3. Inequalities

Solving Linear Inequalities

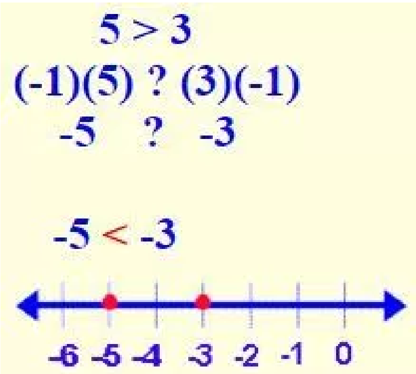
Solving linear inequalities is the same as solving linear equations with one very important exception…

When you **multiply or divide** an inequality by a negative value, it changes the **direction** of the inequality.

Inequalities with one variable:

Consider:

Look at this true statement:
Suppose we multiply both sides by -1.
What is the relationship between these two numbers ?



ANS: -5 is less than -3 because it is further to the left on the number line.

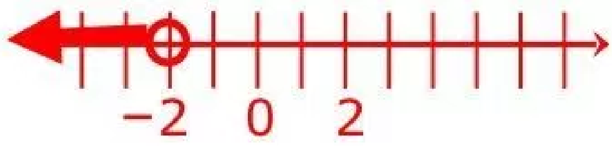
So, we must change the direction of the inequality when we multiply (or divide) by a negative number in order to get the correct answer.

Before we begin our example problems, refresh your memory on what each inequality symbol means. It is helpful to remember that the "open" part of the inequality symbol (the larger part) always faces the larger quantity.

Symbol	Description	Example	Solution Set
>	Greater than, more than	x > 3	All numbers greater than 3; does not include 3
≥	Greater than or equal to, at least	x ≥3	All numbers greater than or equal to 3; includes 3
<	Less than	x < 3	all numbers less than 3; does not include 3
≤	Less than or equal to, no more than	x ≤ 3	All numbers less than or equal to 3; includes 3
≠	Not equal to	x ≠ 3	includes all numbers except 3

1. $-14x > 28$

$x < -2$

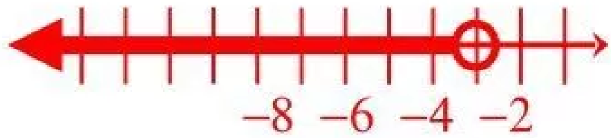


2. $\frac{x}{3} < 15$

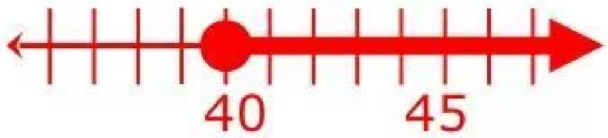
$x < 45$

3. $18 < -6x$

$-3 > x$



4. $\frac{q}{8} \geq 5$ $q \geq 40$



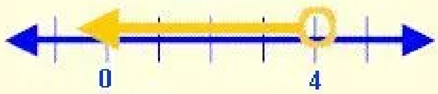
Example 1:

Solve and graph the solution set of: $2x - 6 < 2$

Add 6 to both sides.
Divide both sides by 2.

$$2x - 6 < 2$$
$$2x < 8$$
$$x < 4$$

Open circle at 4 (since x can not equal 4) and an arrow to the left (because we want values less than 4).



Example 2:

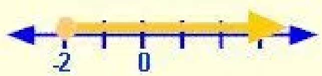
Solve and graph the solution set of: $5 - 3x \leq 13 + x$

Subtract 5 from both sides.
Subtract x from both sides.

$$5 - 3x \leq 13 + x$$
$$-3x \leq 8 + x$$
$$-4x \leq 8$$
$$x \geq -2$$

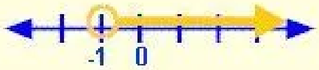
Divide both sides by -4, and don't forget to change the direction of the inequality !
(We divided by a negative.)

Closed circle at -2 (since x can equal -2) and an arrow to the right (because we want values larger than -2).



Example 3:

Solve and graph the solution set of: $3(2x + 4) > 4x + 10$

Multiply out the parentheses.	$3(2x + 4) > 4x + 10$
Subtract $4x$ from both sides.	$6x + 12 > 4x + 10$
Subtract 12 from both sides.	$2x + 12 > 10$
Divide both sides by 2, but don't change the direction of the inequality, since we didn't divide by a negative.	$2x > -2$ $x > -1$
Open circle at -1 (since x can not equal -1) and an arrow to the right (because we want values larger than -1).	

Absolute Value Inequalities

Solving an absolute value inequality problem is similar to solving an absolute value equation.

Start by isolating the absolute value on one side of the inequality symbol, then follow the rules below:

If the symbol is > (or >=) : (or)
If a > 0, then the solutions to |x| > a
are x > a or x < − a.

