\right) + \frac{\cos \pi(0)}{\pi^2}\right]\mathbf{j} - \left(\frac{\cos \pi(0)}{\pi}\right)\mathbf{k}\right)$$

$$= \frac{1}{3}\mathbf{i} - \frac{1}{\pi^2}\mathbf{j} + \frac{1}{\pi}\mathbf{k} - \left(\frac{1}{\pi^2}\mathbf{j} - \frac{1}{\pi}\mathbf{k}\right) \qquad \text{Use } \cos 0 = 1, \cos \pi = -1, \sin \pi = 0$$

$$= \frac{1}{3}\mathbf{i} - \frac{2}{\pi^2}\mathbf{j} + \frac{2}{\pi}\mathbf{k}$$

Therefore, $\int_0^1 \mathbf{r}(t)\,dt = \boxed{\frac{1}{3}\mathbf{i} - \frac{2}{\pi^2}\mathbf{j} + \frac{2}{\pi}\mathbf{k}}$.

Answer 5P.

A projectile is fired with angle of elevation α and initial speed v. So the parametric equations for trajectory of the projectile are,

$x = (v\cos\alpha)t, \quad y = (v\sin\alpha)t - \frac{1}{2}gt^2$. ......(1)

The objective is to find the value of α for which the total distance traveled by the projectile is maximized.

The vector function of the projectile is,

$\mathbf{r}(t) = (v\cos\alpha)t\mathbf{i} + \left((v\sin\alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}$.

Then find the derivative $\mathbf{v}(t) = \mathbf{r}'(t)$ of this vector function:

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

$$= \frac{d}{dt}\mathbf{r}(t)$$

$$= \frac{d}{dt}\left[(v\cos\alpha)t\mathbf{i} + \left((v\sin\alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}\right]$$

$$= \frac{d}{dt}(v\cos\alpha)t\mathbf{i} + \frac{d}{dt}\left((v\sin\alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}$$

$= (v\cos\alpha)\mathbf{i} + \left((v\sin\alpha) - gt\right)\mathbf{j}$. ......(i)

Then, the speed of the projectile, that is, the value of $|\mathbf{v}(t)|$ will be,

$$
\begin{aligned}
|\mathbf{v}(t)| &= |(v\cos\alpha)\mathbf{i} + ((v\sin\alpha) - gt)\mathbf{j}| \\
&= \sqrt{(v\cos\alpha)^2 + ((v\sin\alpha) - gt)^2} \\
&= \sqrt{v^2\cos^2\alpha + \left(v^2\sin^2\alpha + g^2t^2 - 2(v\sin\alpha)gt\right)} \\
&= \sqrt{v^2\cos^2\alpha + v^2\sin^2\alpha + g^2t^2 - 2(v\sin\alpha)gt} \\
&= \sqrt{g^2\left(\frac{v^2}{g^2}\cos^2\alpha + \frac{v^2}{g^2}\sin^2\alpha + t^2 - 2\frac{(v\sin\alpha)}{g}t\right)} \\
&= g\sqrt{\frac{v^2}{g^2}\cos^2\alpha + \frac{v^2}{g^2}\sin^2\alpha + t^2 - 2\frac{(v\sin\alpha)}{g}t} \\
&= g\sqrt{\left(t^2 + \frac{v^2}{g^2}\sin^2\alpha - 2\frac{(v\sin\alpha)}{g}t\right) + \frac{v^2}{g^2}\cos^2\alpha} \\
&= g\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \frac{v^2}{g^2}\cos^2\alpha} \quad \text{......(ii)}
\end{aligned}
$$

The projectile hits the ground when its vertical component is zero, that is, when,

$$
\begin{aligned}
y &= 0 \\
(v\sin\alpha)t - \tfrac{1}{2}gt^2 &= 0 \\
(v\sin\alpha)t &= \tfrac{1}{2}gt^2 \\
t &= \frac{2v}{g}\sin\alpha.
\end{aligned}
$$

Hence, the distance traveled by the projectile is,

$$L(\alpha)=\int_0^{\frac{2v}{g}\sin\alpha}|\mathbf{v}(t)|\,dt$$

$$=\int_0^{\frac{2v}{g}\sin\alpha}\left(g\sqrt{\left(t-\frac{v}{g}\sin\alpha\right)^2+\frac{v^2}{g^2}\cos^2\alpha}\right)dt$$

$$=g\int_0^{\frac{2v}{g}\sin\alpha}\sqrt{\left(t-\frac{v}{g}\sin\alpha\right)^2+\left(\frac{v}{g}\cos\alpha\right)^2}\,dt$$

Use the following formula;

$$\int\sqrt{u^2+a^2}\,du=\frac{1}{2}\left(u\sqrt{u^2+a^2}+a^2\ln\left|u+\sqrt{u^2+a^2}\right|\right)+C$$

with $u=t-\frac{v}{g}\sin\alpha$ and $a=\frac{v}{g}\cos\alpha$ to solve above integral:

$$L(\alpha)=\frac{g}{2}\left[\left(t-\frac{v}{g}\sin\alpha\right)\sqrt{\left(t-\frac{v}{g}\sin\alpha\right)^2+\left(\frac{v}{g}\cos\alpha\right)^2}\right.$$

$$\left.+\left(\frac{v}{g}\cos\alpha\right)^2\ln\left|\left(t-\frac{v}{g}\sin\alpha\right)+\sqrt{\left(t-\frac{v}{g}\sin\alpha\right)^2+\left(\frac{v}{g}\cos\alpha\right)^2}\right|\right]_0^{\frac{2v}{g}\sin\alpha}$$

$$=\frac{g}{2}\left[\left(\left(\frac{v}{g}\sin\alpha\right)\sqrt{\left(\frac{v}{g}\sin\alpha\right)^2+\left(\frac{v}{g}\cos\alpha\right)^2}+\left(\frac{v}{g}\cos\alpha\right)^2\ln\left|\left(\frac{v}{g}\sin\alpha\right)+\right.\right.\right.$$

$$\left.\left.\sqrt{\left(\frac{v}{g}\sin\alpha\right)^2+\left(\frac{v}{g}\cos\alpha\right)^2}\right|\right)-\left(\left(-\frac{v}{g}\sin\alpha\right)\sqrt{\left(-\frac{v}{g}\sin\alpha\right)^2+\left(\frac{v}{g}\cos\alpha\right)^2}\right.$$

$$\left.\left.+\left(\frac{v}{g}\cos\alpha\right)^2\ln\left|\left(-\frac{v}{g}\sin\alpha\right)+\sqrt{\left(-\frac{v}{g}\sin\alpha\right)^2+\left(\frac{v}{g}\cos\alpha\right)^2}\right|\right)\right].$$

Simplify the right hand side of above as follows:

$$L(\alpha)=\frac{g}{2}\left[\left(\frac{v^2}{g^2}\sin\alpha+\frac{v^2}{g^2}\cos^2\alpha\ln\left|\frac{v}{g}(1+\sin\alpha)\right|\right)-\left(-\frac{v^2}{g^2}\sin\alpha+\frac{v^2}{g^2}\cos^2\alpha\ln\left|\frac{v}{g}(1-\sin\alpha)\right|\right)\right]$$

$$=\frac{g}{2}\left[\frac{v^2}{g^2}\sin\alpha+\frac{v^2}{g^2}\cos^2\alpha\ln\left|\frac{v}{g}(1+\sin\alpha)\right|+\frac{v^2}{g^2}\sin\alpha-\frac{v^2}{g^2}\cos^2\alpha\ln\left|\frac{v}{g}(1-\sin\alpha)\right|\right]$$

$$=\frac{g}{2}\left[\frac{2v^2}{g^2}\sin\alpha+\frac{v^2}{g^2}\cos^2\alpha\left(\ln\left|\frac{v}{g}(1+\sin\alpha)\right|-\ln\left|\frac{v}{g}(1-\sin\alpha)\right|\right)\right]$$

$$=\frac{g}{2}\cdot\frac{2v^2}{g^2}\sin\alpha+\frac{g}{2}\cdot\frac{v^2}{g^2}\cos^2\alpha\ln\left|\frac{(v/g)(1+\sin\alpha)}{(v/g)(1-\sin\alpha)}\right|$$

$$=\frac{v^2}{g}\sin\alpha+\frac{v^2}{2g}\left(\cos^2\alpha\right)\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|.$$

The objective is to maximize $L(\alpha)$ for $0\le\alpha\le\frac{\pi}{2}$. So, first find the critical points, that is, find the points where $L'(\alpha)=0$.

