Exercise 11.1

Q1E

(A) A sequence is an ordered list of numbers as $\{a_1, a_2, a_3, \ldots\ldots\ldots, a_n, \ldots\}$
Also a sequence can be defined as a function whose domain is the set of positive integers.

(B) The term a_n approaches 8 as n becomes large.

(C) The term a_n becomes large as n becomes large. We can make a_n as large as we want by taking n sufficiently large.

Q2E

(A) A sequence $\{a_n\}$ in convergent if $\lim_{n\to\infty} a_n$ *exists*

For example

(1) $a_n = \dfrac{n}{n+1}$ *here* $a_n \to 1$ *as* $n \to \infty$

(2) $a_n = \dfrac{1}{2^n}$ *here* $a_n \to 0$ *as* $n \to \infty$

(B) A sequence $\{a_n\}$ is divergent if $\lim_{n\to\infty} a_n$ does not exist.

For example

(1) $a_n = n$ *here* $a_n \to \infty$ $n \to \infty$

(2) $a_n = \dfrac{n^3}{n^2 - 5}$ *here* $a_n \to \infty$ *as* $n \to \infty$

Q3E

Given $a_n = \dfrac{2n}{n^2+1}$

$$a_1 = \frac{2(1)}{(1)^2+1} = 1$$

$$a_2 = \frac{2(2)}{(2)^2+1} = \frac{4}{5}$$

$$a_3 = \frac{2(3)}{(3)^2+1} = \frac{3}{5}$$

$$a_4 = \frac{2(4)}{(4)^2+1}$$
$$= \frac{8}{17}$$
$$a_5 = \frac{2(5)}{(5)^2+1}$$
$$= \frac{5}{13}$$

Therefore, the first five terms of the sequence is

$$\boxed{1, \frac{4}{5}, \frac{3}{5}, \frac{8}{17}, \frac{5}{13}}$$

Q4E

Given $a_n = \frac{3^n}{1+2^n}$

$$a_1 = \frac{3^1}{1+2^1}$$
$$= 1$$
$$a_2 = \frac{3^2}{1+2^2}$$
$$= \frac{9}{5}$$
$$a_3 = \frac{3^3}{1+2^3}$$
$$= 3$$
$$a_4 = \frac{3^4}{1+2^4}$$
$$= \frac{81}{17}$$

$$a_5 = \frac{3^5}{1+2^5}$$
$$= \frac{81}{11}$$

Therefore, the first five terms of the sequence is

$$\boxed{1, \frac{9}{5}, 3, \frac{81}{17}, \frac{81}{11}}$$

Q5E

Given $a_n = \dfrac{(-1)^{n-1}}{5^n}$

$$a_1 = \frac{(-1)^{1-1}}{5^1}$$
$$= \frac{1}{5}$$
$$a_2 = \frac{(-1)^{2-1}}{5^2}$$
$$= \frac{-1}{25}$$
$$a_3 = \frac{(-1)^{3-1}}{5^3}$$
$$= \frac{1}{125}$$
$$a_4 = \frac{(-1)^{4-1}}{5^4}$$
$$= \frac{-1}{625}$$
$$a_5 = \frac{(-1)^{5-1}}{5^5}$$
$$= \frac{1}{3125}$$

Therefore, the first five terms of the sequence is

$$\boxed{\frac{1}{5}, \frac{-1}{25}, \frac{1}{125}, \frac{-1}{625}, \frac{1}{3125}}$$

Q6E

Given $a_n = \cos\left(\dfrac{n\pi}{2}\right)$

$$a_1 = \cos\left(\frac{\pi}{2}\right)$$
$$= 0$$
$$a_2 = \cos\left(\frac{2\pi}{2}\right)$$
$$= -1$$
$$a_3 = \cos\left(\frac{3\pi}{2}\right)$$
$$= 0$$

$$a_4 = \cos\left(\frac{4\pi}{2}\right)$$
$$= 1$$
$$a_5 = \cos\left(\frac{5\pi}{2}\right)$$
$$= 0$$

Therefore, the first five terms of the sequence is

$$\boxed{0, -1, 0, 1, 0}$$

Q7E

Given $a_n = \frac{1}{(n+1)!}$

$$a_1 = \frac{1}{(1+1)!}$$
$$= \frac{1}{2!}$$
$$= \frac{1}{2}$$
$$a_2 = \frac{1}{(2+1)!}$$
$$= \frac{1}{3!}$$
$$= \frac{1}{6}$$
$$a_3 = \frac{1}{(3+1)!}$$
$$= \frac{1}{4!}$$
$$= \frac{1}{24}$$

$$a_4 = \frac{1}{(4+1)!} = \frac{1}{5!} = \frac{1}{120}$$

$$a_5 = \frac{1}{(5+1)!} = \frac{1}{6!} = \frac{1}{720}$$

Therefore, the first five terms of the sequence is

$$\boxed{\frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}}$$

Q8E

Given $a_n = \frac{(-1)^n n}{n!+1}$

$$a_1 = \frac{(-1)^1(1)}{1!+1} = \frac{-1}{2}$$

$$a_2 = \frac{(-1)^2(2)}{2!+1} = \frac{2}{3}$$

$$a_3 = \frac{(-1)^3(3)}{3!+1} = \frac{-3}{7}$$

$$a_4 = \frac{(-1)^4(4)}{4!+1} = \frac{4}{25}$$

$$a_5 = \frac{(-1)^5(5)}{5!+1} = \frac{-5}{121}$$

Therefore, the first five terms of the sequence is

$$\boxed{\frac{-1}{2}, \frac{2}{3}, \frac{-3}{7}, \frac{4}{25}, \frac{-5}{121}}$$

Q9E

Given $a_1 = 1$

$$a_{n+1} = 5a_n - 3$$

Then $a_1 = 1$

$$\begin{aligned} a_2 &= a_{1+1} \\ &= 5a_1 - 3 \\ &= 5 - 3 \\ &= 2 \\ a_3 &= a_{2+1} \\ &= 5a_2 - 3 \\ &= 5(2) - 3 \\ &= 7 \end{aligned}$$

$$\begin{aligned} a_4 &= a_{3+1} \\ &= 5a_3 - 3 \\ &= 5(7) - 3 \\ &= 32 \\ a_5 &= a_4 + 1 \\ &= 5a_4 - 3 \\ &= 5(32) - 3 \\ &= 157 \end{aligned}$$

Therefore, the first five terms of the sequence is

$$\boxed{1, 2, 7.32, 157}$$

Q10E

Given $a_1 = 6$ and

$$a_{n+1} = \frac{a_n}{n}$$

Then $a_1 = 6$

$$\begin{aligned} a_2 &= a_{1+1} \\ &= \frac{a_1}{1} \\ &= \frac{6}{1} \\ &= 6 \\ a_3 &= a_{2+1} \\ &= \frac{a_2}{2} \\ &= \frac{6}{2} \\ &= 3 \end{aligned}$$

$$a_4 = a_{3+1} = \frac{a_3}{3} = \frac{3}{3} = 1$$

$$a_5 = a_{4+1} = \frac{a_4}{4} = \frac{1}{4}$$

Therefore, the first five terms of the sequence is

$$\boxed{6, 6, 3, 1, \frac{1}{4}}$$

Q11E

Given $a_1 = 2$

and $a_{n+1} = \frac{a_n}{1+a_n}$

Then $a_1 = 2$

$$a_2 = a_{1+1} = \frac{a_1}{1+a_1} = \frac{2}{1+2} = \frac{2}{3}$$

$$a_3 = a_{2+1} = \frac{a_2}{1+a_2} = \frac{\frac{2}{3}}{1+\frac{2}{3}} = \frac{2}{5}$$

$$a_4 = a_{3+1}$$
$$= \frac{a_3}{1+a_3}$$
$$= \frac{\frac{2}{5}}{1+\frac{2}{5}}$$
$$= \frac{2}{7}$$

$$a_5 = a_{4+1}$$
$$= \frac{a_4}{1+a_4}$$
$$= \frac{\frac{2}{7}}{1+\frac{2}{7}}$$
$$= \frac{2}{9}$$

Therefore, the first five terms of the sequence is

$$\boxed{2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}}$$

Q12E

Given $a_1 = 2$,

$a_2 = 1$

and $a_{n+1} = a_n - a_{n-1}$

Then $a_1 = 2$

$a_2 = 1$

$$a_3 = a_{2+1}$$
$$= a_2 - a_1$$
$$= 1 - 2$$
$$= -1$$
$$a_4 = a_{3+1}$$
$$= a_3 - a_2$$
$$= -1 - 1$$
$$= -2$$

$$a_5 = a_{4+1}$$
$$= a_4 - a_3$$
$$= -2 + 1$$
$$= -1$$

Therefore, the first five terms of the sequence is

$$\boxed{2, 1, -1, -2, -1}$$

Q13E

Consider the sequence

$$\{a_n\} = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots\right\}$$

Its need to find the general term of the given sequence

From the data, we have that

$$a_1 = 1,\ a_2 = \frac{1}{3},\ a_3 = \frac{1}{5},\ a_4 = \frac{1}{7}, \dots$$

Observe that, the numerators of the fractions have the constant value 1 and denominators of these fractions start with 1 and increased by 2 whenever we got to the next term.

