Numerical Methods

Error = Exact value - Approx. value

Absolute Error = |Exact value - Appro. value|

$$\text{Relative Error } (\%) = \frac{|\text{Exact value} - \text{Approx. Value}|}{\text{Exact value}} \times 100$$

* Solution of Non-linear Equations

- Let $f(x) = 0$ be any non-linear equation, to find the approximate root of $f(x)=0$ choose any two value in between a, b. Particularly adjacent value such a way that there functional value has opposite sign.

$$f(x) = 0$$

$$x = (a, b)$$

$$\Rightarrow f(a) > 0 \; ; \; f(b) < 0$$

$$\boxed{f(a) \cdot f(b) < 0}$$

NOTE:

Intermediate value Theorem:-

- This theorem states that if the function is continuou and $f(a) \cdot f(b) < 0$ then surely there is a root in $[a, b]$ The contra positive of theorem is if the function is continuous and has no roots in $[a, b]$ then $f(a) \cdot f(b) \geq 0$

Approximate root can be found by

(1) Method of bisection
(2) Regula-Falsi method
(3) Secant method
(4) Newton Raphson method (NR method)

used for finding roots of algebraic & transcendental equations.

[1] Bisection method

$$(a, b)$$

$f(a) > 0$, $f(b) < 0$

$$\boxed{x_1 = \frac{a+b}{2}} \quad f(x_1) > 0$$

Replace a by x_1

$$(x_1, b)$$

Again $x_2 = \dfrac{x_1 + b}{2}$

⇒ The rate of convergence in bisection method is very slow.

[2] Regula Falsi method

$$(a, b)$$

$f(a) > 0$, $f(b) < 0$

$$\boxed{x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}}$$

$f(x_1) > 0$

Replace a by x_1

$$(x_1, b)$$

[3] Secant Method

(a, b)

$f(a) > 0, \; f(b) > 0$

$f(a) < 0, \; f(b) < 0$

$$\boxed{x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}}$$

Not applicable when $f(a) = f(b)$

Comparatively bisection method, regula-falsi little fast rate of convergence. Both this method are first order convergence or linear convergence.

If order is more then less error.

✱✱ [4] Newton-Raphson's method (Quadratic convergent)

✱ (Tangent method / Gradient method)

- It has most rapid rate of convergence and convergence of NR method is of 2nd order

let x_0 be an approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root, so that $f(x_1) = 0$. Expanding $f(x_0 + h)$ by Taylor series

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Derivation

$$f(x) = f(x_0) + hf'(x_0) + \cancel{h^2\frac{f''(x_0)}{2!} + \cdots + \infty} = 0$$

$$f(x_0) + hf'(x_0) = 0$$

$$f(x_0) = -hf'(x_0)$$

$$\frac{f(x_0)}{f'(x_0)} = -h = -(x_1 - x_0)$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Method	Order
1. Bisection	→ 1
2. Regula Falsi	→ 1
3. Secant	→ 1.62
4. N.R.	→ 2

Q:- Use N.R. method

1) $x^3 + 2x + 5$

x_0(initial guess) = 1 . Find x_1 = ____.

$$f(x) = x^3 + 2x + 5 = 0$$
$$f'(x) = 3x^2 + 2$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(1+2+5)}{5} = 1 - \frac{8}{5} = \frac{-3}{5}$$

Q:- $f(x) = x^3 + 3x - 7 = 0$, $x_0 = 1$ find x_1

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1 - \frac{(-3)}{6} = 1 + 1/2 = 3/2$$

Q:- $f(x) = e^x - 1 = 0$, $x_0 = -1$ Find $x_1 =$ ___ .

$$f'(x) = e^x$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= -1 - \frac{[e^{-1} - 1]}{e^{-1}}$$

$$x_1 = \frac{-e^{-1} - e^{-1} + 1}{e^{-1}} = \frac{1 - 2e^{-1}}{e^{-1}} = \frac{1 - 2(0.368)}{0.368} = +0.717$$

Q:- $f(x) = x^4 - 3x + 1$, $x_0 = 0$ Find $x_1 =$ ___ .

$$f'(x) = 4x^3 - 3$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{-3} = \underline{1/3}$$

Q:- $f(x) = x^2 - 2x - 1$, $x_0 = 2$ Find $x_2 =$ ___.