First find $L'(\alpha)$:

$$L'(\alpha)=\frac{d}{d\alpha}L(\alpha)$$

$$=\frac{d}{d\alpha}\left(\frac{v^2}{g}\sin\alpha+\frac{v^2}{2g}\left(\cos^2\alpha\right)\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|\right)$$

$$=\frac{v^2}{g}\cos\alpha+\frac{v^2}{2g}\left[\cos^2\alpha\cdot\frac{1-\sin\alpha}{1+\sin\alpha}\cdot\frac{2\cos\alpha}{(1-\sin\alpha)^2}-2\sin\alpha\cos\alpha\cdot\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|\right]$$

$$=\frac{v^2}{g}\cos\alpha+\frac{v^2}{2g}\left[\cos^2\alpha\cdot\frac{2\cos\alpha}{\left(1-\sin^2\alpha\right)}-2\sin\alpha\cos\alpha\cdot\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|\right]$$

$$=\frac{v^2}{g}\cos\alpha+\frac{v^2}{2g}\left[\cos^2\alpha\cdot\frac{2\cos\alpha}{\cos^2\alpha}-2\sin\alpha\cos\alpha\cdot\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|\right]$$

$$=\frac{v^2}{g}\cos\alpha+\frac{v^2}{2g}\left[2\cos\alpha-2\sin\alpha\cos\alpha\cdot\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|\right]$$

$$=\frac{v^2}{g}\cos\alpha+\frac{v^2}{g}\cos\alpha\left[1-\sin\alpha\cdot\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|\right]$$

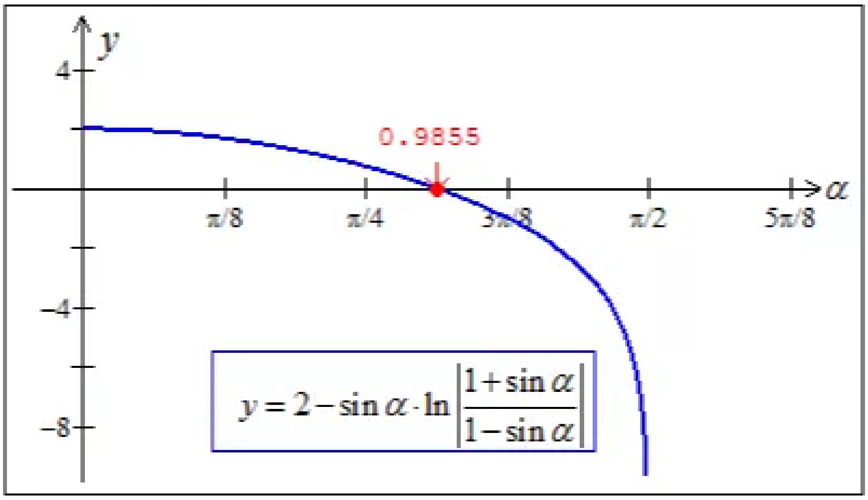
$$=\frac{v^2}{g}\cos\alpha\left(2-\sin\alpha\cdot\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|\right).$$

Solve the equation $L'(\alpha)=0$, and find the value of α.

$$\frac{v^2}{g}\cos\alpha\left(2-\sin\alpha\cdot\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|\right)=0, \text{ or}$$

$$2-\sin\alpha\cdot\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|=0, \text{ because } \cos\alpha\neq 0.$$

Use graphing utility to sketch the graph of $y=2-\sin\alpha\cdot\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|$, and find the points, at which the graph crosses α-axis to find the solution of above equation. See Figure-1.



From figure, it is clear that, the graph crosses the α-axis at $\alpha\approx 0.9855$, or $\alpha\approx 56^\circ$.

So, the critical point is $\alpha\approx 0.9855$.

Compare values of $L(\alpha)$ at the critical point $\alpha\approx 0.9855$ and the endpoints $\alpha=0,\frac{\pi}{2}$:

Because,

$$L(\alpha)=\frac{v^2}{g}\sin\alpha+\frac{v^2}{2g}\left(\cos^2\alpha\right)\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|,$$

so,

$$L(0)=0,\ L\left(\frac{\pi}{2}\right)=\frac{v^2}{g},\ L(0.9855)\approx\frac{1.2v^2}{g}.$$

It gives the maximum value for $\alpha\approx 0.9855$.

Therefore, the distance travelled by the projectile is maximized for $\alpha\approx 0.9855$, or $\boxed{\alpha\approx 56^\circ}$.

Answer 5TFQ.

The given statement is **false**.

We know that if **u** and **v** are two differentiable vector functions, then the derivative of $\mathbf{u}(t)\times\mathbf{v}(t)$ is given by $\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$.

Therefore, we can say that $\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] \neq \mathbf{u}'(t)\times\mathbf{v}'(t)$.

Answer 6CC.

(a) Let C be a plane curve. Then, the curvature of C at a point is a measure of how sensitive its tangent line is to moving the point to other nearby points. There are a number of equivalent ways that this idea can be made precise.

The curvature of a circle of radius R should be large if R is small and small if R is large. Thus the curvature of a circle is defined to be the reciprocal of the radius $\kappa = \frac{1}{R}$.

(b) It is known that the curvature s easier to compute if it is expressed in terms of the parameter t. If $\mathbf{T}(t)$ is the unit tangent vector of the vector $\mathbf{r}(t)$, then the curvature is given by $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$.

(c) If C is a smooth curve given by $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$, then the curvature $\kappa(t)$ of C at t is given by $\kappa(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$.

(d) If C is a smooth curve given by the function $y=f(x)$, then the curvature κ of C at t is given by $\kappa = \frac{|f''(x)|}{\left[1+\left(f'(x)\right)^2\right]^{3/2}}$.

Answer 6E.

The given curve is

$$x = 2-t^3, \qquad y = 2t-1, \qquad z = \ln t$$

(a). Now where the curve meets xz – plane, there $y = 0$

i.e. $2t - 1 = 0$

i.e. $t = \frac{1}{2}$

$x = 2 - t^3$

$=2-\left(\frac{1}{2}\right)^3$

$= \frac{15}{8}$

$y = 2t - 1$

$=0$

$z = \ln t$

$=\ln\left(\frac{1}{2}\right)$

$=-\ln 2$

Therefore C meets xz – plane in point

$$\left(\frac{15}{8}, 0, -\ln 2\right)$$

(b). The vector equation of curve is

$$\vec{r}(t) = <2 - t^3, 2t - 1, \ln t>$$

Then $\vec{r}'(t) = <-3t^2, 2, \frac{1}{t}>$

Now the point (1, 1, 0) corresponds to the parameter $t = 1$ as

$<2 - t^3, 2t - 1, \ln t> = \langle 1,1,0 \rangle \Rightarrow 2 - t^3 = 1, 2t - 1 = 1, \ln t = 0$

Then at t = 1, $\vec{r}'(t) = <-3, 2, 1> = \langle a, b, c \rangle$, which are the direction numbers of tangent line at (1, 1, 0)

Hence the parametric equations of the tangent line at $(x_1, y_1, z_1) =$ (1, 1, 0) is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}, \ \frac{z - z_1}{c} = t$$

$$i.e., \ \frac{x - 1}{-3} = \frac{y - 1}{2}, \ \frac{z - 0}{1} = t$$

i.e. $x = 1 - 3t, \ \ y = 2t + 1, \ z = t$

(c). The normal plane at $(x_1, y_1, z_1) =$ (1, 1, 0) has normal vector $\vec{r}'(1) = <-3, 2, 1> = \langle a, b, c \rangle$, so the equations of normal plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

i.e., $-3(x - 1) + 2(y - 1) + 1(z - 0) = 0$

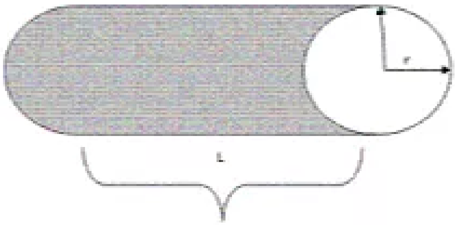
i.e., $-3x + 3 + 2y - 2 + z = 0$

i.e., $-3x + 2y + z + 1 = 0$

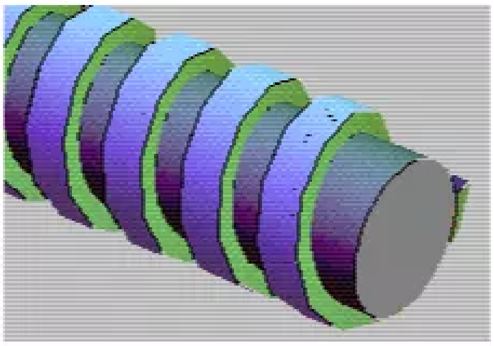
i.e., $3x - 2y - z - 1 = 0$

Answer 6P.

