MATRICES

-A system of mn numbers arranged in the form of rectangular array having 'm' rows and 'n' columns is called a matrix of order m×n.

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

Types of matrix

① Row & Column matrix

-A matrix having a single row is called a row matrix

Eg:- $\begin{bmatrix} 1 & 3 & 5 & 7 \end{bmatrix}$

-A matrix having single column is called a column matrix

Eg:- $\begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

② Square matrix

- When no. of rows = no. of columns then it is called a square matrix.

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 1 \\ 3 & 6 & 5 \end{bmatrix}_{3 \times 3}$$

→ principle diagonal

1, 5, 5 ⇒ principle diagonal elements

TRACE: Sum of principle diagonal elements

$1+5+5=11$

③ Diagonal matrix

- A square matrix in which all off-diagonal elements are zero are called as diagonal matrix.

Eg:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ diag[1 5 5]

Properties of diagonal matrix

1) $\text{diag}[x, y, z] + \text{diag}[p, q, r] = \text{dia}[x+p, y+q, z+r]$
2) $\text{diag}[x, y, z]^{-1} = \text{diag}[1/x, 1/y, 1/z]$
3) Δ of $\text{diag}[x, y, z] = xyz$
4) Eigen value of diagonal matrix $[x\ y\ z]$ is x, y, z

④ Scalar matrix

A scalar matrix is a diagonal matrix with all diagonal elements being equal.

⑤ Unit matrix OR Identity matrix

- A square matrix each of whose diagonal element is 1 and non-diagonal elements are zero, it is known as unit matrix. Denoted by I.

Properties

① $AI = A$ (multiplicative property)

② $I^n = I$

③ $I^{-1} = I$

⑥ Null matrix

- The $m \times n$ matrix whose elements are all zero.
- Null matrix need not be square matrix.

Properties

① $A + O = A$ (additive property)

② $A + (-A) = O$

⑦ Upper triangular matrix

Eg:- $\begin{bmatrix} 1 & 4 & 7 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

If all elements below principle diagonals elements are zero.

⑧ Lower triangular matrix

Eg:- $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 1 & 5 \end{bmatrix}$

If all elements above principle diagonal elements are zero.

NOTE: (1) A square matrix is said to be singular if its determinant is 0 otherwise it is non-singular

(2) If a matrix is either lower triangular or upper triangular then determinant is found by multiplication of principle diagonal elements.

⑨ Idempotent matrix

$A^2 = A$

⑩ Involuntary matrix

$A^2 = I$

Q:- $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$. Find A^3

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-36 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$A^3 = 0$

(11) Nil-potent matrix

-A matrix is said to be nil-potent matrix if class X or index X if $A^x = 0$ & $A^{x-1} \neq 0$

*Equality of two vectors

Two matrix A and B are said to be equal if:-
(1) They are of same order
(2) Each element of A is equal to corresponding element of B.

Matrix addition is commutative and associative.

Matrix subtraction is neither associative nor commutative.

* Multiplication of two matrix

$$[A]_{m \times n} \times [B]_{n \times p} = [c]_{m \times p}$$

equal

-Two matrix can be multiplied only when no. of columns in first is equal to no. of rows of 2nd. Such matrices are called CONFORMERS.

Q:- Consider the matrix $X_{4\times 3}$, $Y_{4\times 3}$, $P_{2\times 3}$. then $[P(X^T Y)^{-1} P^T]^T$ will be of order ___.

$X_{3\times \cancel{4}} \cdot Y_{4\times 3}$

$$(P(Z_{3\times 3})^{-1} P_{3\times 2})^T = P(Z_{3\times 3} \cdot P_{3\times 2})^T$$

$$= P(Z'_{3\times 2})^T$$

$$= [P(Z_{3\times 2})]^T = [P_{2\times 3} \cdot Z_{3\times 2}]^T = P_{2\times 2}$$

$\underline{2\times 2}$

Q:- There are 3 matrix $P_{4\times 2}$, $Q_{2\times 4}$, $R_{4\times 1}$. The minimum no. of multiplication required to compute the matrix $P \times Q \times R$.

Sol^n:- $m \times n \times p = 16$

<u>NOTE</u>: $[A]_{m\times n} \times [B]_{n\times p}$

The min. no. of multiplication req. is $= m \times n \times p$

Case:- 1

$(PQ)R$

$P_{4\times 2} \cdot Q_{2\times 4} \times R_{4\times 1}$

$(4\times 2\times 4) + (4\times 4\times 1)$

$32 + 16 = 48$

Case:- 2

$P(QR)$

$P_{4\times 2} \times Q_{2\times 4} \cdot R_{4\times 1}$

$(4\times 2\times 1) + (2\times 4\times 1)$

$8 + 8 = 16$

* <u>Inverse of a matrix</u>

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

adj A = transponse of a co-factor matrix

Q:- Find A^{-1} of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Sol^n:- $A \text{ (co-factor)} = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & 1 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{bmatrix}$$

$$|A| = 0 - 1(1-9) + 2(1-6)$$
$$= 8 - 10 = -2$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{bmatrix}$$

Q:- Find A^{-1}, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Q:- The max. value of determinant among all 2X2 real symmetric matrix with trace 14 is ___.