$f'(x) = 2x - 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2 - \frac{(-1)}{2} = 2 + 1/2 = 5/2 = 2.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.5 - \frac{0.25}{3}$$

$x_2 = 2.4167$

Q:- A root of eq^n $f(x) = (x-1)^2 + x - 3 = 0$ is used to find NR method. This method fails in very first iteration of initial guess $x_0 =$ ___.

$f'(x) = 2(x-1) + 1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f'(x) \neq 0$

$2(x-1) + 1 \neq 0$

$2(x-1) \neq -1$

$(x-1) \neq -1/2$

$x \neq 1/2$

If $x_0 = 1/2$ then N.R. method fails.

Q:- The N.R. iteration $x_{n+1} = \frac{1}{2}\left[x_n + \frac{R}{x_n}\right]$ can be used to compute

(a) R^2 (b) $\sqrt{R}$ (c) $\log R$ (d) $1/R$

$\Rightarrow$ At convergence $x_n = x_{n+1} = \alpha$

$$\alpha = \frac{1}{2}\left[\alpha + \frac{R}{\alpha}\right]$$

$$2\alpha = \frac{R+\alpha^2}{\alpha}$$

$$2\alpha^2 = R + \alpha^2$$

$$\alpha^2 = R$$

$$\boxed{\alpha = \sqrt{R}}$$

Q:- $x_{n+1} = \frac{1}{2}\left[x_n + \frac{117}{x_n}\right]$ converges at ______

$\sqrt{117}$

NOTE:- $x_{n+1} = \frac{1}{2}\left[x_n + \frac{R}{x_n}\right]$ converges by N.R. at $\sqrt{R}$

Q:- $x = 10\cos x$. Use N.R. $x_0 = \pi/4$ Find $x_1 =$ ______ (upto 2decimal accuracy)

$f(x) = x - 10\cos x = 0$

$f'(x) = +10\sin x + 1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi/4 - \frac{\left[\pi/4 - 10 \cdot 1/\sqrt{2}\right]}{1 + 10/\sqrt{2}} = \pi/4 - \frac{(\pi/4 - 7.071)}{8.071}$$

$= 0.7853 + 0.7787$

$= 1.564$

$\cancel{x_1 = \pi/4 - \frac{10/\sqrt{2}}{-10/\sqrt{2}} = \pi/4 + 1 = 1.78}$

Q:- $3x - e^{x} + \sin x = 0$, $x_0 = 0.333$. Find $x_1 = \_\_\_$

$f'(x) = 3 - e^{x} + \cos x$

$f(x) = 0.999 - 1.395 + 0.3268$

$f(x) = -0.0692$

$$\cancel{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

$$\cancel{= 0.333 - \frac{(-0.0692)}{2.55}}$$

$$\cancel{= 0.333 + 0.0271}$$

$$\cancel{x_1 = 0.3658}$$

$$\cancel{f'(x) = 3 - e^{0.333} + \cos(19.07)}$$

$$\cancel{= 3 - 1.395 + 0.945}$$

$$\cancel{f'(x) = 2.55}$$

0.36

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.333 - \left[\frac{3(0.333) - e^{0.333} + \sin(0.333)}{3 - e^{0.333} + \cos(0.333)}\right]$$

$$= 0.333 - \left[\frac{0.999 - 1.3951 + 0.3269}{3 - 1.3951 + 0.9450}\right]$$

$$= 0.333 - \left[\frac{-0.0692}{2.5499}\right]$$

$$= 0.333 + 0.0271$$

$$\boxed{x_1 = 0.3601}$$

degree $(1^\circ) = \frac{\pi}{180} \times 1$ (radian)

$0.333 \rightarrow$ radian

so, $0.333 \times \frac{180}{\pi} = 19.072$

$\sin(19.072) = 0.3267$

Q:- Solution of the variable x_1 & x_2 for following eq^n is employed by N.R. method. Equation

(i) $10x_2 \sin x_1 - 0.8 = 0$

(ii) $10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0$

Assuming initial guess $x_1 = 0$, $x_2 = 1$ the Jacobian matrix is

(a) $\begin{bmatrix} 10 & -0.8 \\ 0 & -0.6 \end{bmatrix}$ (b) $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -0.8 \\ 10 & -0.6 \end{bmatrix}$ (d) $\begin{bmatrix} 10 & 0 \\ 10 & -10 \end{bmatrix}$

Sol^n:-

Jacobian → it gives relationship bet^n any two

$u(x_1, x_2)$

$v(x_1, x_2)$

Jacobian matrix $\begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix}$

$$J = \begin{bmatrix} 10x_2 \cos x_1 & 10 \sin x_1 \\ 10x_2 \sin x_1 & 20x_2 - 10\cos x_1 \end{bmatrix}$$

$$J = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

Q:- The bisection method is used to compute a zero of the function $f(x) = x^4 - x^3 - x^2 - 4$ in the interval $[1, 9]$. The method converges in solution after _____ iteration.