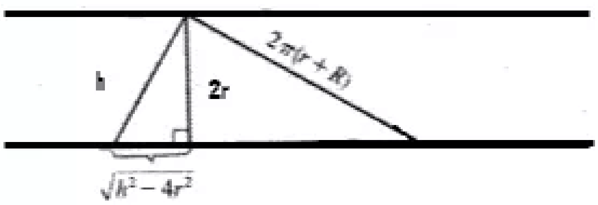
A cable has radius r and length L



the cable is wound around a spool with radius R without overlapping.



To construct a diagram for the datas, let h be the vertical distance between coils and we imagine that the top of the cable form a kind of helix of radius R + r.



From similar triangles

$$\frac{2r}{\sqrt{h^2-4r^2}} = \frac{2\pi(r+R)}{h}$$

$$\text{hr} = \pi\left(r+R\right)\sqrt{h^2-4r^2}$$

$$(\text{hr})^2 = \left(\pi\left(r+R\right)\sqrt{h^2-4r^2}\right)^2$$

$$h^2r^2 = \pi^2(r+R)^2\left(h^2-4r^2\right)$$

$$h^2(r^2-\pi^2(r+R)^2\Big) = -4\pi^2(r+R)^2r^2$$

$$h^2 = \frac{-4\pi^2r^2(r+R)^2}{\left(r^2-\pi^2(r+R)^2\right)}$$

$$h^2 = \frac{4\pi^2 r^2 (r+R)^2}{\left(-r^2 + \pi^2 (r+R)^2\right)}$$

$$h = \frac{2\pi\, r\, (r+R)}{\sqrt{\pi^2 (r+R)^2 - r^2}}$$

We parametrixe the helix using

$x(t) = (R + r)\text{cost}$, $y(t) = (R + r)\text{sint}$, then we have that $z\left(t\right) = \frac{h}{2\pi} t$

and

$x'(t) = -(R + r)\text{sint}$, $y'(t) = (R + r)\text{cost}$, then we have that $z'\left(t\right) = \frac{h}{2\pi}$

Now we calculate the length for a complete cycle

$$l = \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\ \mathrm{d}\,t$$

$$l = \int_0^{2\pi} \sqrt{[-(r+R)\text{cost}]^2 + [(r+R)\text{sint}]^2 + \left[\frac{h}{2\pi}\right]^2}\ \mathrm{d}\,t$$

$$l = \int_0^{2\pi} \sqrt{(r+R)^2 \left(\text{cost}^2 + \text{sint}^2\right) + \left[\frac{h}{2\pi}\right]^2}\ \mathrm{d}\,t$$

$$l = \int_0^{2\pi} \sqrt{[R+r]^2 + \left[\frac{h}{2\pi}\right]^2}\ \mathrm{d}\,t$$

$$l = \left[\sqrt{[R+r]^2 + \left[\frac{h}{2\pi}\right]^2} \; t \right]_0^{2\pi}$$

$$l = 2\pi \sqrt{[R+r]^2 + \left[\frac{h}{2\pi}\right]^2}$$

Using the value of h, $h = \dfrac{2\pi\, r\,(r+R)}{\sqrt{\pi^2(r+R)^2 - r^2}}$

$$l = 2\pi \sqrt{[R+r]^2 + \left[\frac{2\pi r(r+R)}{2\pi\sqrt{\pi^2(r+R)^2 - r^2}}\right]^2}$$

$$l = 2\pi \sqrt{[R+r]^2 \left[1 + \frac{r^2}{\pi^2(R+r)^2 - r^2}\right]}$$

$$l = \frac{2\pi^2(R+r)^2}{\sqrt{\pi^2(R+r)^2 - r^2}}$$

The number of complete cylcles is given by [[L / 1]] , so we get that the shortest length along the spool that is covered by the cable is

$$h\left[\left[\frac{L}{l}\right]\right] = \frac{2\pi\, r\,(r+R)}{\sqrt{\pi^2(r+R)^2 - r^2}} \left[\left[L \div \frac{2\pi^2(R+r)^2}{\sqrt{\pi^2(R+r)^2 - r^2}}\right]\right]$$

$$h\left[\left[\frac{L}{l}\right]\right] = \frac{2\pi\, r\,(r+R)}{\sqrt{\pi^2(r+R)^2 - r^2}} \left[\left[\frac{L\sqrt{\pi^2(R+r)^2 - r^2}}{2\pi^2(R+r)^2}\right]\right]$$

6TFQ.

The given statement is **false**.

Let $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$. Then, $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$ and $|\mathbf{r}'(t)| = \sqrt{1+4t^2}$.

We have $|\mathbf{r}(t)| = \sqrt{t^2+t^4}$.

Now, find $\frac{d}{dt}|\mathbf{r}(t)|$.

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{d}{dt}\left(\sqrt{t^2+t^4}\right)$$
$$= \frac{1}{2}\frac{2t+4t^3}{\sqrt{t^2+t^4}}$$

We note that $\frac{d}{dt}|\mathbf{r}(t)| \neq |\mathbf{r}'(t)|$.

Therefore, we can say that $\frac{d}{dt}|\mathbf{r}(t)| \neq |\mathbf{r}'(t)|$.

<**Answer 7CC.**

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(b) The plane determined by the normal and binormal vectors **N** and **B** at a point P on a curve C is called the normal plane of C at P. It consists of all lines that are orthogonal to the tangent vector **T**. The plane determined by the vectors **T** and **N** is called the osculating plane of C at P.

The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C and has radius $\rho = 1/\kappa$ is called the osculating circle.

Answer 7E.

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Answer 7P.

To find out whether this curve is in a plane, see if any arbitrary point on the curve will satisfy the same plane equation. There are several different ways to do this; we choose to work with the parametric equation for a plane,

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0$$

Where $\mathbf{n}$ is a vector normal to the plane and $\mathbf{r}_0$ is the position vector for point on the plane.

If we can find a normal vector $\mathbf{n}$ such that any point on the curve satisfies this plane equation, then every point on the curve is in that plane. Our strategy is to assume the curve is in a plane, find the normal vector to that plane, and then prove that that plane contains every other point on the curve.

If the curve is in a plane, then any three points on the curve are coplanar and can be used to find the normal vector to the plane. We find three points on the curve by plugging in $t=0$, $t=1$, and $t=-1$.

For $t=0$:

$$a_1(0)^2+b_1(0)+c_1=c_1$$

$$a_2(0)^2+b_2(0)+c_2=c_2$$

$$a_3(0)^2+b_3(0)+c_3=c_3$$

For $t=1$:

$$a_1(1)^2+b_1(1)+c_1=a_1+b_1+c_1$$

$$a_2(1)^2+b_2(1)+c_2=a_2+b_2+c_2$$

$$a_3(1)^2+b_3(1)+c_3=a_3+b_3+c_3$$

For $t=-1$:

$$a_1(-1)^2+b_1(-1)+c_1=a_1-b_1+c_1$$

$$a_2(-1)^2+b_2(-1)+c_2=a_2-b_2+c_2$$

$$a_3(-1)^2+b_3(-1)+c_3=a_3-b_3+c_3$$

So we have three points, which we can write as their position vectors as $\mathbf{u}=\langle c_1,c_2,c_3\rangle$, $\mathbf{v}=\langle a_1+b_1+c_1,a_2+b_2+c_2,a_3+b_3+c_3\rangle$, and $\mathbf{w}=\langle a_1-b_1+c_1,a_2-b_2+c_2,a_3-b_3+c_3\rangle$.

We use these three points to find two displacement vectors in the plane by subtracting their coordinates:

$$\begin{aligned}\mathbf{p}&=\mathbf{v}-\mathbf{w}\\&=(2b_1,2b_2,2b_3)\end{aligned}$$

$$\begin{aligned}\mathbf{q}&=\mathbf{v}-\mathbf{u}\\&=(a_1+b_1,a_2+b_2,a_3+b_3)\end{aligned}$$

We now have two vectors in the plane. In order to find a normal vector to the plane, we take the cross product of these two vectors, as the cross product of two vectors is always perpendicular to both of them, i.e., perpendicular to the plane those two vectors are in, which is exactly what we want.