The second term has denominator 3, the third term has denominator 5; in general, the n^{th} term will have denominator $2n-1$.the sign of the terms are positive.

Therefore, general term of the given sequence is

$$a_n = \frac{1}{2n-1}$$

Q14E

We are given that

$$a_1 = 1,$$

$$a_2 = \frac{-1}{3},$$

$$a_3 = \frac{1}{9},$$

$$a_4 = \frac{-1}{27},$$

$$a_5 = \frac{1}{81}$$

Notice that all the numerators of these fractions is 1. In general, the n^{th} term will have numerator 1. The denominators of these factors are power of 3, so a_n has denominator 3^{n-1}. The sign of the terms are alternately positive and negative, so we need to multiply by a power of -1. Here we want to start with a positive term and so we use $(-1)^{n-1}$

Therefore

$$\boxed{a_n = (-1)^{n-1}\frac{1}{3^{n-1}}}$$

Q15E

We are given that

$$a_1 = -3$$

$$a_2 = 2$$

$$a_3 = \frac{-4}{3}$$

$$a_4 = \frac{8}{9}$$

$$a_5 = \frac{-16}{27}$$

Notice that all the numerators of these fractions are power of 2. In general, the n^{th} term will have numerator 2^{n-1}. The denominators of these factors are power of 3, and the first term is $\frac{1}{3^{-1}}$, so a_n has denominator 3^{n-2}. The sign of the terms are alternately positive and negative, so we need to multiply by a power of -1. Here we want to start with a negative term and so we use $(-1)^n$

Therefore

$$\boxed{a_n = (-1)^n \frac{2^{n-1}}{3^{n-2}}}$$

Q16E

We are given that

$$a_1 = 5$$

$$a_2 = 8$$

$$a_3 = 11$$

$$a_4 = 14$$

$$a_5 = 17$$

Here first term is 5 and increase by 3 whenever we go the next term.
Therefore,

$$a_n = 5 + 3(n-1)$$

$$\boxed{= 3n + 2}$$

Q17E

We can given that

$$a_1 = \frac{1}{2}$$

$$a_2 = \frac{-4}{3}$$

$$a_3 = \frac{9}{4}$$

$$a_4 = \frac{-16}{5}$$

$$a_5 = \frac{25}{6}$$

Notice that the numbers of the n terms is the square of n. so a_n has numerator n^2 . The denominators of these factors start with 2 and increase by 1 whenever we go 10 the next term, in general, the n^{th} term will have numbers n+1. The Light of terms are alternately positive and negative, so we need to multiply by a power of -1. Here we want to start with a positive term and so we use $(-1)^{n-1}$

Therefore

$$\boxed{a_n = (-1)^{n-1} \frac{n^2}{(n+1)}}$$

Q18E

We are given that

$a_1 = 1, a_2 = 0, a_3 = -1, a_4 = 0, a_5 = 1, a_6 = 0, a_7 = -1, a_8 = 0 \ldots$

Notice that all even terms are zero and the sign of the odd terms are alternately positive and negative, so we need to multiply by a power of -1.

Therefore

$$a_n = \begin{cases} (-1)^{\frac{n-1}{2}} & \text{when n is odd} \\ 0 & \text{when n is even} \end{cases}$$

Q19E

Given $a_n = \dfrac{3n}{1+6n}$

$$\therefore a_1 = \frac{3(1)}{1+6} = \frac{3}{7} = 0.4286$$

$$a_2 = \frac{3(2)}{1+6(2)} = \frac{6}{13} = 0.4615$$

$$a_3 = \frac{3(3)}{1+6(3)} = \frac{9}{19} = 0.4736$$

$$a_4 = \frac{3(4)}{1+6(4)} = \frac{12}{25} = 0.4800$$

$$a_5 = \frac{3(5)}{1+6(5)} = \frac{15}{31} = 0.4838$$

$$a_6 = \frac{3(6)}{1+6(6)} = \frac{18}{37} = 0.4864$$

$$a_7 = \frac{3(7)}{1+6(7)} = \frac{21}{43} = 0.4883$$

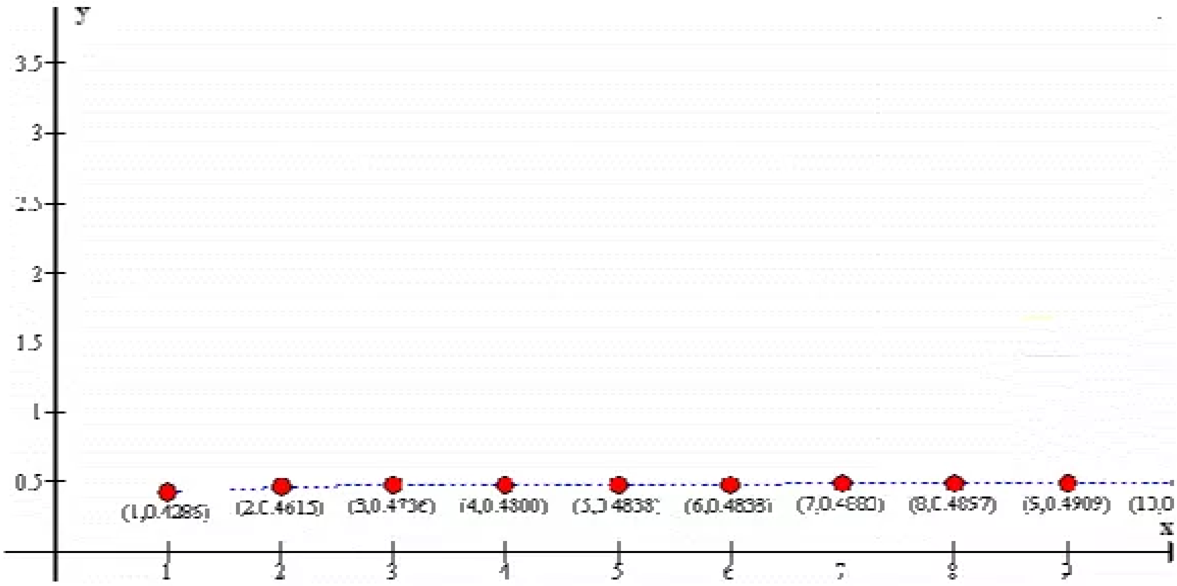
$$a_8 = \frac{3(8)}{1+6(8)} = \frac{27}{55} = 0.4897$$

$$a_9 = \frac{3(9)}{1+6(9)} = \frac{12}{25} = 0.4909$$

$$a_{10} = \frac{3(10)}{1+6(10)} = \frac{30}{61} = 0.4918$$

Yes, this sequence appears to have the limit.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{3n}{1+6n} \\ &= \lim_{n\to\infty} \frac{3}{\frac{1}{n}+6} \\ &= \frac{3}{6} \\ &= \frac{1}{2}\end{aligned}$$



Q20E

Any list of numbers that is written in definite orders is called a sequence.