Sol:- $x_1 = \frac{a+b}{2} = \frac{1+9}{2} = 5$

$f(5) = 625 - 125 - 25 - 4$

$f(5) = 471$

$(5, 9)$ $x_2 = 7$

$f(7) = 2401 - 343 - 49 - 4$

$f(7) = 2005$

$(7, 9)$ $x_3 = 8$

$f(8) = 4096 - 512 - 64 - 4$

$f(8) = 3516 > 0$

$[1, 9]$

$a = 1,\ b = 9$

$f(a) < 0,\ f(b) > 0$

$x_1 = \frac{1+9}{2} = 5$

$f(5) > 0$

$[1, 5]$

$x_2 = \frac{1+5}{2} = \frac{6}{2} = 3$

$f(3) > 0$

$\rightarrow [1, 3]$

$x_3 = \frac{1+3}{2} = \frac{4}{2} = 2$

$f(2) = 0$

3rd iteration $f(x) = 0$

Q:- Given $a > 0$ we wish to calculate reciprocal value $\frac{1}{a}$ by N.R. method for $f(x) = 0$

(1) The NR algorithm for $f(x)$ will be.

(a) $x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$

(b) $x_{k+1} = \left(x_k + \frac{a}{2} x_k^2\right)$

(c) $x_{k+1} = 2x_k - a x_k^2$

(d) $x_{k+1} = x_k - \frac{a}{2} x_k^2$

(2) For $a = 7$ and starting with $x_0 = 0.2$
The first two iteration will be

Sol:-

$\cancel{f(x) = (x - 1/a) = 0}$

$\cancel{f'(x) = 1}$

At convergence,

$x_{k+1} = x_k = \alpha$

option (c) $\alpha = 2\alpha - a\alpha^2$

$-\alpha = -a\alpha^2$

$\boxed{\alpha = 1/a}$

(2) $a = 7$, $x_0 = 0.2$

$x_1 = 2x_0 - a x_0^2$

$= 2(0.2) - 7(0.2)^2$

$= 0.4 - 7(0.04)$

$\boxed{x_1 = 0.12}$

$x_2 = 2x_1 - a x_1^2$

$= 2(0.12) - 7(0.12)^2$

$= 0.24 - 7(0.0144)$

$\boxed{x_2 = 0.1392}$

Q:- Given that one of the root of eqn $x^3 - 10x^2 + 31x - 30 = 0$ is 5 Find other two roots.

$\cancel{x^2(x-10)}$

$$
\begin{array}{r|l}
 & x^2 - 5x + 6 \\
\hline
x - 5 & x^3 - 10x^2 + 31x - 30 \\
 & x^3 - 5x^2 \\
 & - \quad + \\
\hline
 & -5x^2 + 31x \\
 & -5x^2 + 25x \\
 & + \quad - \\
\hline
 & 6x - 30 \\
 & 6x - 30 \\
\hline
 & 0
\end{array}
$$

$x = 2, 3$

Q:- The quadratic eq^n $x^2-4x+4=0$ is to be solved numerically starting with initial guess $x_0=3$. The NR method is applied once to get new estimate and then secant method is applied using initial and this new estimate. The estimated value of root after the application of secant method is ____.

Sol^n:- NR-method

$f(x)=x^2-4x+4$

$f'(x)=2x-4$

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$

$$=3-\frac{[9-12+4]}{2}$$

$$x_1=3-\frac{(1)}{2}=5/2$$

$(3, 5/2)$

$(5/2, 3)$

a b

$(3,5/2)$

$$x'=\frac{af(b)-bf(a)}{f(b)-f(a)}$$

$$=\frac{3\cdot\left[\frac{25}{4}-4\cdot\frac{5}{2}+4\right]-5/2\,[9-12+4]}{\left(\frac{25}{4}-\frac{20}{2}+4\right)-(9-12+4)}$$

$$\frac{3(1)-5/2(3/4)}{1-3/4}$$

$$\frac{3-15/8}{1/4}$$

$$x'=\frac{3\left[\frac{25-40+16}{4}\right]-\frac{5}{2}(1)}{}$$

$$=\frac{3\cdot 1/4-5/2}{-3/4}$$

$$\frac{1}{4}-1$$

$$=\frac{\frac{3-10}{4}}{-3/4}$$

$$x'=7/3$$

Q:- The real root of eq^n $5x - 2\cos x - 1 = 0$ (up two decimal accuracy) is ______.