Find the cross product by using a determinant. The cross product of two vectors $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$ can be written as the determinant

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$$

So the cross product of our two displacement vectors $\mathbf{p}$ and $\mathbf{q}$ is

$$\mathbf{p}\times\mathbf{q}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2b_1 & 2b_2 & 2b_3\\ a_1+b_1 & a_2+b_2 & a_3+b_3\end{vmatrix}$$

$$\begin{aligned}&=[(2b_2a_3+2b_2b_3)-(2b_3a_2+2b_3b_2)]\mathbf{i}\\ &-[(2b_1a_3+2b_1b_3)-(2b_3a_1+2b_3b_1)]\mathbf{j}\\ &+[(2b_1a_2+2b_1b_2)-(2b_2a_1+2b_2b_1)]\mathbf{k}\end{aligned}$$

$$=(2b_2a_3-2b_3a_2)\mathbf{i}-(2b_1a_3-2b_3a_1)\mathbf{j}+(2b_1a_2-2b_2a_1)\mathbf{k}$$

$$=(2b_2a_3-2b_3a_2)\mathbf{i}+(2b_3a_1-2b_1a_3)\mathbf{j}+(2b_1a_2-2b_2a_1)\mathbf{k}$$

$$=(2b_2a_3-2b_3a_2)\mathbf{i}-(2b_1a_3-2b_3a_1)\mathbf{j}+(2b_1a_2-2b_2a_1)\mathbf{k}$$

$$=(2b_2a_3-2b_3a_2)\mathbf{i}+(2b_3a_1-2b_1a_3)\mathbf{j}+(2b_1a_2-2b_2a_1)\mathbf{k}$$

This vector is a normal vector to the plane. Because constant multiples of vectors lie in the same direction as that vector, we can divide through by 2 and still have a vector in the same direction (normal to the plane). We call this our normal vector:

$\mathbf{n}=(b_2a_3-b_3a_2)\mathbf{i}+(b_3a_1-b_1a_3)\mathbf{j}+(b_1a_2-b_2a_1)\mathbf{k}$ …… (1)

If the curve is in a plane, this vector is normal to that plane.

Now that we have a normal vector, we can write an equation for the plane. Going back to the points we found in the course of finding the normal vector, we choose $\mathbf{u}=\langle c_1,c_2,c_3\rangle$, when $t=0$, as the position vector for a specific point on the plane, and plug this and the normal vector into the parametric plane equation.

$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0)=0$

$\boxed{\langle(b_2a_3-b_3a_2),(b_3a_1-b_1a_3),(b_1a_2-b_2a_1)\rangle\cdot(\mathbf{r}-\langle c_1,c_2,c_3\rangle)=0}$ …… (1)

If the curve is in a plane, this is its parametric equation.

But now we have to prove that for any point t, the point on the curve given by $\mathbf{r}(t)$ satisfies the equation given in (1). We do this by plugging in an arbitrary point $\mathbf{r}(t)=\langle a_1t^2+b_1t+c_1,a_2t^2+b_2t+c_2,a_3t^2+b_3t+c_3\rangle$ and showing that it satisfies this equation. We plug it in and show that the dot product in (1) equals 0 no matter what t we are at.

The formula for calculating the dot product of two vectors $\mathbf{a}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{b}=\langle b_1,b_2,b_3\rangle$ is to multiply component-wise and add the products:

$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$

So plugging an arbitrary $\mathbf{r}(t)$ into (1) and calculating the dot product gives us:

$$\langle (b_2a_3 - b_3a_2), (b_3a_1 - b_1a_3), (b_1a_2 - b_2a_1)\rangle \cdot \left(\langle a_1t^2 + b_1t + c_1, a_2t^2 + b_2t + c_2, a_3t^2 + b_3t + c_3\rangle - \langle c_1, c_2, c_3\rangle\right)$$

$$= \langle (b_2a_3 - b_3a_2), (b_3a_1 - b_1a_3), (b_1a_2 - b_2a_1)\rangle \cdot \left\langle (a_1t^2 + b_1t), (a_2t^2 + b_2t), (a_3t^2 + b_3t)\right\rangle$$

$$= (b_2a_3 - b_3a_2)(a_1t^2 + b_1t) + (b_3a_1 - b_1a_3)(a_2t^2 + b_2t) + (b_1a_2 - b_2a_1)(a_3t^2 + b_3t)$$

$$= b_2a_3a_1t^2 + b_2a_3b_1t - b_3a_2a_1t^2 - b_3a_2b_1t + b_3a_1a_2t^2 + b_3a_1b_2t$$

$$- b_1a_3a_2t^2 - b_1a_3b_2t + b_1a_2a_3t^2 + b_1a_2b_3t - b_2a_1a_3t^2 - b_2a_1b_3t$$

$$= 0$$

In the second-to-last line, the color coding shows the matching terms that cancel each other out and the dot product indeed equals 0, as desired. We have just shown that any arbitrary point $\mathbf{r}\,(t)$ satisfies this plane equation.

Since any point $\mathbf{r}\,(t)$ satisfies equation (1), it is indeed an equation for the plane containing this curve.

Answer 7TFQ.

The given statement is **false**.

The curvature of a curve C at a given point can be defined as the rate of change of the unit tangent vector $\mathbf{T}\,(t)$ with respect to the arc length $s\,(t)$. Thus, we get $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$.

Therefore, we can say that $\kappa \neq \left|\frac{d\mathbf{T}}{dt}\right|$.

Answer 8CC.

(a) If x and y are twice differentiable functions of t, and $\mathbf{r}$ is a vector-valued function given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then the velocity vector is given by
$\mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

The speed of the object is given by $\|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$.

The acceleration is given by $\mathbf{a}(t) = \mathbf{r}'(t)$ or $\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$.

(b) We have the tangential component of acceleration as $\mathbf{a_T}$ and the normal components of acceleration as $\mathbf{a_N}$. Then, the acceleration is given by
$\mathbf{a} = \mathbf{a_T} + \mathbf{a_N}$.

Answer 8E.

$$\vec{r}(t) = \langle 2t^{3/2}, \cos 2t, \sin 2t \rangle$$

Then $\vec{r}\,'(t) = \langle 3t^{1/2}, -2\sin 2t, 2\cos 2t \rangle$

And $\left|\vec{r}\,'(t)\right| = \sqrt{\left(3t^{1/2}\right)^2 + (-2\sin 2t)^2 + (2\cos 2t)^2}$

$$= \sqrt{9t + 4\sin^2 2t + 4\cos^2 2t}$$

$$= \sqrt{9t + 4}$$

Answer 8TFQ.

The given statement is **false**.

We know that the binomial vector $\mathbf{B}(t)$ is perpendicular to both the tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$. Then, $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$. Since cross multiplication of vectors is not commutative, we can say that $\mathbf{B}(t) = \mathbf{N}(t) \times \mathbf{T}(t)$.

Answer 9CC.

The Kepler's law of planetary motion includes three parts:

a) The orbit of every planet is an ellipse with the Sun at one of the two foci.

b) A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

c) The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

Answer 9E.

The equation of helix is

$$\vec{r}_1(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$$

Then $\vec{r}_1'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$

And $\left|\vec{r}_1'(t)\right| = \sqrt{\sin^2 t + \cos^2 t + 1}$

$$= \sqrt{2}$$

Then the equation of tangent is

$$\vec{T}_1(t) = \frac{\vec{r}_1'(t)}{\left|\vec{r}_1'(t)\right|}$$

$$= < \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$$

The point (1, 0, 0) corresponds to parameter $t = 0$

Then the tangent at (1, 0, 0) is

$$\vec{T}_1(0) = < 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$$

The equation of curve is

$$\vec{r}_2(t) = (1+t)\hat{i} + t^2\hat{j} + t^3\hat{k}$$

Then $\vec{r}_2'(t) = \hat{i} + 2t\,\hat{j} + 3t^2\hat{k}$

And $\left|\vec{r}_2'(t)\right| = \sqrt{1 + 4t^2 + 9t^4}$

The point (1, 0, 0) corresponds to parameter $t = 0$

The equation of tangent to curve is

$$\vec{T}_2(t) = \frac{\vec{r}_2'(t)}{\left|\vec{r}_2'(t)\right|}$$

Then tangent at (1, 0, 0) is

$$\vec{T}_2(0) = < 1, 0, 0 >$$

Now the angle between the two curves are equal to the angle between their respective tange4nts at the point of inter section, let it be θ

Then $\cos\theta = \dfrac{\vec{T}_1(0).\vec{T}_2(0)}{\left|\vec{T}_1(0)\right|\left|\vec{T}_2(0)\right|}$

$= \dfrac{<0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} > . <1,0,0>}{\sqrt{1}.\sqrt{1}}$

$\cos\theta = 0$

i.e. $\theta = \cos^{-1}(0)$

i.e. $\theta = \boxed{\dfrac{\pi}{2}}$

Answer 9TFQ.

The given statement is **true**.

We know that if f is twice continuously differentiable, then at an inflection point of the curve $y = f(x)$ the curvature is 0.

Answer 10E.

