Relations and Functions

- A relation R from a set A to a set B is a subset of $A \times B$ obtained by describing a relationship between the first element a and the second element b of the ordered pairs in $A \times B$. That is, $R \subseteq \{(a, b) \in A \times B, a \in A, b \in B\}$

- The domain of a relation R from set A to set B is the set of all first elements of the ordered pairs in R.

- The range of a relation R from set A to set B is the set of all second elements of the ordered pairs in R. The whole set B is called the co-domain of R. Range $\subseteq$ Co-domain

- A relation R in a set A is called an empty relation, if no element of A is related to any element of A. In this case, $R = \Phi \subset A \times A$

  Example: Consider a relation R in set $A = \{3, 4, 5\}$ given by $R = \{(a, b): a^b < 25$, where $a, b \in A\}$. It can be observed that no pair (a, b) satisfies this condition. Therefore, R is an empty relation.

- A relation R in a set A is called a universal relation, if each element of A is related to every element of A. In this case, $R = A \times A$

  Example: Consider a relation R in the set $A = \{1, 3, 5, 7, 9\}$ given by $R = \{(a, b): a + b$ is an even number$\}$.
  Here, we may observe that all pairs (a, b) satisfy the condition R. Therefore, R is a universal relation.

- Both the empty and the universal relation are called trivial relations.

- A relation R in a set A is called reflexive, if $(a, a) \in R$ for every $a \in R$.

  Example: Consider a relation R in the set A, where $A = \{2, 3, 4\}$, given by $R = \{(a, b): a^b = 4$, 27 or 256$\}$. Here, we may observe that $R = \{(2, 2), (3, 3)$, and $(4, 4)\}$. Since each element of R is related to itself (2 is related 2, 3 is related to 3, and 4 is related to 4), R is a reflexive relation.

- A relation R in a set A is called symmetric, if $(a_1, a_2) \in R \Rightarrow (a_2, a_1) \in R, \forall\ (a_1, a_2) \in R$

  Example: Consider a relation R in the set A, where A is the set of natural numbers, given by $R = \{(a, b): 2 \le ab < 20\}$. Here, it can be observed that $(b, a) \in R$ since $2 \le ba < 20$ [since for natural numbers a and b, $ab = ba$]
  Therefore, the relation R is symmetric.

- A relation R in a set A is called transitive, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R \Rightarrow (a_1, a_3) \in R$ for all $a_1, a_2, a_3 \in A$

  Example: Let us consider a relation R in the set of all subsets with respect to a universal set U given by $R = \{(A, B): A$ is a subset of $B\}$
  Now, if A, B, and C are three sets in R, such that $A \subset B$ and $B \subset C$, then we also have $A \subset C$. Therefore, the relation R is a symmetric relation.

- A relation R in a set A is said to be an equivalence relation, if R is altogether reflexive, symmetric, and transitive.

  Example: Let (a, b) and (c, d) be two ordered pairs of numbers such that the relation between them is given by $a + d = b + c$. This relation will be an equivalence relation. Let us prove this.
  (a, b) is related to (a, b) since $a + b = b + a$. Therefore, $R$is reflexive.
  If (a, b) is related to (c, d), then $a + d = b + c \Rightarrow c + b = d + a$. This shows that (c, d) is related to (a, b). Hence, R is symmetric.
  Let (a, b) is related to (c, d); and (c, d) is related to (e, f), then $a + d = b + c$ and $c + f = d + e$.
  Now, $(a + d) + (c + f) = (b + c) + (d + e) \Rightarrow a + f = b + e$. This shows that (a, b) is related to (e, f). Hence, R is transitive.
  Since R is reflexive, symmetric, and transitive, R is an equivalence relation.

- Given an arbitrary equivalence relation R in an arbitrary set X, R divides X into mutually disjoint subsets Ai called partitions or subdivisions of X satisfying:

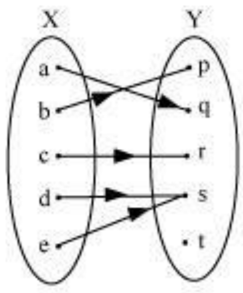
  - All elements of Ai are related to each other, for all i.
  - No element of Ai is related to any element of Aj , $i \neq j$
  - ⍰$Aj = X$ *and* $Ai \cap Aj = \emptyset$, $i \neq j$

  The subsets Ai are called equivalence classes.

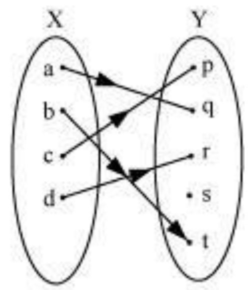
- A function f from set X to Y is a specific type of relation in which every element x of X has one and only one image y in set Y. We write the function f as $f: X \rightarrow Y$, where $f(x) = y$

- A function $f: X \rightarrow Y$ is said to be one-one or injective, if the image of distinct elements of X under f are distinct. In other words, if $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$. If the function f is not one-one, then f is called a many-one function.

  The one-one and many-one functions can be illustrated by the following figures:



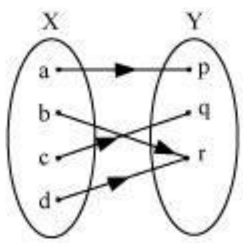
many-one



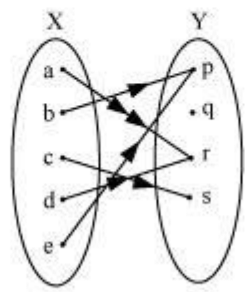
one-one

- A function f: $X \to Y$ can be defined as an onto (surjective) function, if $\forall\, y \in Y$, there exists $x \in X$ such that $f(x) = y$.