initial guess

$x_0 = 0$

$f(x) = 5x - 2\cos x - 1$

$f'(x) = 5 + 2\sin x$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 0 - \frac{(-3)}{5}$

$x_1 = 3/5 = 0.6$

second guess

$x_0' = 0.6$

$x_2 = 0.6 - \frac{f(0.6)}{f'(0.6)}$

$= 0.6 - \frac{(3 - 2(0.825) - 1)}{6.129}$

$= 0.6 - \frac{0.343}{6.129}$

$x_2 = 0.543$

third guess

$x_0'' = 0.543$

$x_3 = 0.543 - \frac{[0.00267]}{6.03}$

$x_3 = 0.542$

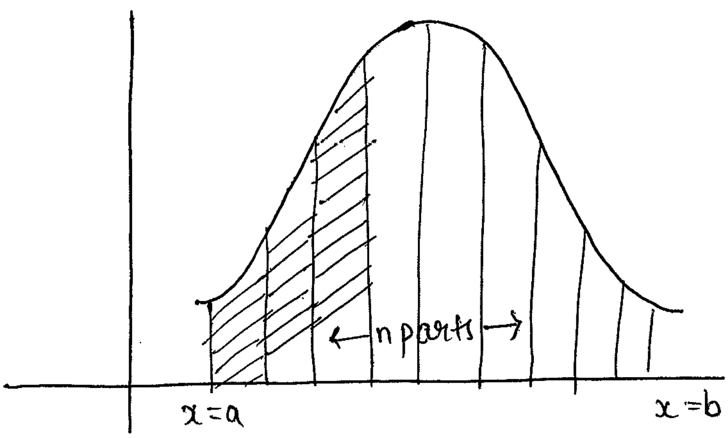
$\boxed{0.54}$

Q:- The series $x_{n+1} = \frac{x_n}{2} + \frac{9}{8x_n}$, $x_0 = 0.5$ obtain from N.R. The series converges to ___.

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{R}{x_n}\right]$$

$$\sqrt{R} = \sqrt{9/4} \simeq \underline{\underline{3/2}}$$

* Numerical Integration *



$$\left[h = \frac{b-a}{n}\right]$$

* Trapezoidal method *

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\dots)\right]$$

- The accuracy of trapezoidal method or order of integration of trapezoidal method is order of x^2 [2nd decimal difference]

* Simpson's $1/3^{rd}$ method *

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\dots) + 2(y_2+y_4+\dots)\right]$$

- The accuracy of Simpson's $1/3^{rd}$ rule is $O(h^4)$ [4th decimal difference]

$\downarrow$ o (alphabet)

* The accuracy of Simpson's $3/8^{th}$ rule is $O(h^5)$

$$\Delta = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right] \quad \text{(cubic polynomial)}$$

$\downarrow$ AREA

Q:- $\int_0^1 \frac{1}{1+x^2}\,dx$ by dividing into 4 equal integral and hence find value π using

(1) Trapezoidal method

(2) Simpson $1/3^{rd}$ method

Sol^n:-

x	0	0.25	0.5	0.75	1
y	1	0.9411	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$\left[\tan^{-1}x\right]_0^1 = \frac{0.25}{2}\left[1.5+2(0.9411+0.8+0.64)\right]$$

$$\pi/4 - 0 = 0.125\left[1.5+2(2.3811)\right]$$

$$\frac{\pi}{4} = 0.78277$$

$$\boxed{\pi = 3.1311}$$

(2) Using Simpson's $1/3^{rd}$

$$\int_0^1 \frac{1}{1+x^2} = \frac{h}{3}\left[y_0+y_4+4(y_1+y_3)+2y_2\right]$$

$$\left[\tan^{-1}\right]_0^1 = \frac{0.25}{3}\left[1+0.5+4(0.9411+0.64)+2(0.8)\right]$$

$$\frac{\pi}{4} = \frac{0.25}{3}\left[8.8244\right]$$

$$\frac{\pi}{4} = \cancel{0.735367} \quad 0.785052$$

$$\pi = 3.14156$$

Q:- The calculator has accuracy upto 8 digits after decimal place. The value of $\int_{0}^{2\pi} \sin x \, dx$ when evaluated using trapezoidal method with 8 equal intervals to ~~find~~ 5 significant digit is ____.