The terms of a sequence can be expressed by a general formula and each term can be obtained by the substitution of the term number in the general formula.

Consider the sequence shown below;

$$a_n = 2 + \frac{(-1)^n}{n}$$

Determine the value of a_n for $n = 1$:

$$\begin{aligned}a_1 &= 2 + \frac{(-1)}{1} \\ &= 2 - 1 \\ &= 1.0000\end{aligned}$$

Determine the value of a_n for $n=2$:

$$a_2 = 2 + \frac{(-1)^2}{2}$$
$$= 2 + \frac{1}{2}$$
$$= \frac{5}{2}$$
$$= 2.5000$$

Determine the value of a_n for $n=3$:

$$a_3 = 2 + \frac{(-1)^3}{3}$$
$$= 2 - \frac{1}{3}$$
$$= \frac{5}{3}$$
$$= 1.6667$$

Determine the value of a_n for $n=4$:

$$a_4 = 2 + \frac{(-1)^4}{4}$$
$$= 2 + \frac{1}{4}$$
$$= \frac{9}{4}$$
$$= 2.2500$$

Determine the value of a_n for $n=5$:

$$a_5 = 2 + \frac{(-1)^5}{5}$$
$$= 2 - \frac{1}{5}$$
$$= \frac{9}{5}$$
$$= 1.8000$$

Determine the value of a_n for $n=6$:

$$a_6 = 2+\frac{(-1)^6}{6}$$
$$= 2+\frac{1}{6}$$
$$= \frac{13}{6}$$
$$= 2.1667$$

Determine the value of a_n for $n=7$:

$$a_7 = 2+\frac{(-1)^7}{7}$$
$$= 2-\frac{1}{7}$$
$$= \frac{13}{7}$$
$$= 1.8571$$

Determine the value of a_n for $n=8$:

$$a_8 = 2+\frac{(-1)^8}{8}$$
$$= 2+\frac{1}{8}$$
$$= \frac{17}{8}$$
$$= 2.1250$$

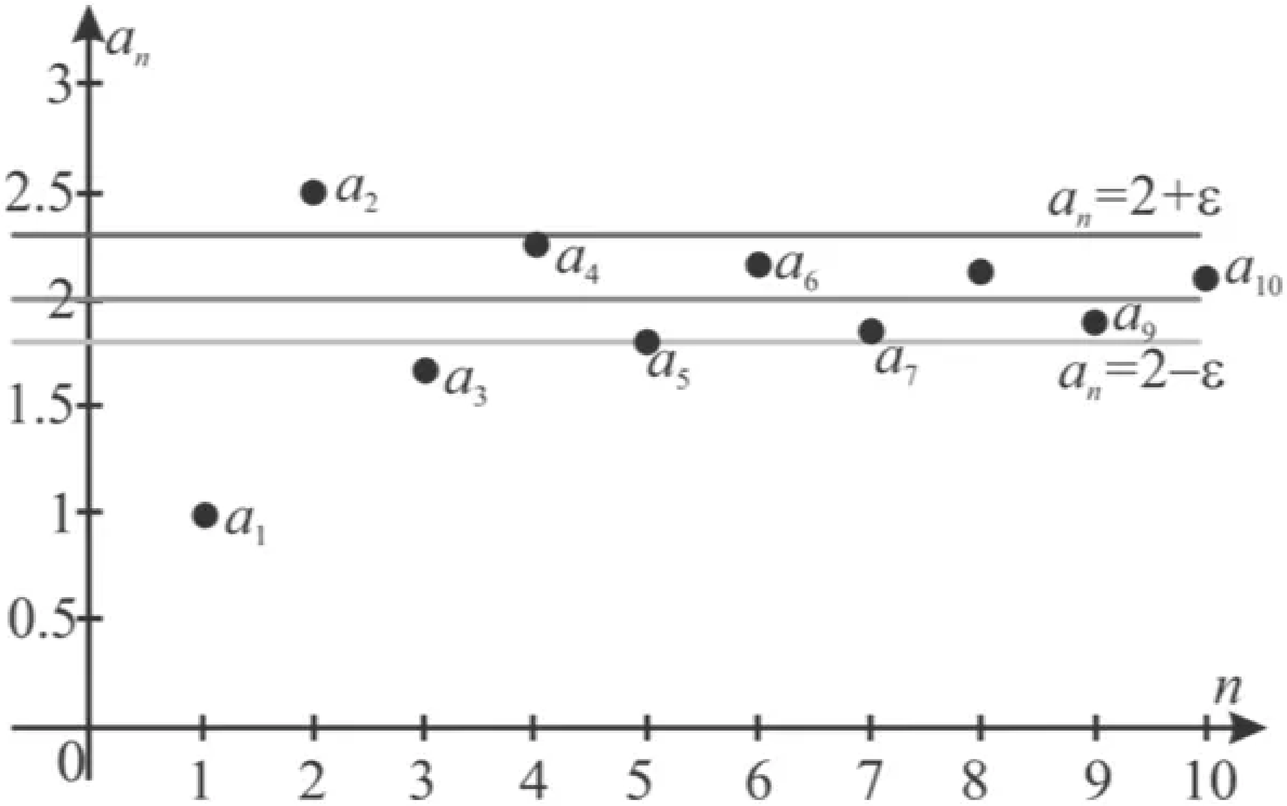
Determine the value of a_n for $n=9$:

$$a_9 = 2+\frac{(-1)^9}{9}$$
$$= 2-\frac{1}{9}$$
$$= \frac{17}{9}$$
$$= 1.8889$$

Determine the value of a_n for $n = 10$:

$$a_{10} = 2 + \frac{(-1)^{10}}{10}$$
$$= 2 + \frac{1}{10}$$
$$= \frac{21}{10}$$
$$= 2.1000$$

Use the above ten terms and graph sequence as shown below:



Observe the graph to find that all terms of the sequence approaches to a value.

So, the sequence has a limit.

Determine the limit of the sequence.

Consider the cases for n as odd and even.

Take n as odd, so $(-1)^n = -1$.

$$a_n = 2 - \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(2 - \frac{1}{n}\right)$$
$$= 2 - 0$$
$$= 2$$

Take n as even, so $(-1)^n = 1$.

$$a_n = 2 + \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(2 + \frac{1}{n}\right)$$
$$= 2 + 0$$
$$= 2$$

Hence, limit of the sequence is $\boxed{2}$.