The equation of curve is

$\vec{r}(t) = e^t\hat{i} + e^t \sin t\,\hat{j} + e^t \cos t\,\hat{k}$

Then $\vec{r}\,'(t) = e^t\hat{i} + e^t(\sin t + \cos t)\hat{j} + e^t(\cos t - \sin t)\hat{k}$

And $\left|\vec{r}\,'(t)\right| = \sqrt{e^{2t} + e^{2t}(\sin t + \cos t)^2 + e^{2t}(\cos t - \sin t)^2}$

$= \sqrt{e^{2t} + e^{2t} + e^{2t}}$

$= \sqrt{3e^{2t}}$

$= e^t\sqrt{3}$

Now $\dfrac{ds}{dt} = \left|\vec{r}\,'(t)\right| = \sqrt{3}e^t$

The point (1, 0, 1) corresponds to parameter $t = 0$

Then $s = s(t) = \int_0^t \left|\vec{r}\,'(u)\right| du$

$= \int_0^t \sqrt{3}e^4 du$

$= \sqrt{3}\left[e^4\right]_0^t$

$= \sqrt{3}\left[e^t - 1\right]$

Then $t=\ln\left(\frac{s}{\sqrt{3}}+1\right)$

Or $t=\ln\left(s+\sqrt{3}\right)-\frac{1}{2}\ln 3$

Therefore $t=\ln\left(\frac{s}{\sqrt{3}}+1\right)$

Hence the required re parameterization is obtained by substituting for t

$$\vec{r}(t(s))=e^{\ln\left(\frac{s}{\sqrt{3}}+1\right)}\hat{i}+e^{\ln\left(\frac{s}{\sqrt{3}}+1\right)}\sin\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right)\hat{j}+$$
$$e^{\ln\left(\frac{s}{\sqrt{3}}+1\right)}\cos\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right)\hat{k}$$

i.e.
$$\vec{r}(t(s))=\left(1+\frac{s}{\sqrt{3}}\right)\hat{i}+\left(1+\frac{s}{\sqrt{3}}\right)\sin\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right)\hat{j}$$
$$+\left(1+\frac{s}{\sqrt{3}}\right)\cos\left(\ln\left(\frac{s}{\sqrt{3}}+1\right)\right)\hat{k}$$

Answer 10TFQ.

The given statement is **true**.

We know that the curvature of a straight line is always zero because the tangent vector is a constant.

Therefore, we can say that a curve with $\kappa(t)=0$ represents a straight line.

Answer 11E.

Consider the following curve:

$$\mathbf{r}(t)=\left\langle \frac{1}{3}t^3,\frac{1}{2}t^2,t\right\rangle$$

a)

Find the unit tangent vector.

Recollect the unit tangent vector.

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Differentiate of $\mathbf{r}(t)$ with respect to t.

$$\begin{aligned}\mathbf{r}'(t)&=\frac{d}{dt}\left\langle \frac{1}{3}t^3,\frac{1}{2}t^2,t\right\rangle\\&=\left\langle \frac{1}{3}\frac{d}{dt}t^3,\frac{1}{2}\frac{d}{dt}t^2,\frac{d}{dt}t\right\rangle\\&=\left\langle \frac{1}{3}(3t^2),\frac{1}{2}(2t),1\right\rangle\\&=\left\langle t^2,t,1\right\rangle\end{aligned}$$

Find the magnitude of the vector $\mathbf{r}'(t)$.

$$|\mathbf{r}'(t)| = \sqrt{(t^2)^2 + t^2 + 1}$$
$$= \sqrt{t^4 + t^2 + 1}$$

The unit tangent vector is follows:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$
$$= \frac{\langle t^2, t, 1\rangle}{\sqrt{t^4 + t^2 + 1}}$$

Thus, the unit tangent vector is $\boxed{\mathbf{T}(t) = \frac{\langle t^2, t, 1\rangle}{\sqrt{t^4 + t^2 + 1}}}$.

Find the unit normal vector.

Recollect the formula:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

The derivative of the tangent vector to obtain,

$$\mathbf{T}'(t) = \frac{d}{dt}\left[\langle t^2, t, 1\rangle (t^4 + t^2 + 1)^{-\frac{1}{2}}\right]$$
$$= \langle t^2, t, 1\rangle \frac{d}{dt}(t^4 + t^2 + 1)^{-\frac{1}{2}} + (t^4 + t^2 + 1)^{-\frac{1}{2}} \frac{d}{dt}\langle t^2, t, 1\rangle$$
$$= -\frac{1}{2}(t^4 + t^2 + 1)^{-3/2}(4t^3 + 2t)\langle t^2, t, 1\rangle + (t^4 + t^2 + 1)^{-1/2}\langle 2t, 1, 0\rangle$$
$$= \frac{-2t^3 - t}{(t^4 + t^2 + 1)^{3/2}} \cdot \langle t^2, t, 1\rangle + \frac{1}{(t^4 + t^2 + 1)^{1/2}}\langle 2t, 1, 0\rangle$$
$$= \frac{-2t^3 - t}{(t^4 + t^2 + 1)^{3/2}} \cdot \langle t^2, t, 1\rangle + \frac{(t^4 + t^2 + 1)}{(t^4 + t^2 + 1)^{3/2}}\langle 2t, 1, 0\rangle$$
$$= \frac{\langle -2t^5 - t^3, -2t^4 - t^2, -2t^3 - t\rangle}{(t^4 + t^2 + 1)^{3/2}} + \frac{\langle 2t^5 + 2t^3 + 2t, t^4 + t^2 + 1, 0\rangle}{(t^4 + t^2 + 1)^{3/2}}$$
$$= \frac{\langle -2t^5 - t^3 + 2t^5 + 2t^3 + 2t, -2t^4 - t^2 + t^4 + t^2 + 1, -2t^3 - t\rangle}{(t^4 + t^2 + 1)^{3/2}}$$

$$=\frac{\langle t^3+2t,\ -t^4+1,\ -2t^3-t\rangle}{\left(t^4+t^2+1\right)^{3/2}}$$

To find the magnitude of the tangent vector to obtain,

$$\left|\vec{T}'(t)\right|=\sqrt{\frac{\left(t^3+2t\right)^2}{\left(t^4+t^2+1\right)^3}+\frac{\left(-t^4+1\right)^2}{\left(t^4+t^2+1\right)^3}+\frac{\left(2t^3+t\right)^2}{\left(t^4+t^2+1\right)^3}}$$

$$=\sqrt{\frac{t^6+4t^2+4t^4+t^3+1-2t^4+4t^6+t^2+4t^4}{\left(t^4+t^2+1\right)^3}}$$

$$=\sqrt{\frac{t^8+5t^6+6t^4+5t^2+1}{\left(t^4+t^2+1\right)^3}}$$

The normal vector will be,

$$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{\left|\mathbf{T}'(t)\right|}$$

$$=\frac{\langle t^3+2t,-t^4+1,-2t^3-t\rangle}{\left(t^4+t^2+1\right)^{3/2}}\cdot\frac{\left(t^4+t^2+1\right)^{3/2}}{\sqrt{t^8+5t^6+6t^4+5t^2+1}}$$

$$=\boxed{\frac{\langle t^3+2t,-t^4+1,-2t^3-t\rangle}{\sqrt{t^8+5t^6+6t^4+5t^2+1}}}$$

c)

Find the curvature.

Recollect the curvature formula.

$$\kappa=\frac{\left|\mathbf{T}'(t)\right|}{\left|\mathbf{r}'(t)\right|}$$

Evaluate the curvature is shown below:

$$\kappa=\frac{\left|\mathbf{T}'(t)\right|}{\left|\mathbf{r}'(t)\right|}$$

$$=\frac{\sqrt{t^8+5t^6+6t^4+5t^2+1}}{\left(t^4+t^2+1\right)^{3/2}}\cdot\frac{1}{\sqrt{t^4+t^2+1}}$$

$$=\boxed{\frac{\sqrt{t^8+5t^6+6t^4+5t^2+1}}{\left(t^4+t^2+1\right)^2}}.$$

Answer 11TFQ.

The given statement is **false**.

Let $\mathbf{r}(t) = t\mathbf{i} + \sqrt{1-t^2}\mathbf{j}$ and find $|\mathbf{r}(t)|$.

$$\begin{aligned}|\mathbf{r}(t)| &= \sqrt{(t)^2 + \left(1-t^2\right)^2} \\ &= \sqrt{t^2+1-t^2} \\ &= 1\end{aligned}$$

Now, we have $\mathbf{r}'(t) = \mathbf{i} - \dfrac{t}{\sqrt{1-t^2}}\mathbf{j}$.

Determine $|\mathbf{r}'(t)|$.

$$\begin{aligned}|\mathbf{r}'(t)| &= \sqrt{1^2 + \left(\frac{t}{\sqrt{1-t^2}}\right)^2} \\ &= \sqrt{1^2 + \frac{t^2}{1-t^2}} \\ &= \sqrt{\frac{1}{1-t^2}}\end{aligned}$$

We have $|\mathbf{r}'(t)| = \sqrt{\dfrac{1}{1-t^2}}$.