Sol:-

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
y	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	-1	$-1/\sqrt{2}$	0

$$h = \frac{b-a}{n} = \frac{2\pi - 0}{8} \qquad f(x) = \sin x$$

$$h = \pi/4$$

$$\int_0^1 \sin x \, dx = \frac{h}{2}\left[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$\int_0^1 \sin x \, dx = 0$$

$$\int_0^1 \sin x \, dx = \underline{\underline{0.00000}}$$

Q:- The estimate of $\int_{0.5}^{1.5} \frac{dx}{x}$ obtained by Simpson's rule with three point function evaluate exceeds exact value by ____.

$$\text{Exact value} = \left[\log x\right]_{0.5}^{1.5} = 1.0986$$

x	0.5	1	1.5
y	2	1	0.6667
	$\uparrow y_0$	$\uparrow y_1$	$\uparrow y_2$

n_{pt} is no. of points

n_i is no. of intervals

$n_i = n_{pt} - 1$

$n = 3 - 1 = 2$

$h = \frac{b-a}{n} = \frac{1.5-0.5}{2}$

$= \frac{1}{2}$

$= \frac{h}{3}\left[(y_0 + y_2) + 4y_1\right]$

$= 1.11111$

Exceeds by = $\boxed{0.0125}$

Q:- Torque exerted over a cycle listed in table. Flywheel energy in J/unit cycle using Simpson's rule is ______.

Angle (degree)	0	60	120	180	240	300	360
Torque N-m	0	1066	-323	0	323	-355	0

$$E = \int_0^{2\pi} T \cdot d\theta = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{\pi}{3 \cdot 3}\left[(0+0) + 4(1066 + 0 - 355) + 2[-323 + 323]\right]$$

$h = 60^\circ \times \frac{\pi}{180}$

$h = \frac{\pi}{3}$

$$= \frac{\pi}{9}[2844]$$

$$= 992.743 \text{ J/unitcycle}$$

Q:- Numerical integration using trapezoidal rule give the best result for single variable function which is ___

Sol^n:-

(a) parabolic (c) linear ✓

(b) hyperbolic (d) logarithm

Q:- The value of integration ~~2.524~~ $\int_{2.5}^{4} \ln(x)\,dx$ calculated with trapezoidal rule with 5 sub intevals is ___.

~~$h = \frac{4-2.5}{5} = \frac{6.5}{5} = 1.3$~~

~~x~~	~~2.5~~	~~3.8~~	~~5.1~~	~~6.4~~	~~7.7~~
~~y~~					

$$h = \frac{b-a}{n} = \frac{4-2.5}{5} = 0.3$$

x	2.5	2.8	3.1	3.4	3.7	4
y	0.9163	1.0296	1.1314	1.2237	1.3083	1.3863
	y_0	y_1	y_2	y_3	y_4	y_5

$$\frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.3}{2}\left[\qquad\qquad\qquad\right] = 1.7533$$

*Numerical Solution of Ordinary Diff. Equation

[1] Taylor's series method

[2] Picard's method

[3] Runga Kutta method

[4] Euler's method

[5] Modified euler's method

Q:-

	A		B
P-1	Numerical Integration	M-1	NR method
P-2	Sol^n to transidential Equation	M-2	RK method
P-3	Sol^n to system of linear equation	M-3	Simpson's 1/3
P-4	Sol^n to differential eq^n	M-4	Gaucs elimination method

(roots are everlasting no. not possible to find)

Sol^n:- P1 - M3

P2 - M1

P3 - M4

P4 - M2

* Runga-Katta(RK)method *

If $\frac{dy}{dx} = f(x, y)$, $x = x_0$, $y = y_0$

$$y = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$h = x - x_0$$

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f(x_0 + h/2, y_0 + k_1/2)$$

$$k_3 = h f(x_0 + h/2, y_0 + k_2/2)$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

Q:- Consider an O.D.E. $\frac{dx}{dt} = 4t + 4$ if $x = x_0$ at $t = 0$, the increment in x calculated using multistep method (Ranga-katta) with step size $\Delta t = 0.2$ is ______.