Q21E

Given $a_n = 1 + \left(-\frac{1}{2}\right)^n$

$$\therefore a_1 = 1 + \left(-\frac{1}{2}\right)^1 = \frac{1}{2} = 0.5000$$

$$a_2 = 1 + \left(-\frac{1}{2}\right)^2 = \frac{5}{4} = 1.2500$$

$$a_3 = 1 + \left(-\frac{1}{2}\right)^3 = \frac{7}{8} = 0.8750$$

$$a_4 = 1 + \left(-\frac{1}{2}\right)^4 = \frac{17}{16} = 1.0625$$

$$a_5 = 1 + \left(-\frac{1}{2}\right)^5 = \frac{31}{32} = 0.9688$$

$$a_6 = 1 + \left(-\frac{1}{2}\right)^6 = \frac{65}{64} = 1.0156$$

$$a_7 = 1 + \left(-\frac{1}{2}\right)^7 = \frac{127}{128} = 0.9922$$

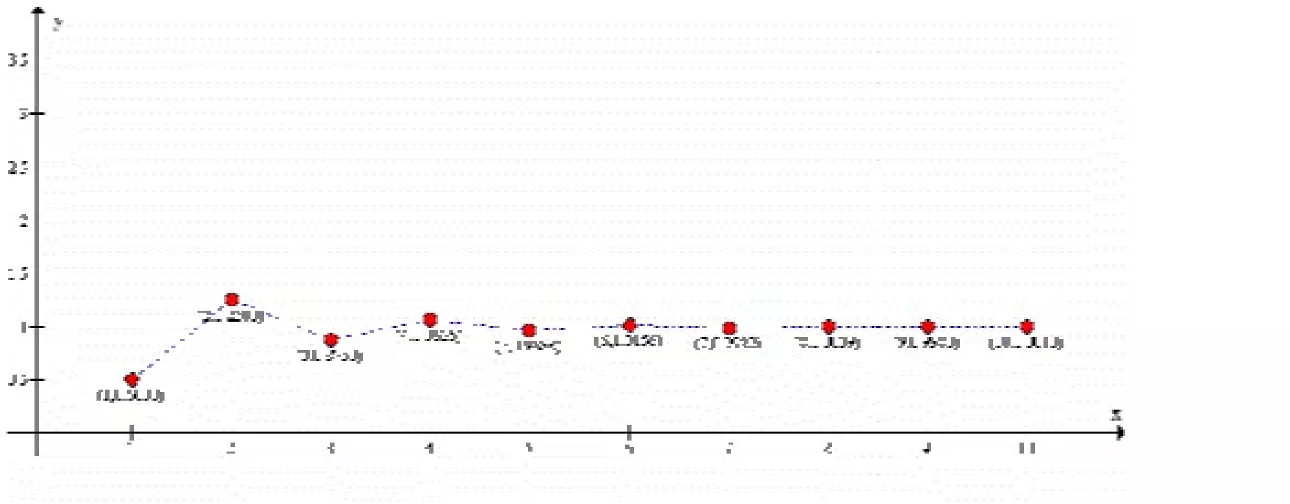
$$a_8 = 1 + \left(-\frac{1}{2}\right)^8 = \frac{257}{256} = 1.0039$$

$$a_9 = 1 + \left(-\frac{1}{2}\right)^9 = \frac{511}{512} = 0.9980$$

$$a_{10} = 1 + \left(-\frac{1}{2}\right)^{10} = \frac{1025}{1024} = 1.0010$$

Yes, this sequence appears to have the limit

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[1 + \left(-\frac{1}{2}\right)^n\right]$$
$$= 1 + 0$$
$$= 1$$



Q22E

Given sequence $a_n = 1 + \frac{10^n}{9^n}$

$$\therefore a_1 = 1 + \left(\frac{10}{9}\right)^1 = 2.1112$$

$$a_2 = 1 + \left(\frac{10}{9}\right)^2 = 2.2347$$

$$a_3 = 1 + \left(\frac{10}{9}\right)^3 = 2.3720$$

$$a_4 = 1 + \left(\frac{10}{9}\right)^4 = 2.5246$$

$$a_5 = 1 + \left(\frac{10}{9}\right)^5 = 2.6941$$

$$a_6 = 1 + \left(\frac{10}{9}\right)^6 = 2.8825$$

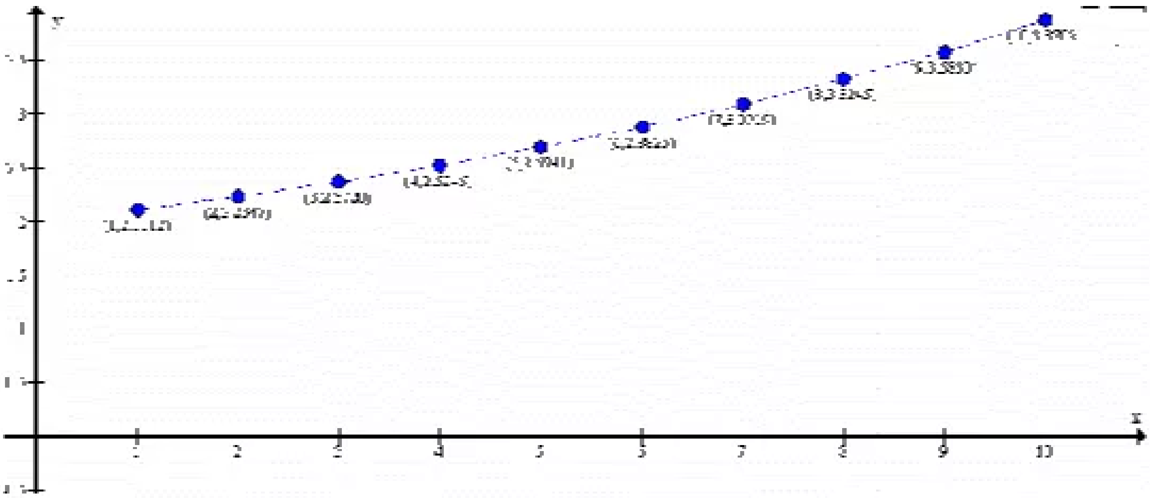
$$a_7 = 1 + \left(\frac{10}{9}\right)^7 = 3.0919$$

$$a_8 = 1 + \left(\frac{10}{9}\right)^8 = 3.3245$$

$$a_9 = 1 + \left(\frac{10}{9}\right)^9 = 3.5830$$

$$a_{10} = 1 + \left(\frac{10}{9}\right)^{10} = 3.8702$$

No, this series not appears to have a limit.
Since as n tends to infinity, $a_n \to \infty$



Q23E

Consider the sequence,

$a_n = 1-(0.2)^n$.

Notice that this sequence can be split up into two different parts: 1 and $-(0.2)^n$.

The first part does not change with n, and the second part goes to 0 since it is a geometric sequence with ratio $0.2 < 1$.

Let $b_n = 1$, a constant sequence, and $c_n = -(0.2)^n$.

Then, the sum of the two sequences can be taken as,

$a_n = b_n + c_n$.

Take limits on both sides, then the equation becomes,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (b_n + c_n)$

$= \lim_{n\to\infty} b_n + \lim_{n\to\infty} c_n$ By the addition rule for limits

$= \lim_{n\to\infty} 1 + \lim_{n\to\infty} -(0.2)^n$ Use $b_n = 1$and $c_n = -(0.2)^n$

$= 1+0$ Use $b_n = 1$always and $c_n \to 0$as discussed above.

$= 1$

Here, the limit value is 1.

Therefore, the sequence converges by the geometrical series test.

Hence, the limit value of the sequence is

$\lim_{n\to\infty} a_n = \boxed{1}$.

Q24E

Consider the sequence

$$a_n = \frac{n^3}{n^3+1}$$

Its need to determine whether the sequence converges or diverges, if it converges, find the limit.

To determine whether the sequence converges or diverges follow the monotonic sequence theorem.

Monotonic Sequence Theorem states that,

"Every bounded, monotonic sequence is convergent."

Verify that whether the sequence $\{a_n\}$ is bounded or not:

Observe that, $0 < n^3 < n^3 + 1 \quad \forall n \in N$

$$\Rightarrow \frac{0}{n^3+1} < \frac{n^3}{n^3+1} < \frac{n^3+1}{n^3+1} \quad \forall n \in N$$

$$\Rightarrow 0 < \frac{n^3}{n^3+1} < 1 \quad \forall n \in N$$

$$\Rightarrow 0 < a_n < 1 \quad \forall n \in N$$

That is, the sequence a_n is bounded.

Verify that whether the sequence $\{a_n\}$ is monotonic or not:

For this, consider

$$a_{n+1} - a_n = \frac{(n+1)^3}{1+(n+1)^3} - \frac{n^3}{1+n^3}$$

$$= \frac{(n+1)^3\left[1+n^3\right] - n^3\left[1+(n+1)^3\right]}{\left[1+(n+1)^3\right]\left[1+n^3\right]}$$

$$= \frac{(n+1)^3 - n^3}{\left[1+(n+1)^3\right]\left[1+n^3\right]} \quad > 0 \quad \forall n \in N$$

Therefore, $a_{n+1} - a_n > 0 \quad \forall n \in N$

So, sequence $\{a_n\}$ is increasing sequence.

Therefore, it is monotonic sequence.

Since sequence $\{a_n\}$ is bounded and monotonic, by monotonic sequence theorem

$\{a_n\}$ is $\boxed{\text{convergent}}$.