Therefore, we can say that if $|\mathbf{r}(t)| = 1$, then it is not necessary that $|\mathbf{r}'(t)|$ be a constant.

Answer 12E.

The parametric equations of ellipse are:

$$x = 3\cos t,\quad y = 4\sin t$$

Then vector equation is

$$\vec{r}(t) = <3\cos t, 4\sin t>$$

Then $\quad \vec{r}\,'(t) = <-3\sin t, 4\cos t>$

And $\quad \left|\vec{r}\,'(t)\right| = \sqrt{9\sin^2 t + 16\cos^2 t}$

$$\vec{r}\,''(t) = <-3\cos t, -4\sin t>$$

Then curvature is $k = \dfrac{\left|\vec{r}\,'(t)\times\vec{r}\,''(t)\right|}{\left|\vec{r}\,'(t)\right|^3}$

$$= \frac{12}{\left(9\sin^2 t + 16\cos^2 t\right)^{3/2}}$$

Now point (3, 0) corresponds to parameter t = 0

Then the curvature at (3, 0) is

$$k = \frac{12}{\left(9\sin^2 0 + 16\cos^2 0\right)^{3/2}}$$

i.e. $$k = \frac{12}{16^{3/2}}$$

$$= \frac{12}{4 \times 4 \times 4}$$

$$= \boxed{\frac{3}{16}}$$

Now the point (0, 4) corresponds to parameter $t = \pi/2$

Then the curvature at (0, 4) is

$$k = \frac{12}{(9+0)^{3/2}}$$

$$= \frac{12}{3 \times 3 \times 3}$$

$$= \boxed{\frac{4}{9}}$$

Answer 12TFQ.

Given $|\mathbf{r}(t)| = 1$, we can say that $|\mathbf{r}(t)|^2 = 1$.

Then, $\mathbf{r}(t) \cdot \mathbf{r}(t) = 1$.

On differentiating both sides of the equation, we get $\frac{d}{dt}\left[\mathbf{r}(t) \cdot \mathbf{r}(t)\right] = 0$.

Rewriting the equation we get

$$2\frac{d\left[\mathbf{r}(t)\right]}{dt} \cdot \mathbf{r}(t) = 0$$

$$2\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$$

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

Therefore, we can say that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Thus the given statement is **true**.

Answer 13E.

The given curve is $y = x^4$

Then $y' = 4x^3$

And $y'' = 12x^2$

The curvature is given by

$$k = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}}$$

$$= \frac{|12x^2|}{\left(1+16x^6\right)^{3/2}}$$

At point (1, 1), the curvature is

$$k = \frac{12}{(1+16)^{3/2}}$$

$$= \boxed{\frac{12}{(17)^{3/2}}}$$

Answer 13TFQ.

The given statement is **true**.

The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which $\mathbf{N}$ points), and has radius $\rho = \frac{1}{\kappa}$ is called the osculating circle of C at P. It is the circle that best describes how C behaves near P as it shares the same tangent, normal, and curvature at P.

Answer 14E.

The given curve is

$$y = x^4 - x^2$$

Then $y' = 4x^3 - 2x$

$$y'' = 12x^2 - 2$$

The radius of curvature is

$$k(x) = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}}$$

$$= \frac{|12x^2 - 2|}{\left[1+\left(4x^3-2x\right)^2\right]^{3/2}}$$

At origin $k(0) = \frac{|-2|}{[1+0]^{3/2}} = 2$

Then the radius of osculating circle at (0, 0) is

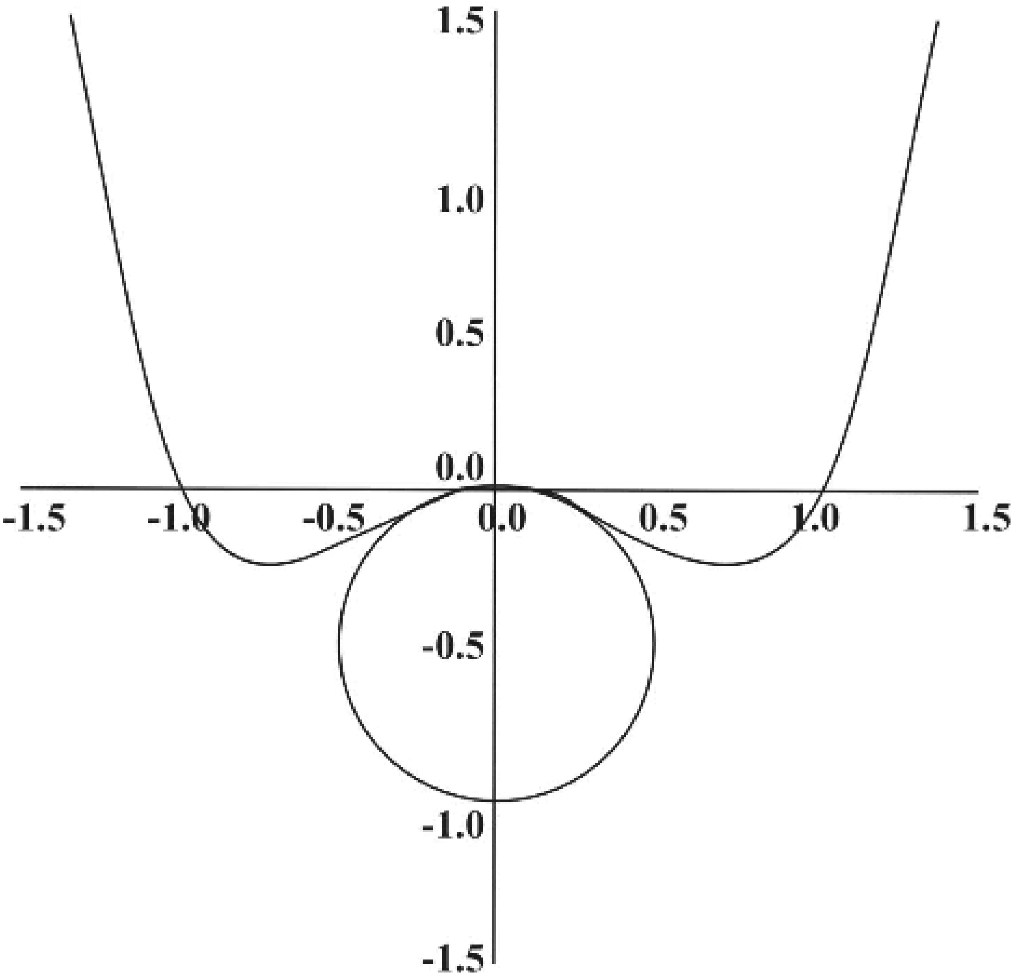
$$\rho = \frac{1}{k(0)} = \frac{1}{2}$$

From graph the centre of the circle is: $\left(0, \dfrac{-1}{2}\right)$

Then the equation of osculating circle at (0, 0) is

$$(x-0)^2 + \left(y+\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

i.e. $\boxed{x^2 + \left(y+\frac{1}{2}\right)^2 = \frac{1}{4}}$



Answer 14TFQ.

Suppose that $x = r(t)$ and $r'(t) = r'(s)$ be two parameterization of the same curve then a point on the curve can be represented by $t = t_0$ and $s = s_1$.

But, the tangent vector is going to be the same since only one tangent vector can be drawn to a curve.

The given statement is **true**.

Answer 15E.