If a < 0, all real numbers will satisfy . |x| > a

Think about it: absolute value is always positive (or zero), so, of course, it is greater than any negative number.

If the symbol is < (or <=) : (and)
If a > 0, then the solutions to |x| < a
are x < a and x > − a.
Also written: − a < x < a.

If a < 0, there is no solution to .|x| < a

Think about it: absolute value is always positive (or zero), so, of course, it cannot be less than a negative number.

Remember:

When working with any absolute value inequality,
you must create two cases.
If <, the connecting word is "and".
If >, the connecting word is "or".

To set up the two cases:

x < a
Case 1: Write the problem without the absolute value sign, and solve the inequality.

x > -a
Case 2: Write the problem without the absolute value sign, reverse the inequality, negate the value NOT under the absolute value, and solve the inequality.

Example 1: (solving with "greater than")

Solve: $|x-20|>5$

Case 1:		Case 2:
$x-20>5$	**or**	$x-20<-5$
$x>25$		$x<15$

$x<15$ or $x>25$

Example 2: (solving with "less than or equal to")

Solve: $|x-3|\le 4$

Case 1:		Case 2:
$x-3\le 4$	**and**	$x-3\ge -4$
$x\le 7$		$x\ge -1$

$x\ge -1$ and $x\le 7$
also written as:
$-1\le x\le 7$

Example 3: (isolating the absolute value first)

Solve: $|3+x|-4<0$

Case 1:		Case 2:
$|3+x|<4$	**and**	$|3+x|<4$
$3+x<4$		$3+x>-4$
$x<1$		$x>-7$

$x<1$ and $x>-7$
also written as:
$-7<x<1$

Example 4: (compound inequalities)

Separate a compound inequality into two separate problems.

Solve: $5<|x+1|<7$

$5<|x+1|$

Case 1:		Case 2:
$5<x+1$	**or**	$-5>x+1$
$4<x$		$-6>x$

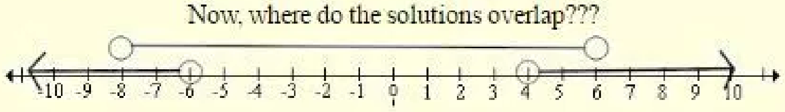
$x>4$ or $x<-6$

$|x+1|<7$

Case 1:		Case 2:
$x+1<7$	**and**	$x+1>-7$
$x<6$		$x>-8$

$-8<x<6$

Now, where do the solutions overlap???



$-8<x<-6$ as well as $4<x<6$

Example 5: (all values work)

Solve: $|x+4|>-3$

Case 1:		Case 2:
$x+4>-3$	**or**	$x+4<3$
$x>-7$		$x<-1$

$x>-7$ or $x<-1$
Answer: $x\in\Re$

WHOA!!!!
Don't solve this problem!

You already know the answer!
Absolute value is ALWAYS positive (or zero), so it is always > -3.
All values work!

Solving Rational Inequalities

A rational inequality is an inequality which contains a rational expression. When solving these rational inequalities, there are steps that lead us to the solution.

To solve Rational Inequalities:

(1) Write the inequality as an equation, and solve the equation.
(2) Determine any values that make the denominator equal 0.
(3) On a number line, mark each of the critical values from steps 1 and 2. These values will create intervals on the number line.
(4) Select a test point in each interval, and check to see if that test point satisfies the inequality. (Find the intervals which satisfy the inequality).
(5) Mark the number line to reflect the values and intervals that satisfy the inequality.
(6) State your answer using the desired form of notation.

Example 1: Solve $\frac{x-4}{x+5} < 4$

$$\frac{x-4}{x+5} = 4$$

Create an equation. Change < to =, a solve. Notice that if $x = -5$, the denominator is 0.

$$\frac{(x+5)}{1}\cdot\frac{x-4}{x+5} = 4(x+5)$$

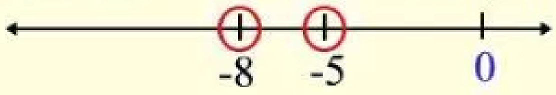
Multiply both sides by $(x + 5)$ to eliminate the fraction. You could also "cross-multiply". In a proportion the product of the means equals the product of the extremes.

$$x - 4 = 4x + 20$$
$$3x = -24$$
$$x = -8$$

Critical values are $x = -8$ and $x = -5$

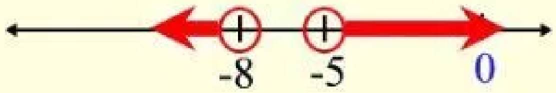
On the number line, plot -5 and -8. Since -5 cannot be used, it is an open circle.