To find limit of $\{a_n\}$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{n^3+1}$$

Divide the numerator and denominator by the highest power of n

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^3}}$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n^3}}$$

$$= \frac{1}{1+0}$$

$$= 1$$

Thus $\lim_{n\to\infty} \frac{n^3}{n^3+1} = \boxed{1}$

Q25E

We have $a_n = \dfrac{3+5n^2}{n+n^2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3+5n^2}{n+n^2}$$

Divide the numerator and denominator by n^2

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{3}{n^2}+5}{\frac{1}{n}+1} = \frac{0+5}{0+1} = 5$$

Thus a_n converges and $\boxed{\lim_{n\to\infty} a_n = 5}$

Q26E

Let $\{a_n\}$ be the sequence.

The nth term of sequence is defined as $a_n = \frac{n^3}{n+1}$.

The objective is to determine whether the sequence convergent or not.

Recollect that if $\lim_{n\to\infty} a_n$ exists then sequence converges otherwise diverges.

Consider the expression, $\lim_{n\to\infty} \frac{n^3}{n+1}$.

Divide both numerator and denominator by highest power of n that occurs in denominator.

$$\lim_{n\to\infty} \frac{n^3}{n+1} = \lim_{n\to\infty} \frac{\frac{n^3}{n}}{\frac{n+1}{n}}$$

$$= \lim_{n\to\infty} \frac{n^2}{1+\frac{1}{n}}$$

$$= \frac{\lim_{n\to\infty} n^2}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}} \text{ Since } \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

$$= \frac{\infty}{1+0}$$

$$= \infty$$

That is $\lim_{n\to\infty} \frac{n^3}{n+1} = \infty$.

So, $\lim_{n\to\infty} \frac{n^3}{n+1}$ does not exist.

Therefore, the sequence defined by $a_n = \frac{n^3}{n+1}$ diverges.

Q27E

Consider the sequence

$a_n = e^{1/n}$

Its need to determine whether the sequence is converges or diverges. If it converges, find the limit.

The terms of the given sequence are

$e, e^{1/2}, e^{1/3}, e^{1/4}, e^{1/5}, \cdots$

We have that

$\cdots < e^{1/5} < e^{1/4} < e^{1/3} < e^{1/2} < e$

From the above relation, we observe that, terms of the sequence $a_n = e^{1/n}$ are decreasing as n increases.

Thus, given sequence is a decreasing sequence and it is bounded above by e.

Hence, $a_n = e^{1/n}$ is a monotonic sequence ...... (1)

Since exponential of a quantity is always greater than zero, so it is bounded below by zero.

Thus, we get

$0 < e^{1/n} < e$ for all $n \in N$

That is, the sequence $a_n = e^{1/n}$ is a bounded sequence. ...... (2)

From (1) and (2), we observe that the sequence $a_n = e^{1/n}$ is a bounded and monotonic sequence.

By using the fact that, "Every bounded and monotonic sequence is convergent" we confirm that $a_n = e^{1/n}$ is a $\boxed{\text{convergent sequence}}$.

To find the limit of the sequence $a_n = e^{1/n}$:

Suppose that $f(x) = e^x$ and $b_n = \frac{1}{n}$

Observe that $f(x) = e^x$ is everywhere continuous function.

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n}$

$= 0$

Recall the theorem that, if $\lim_{n\to\infty} b_n = L$ and the function f is continuous at L, then

$\lim_{n\to\infty} f(b_n) = f(L)$

Since $\lim_{n\to\infty} b_n = 0$, $f(x) = e^x$ is everywhere continuous function in particular it is continuous at 0, by the above theorem we confirm that

$\lim_{n\to\infty} e^{1/n} = \lim_{n\to\infty} f(b_n)$ Use $b_n = 1/n$

$= f(0)$ Use $\lim_{n\to\infty} b_n = 0$

$= e^0$ Replace x by 0 in $f(x) = e^x$

$= 1$

Thus $\lim_{n\to\infty} e^{1/n} = \boxed{1}$.

.

Q28E

Consider the following sequence:

$$a_n = \frac{3^{n+2}}{5^n}$$

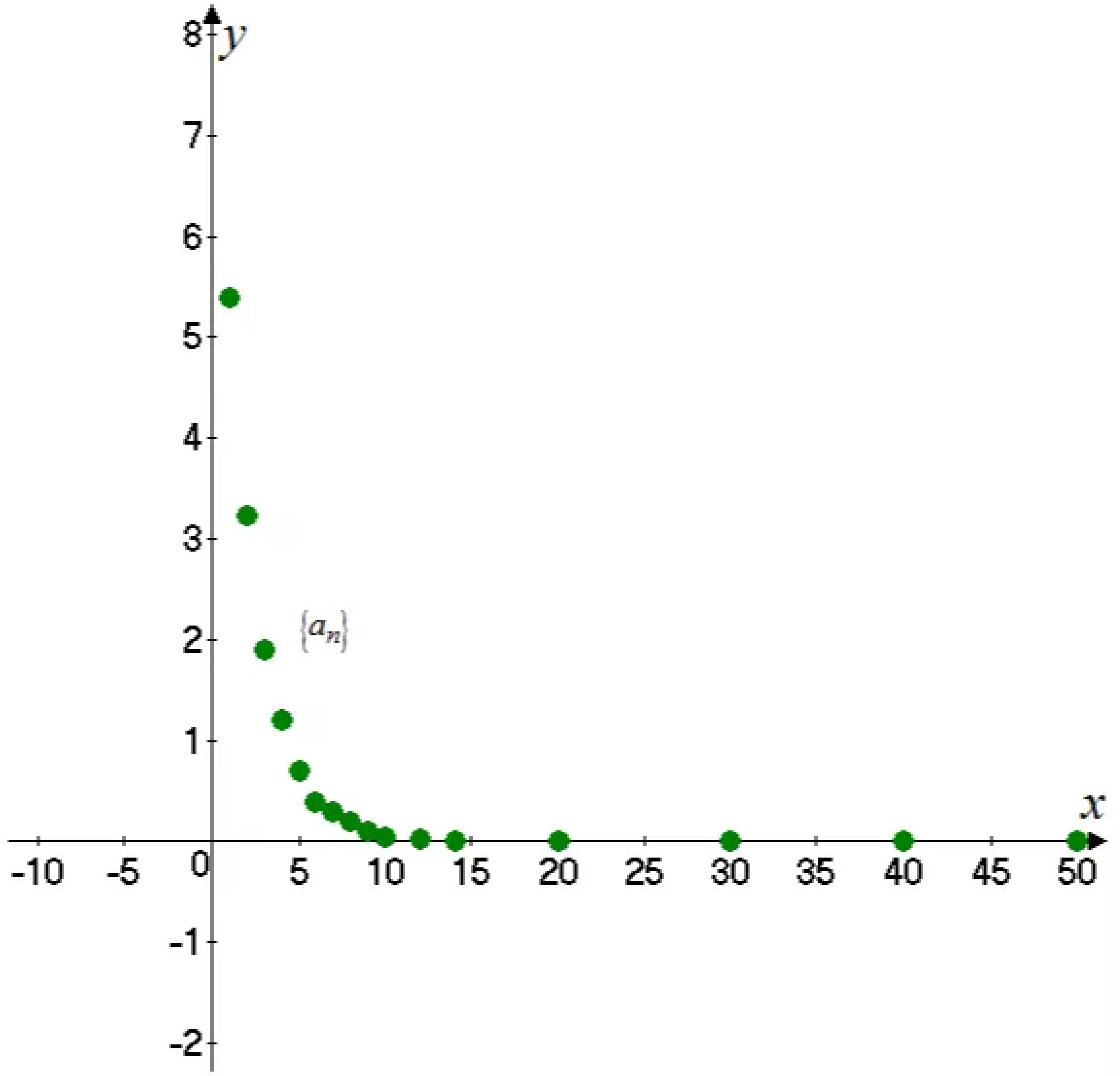
Write the terms of the sequence:

$$\{5.4, 3.24, 1.9, 1.2, 0.7, 0.4, 0.3, 0.2, 0.1, 0.09, 0.05, 0.03, 0.02, 0.01, 0.007, 0.004, 0.002, \ldots\}$$

The table below shows the sequence of terms:

n	$a_n = \frac{3^{n+2}}{5^n}$
1	5.4
2	3.24
3	1.9
4	1.2
5	0.7
⋮	⋮
10	0.0544
20	0.00032
30	0.0000019
40	0.000000012
50	0.000000000072

The graph of the sequence $\{a_n\} = \left\{\frac{3^{n+2}}{5^n}\right\}$ is as shown below:



From the graph, notice that a_n approaches 0.