~~(x_0, t)~~

$x - x_0 = h = 0.2$

~~$x_0 =$~~

$k_1 = h f(x_0, t)$

$= 0.2[4]$

$k_1 = 0.8$

$\frac{dx}{dt} = 4t + 4$ only depend on t

(t, x_0)

Comparing $\frac{dy}{dx} = f(x, y)$

$h = 0.2$

$k_1 = 0.2[4] = 0.8$

$$K_2 = h f\left(t_0 + h/2,\ x_0 + \frac{k_1}{2}\right)$$

$$= 0.2\ f\left(0 + 0.1,\ 0.4\right)$$

$$k_2 = 0.88$$

$$k_3 = h f\left(t_0 + h/2,\ x_0 + \frac{k_2}{2}\right)$$

$$= 0.2\ f\left(\left(0 + \frac{0.2}{2}\right),\ 0\right)$$

$$k_3 = 0.88$$

$$k_4 = hf\left(t_0 + h,\ x_0 + k_3\right)$$

$$k_4 = 0.96$$

By Formula,

$$y = x_0 + \frac{t}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$$= 0 + \frac{1}{6}\left(0.8 + 2(0.88) + 2(0.88) + 0.96\right)$$

$$\boxed{y = 0.88}$$

*Numerical Solution of System of Linear Equation

(1) Matrix Inversion method

(2) Cramer's rule

(3) Gauss elimination method

(4) Gauss-Jordar method

(5) Gauss Sedel method

(6) Gauss Jacobi iteritgive method

(7) Gout's and Oolittle's method

Q:- Gauss Sedel method is used to solve following eq^n $x_1 + 2x_2 + 3x_3 = 5$

$2x_1 + 3x_2 + x_3 = 1$

$3x_1 + 2x_2 + x_3 = 3$. Assuming initial guess

$x_1 = x_2 = x_3 = 0$. The value of x_3 after 1st

iteration is ______.

Sol^n:-

NOTE:- Pivoting is required whenever you need in question highes co-efficient (x) → starts with

$\Rightarrow 3x_1 + 2x_2 + x_3 = 3$

$$x_1 = \frac{3 - 2x_2 - x_3}{3}$$

$\boxed{x_1 = 1}$ $[x_2 = 0, x_3 = 0]$

$\Rightarrow 2x_1 + 3x_2 + x_3 = 1$

$2 + 3x_2 = 1$

$3x_2 = 1 - 2$

$\boxed{x_2 = -1/3}$

$\Rightarrow x_1 + 2x_2 + 3x_3 = 5$

$$x_3 = \frac{5 - x_1 - 2x_2}{3}$$

$$x_3 = \frac{5 - 1 + 2/3}{3} = \frac{12+2}{9} = \frac{14}{9}$$

$\boxed{x_3 = 14/9}$ $\boxed{x_3 = 1.5555}$

* Method of Factorisation [Dolittle's method]

$$a_{11}x + a_{12}y + a_{13}z = b_1 \quad \text{---} (1)$$

$$a_{21}x + a_{22}y + a_{23}z = b_2 \quad \text{---} (2)$$

$$a_{31}x + a_{32}y + a_{33}z = b_3 \quad \text{---} (3)$$

$$\boxed{AX = B} \quad \text{---} (1)$$

$A = LU$

L = lower unit triangular matrix

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

U = upper triangular matrix

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

* Crout's method

L = Lower triangular matrix

$$= \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

U = upper triangular matrix

$$= \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$LUX = B$$

$$UX = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\boxed{LY = B}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Finding y_1, y_2, y_3 and substituting in $UX = Y$ to get value of x, y and z

Q:- Solve the equation

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$ by factorization method

Ans:- $A = LU$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$= \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{11} \cdot L_{12} & U_{12} \cdot L_{21} + U_{22} & U_{13} L_{21} + U_{23} \\ L_{31} \cdot U_{11} & L_{31} \cdot U_{12} + L_{32} \cdot U_{22} & L_{31} \cdot U_{13} + L_{32} U_{23} + U_{33} \end{bmatrix}$$