The inequality is strictly "less than", so the -8 is also an open circle.



Test a point in each of the three intervals formed:

test point -9	test point -6	test point 0
$\frac{-9-4}{-9+5} < 4$	$\frac{-6-4}{-6+5} < 4$	$\frac{0-4}{0+5} < 4$
$\frac{-13}{4} = 3\frac{1}{4} < 4$ True!	$\frac{-10}{-1} = +10 < 4$ False!	$\frac{-4}{+5} < 4$ True!



Stated as an inequality, the solution is:

$$x < -8 \text{ or } x > -5$$

Stated in interval notation, the solution is:

$$(-\infty, -8) \cup (-5, +\infty)$$

When the numerator of the inequality is a quadratic expression, combine the Quadratic Inequality method of solution with this Rational Inequality method.

Check out this example.

Example 2: Solve $\dfrac{x^2-2x-15}{x-2} \geq 0$

$$\frac{x^2-2x-15}{x-2} \geq 0$$

Separate the numerator, form an equation, and solve this quadratic equation.

$$x^2-2x-15$$

Factor, and find the solutions or critical values for the numerator.

Also keep in mind that the denominator has $x = 2$ as a critical value as well.

$$x^2-2x-15=0$$
$$(x-5)(x+3)=0$$
$$x-5 \text{ or } x=-3$$

Since $x = 2$ creates an undefined expression, it is drawn as an open circle on the number line.

Place the critical values on a number line. Since the inequality is *greater than or equal to*, the $x = 5$ or $x = -3$ are drawn on the number line as a solid circle, whicl means to include them as part of the answer. Test the intervals that are formed.

test $x = -4$	test $x = 0$	test $x = 3$	test $x = 6$
$\dfrac{(-4)^2 - 3(-4) - 15}{(-4) - 2} \geq 0$	$\dfrac{(0)^2 - 3(0) - 15}{(0) - 2} \geq 0$	$\dfrac{(3)^2 - 3(3) - 15}{(3) - 2} \geq 0$	$\dfrac{(6)^2 - 3(6) - 15}{(6) - 2} \geq 0$
$\dfrac{16 + 12 - 15}{-6} = \dfrac{13}{-6} \geq 0$	$\dfrac{-15}{-2} = \dfrac{15}{2} \geq 0$	$\dfrac{9 - 9 - 15}{1} = \dfrac{-15}{1} \geq 0$	$\dfrac{36 - 18 - 15}{4} = \dfrac{3}{4} \geq 0$
False !!	True !!	False !!	True !!

The solution is:

$$-3 \leq x < 2 \text{ or } 5 \leq x < \infty$$

In interval notation, the solution is:

$$[-3, 2) \cup [5, \infty)$$

Set-builder & Interval Notation

A **set** is a collection of unique elements. Elements in a set do not "repeat".

Methods of Describing Sets:

Sets may be described in many ways: by roster, by set-builder notation, by interval notation, by graphing on a number line, and/or by Venn diagrams. For graphing on a number line, see Linear Inequalities. For Venn diagrams, see Working with Sets and Venn Diagrams.

Set Builder and Interval Notation

Set Builder Notation – is a mathematical shorthand for accurately stating a specific group of numbers.

$\mathbb{Z}$ - the set of integers $\mathbb{N}$ - the set of natural numbers

$\mathbb{R}$ - the set of real numbers $\mathbb{Q}$ - the set of rational numbers

Example 1: $\{x \in \mathbb{Z} \mid -4 \leq x < 3\} = \{-4, -3, -2, -1, 0, 1, 2\}$

x is a member of the set of integers such that x is greater than or equal to -4 and less than 3.

Interval Notation – a similar method to set builder notation that uses brackets instead of inequality signs.

"[" or "]" - same as $\leq$ or $\geq$ and "(" or ")" - same as $<$ or $>$.

Example 2: $\{x \in \Re \mid -4 \leq x < 3\}$

By roster: A roster is a list of the elements in a set, separated by commas and surrounded by French curly braces.

$\{2, 3, 4, 5, 6\}$	is a roster for the set of integers from 2 to 6, inclusive.
$\{1, 2, 3, 4, \ldots\}$	is a roster for the set of positive integers. The three dots indicate that the numbers continue in the same pattern indefinitely. (Those three dots are called an **ellipsis**.)
Rosters may be awkward to write for certain sets that contain an infinite number of entries.	

By set-builder notation: Set-builder notation is a mathematical shorthand for precisely stating all numbers of a specific set that possess a specific property.

$\mathbb{R}$ = real numbers; $\mathbb{Z}$ = integer numbers; $\mathbb{N}$ = natural numbers.