Thus $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^{n+2}}{5^n}$ exists; that is, the sequence $\{a_n\} = \left\{\frac{3^{n+2}}{5^n}\right\}$ is convergent.

Find the limit of the sequence $\{a_n\}$ as shown below:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\frac{3^{n+2}}{5^n}\right]$$
$$= \lim_{n\to\infty}\left[\frac{3^n \cdot 3^2}{5^n}\right]$$
$$= 9\lim_{n\to\infty}\left[\frac{3^n}{5^n}\right]$$
$$= 9\lim_{n\to\infty}\left(\frac{3}{5}\right)^n$$
$$= 9\cdot 0$$
$$= 0$$

Therefore, the limit of the sequence $\{a_n\} = \left\{\frac{3^{n+2}}{5^n}\right\}$ is $\boxed{0}$.

Q29E

Consider the sequence

$$a_n = \tan\left(\frac{2n\pi}{1+8n}\right)$$

The objective is to determine whether the sequence with $a_n = \tan\left(\frac{2n\pi}{1+8n}\right)$ is converges or diverges.

Continuity and Convergence Theorem:

If $\lim_{n\to\infty} a_n = L$ and the function *f* is continuous at *L*, then $\lim_{n\to\infty} f(a_n) = f(L)$.

As the tangent function is continuous at $\frac{\pi}{4}$, so the theorem can be applied.

According to the theorem, write the limit as follows:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\tan\left(\frac{2n\pi}{1+8n}\right)\right)$$
$$= \tan\left(\lim_{n\to\infty}\left(\frac{2n\pi}{1+8n}\right)\right)$$
$$= \tan\left(\lim_{n\to\infty}\left(\frac{2n\pi}{n\left(\frac{1}{n}+8\right)}\right)\right)$$
$$= \tan\left(\lim_{n\to\infty}\left(\frac{2\pi}{\left(\frac{1}{n}+8\right)}\right)\right)$$
$$= \tan\left(\frac{\pi}{4}\right)$$
$$= 1$$

Thus, $a_n = \tan\left(\frac{2n\pi}{1+8n}\right)$ converges to 1, as $n \to \infty$.

Q30E

Consider the following sequence:

$$a_n = \sqrt{\frac{n+1}{9n+1}}$$

To determine if a sequence converges, take the limit of each term as $n \to \infty$.

If this limit exists, then the sequence converges otherwise the sequence diverges.

Now, $\lim_{n\to\infty}\sqrt{\frac{n+1}{9n+1}} = \frac{\infty}{\infty}$.

Using the power law, rewrite the limit as under:

$$\lim_{n\to\infty}\sqrt{\frac{n+1}{9n+1}} = \sqrt{\lim_{n\to\infty}\frac{n+1}{9n+1}}$$
$$= \frac{\infty}{\infty}$$

Since the limit is undefined, use L'Hospital's Rule.

According to L'Hospital's Rule, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\infty}{\infty}$, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\frac{d}{dn} f(n)}{\frac{d}{dn} g(n)}$.

Take the derivative of both the numerator and denominator.

$\frac{d}{dn}\left(a \cdot x^n\right) = a \cdot nx^{n-1}$ And

$\frac{d}{dn} c = 0$, where c is a constant

Thus, $\frac{d}{dn}(n+1) = 1$

$\frac{d}{dn}(9n+1) = 9$

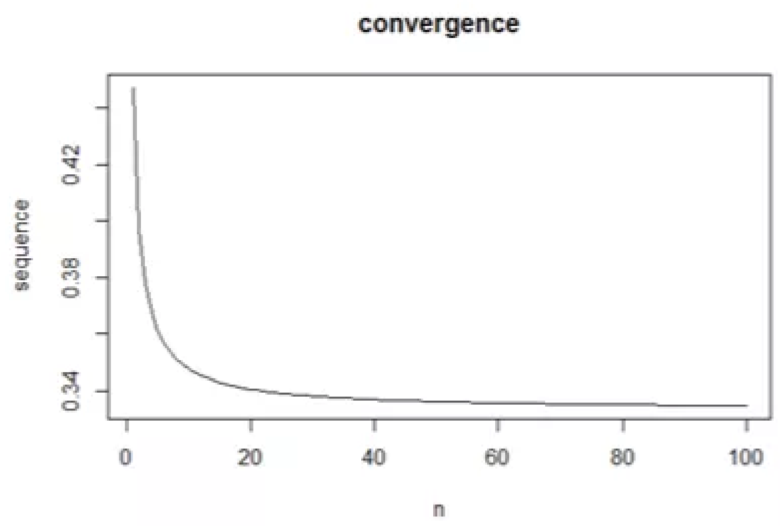
Therefore, $\lim_{n\to\infty} \sqrt{\frac{n+1}{9n+1}} = \sqrt{\lim_{n\to\infty} \frac{n+1}{9n+1}}$.

$= \sqrt{\frac{1}{9}}$

$= \frac{1}{3}$

Thus, the sequence converges, and its limit is $\boxed{\frac{1}{3}}$.

Sketch a graph to show the convergence as shown below:



Q31E

Given sequence

$$a_n = \frac{n^2}{\sqrt{n^3+4n}}$$

We divide numerator and denominators by n^2

Therefore

$$\lim_{t\to\infty} a_n = \lim_{t\to\infty} \frac{n^2/n^2}{\sqrt{\frac{n^3}{n^4}+\frac{4n}{n4}}}$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{\frac{1}{n}+\frac{4}{n^3}}}$$

$$= \frac{1}{\sqrt{0+0}}$$

$$= \infty$$

Hence the given sequence is divergent.

Q32E

Given sequence

$$a_n = e^{\frac{2n}{(n+2)}}$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\left(\frac{2n}{n+2}\right)}$$

$$= e^{\lim_{n\to\infty}\left(\frac{2n}{n+2}\right)}$$

$$= e^{\lim_{n\to\infty}\left(\frac{2}{1+\frac{2}{n}}\right)}$$

$$= e^{\frac{2}{1+0}}$$

$$= e^2$$

The sequence $\{a_n\}$ converge to e^2

Q33E

Consider the sequence

$$a_n = \frac{(-1)^n}{2\sqrt{n}}$$

Write out the terms of the sequence a_n, we obtain

$$\left\{\frac{(-1)^1}{2\sqrt{1}}, \frac{(-1)^2}{2\sqrt{2}}, \frac{(-1)^3}{2\sqrt{3}}, \frac{(-1)^4}{2\sqrt{4}}, \frac{(-1)^5}{2\sqrt{5}}, \ldots\right\}$$
$$= \left\{-\frac{1}{2}, \frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{4}}, -\frac{1}{2\sqrt{5}}, \ldots\right\}$$

Ratioo between any two succeesive terms of the above sequence is always less than 1.
For example, ratio between frist two successive terms is

$$\frac{\frac{1}{2\sqrt{2}}}{-\frac{1}{2}} = -\frac{1}{\sqrt{2}} < 1$$

Hence it is **convergent sequence**.

Calculate the **limit of the sequence**:

First calculate the limit of the absolute value sequence

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{2\sqrt{n}}\right| = \lim_{n\to\infty}\frac{1}{2\sqrt{n}}$$
$$= 0 \qquad \text{As } n\to\infty,\ 1/\sqrt{n}\to 0$$

Recall the teorem that

" If $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$ "

By the above theorem,

$$\lim_{n\to\infty}\frac{(-1)^n}{2\sqrt{n}} = 0$$

Thus, the sequence **converges to** $\boxed{0}$.