The given equation of curve is

$$x = \sin 2t,\ \ y = t,\ z = \cos 2t$$

Then the vector equation is

$$\vec{r}(t) = < \sin 2t, t, \cos 2t >$$

$$\vec{r}\,'(t) = < 2\cos 2t, 1, -2\sin 2t >$$

$$\left|\vec{r}\,'(t)\right| = \sqrt{4\cos^2 2t + 1 + 4\sin^2 2t}$$

$$= \sqrt{5}$$

The unit normal vector is

$$\vec{T}(t) = \vec{r}\,'(t) \Big/ \left|\vec{r}\,'(t)\right|$$

$$\vec{T}(t) = \frac{1}{\sqrt{5}} < 2\cos 2t, 1, -2\sin 2t >$$

And $\quad \vec{T}\,'(t) = \frac{1}{\sqrt{5}} < -4\sin 2t, 0, -4\cos 2t >$

Then $\quad \left|\vec{T}\,'(t)\right| = \sqrt{\frac{16}{5}\sin^2 2t + 0 + \frac{16}{5}\cos^2 2t}$

$$= \sqrt{\frac{16}{5}}$$

$$= \frac{4}{\sqrt{5}}$$

Now the point (0, π, 1) corresponds to parameter t = π

And $\quad \vec{N}(t) = \frac{\vec{T}\,'(t)}{\left|\vec{T}\,'(t)\right|}$

i.e. $\quad \vec{N}(t) = \frac{1}{4} < -4\sin 2t, 0, -4\cos 2t >$

$$= < -\sin 2t, 0, -\cos 2t >$$

The osculating plane at (0, π, 1) contains the vectors $\vec{T}$ and $\vec{N}$, so its normal vector is $\vec{T} \times \vec{N} = \vec{B}$

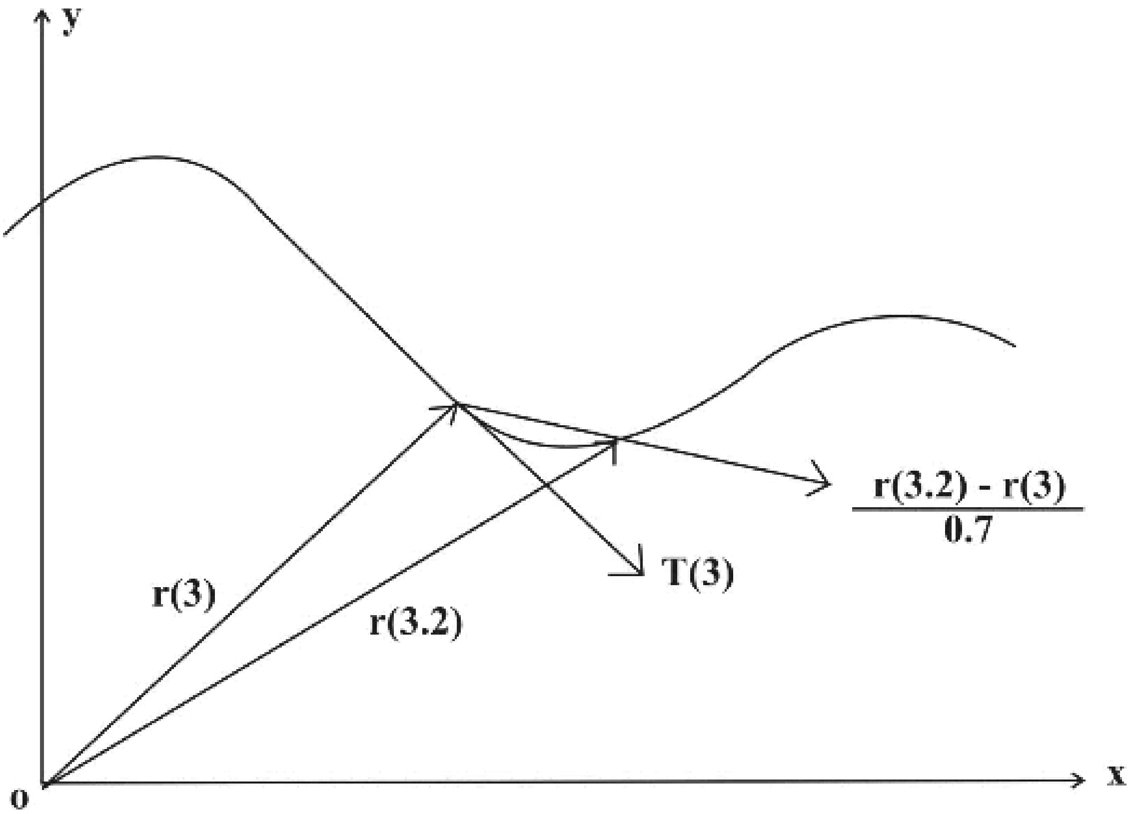
$$\vec{B}(t) = \vec{T} \times \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2\cos 2t}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{-2\sin 2t}{\sqrt{5}} \\ -\sin 2t & 0 & -\cos 2t \end{vmatrix}$$

$$= \hat{i}\left(\frac{-\cos 2t}{\sqrt{5}}\right) - \hat{j}\left(\frac{-2}{\sqrt{5}}\cos^2 2t - \frac{2}{\sqrt{5}}\sin^2 2t\right) + \hat{k}\left(\frac{\sin 2t}{\sqrt{5}}\right)$$

$$= \frac{-\cos 2t}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{j} + \frac{\sin 2t}{\sqrt{5}}\hat{k}$$

Answer 16E.

(A)



(B)

$$\vec{v}(3) = \lim_{h \to 0} \frac{\vec{r}(3+h) - \vec{r}(3)}{h}$$

(C)

$$\vec{T}(3) = \frac{\vec{r}\,'(3)}{|\vec{r}\,'(3)|}$$

Answer 17E.

The position function is

$$\vec{r}(t) = t \ln t \hat{i} + t \hat{j} + e^{-t} \hat{k}$$

Then velocity $\vec{v}(t) = \vec{r}\,'(t)$

$$= (\ln t + 1)\hat{i} + \hat{j} - e^{-t} \hat{k}$$

$$\text{Speed} = \left|\vec{v}(t)\right|$$
$$= \sqrt{(\ln t + 1)^2 + (1)^2 + (e^{-t})^2}$$
$$= \sqrt{1 + (\ln t)^2 + 2\ln t + 1 + e^{-2t}}$$
$$= \sqrt{(\ln t)^2 + 2\ln t + e^{-2t} + 2}$$

And acceleration $\vec{a}(t) = \vec{v}'(t)$

$$= \vec{r}''(t)$$
$$= \boxed{\frac{1}{t}\hat{i} + e^{-t}\hat{k}}$$

Answer 15E.

$$\vec{a}(t) = 6t\hat{i} + 12t^2\hat{j} - 6t\hat{k}$$

We know $\vec{a}(t) = \vec{v}'(t)$

Then $\vec{v}(t) = \int \vec{a}(t)\,dt$

$$= \int <6t,\ 12t^2,\ -6t> dt$$
$$= <3t^2, 4t^3, -3t^2> + \vec{C}$$

Where $\vec{C}$ is arbitrary constant vector of integration

It is given that $\vec{v}(0) = \hat{i} - \hat{j} + 3\hat{k}$

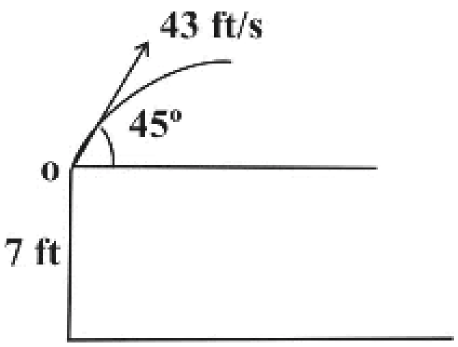
Thus $\hat{i} - \hat{j} + 3\hat{k} = 0 + \vec{C}$

Then $\vec{C} = \hat{i} - \hat{j} + 3\hat{k}$

i.e. $\vec{v}(t) = 3t^2\hat{i} + 4t^3\hat{j} - 3t^2\hat{k} + (\hat{i} - \hat{j} + 3\hat{k})$

$$\vec{v}(t) = (3t^2 + 1)\hat{i} + (4t^3 - 1)\hat{j} + (-3t^2 + 3)\hat{k}$$

Answer 19E.



We know for a projectile, the position at any time t is given by

$$\vec{r}(t) = (v_0 \cos\alpha)t\hat{i} + \left[(v_0 \sin\alpha)t - \frac{1}{2}gt^2\right]\hat{j}$$

Or $x = (v_0 \cos\alpha)t, \quad y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$

(A)

When t = 2seconds

Then $x = 43\times\cos 45°\times 2$

$= 60.81$ ft

And $y = 43\times\sin 45°\times 2 - \frac{1}{2}\times 32\times(2)^2$

$= 60.81 - 64$

$= -3.18$ ft

Then distance above the ground is

$7 + (-3.18)\,ft = 3.8$ ft

Hence after 2 seconds the shot is about 3.8 ft above the ground at 60.8 ft from the athlete

(B)

We know the maximum height of projectile is given by

$$h_{max} = \frac{(v_0 \sin\alpha)^2}{2g}$$

(Which is the maximum height above horizontal through the point 0 from where the shot is thrown and is parallel is ground)

Then above horizontal $h_{max} = \frac{(43\sin 45°)^2}{2\times 32}$

$= 14.44$ ft

Thus the height above the ground is

$(14.44 + 7)$ft

$= \boxed{21.4 \text{ ft}}$

(C)

Taking point 0 as origin we see that $y = -7$ ft

Then $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$ gives

$-7 = (43\times\sin 45°)t - \frac{1}{2}(32)t^2$

$-7 = 30.4t - 16t^2$

i.e. $16t^2 - 30.4t - 7 = 0$

i.e. $t = \frac{30.4 \pm \sqrt{(30.4)^2 + 4(16)(7)}}{2(16)}$

$= \frac{30.4 \pm 37.04}{32}$

$= -0.2s,\ 2.107s$

Then time of flight is 2.107s

Thus range of shot $= (v_0\cos\alpha)t - (43\cos 45°)(2.107) = 64.06$ft

Hence shot lands 64.06ft from the athlete

Answer 20E.