Sol^n:- $\begin{vmatrix} a & c \\ c & b \end{vmatrix}$ $\quad a + b = 14 \quad$ $\begin{vmatrix} x & y \\ y & 14-x \end{vmatrix}$

$$\Delta = ab - c^2$$

$$\Delta = a(14-a) - c^2$$

$$\Delta = 14a - a^2 - c^2$$

$$\frac{d\Delta}{da} = 14 - 2a = 0$$

$a = 7$ $\qquad \dfrac{d^2\Delta}{da^2} = -2 < 0 \leftarrow$ maxima.

*RANK of a MATRIX.

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}_{3\times3} = A$$

$\Delta = (3\times3) = 0$ then $\rho(A) \neq 3$
< 3

$\begin{vmatrix} a & d \\ b & e \end{vmatrix} \neq 0$ $\rho(A) = 2$

$$\begin{bmatrix} a & e & i \\ b & f & j \\ c & g & k \\ d & h & l \end{bmatrix}_{4\times3} = A$$

$\rho(A) = $ min. of 4 and 3
$=$ atleast 3

If all the minors of order $r+1$ are zero, but there is atleast one non-zero minor of order r if exists it is called RANK of a matrix. and it is denoted by $\rho(A) = r$

Properties of RANK

1) If A is a null matrix then $\rho(A) = 0$

2) If A is a non-zero matrix then rank of A ≥ 1
$\rho(A) \geq 1$

3) If I be the unit matrix or identity matrix of $n\times n$ then $\rho(I) = n$

4) If A ~~be~~ is the matrix of order $m\times n$ then
$\rho(A) \leq$ min of m or n

Q:- $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 5 & 7 \end{bmatrix}_{2\times 4}$. Find Rank

Sol:- $0 - 4 = -4 \neq 0$ Rank = 2

Q:- $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3}$

$4 - 4 = 0$

$12 - 12 = 0$ Rank = 1

Q:- $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$1(20-12) - 2(5-4) + 3(6-8)$

$8 - 2 + (-6) = 0 . < 3$

$4 - 2 = 2 \neq 0$

Rank = 2

Q:- $\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}_{3\times 4}$

$2(24-0) - 3(60-0) + 4(25+8)$

$48 - 180 + 132 = 0$

$32 \times 4 = 128$

$\boxed{\text{Rank} = 2}$

$\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & -1 \\ 5 & 12 & -1 \end{vmatrix} = 3(+12) - 4(-2+5) - 1(24)$

$= 36 - 12 - 24$

$= 36 - 36 = 0$

Q:- $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$1 \quad 4(5-21) - 3(10-24) + 2(14-8)$

$-64 - 3(-12) + 2(6)$

$-64 + 48 + 12 = 0$

$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \\ 3 & 2 & 1 \end{vmatrix} = 1(4-6) - 2(2-9) + 3(4-12)$

$= -2 + 14 - 24 \neq 0$

Rank = 3

$$2\begin{vmatrix} 2 & 3 & 2 \\ 3 & 1 & 3 \\ 6 & 7 & 5 \end{vmatrix} = 2\left(2(5-21) - 3(15-18) + 2(21-6)\right)$$

$$= 2(-32+9+2(15))$$

$$= 2(-\cancel{2}+9) = 2(+7) = 14$$

$R_4 - (R_3 + R_2 + R_1)$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 8 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Rank} < 4 \qquad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

Rank = 3

NOTE:
(1) $\rho(A^TA) = \rho(A)$

(2) Dimension of Null Space = Order − Rank

(3) Nullity = no. of column − Rank

Q:- The dimension of null space of matrix is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}_{3\times3}$

Find out dimension of null space

(a) 0 (b) 1 (c) 2 (d) 3

$0(1-0) - 1(-1-0) + 1(0-1)$

$\cancel{0} + 1 - 1 = \cancel{0}0 \cancel{\neq} 0 < 3 \quad \text{Rank} = \cancel{3}\ 2$

Di. of null space $=$ Order − Rank $= 3 - 2 = 1$

* Consistency and inconsistency of system of equations

- Any system is said to be consistence system if it has a solution.
- Inconsistence system has no solution.