Q34E

Consider the sequence a_n.

$$a_n = \frac{(-1)^{n+1} n}{n+\sqrt{n}}$$

Now, find $\lim_{n\to\infty} a_n$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1} n}{n+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{(-1)^{n+1}\frac{n}{n}}{\frac{n}{n}+\frac{\sqrt{n}}{n}}$$ Divide both numerator and denominator by n

$$= \lim_{n\to\infty} \frac{(-1)^{n+1}}{1+\frac{1}{\sqrt{n}}}$$ Simplify

$$= \lim_{n\to\infty} \left((-1)^{n+1} \cdot \frac{1}{1+\frac{1}{\sqrt{n}}} \right)$$ Since $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

$$= \left(\lim_{n\to\infty} (-1)^{n+1} \right) \cdot \left(\lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}} \right)$$

But $\lim_{n\to\infty} a_n$ does not exists.

Hence, the given sequence **does not converge**.

Q35E

Here $a_n = \cos\left(\frac{n}{2}\right)$

As n increases and tends to infinity. The value of $\cos\left(\frac{n}{2}\right)$ oscillates between -1 and +1 and does not tend to a unique value.

Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(n/2) =$ Does not exist.

So, Given sequence is **Divergent**

Q36E

Given $a_n = \cos\left(\frac{2}{n}\right)$

As the value of n increases, the value of 2/n decreases continuously and $\lim_{n\to\infty} \frac{2}{n} = 0$

And so, $\lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{2}{n}\right) = \cos 0 = 1$

i.e. $\lim_{n\to\infty} \cos\left(\frac{2}{n}\right)$ tends to unique value 1.

Hence, the given sequence is $\boxed{\text{convergent}}$ and it's $\boxed{\text{Limit is } 1}$.

Q37E

We have $a_n = \frac{(2n-1)!}{(2n+1)!}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!}$$

$$= \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots(2n-1)}{1\cdot 2\cdot 3\cdots(2n-1)(2n)(2n+1)} \qquad (\text{since } n! = 1\cdot 2\cdot 3\cdots n)$$

$$= \lim_{n\to\infty} \frac{1}{(2n+1)(2n)} \qquad (\text{cancelling the terms upto } (2n-1))$$

$$= \frac{1}{\lim_{n\to\infty}(2n+1).\lim_{n\to\infty}(2n)}$$

$$= \frac{1}{\infty}$$

$$= 0$$

Thus the sequence a_n is convergent and $\boxed{\lim_{n\to\infty} a_n = 0}$

Q38E

We have $a_n = \frac{\ln n}{\ln 2n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{\ln 2n} \qquad [\text{Form of } (\infty/\infty)]$$

Applying L- Hospitals Rule

$$\lim_{n\to\infty} \frac{\ln x}{\ln 2x} = \lim_{n\to\infty} \frac{1/x}{(1/2x).2}$$

$$= \lim_{n\to\infty} 1$$

$$= 1$$

Thus a_n is convergent and $\boxed{\lim_{n\to\infty} a_n = 1}$

Q39E

We have $a_n = \dfrac{e^n + e^{-n}}{e^{2n} - 1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n + e^{-n}}{e^{2n} - 1}$$

Divide numerator and demimonde by e^{2n}

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n / e^{2n} + e^{-n} / e^{2n}}{1 - 1/e^{2n}}$$

$$= \lim_{n\to\infty} \frac{e^{-n} + e^{-3n}}{1 - e^{-2n}}$$

$$= \frac{0+0}{1-0} \qquad \left[\text{Since } e^{-x} \to 0 \text{ as } x \to \infty\right]$$

$$= 0$$

Thus a_n is convergent and $\boxed{\lim_{n\to\infty} a_n = 0}$

Q40E

Given sequince

$$a_n = \frac{\tan^{-1}(n)}{n}$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\tan^{-1}(n)}{n}$$

$$= \left(\lim_{n\to\infty} \tan^{-1}(n)\right) \cdot \left(\lim_{n\to\infty} \frac{1}{n}\right)$$

$$= \left(\frac{\pi}{2}\right) \cdot (0)$$

$$= 0$$

The given sequence converges to 0.

Q41E

We have $a_n = n^2 e^{-n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 e^{-n}$$

$$= \lim_{n\to\infty} \frac{n^2}{e^n} \qquad \left[\text{Form of } (\infty/\infty)\right]$$

Applying L- Hospital's Rule

$$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}$$

$$=\lim_{x\to\infty}\frac{2}{e^x} \quad \text{[Again by L- Hospital's Rule]}$$

$$=\frac{2}{\infty}$$

$$=0$$

Then $\lim_{n\to\infty}\frac{n^2}{e^n}=0$

Thus a_n is convergent and $\boxed{\lim_{n\to\infty}a_n=0}$

Q42E

We have $a_n=\ln(n+1)-\ln n$

$$=\ln\left(\frac{n+1}{n}\right)$$

$$=\ln\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)$$

$$=\ln(1+0)$$

$$=\ln(1)$$

$$=0$$

Thus a_n is convergent and $\boxed{\lim_{n\to\infty}a_n=0}$

We have $a_n=\frac{\cos^2 n}{2^n}$

Since $-1\le\cos x\le 1$ for all values of x

So $0\le\cos^2 x\le 1$ for all values of x

Then $0\le\cos^2 n\le 1$ for all values of n

Therefore $0\le\frac{\cos^2 n}{2^n}\le\frac{1}{2^n}$ for all values of n

Since $\lim_{n\to\infty}\frac{1}{2^n}=0$ and $\frac{1}{2^n}$ convergent

Then by Squeeze Theorem, the sequence $a_n=\frac{\cos^2 n}{2^n}$ must be $\boxed{\text{convergent}}$ and

$\boxed{\lim_{n\to\infty}a_n=0}$

Q44E

Consider the sequence

$a_n = \sqrt[n]{2^{1+3n}}$ ...... (1)

Its need to determine whether the sequence is converges or diverges. If it converges, find the limit.

On rewriting the given sequence, we have that

$\sqrt[n]{2\cdot 2^{3n}} = \sqrt[n]{2\cdot\left(2^3\right)^n}$

$= \sqrt[n]{2\cdot 8^n}$

$= \left(8^n\right)^{1/n} 2^{1/n}$

$= 8\cdot 2^{1/n}$

Thus $\sqrt[n]{2^{1+3n}} = 8\cdot 2^{1/n}$ ...... (2)

The terms of the sequence $a_n = 8\cdot 2^{1/n}$ are

$8\cdot 2^1, 8\cdot 2^{1/2}, 8\cdot 2^{1/3}, 8\cdot 2^{1/4}, 8\cdot 2^{1/5}, \cdots$

We have that

$\cdots < 8\cdot 2^{1/5} < 8\cdot 2^{1/4} < 8\cdot 2^{1/3} < 8\cdot 2^{1/2} < 8\cdot 2$

From the above relation, we observe that, terms of the sequence $a_n = 8\cdot 2^{1/n}$ are decreasing as n increases.

Thus, given sequence is a decreasing sequence and it is bounded above by $16\ (=8\cdot 2)$.

Hence, $a_n = 8\cdot 2^{1/n}$ is a monotonic sequence ...... (3)

Since exponential of a quantity is always greater than zero, so the sequence $a_n = 8\cdot 2^{1/n}$ is bounded below by zero.

Thus, we get

$0 < 8\cdot 2^{1/n} < 16$ for all $n \in N$

That is, the sequence $a_n = 8\cdot 2^{1/n}$ is a bounded sequence. ...... (4)

From (3) and (4), we observe that the sequence $a_n = 8\cdot 2^{1/n}$ is a bounded and monotonic sequence.

By using the fact that, "Every bounded and monotonic sequence is convergent" we confirm that $a_n = 8\cdot 2^{1/n}$ is a convergent sequence. ...... (5)

From (2) and (5), we observe that $a_n = \sqrt[n]{2^{1+3n}}$ is a $\boxed{\text{convergent sequence}}$.