$\rightarrow U_{11} = 2$

$U_{12} = 3$

$U_{13} = 1$

$\rightarrow L_{21} U_{11} = 1$

$\therefore L_{21} = 1/2$

$\rightarrow L_{21} U_{12} + U_{22} = 2$

$\therefore U_{22} = 2 - \frac{1}{2} \cdot 3 = \frac{1}{2}$

$\rightarrow L_{21} \cdot U_{13} + U_{23} = 3$

$\therefore U_{23} = 3 - \frac{1}{2} \cdot 1 = 5/2$

$\rightarrow L_{31} \cdot U_{11} = 9$

$\therefore L_{31} = 3/2$

$\rightarrow L_{31} \cdot U_{12} + L_{32} \cdot U_{22} = 1$

$3/2 \cdot 3 + L_{32} \cdot 1/2 = 1$

$\frac{1}{\cancel{2}} L_{32} = 1 - \frac{9}{2} = -\frac{7}{\cancel{2}}$

$L_{32} = -7$

$\rightarrow L_{31} \cdot U_{13} + L_{32} \cdot U_{23} + U_{33} = 2$

$3/2 \cdot 1 + (-7) \cdot 5/2 + U_{33} = 2$

$U_{33} = 2 - \frac{3}{2} + \frac{35}{2}$

$U_{33} = 18$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

$$AX = B$$

$$LUX = B$$

$$UX = Y$$

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$\rightarrow y_1 = 9$

$\rightarrow \frac{y_1}{2} + y_2 = 6$

$$y_2 = 6 - \frac{9}{2} = 3/2$$

$\rightarrow 3/2\, y_1 - 7 y_2 + y_3 = 8$

$$\frac{3}{2} \cdot 9 - 7 \cdot \frac{3}{2} + y_3 = 8$$

$$y_3 = 8 - \frac{27}{2} + \frac{21}{2}$$

$$y_3 = 5$$

Now, $UX = Y$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

$\rightarrow 18z = 5$

$$\boxed{z = 5/18}$$

$\rightarrow 1/2\, y + 5/2\, z = 3/2$

$$y + 5 \cdot \frac{5}{18} = 3/\cancel{2}$$

$$y = 3/\cancel{2} - \frac{25}{18} = \frac{\cancel{27}\,54 - 25}{18} = \frac{\cancel{2}\,29}{18}$$

$\rightarrow 2x + 3y + z = 9$

$$2x + \cancel{3} \cdot \frac{29}{\cancel{18}\,6} + \frac{5}{18} = 9$$

$$\therefore \boxed{x = 35/18}$$

Q:- A matrix A is 2x2 matrix $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$. is decomposed into a product of lower triangular matrix L and upper triangular matrix U. The properly decomposed L and U matrix respectively are

(a) $\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

Sol^n:- Multiply L and U from all option & you will get directly A from option (c)

Q:- In the LU decomposition of matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$, if the diagonal element of U are both 1, then the lower diagonal entry L_{22} of L is ____.

$$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} 1 & U_{21} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{11}U_{21} \\ L_{21} & L_{21}U_{21} + L_{22} \end{bmatrix}$$

$$L_{21} \cdot U_{21} + L_{22} = 9$$

Now, $\underline{L_{11} = 2}$ & $\underline{L_{21} = 4}$

$L_{11} \cdot U_{21} = 2$ $\quad \therefore L_{21} \cdot U_{21} + L_{22} = 9$

$\underline{U_{21} = 1}$ $\quad 4 \cdot 1 + L_{22} = 9$

$$\underline{L_{22} = 5}$$

NOTE: There is no advantage using Crout's method over Dolittle, so either method is used.

*Euler's method [2nd method to find D.E.]