$$a_1x + b_1y + c_1z = d_1 \quad \text{———} (1)$$

$$a_2x + b_2y + c_2z = d_2 \quad \text{———} (2)$$

$$a_3x + b_3y + c_3z = d_3 \quad \text{———} (3)$$

If $d_1 = d_2 = d_3 = 0$ $\Rightarrow$ then system is homogeneous

$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ (coefficient matrix), $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ (constant matrix), $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (variable matrix)

$$\boxed{AX = B}$$

$AB = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ (augmented matrix)

Case:- I If $\rho(A) = \rho(AB) =$ no. of unknowns

$\Rightarrow$ unique solution

Case: II If $\rho(A) = \rho(AB) <$ no. of unknowns

$\Rightarrow$ infinite solution / many solution

Case:- III If $\rho(A) \neq \rho(AB)$

$\Rightarrow$ no solution

Q:- If $x+y+z=3$, $x+2y+3z=4$ and $x+4y+z=6$

$$A=\begin{bmatrix}1&1&1\\1&2&3\\1&4&1\end{bmatrix} = 1(2-12)-1(1-3)+1(4-2)$$

$$= -10+2+2 \neq 0 \quad \text{Rank } \rho(A)=3$$

$$AB=\left[\begin{array}{ccc|c}1&1&1&3\\1&2&3&4\\1&4&1&6\end{array}\right]_{3\times 4} \quad \text{Rank } \rho(AB)=3$$

$$\begin{vmatrix}1&1&3\\2&3&4\\4&1&6\end{vmatrix} = 1(18-4)-1(12-16)+3(2-12)$$

$$=14+4-30 \neq 0$$

$\Rightarrow$ unique solution

Q:- $x-2y+3z=2$

$2x-3z=3$

$x+y+z=0$

$$A=\begin{bmatrix}1&-2&3\\2&0&-3\\1&1&1\end{bmatrix} = 1(0+3)+2(2+3)+3(2)$$

$$=3+10+6 \neq 0$$

Rank $=3$

$\Rightarrow$ unique solution

Q:- $4x-2y+6z=8$

$x+y-3z=-1$

$15x-3y+9z=21$

$$\begin{vmatrix}4&-2&6\\1&1&-3\\15&-3&9\end{vmatrix} = 4(9-9)+2(9+45)+6(-3-15)$$

$$=0+108-108=0$$

$$\begin{vmatrix}4&-2\\1&1\end{vmatrix} = 4+2=6 \neq 0 \quad \text{Rank}=2$$

$$AB = \begin{bmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 12 \end{bmatrix}$$

$$\begin{vmatrix} -2 & 6 & 8 \\ 1 & -3 & -1 \\ -3 & 9 & 12 \end{vmatrix} = -2(-36+9) - 6(12-3) + 8(9-9)$$

$$= -2(-25) - 6(9) + 0$$

$$= 50 - 54 = -4 \neq 0$$

Rank = ~~4~~ 3

$\Rightarrow$ Infinite solution

~~Solution.~~

Q:- For what value of λ & μ does system of eq^n

$x + y + z = 6$ have ① no sol^n

$x + 2y + 3z = 10$ ② unique sol^n

$x + 2y + \lambda z = \mu$ ③ more than one sol^n

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = 1(2\lambda - 6) - 1(\lambda - 3) + 1(2-2)$$

$$= 2\lambda - 6 - \lambda + 3$$

$$= \lambda - 3$$

$\lambda = 3$ for rank $\neq 3$

$$AB \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 & 6 \\ 2 & 3 & 10 \\ 2 & \lambda & \mu \end{vmatrix} = 1(3\mu - 10\lambda) - 1(2\mu - 20) + 6(2\lambda - 6)$$

$$= \mu + 2\lambda - 16 = \mu + 6 - 16 = \mu - 10$$

when there are unknown try to make adjacent elements zero (0).

$$AB = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_3 : R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

① unique sol^n $\rho(A)=\rho(AB)=$ no. of unknowns

$\lambda - 3 \neq 0$, $\mu \in R$

③ no sol^n

$\lambda = 3$, $\mu \neq 10$

② many sol^n $\rho(A)=\rho(AB) <$ no. of unknowns

$\lambda - 3 = 0$, $\mu - 10 = 0$

$\lambda = 3$, $\mu = 10$

① Symmetric Matrix

$A^T = A$

② Skew · symmetric matrix

$A^T = -A$

☆ ③ Orthogonal matrix

☆ $AA^T = I = AA^{-1}$

Conjugate of matrix (complex)

$$A = \begin{bmatrix} 2-i & 6+i & 3 \\ 3 & 5 & 7-i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 2+i & 6-i & 3 \\ 3 & 5 & 7+i \end{bmatrix}$$

$\bar{A} = A$, if purely real

$\bar{A} = -A$, if purely imagin.