  The onto and many-one (not onto) functions can be illustrated by the following figures:

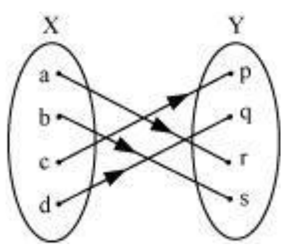


Onto



Not onto

- A function f: $X \to Y$ is said to be bijective, if it is both one-one and onto. A bijective function can be illustrated by the following figure:



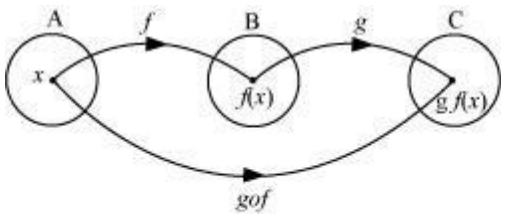
Example: Show that the function f: $\mathbf{R} \to \mathbf{N}$ given by $f(x) = x^3 - 1$ is bijective.

Solution: Let $x_1, x_2 \in \mathbf{R}$
For $f(x_1) = f(x_2)$, we have
$x_1^3 - 1 = x_2^3 - 1$
$\Rightarrow x_1^3 = x_2^3$
$\Rightarrow x_1 = x_2$
Therefore, f is one-one.
Also, for any y in $\mathbf{N}$, there exists $\sqrt[3]{y+1}$ in $\mathbf{R}$ such that

$f\left(\sqrt[3]{y+1}\right) = \left(\sqrt[3]{y+1}\right)^3 - 1 = y$.

Therefore, f is onto.
Since f is both one-one and onto, f is bijective.

- **Composite function:** Let f: $A \to B$ and g: $B \to C$ be two functions. The composition of f and g,i.e. gof, is defined as a function from A to C given by $gof\,(x) = g\,(f\,(x))$, $\forall\, x \in A$

  Example: Find gof and fog, if f: $\mathbf{R} \to \mathbf{R}$ and g: $\mathbf{R} \to \mathbf{R}$ are given by $f\,(x) = x^2 - 1$ and $g\,(x) = x^3 + 1$.

  Solution:

  $gof(x) = g(f(x))$

  $= g\left(x^2 - 1\right)$

  $= \left(x^2 - 1\right)^3 + 1$

  $= x^6 - 1 - 3x^4 + 3x^2 + 1$

  $= x^2\left(x^4 - 3x^2 + 3\right)$

  $fog(x) = f(g(x))$

  $= f\left(x^3 + 1\right)$

  $= \left(x^3 + 1\right)^2 - 1$

  $= x^6 + 2x^3 + 1 - 1$

  $= x^3\left(x^3 + 2\right)$

- A function f: $X \to Y$ is said to be invertible, if there exists a function g: $Y \to X$ such that $gof = I_X$ and $fog = I_Y$. In this case, g is called inverse of f and is written as $g = f^{-1}$

- A function f is invertible, if and only if f is bijective.

Example: Show that f: $\mathbf{R}^+ \cup \{0\} \rightarrow \mathbf{N}$ defined as $f(x) = x^3 + 1$ is an invertible function. Also, find f^{-1}.

Solution: Let $x_1, x_2 \in \mathbf{R}^+ \cup \{0\}$ and $f(x_1) = f(x_2)$

$\therefore x_1^3 + 1 = x_2^3 + 1$

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

Therefore, f is one-one.

Also, for any y in $\mathbf{N}$, there exists $\sqrt[3]{y-1} \in \mathbf{R}^+ \cup \{0\}$ such that $f\left(\sqrt[3]{y-1}\right) = y$.

$\therefore f$ is onto.

Hence, f is bijective.

This shows that, f is invertible.

Let us consider a function g: $\mathbf{N} \rightarrow \mathbf{R}^+ \cup \{0\}$ such that $g(y) = \sqrt[3]{y-1}$

Now,

$$gof(x) = g(f(x)) = g\left(x^3+1\right) = \sqrt[3]{\left(x^3+1\right)-1} = x$$

$$fog(y) = f(g(y)) = f\left(\sqrt[3]{y-1}\right) = \left(\sqrt[3]{y-1}\right)^3 + 1 = y$$

Therefore, we have

$gof(x) = I_{R^+ \cup \{0\}}$ and $fog(y) = I_N$

$\therefore f^{-1}(y) = g(y) = \sqrt[3]{y-1}$

- **Relation:** A relation R from a set A to a set B is a subset of the Cartesian product A × B, obtained by describing a relationship between the first element x and the second element y of the ordered pairs (x, y) in A × B.

- The image of an element x under a relation R is y, where $(x, y) \in R$

- **Domain:** The set of all the first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.

- **Range and Co-domain:** The set of all the second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the co-domain of the relation R. Range $\subseteq$ Co-domain

Example: In the relation X from **W** to **R**, given by X = $\{(x, y): y = 2x + 1; x \in W, y \in R\}$, we obtain X = {(0, 1), (1, 3), (2, 5), (3, 7) …}. In this relation X, domain is the set of all whole numbers, i.e., domain = {0, 1, 2, 3 …}; range is the set of all positive odd integers, i.e., range = {1, 3, 5, 7 …}; and the co-domain is the set of all real numbers. In this relation, 1, 3, 5 and 7 are called the images of 0, 1, 2 and 3 respectively.

- The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B.

  If $n(\mathrm{A}) = p$ and $n(\mathrm{B}) = q$, then $n(\mathrm{A} \times \mathrm{B}) = pq$ and the total number of relations is 2^{pq}.