- Consider the D.E. $\frac{dy}{dx} = f(x, y)$

Also known as Runga-katta 1st order

$y(x_0) = y_0 \Rightarrow$ initial condition

$y(x_n) = y_n \cdot \varepsilon$

$$\boxed{y_n = y(x_n) = y_{n-1} + hf(x_{n-1}, y_{n-1})}$$

where $h = x_n - x_{n-1}$

$\underline{n=1} \Rightarrow \quad y_1 = y_0 + hf(x_0, y_0)$

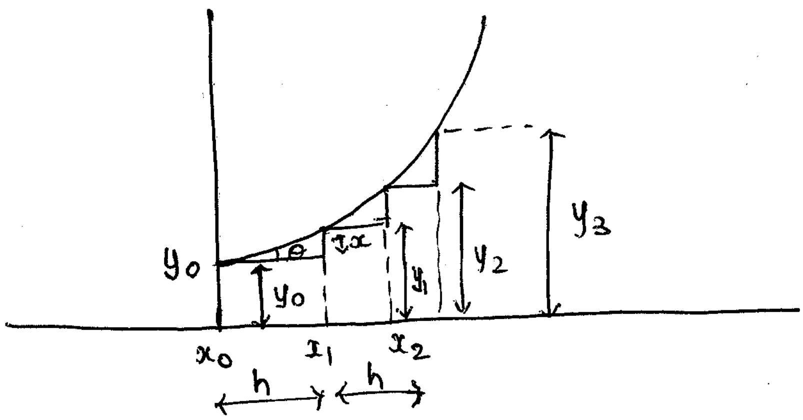
$\underline{n=2} \Rightarrow \quad y_2 = y_1 + hf(x_1, y_1)$

PROOF: suppose $f(x, y) = \frac{dy}{dx} = y$

$y(0) = 1$

$\downarrow$

$f(x) = e^x$



Here, $y_1 = y_0 + x$

but $\tan\theta = \frac{x}{h}$

$x = h\tan\theta$

$\tan\theta = \frac{dy}{dx} = f(x, y)$

$x = h \cdot f(x_0, y_0)$

$\downarrow$

$$\boxed{y_1 = y_0 + hf(x_0, y_0)}$$

So, $\boxed{y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})}$

NOTE:

- In euler's method, we approximate the curve of solution by tangent in each interval, that is by sequence of short lines.
- Unless h is small, the error is bound to be quite significant.
- This sequence of line may also deviate considerably from the curve of solution. Hence there is modification of this method by modified Euler's method, which is more accurate.

③ Modified Euler's method

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

Q:- Consider the eq^n $\frac{du}{dt} = 3t^2+1$, with $u=0$ at $t=0$ This is numerically solved using Euler's method with step size $\Delta t = 2$. The absolute error in the solution in end of first step is ____.

$\rightarrow h = 2$

By Euler's eq^n

$u_0 = t_0 = 0$

$u_1 = u_0 + hf(0,0)$

$u_1 = 0 + 2(1)$

$u_1 = 2$

$\rightarrow$ Exact value

$du = (3t^2+1)\,dt$

$\int du = \int (3t^2+1)\,dt$

limit = 2
because 1st iteration

$u = (t^3 + t)\Big|_{t=2}$

$u = 8 + 2 = 10$

Absolute error = |Exact − Approx.| = |10 − 2| = 8

Q:- Consider the 1st order initial value problem $y' = y + 2x - x^2$, $y(0) = 1$ $(0 \le x < \infty)$ with exact solution $y(x) = x^2 + e^x$ for $x = 0.1$. The percentage difference between exact solution of sol^n obtain using ~~signal~~ single iteration of 2nd order RK method with step size h+0.1 is ____.

$\rightarrow \quad \frac{dy}{dx} = y + 2x - x^2$

$k_1 = hf(x_0, y_0)$
$= hf(0, 1)$
$= 0.1\,[1]$
$k_1 = 0.1$

$k_2 = hf(x_0 + h, y_0 + k_1)$
$= 0.1\, f(0.1, 1.1)$
$= 0.1\,[1.1 + 0.2 - 0.01]$
$k_2 = 0.129$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$= 1 + \frac{1}{2}[0.1 + 0.129]$$

$$y_1 = 1.1145$$

Exact value

$$y_1 = (0.1)^2 + e^{0.1}$$

$$y_1 = 1.1151$$

$$\therefore \text{Error} = \frac{1.1151 - 1.1145}{1.1151} \times 100\%$$

$$\text{Error} = 0.0538\%$$

Q:- Given $\frac{dy}{dx} - y = x$; $y(0) = 0$ by euler's first order differential eqn method with step size of 0.1, the value of $y(0.3)$ is ——. (Initial condn $y(0) = 0$)

⇒ Euler's method

	x	y	$y_{new} = y_{old} + h\,f(x_{old}, y_{old})$
x_0	0	0	$y_1 = 0 + 0.1(0+0) = 0$
x_1	0.1	0	$y_2 = 0 + 0.1(0.1+0) = 0.01$
x_2	0.2	0.01	$y_3 = 0.01 + 0.1(0.2+0.01)$
x_3	$\boxed{0.3}$		$= 0.01 + 0.1(0.21)$
	0.4		$= 0.01 + 0.021$

$$\boxed{y_3 = 0.031}$$