$\{x \in \mathbb{Z} \mid 2 \leq x \leq 6\}$	is set-builder notation for the set of integers from 2 to 6, inclusive. $\in$ = "is an element of" Z = the set of integers | = the words "such that" The statement is read, "all x that are elements of the set of integers, such that, x is between 2 and 6 inclusive."
$\{x \in \mathbb{Z} \mid x > 0\}$	The statement is read, "all x that are elements of the set of integers, such that, the x values are greater than 0, positive." (The positive integers can also be indicated as the set Z^+.)
It is also possible to use a colon (:), instead of the | , to represent the words "such that". $\{x \in \mathbb{Z} \mid 2 \leq x \leq 6\}$ is the same as $\{x \in \mathbb{Z} : 2 \leq x \leq 6\}$	

By interval notation: An **interval** is a connected subset of numbers. **Interval notation** is an alternative to expressing your answer as an inequality. Unless specified otherwise, we will be working with real numbers.

When using interval notation, the symbol:
(means "not included" or "open".
[means "included" or "closed".

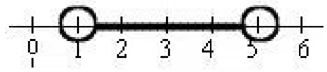
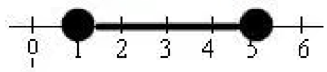
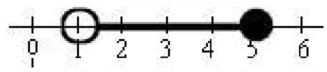
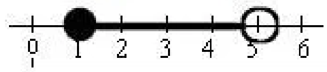
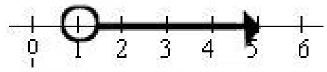
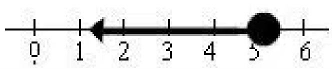
$2 \leq x < 6$	as an inequality.
$[2, 6)$	in interval notation.

Intervals

There are four types of interval:

1. **Open interval:** Let a and b be two real numbers such that a<b, then the set of all real numbers lying strictly between a and b is called an open interval and is denoted by]a, b[or (a, b). Thus,]a, b[or (a, b)= { x ∊ R: a < x < b}.
2. **Closed interval:** Let a and b be two real numbers such that a<b, then the set of all real numbers lying between a and b including a and b is called a closed interval and is denoted by [a, b]. Thus, [a, b] = { x ∊ R: a ≤ x ≤ b}.
3. **Open-Closed interval:** It is denoted by]a, b] or (a, b] and]a, b] or (a, b] = { x ∊ R: a < x ≤ b}.
4. **Closed-Open interval:** It is denoted by [a, b[or [a, b) and [a, b[or [a, b) = { x ∊ R: a ≤ x < b}.

The chart below will show you all of the possible ways of utilizing interval notation.

Interval Notation: (description)	(diagram)
Open Interval: (a, b) is interpreted as $a < x < b$ where the endpoints are NOT included. (While this notation resembles an ordered pair, in this context it refers to the interval upon which you are working.)	$(1, 5)$ 
Closed Interval: $[a, b]$ is interpreted as $a \leq x \leq b$ where the endpoints are included.	$[1, 5]$ 
Half-Open Interval: $(a, b]$ is interpreted as $a < x \leq b$ where a is not included, but b is included.	$(1, 5]$ 
Half-Open Interval: $[a, b)$ is interpreted as $a \leq x < b$ where a is included, but b is not included.	$[1, 5)$ 
Non-ending Interval: (a, ∞) is interpreted as $x > a$ where a is not included and infinity is always expressed as being "open" (not included).	$(1, \infty)$ 
Non-ending Interval: $(-\infty, b]$ is interpreted as $x \leq b$ where b is included and again, infinity is always expressed as being "open" (not included).	$(-\infty, 5]$ 

For some intervals it is necessary to use combinations of interval notations to achieve the desired set of numbers. Consider how you would express the interval **"all numbers except 13".**

As an inequality:	$x < 13$ or $x > 13$
In interval notation:	$(-\infty, 13) \cup (13, \infty)$
Notice that the word "or" has been replaced with the symbol "U", which stands for "union".	

Consider expressing in interval notation, the set of numbers which contains all numbers less than 0 and also all numbers greater than 2 but less than or equal to 10.

As an inequality:	$x < 0$ or $2 < x \leq 10$
In interval notation:	$(-\infty, 0) \cup (2, 10]$

As you have seen, there are many ways of representing the same interval of values. These ways may include word descriptions or mathematical symbols.

The following statements and symbols ALL represent the same interval:	
WORDS:	**SYMBOLS:**
"all numbers between positive one and positive five, including the one and the five."	$1 \leq x \leq 5$
"x is less than or equal to 5 and greater than or equal to 1"	$\{x \in \mathbb{R} \mid 1 \leq x \leq 5\}$
"x is between 1 and 5, inclusive"	$[1,5]$