The position function of the particle is

$$r(t) = t\hat{i} + 2t\hat{j} + t^2\hat{k}$$

Then $r'(t) = \hat{i} + 2\hat{j} + 2t\hat{k}$

$$r''(t) = 2\hat{k}$$

$$|r'(t)| = \sqrt{1+4+4t^2}$$

$$= \sqrt{5+4t^2}$$

Then the tangential component of acceleration is

$$a_T = \frac{r'(t) \times r''(t)}{|r'(t)|}$$

$$= \frac{<1,2,2t>.<0,0,2>}{\sqrt{5+4t^2}}$$

i.e. $a_T = \dfrac{4t}{\sqrt{5+4t^2}}$

Since $r'(t) \times r''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2t \\ 0 & 0 & 2 \end{vmatrix}$

$$= \hat{i}(4) - \hat{j}(2) + \hat{k}(0)$$

$$= 4\hat{i} - 2\hat{j}$$

And $|r'(t) \times r''(t)| = \sqrt{16+4}$

$$= \sqrt{20}$$

$$= 2\sqrt{5}$$

Thus the normal component of acceleration is

$$a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$$

$$= \frac{2\sqrt{5}}{\sqrt{5+4t^2}}$$

i.e. $a_N = \dfrac{2\sqrt{5}}{\sqrt{5+4t^2}}$

Answer 21E.

The position vector is given by

$$r(t) = t\,R(t)$$

(A)

Then $r'(t) = R(t) + tR'(t)$ ------------------ (1)

Now $R'(t) = v_d$ (given)

And $R(t) = \cos wt\hat{i} + \sin wt\hat{j}$

Therefore $v(t) = r'(t) - \cos wt\hat{i} + \sin wt\hat{j} + t v_d$

(B)

From (1) $r'(t) = R(t) + tR'(t)$

Then $r''(t) = R'(t) + R'(t) + tR''(t)$

$= 2R'(t) + tR''(t)$

Since $R'(t) = v_d$

And $R''(t) = a_d$

Therefore acceleration $a = r''(t)$

i.e. $a = 2v_d + t a_d$

(C)

$r(t) = e^{-t}\cos wt\hat{i} + e^{-t}\sin wt\hat{j}$

$= e^{-t}\left(\cos wt\hat{i} + \sin wt\hat{j}\right)$

$= e^{-t}R(t)$

Then $r'(t) = -e^{-t}R(t) + e^{-t}R'(t)$

$= -e^{-t}R(t) + e^{-t}v_d$

And $r''(t) = e^{-t}R(t) - e^{-t}R'(t) - e^{-t}R'(t) + e^{-t}R''(t)$

$= e^{-t}R(t) - 2e^{-t}R'(t) + e^{-t}R''(t)$

Hence $a = e^{-t}R - 2e^{-t}v_d + e^{-t}a_d$

Therefore carioles acceleration is $e^{-t}R - 2e^{-t}v_d$

Answer 23E.

(a) We have $\mathbf{r}(t) = R\cos\omega t\mathbf{i} + R\sin\omega t\mathbf{j}$.

Differentiating with respect to t we have

$\mathbf{v}(t) = \mathbf{r}'(t)$

$= -R\omega\sin\omega t\mathbf{i} + R\omega\cos\omega t\mathbf{j}$

$= \omega R\left(-\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j}\right)$

Finding $\mathbf{v}(t)\cdot\mathbf{r}(t)$.

$$\begin{aligned}\mathbf{v}(t)\cdot\mathbf{r}(t) &= (-R\omega\sin\omega t\mathbf{i} + R\omega\cos\omega t\mathbf{j})\cdot(R\cos\omega t\mathbf{i} + R\sin\omega t\mathbf{j}) \\ &= -R^2\omega\sin\omega t\cos\omega t + R^2\omega\sin\omega t\cos\omega t \\ &= 0\end{aligned}$$

We can thus say that $\mathbf{v}(t)$ is perpendicular to $\mathbf{r}(t)$.
Therefore, we can conclude that $\mathbf{v}(t)$ is the tangent to the circle and points in the direction of motion.

(b) Find $|\mathbf{v}(t)|$.

$$\begin{aligned}|\mathbf{v}(t)| &= \sqrt{(-R\omega\sin\omega t)^2 + (R\omega\cos\omega t)^2} \\ &= \sqrt{R^2\omega^2\sin^2\omega t + R^2\omega^2\cos^2\omega t} \\ &= \sqrt{R^2\omega^2} \\ &= R\omega\end{aligned}$$

Since $2\pi R$ is the distance traveled in one complete revolution, we get the period as $T = \dfrac{2\pi R}{|\mathbf{v}(t)|}$.

Plug in $|\mathbf{v}(t)|$ with $R\omega$.

$$\begin{aligned}T &= \frac{2\pi R}{R\omega} \\ &= \frac{2\pi}{\omega}\end{aligned}$$

Thus, we get T as $\dfrac{2\pi}{\omega}$.

(c) Now, find $\mathbf{a}(t)$ given by $\dfrac{d}{dt}[\mathbf{v}(t)]$.

$$\begin{aligned}\mathbf{a}(t) &= \frac{d}{dt}[-R\omega\sin\omega t\mathbf{i} + R\omega\cos\omega t\mathbf{j}] \\ &= -R\omega^2\cos\omega t\mathbf{i} - R\omega^2\sin\omega t\mathbf{j} \\ &= -\omega^2(R\cos\omega t\mathbf{i} + R\sin\omega t\mathbf{j})\end{aligned}$$

We note that $\mathbf{a}(t) = -\omega^2\mathbf{r}(t)$.
Thus, we can say that $\mathbf{a}(t)$ is proportional to $\mathbf{r}(t)$ and points towards the origin.

Find $\left|\mathbf{a}(t)\right|$.

$$\begin{aligned}\left|\mathbf{a}(t)\right| &= \left|-\omega^2\right|\left|\mathbf{r}(t)\right| \\ &= \omega^2\left(\sqrt{(R\sin\omega t)^2+(R\cos\omega t)^2}\right) \\ &= \omega^2\sqrt{R^2\sin^2\omega t+R^2\cos^2\omega t} \\ &= \omega^2 R\end{aligned}$$

Therefore, we get $\boxed{\left|\mathbf{a}(t)\right| = \omega^2 R}$.

(d) We know that

$\mathbf{F} = m\mathbf{a}(t)$.

Then, we get $\mathbf{F}(t) = -m\omega^2\mathbf{r}(t)$.

Finding $\left|\mathbf{F}(t)\right|$.

$$\begin{aligned}\left|\mathbf{F}(t)\right| &= \left|-m\omega^2\right|\left|\mathbf{r}(t)\right| \\ &= m\omega^2\left(\sqrt{(R\sin\omega t)^2+(R\cos\omega t)^2}\right) \\ &= m\omega^2\sqrt{R^2\sin^2\omega t+R^2\cos^2\omega t} \\ &= m\omega^2 R\end{aligned}$$

Therefore, we get $\boxed{\left|\mathbf{F}(t)\right| = \dfrac{m\left|\mathbf{v}(t)\right|^2}{R}}$.

Answer 24E.

(a) Given equations $|F|\cos\theta = mg$ ....(1) and

$|F|\sin\theta = \dfrac{mv_R^2}{R}$ ....(2).

Divide equation (2) by equation (1).

$$\frac{|F|\sin\theta}{|F|\cos\theta} = \frac{\frac{mv_R^2}{R}}{mg}$$

$$\tan\theta = \frac{v_R^2}{gR}$$

Thus, we get $\boxed{v_R^2 = gR\tan\theta}$.

(b) Plug in g with 9.8, R with 400, and θ with $12°$ in $v_R^2 = gR\tan\theta$.

$$v_R^2 = (9.8)(400)\tan 12°$$

$$v_R = \sqrt{834.96}$$

$$= 28.89$$

Thus, we get the velocity as $\boxed{28.89 \text{ m/s}}$.

(c) It is given that v_R is increase by 50%. Then, we get $v_R = 43.299$.

Substitute 9.8 for g, v_R with 43.299, and θ with $12°$ in $v_R^2 = gR\tan\theta$.

$$(43.299)^2 = (9.8)(R)\tan 12°$$

$$R = \frac{(43.299)^2}{(9.8)\tan 12°}$$

$$\approx 900$$

Thus, we get R as approximately $\boxed{900 \text{ ft}}$.