To find the limit of the sequence $a_n = 8\cdot 2^{1/n}$:

Suppose that $f(x) = 8\cdot 2^x$ and $b_n = \frac{1}{n}$

Observe that $f(x) = 8\cdot 2^x$ is everywhere continuous function.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n}$$

$$= 0$$

Recall the theorem that, if $\lim_{n\to\infty} b_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(b_n) = f(L)$$

Since $\lim_{n\to\infty} b_n = 0$, $f(x) = 8\cdot 2^x$ is everywhere continuous function in particular it is continuous at 0, by the above theorem we confirm that

$\lim_{n\to\infty} 8\cdot 2^{1/n} = \lim_{n\to\infty} f(b_n)$ Use $b_n = 1/n$

$= f(0)$ Use $\lim_{n\to\infty} b_n = 0$

$= 8\cdot 2^0$ Replace x by 0 in $f(x) = 8\cdot 2^x$

$= 8$

Thus $\lim_{n\to\infty} 8\cdot 2^{1/n} = 8$ ...... (6)

From (2) and (6), we observe that $\lim_{n\to\infty} \sqrt[n]{2^{1+3n}} = \boxed{8}$

Q45E

Consider the sequence,

$$a_n = n\sin\left(\frac{1}{n}\right)$$

The objective is to determine whether the given sequences convergent or divergent.

A sequence is convergent if $\lim_{n\to\infty} a_n$ exists and if does not exists the sequence is said to be divergent.

The sequence $a_n = n\sin\left(\frac{1}{n}\right)$ is convergent or divergent can be found out as follows:

Apply the limit as n tends to infinity to the given sequence.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\sin\left(\frac{1}{n}\right)$$
$$= \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \qquad \text{(Rewrite)}$$

Use L'hospital Rule since the sequence has $\frac{0}{0}$ form when apply the limit n tends to infinity.

Therefore,

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{-\frac{1}{n^2}\cos\left(\frac{1}{n}\right)}{-\frac{1}{n^2}}$$
$$= \lim_{n\to\infty} \cos\left(\frac{1}{n}\right)$$
$$= \cos(0)$$
$$= \boxed{1}$$

The limit exists.

Hence, the sequence $a_n = n\sin\left(\frac{1}{n}\right)$ is **convergent**.

The limit is $\boxed{1}$.

Q46E

Given sequence

$$a_n = 2^{-n}\cos n\pi$$

We know that if

$|a_n| \le b_n,\ b_n \to 0$, then $a_n \to 0$

We have

$$\left|\frac{\cos n\pi}{2^n}\right| \le \frac{1}{2^n} \text{ and } \frac{1}{2^n} \to 0$$

Hence $\lim_{n\to\infty} \frac{\cos n\pi}{2^n} = 0$

Therefore given sequence converges to 0.

Q47E

Consider the sequence, $a_n = \left(1+\frac{2}{n}\right)^n$

The objective is to determine whether the sequence converges.

If $\{a_n\}$ is a sequence and $\lim_{n\to\infty} a_n$ exist then sequence converges.

Use the following formula:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Let $j = \frac{n}{2}$. Then $j \to \infty$ as $n \to \infty$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$$

$$= \lim_{j\to\infty}\left(1+\frac{1}{j}\right)^{2j}$$

$$= \lim_{j\to\infty}\left(1+\frac{1}{j}\right)^j \cdot \left(1+\frac{1}{j}\right)^j$$

$$= \lim_{j\to\infty}\left(1+\frac{1}{j}\right)^j \cdot \lim_{j\to\infty}\left(1+\frac{1}{j}\right)^j$$ Multiplication property of limits

$= e^2$ Since $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

That is $\lim_{n\to\infty} a_n = e^2$

Therefore the sequence $a_n = \left(1+\frac{2}{n}\right)^n$ converges.

Q48E

We have $a_n = \frac{\sin 2n}{1+\sqrt{n}}$

Taking limit as $n \to \infty$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\sin 2n}{1+\sqrt{n}}$$

$$= \frac{\lim_{n\to\infty}\sin 2n}{\lim_{n\to\infty}\left(1+\sqrt{n}\right)}$$

Since value of sin2n oscillates between -1 and 1 for any value of n which is any finite number and $\left(1+\sqrt{n}\right)\to\infty \quad as \quad n\to\infty$

So $\quad \lim_{n\to\infty} a_n = \dfrac{(\text{finite value})}{\lim_{n\to\infty}\left(1+\sqrt{n}\right)}$

$= 0$

So $\quad \boxed{\lim_{n\to\infty} a_n = 0}$

Since limit of sequence exists so given sequence is $\boxed{convergent}$

Q49E

Consider the sequence

$$a_n = \ln\left(2n^2+1\right)-\ln\left(n^2+1\right)$$

Its need to determine whether the sequence is converges or diverges. If it converges, find the limit.

On rewriting the given sequence, we have that

$$a_n = \ln\left(\frac{2n^2+1}{n^2+1}\right) \text{ Use } \ln f(x)-\ln g(x)=\ln\frac{f(x)}{g(x)}$$

To determine whether the given sequence is increasing or decreasing, consider the function

$$f(x)=\ln\left(\frac{2x^2+1}{x^2+1}\right)$$

Then

$$f'(x)=\frac{1}{\left(\frac{2x^2+1}{x^2+1}\right)}\frac{d}{dx}\left(\frac{2x^2+1}{x^2+1}\right)$$

$$=\frac{\left(x^2+1\right)(4x)-\left(2x^2+1\right)(2x)}{\left(x^2+1\right)^2}\cdot\frac{x^2+1}{2x^2+1}$$

$$=\frac{2x}{\left(x^2+1\right)^2}\cdot\frac{x^2+1}{2x^2+1}>0 \quad \forall x\in N$$

So, $f'(x)>0 \quad \forall \quad x\in[1,\infty)$

Thus f in increasing on $[1,\infty)$ and so $f(n)<f(n+1)$

Therefore $\{a_n\}$ is increasing sequence.

Hence, $\{a_n\}$ is a monotonic sequence ...... (1)

Observe that

$2n^2+1>n^2+1$ for all $n\in N$

$\frac{2n^2+1}{n^2+1}>1$ for all $n\in N$

$\ln\left(\frac{2n^2+1}{n^2+1}\right)>\ln 1$ for all $n\in N$

$\Rightarrow\ 0<\ln\left(\frac{2n^2+1}{n^2+1}\right)$ for all $n\in N$

That is, the sequence $a_n=\ln\left(\frac{2n^2+1}{n^2+1}\right)$ is a bounded below by 0. ...... (2)

Also, observe that

$\frac{n^2}{n^2+1}<1$ for all $n\in N$

$\Rightarrow\ 1+\frac{n^2}{n^2+1}<1+1$ for all $n\in N$

$\Rightarrow\ \frac{n^2+1+n^2}{n^2+1}<1+1$ for all $n\in N$

$\Rightarrow\ \frac{2n^2+1}{n^2+1}<2$ for all $n\in N$

$\Rightarrow\ \ln\left(\frac{2n^2+1}{n^2+1}\right)<\ln 2$ for all $n\in N$

$\Rightarrow\ a_n<\ln 2$ for all $n\in N$

That is, the sequence $a_n=\ln\left(\frac{2n^2+1}{n^2+1}\right)$ is a bounded above by $\ln 2$. ...... (3)

From (2) and (3), we observe that

$0<\ln\left(\frac{2n^2+1}{n^2+1}\right)<\ln 2$ for all $n\in N$

That is, the sequence $a_n=\ln\left(\frac{2n^2+1}{n^2+1}\right)$ is a bounded sequence. ...... (4)

From (1) and (4), we observe that the sequence $a_n=\ln\left(\frac{2n^2+1}{n^2+1}\right)$ is a bounded and monotonic sequence.

By using the fact that, "Every bounded and monotonic sequence is convergent" we confirm that $a_n=\ln\left(\frac{2n^2+1}{n^2+1}\right)$ is a convergent sequence.

To find the limit of the sequence $a_n = \ln\left(\frac{2n^2+1}{n^2+1}\right)$:

Suppose that $g(x) = \ln x$ and $b_n = \frac{2n^2+1}{n^2+1}$

Observe that $g(x) = \ln x$ is continuous on $(0,\infty)$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2n^2+1}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{n^2\left(2+\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{2+\frac{1}{n^2}}{1+\frac{1}{n^2}}$$

$$= \frac{2+0}{1+0}$$

$$= 2$$

Thus $\lim_{n\to\infty} b_n = 2$

Recall the theorem that, if $\lim_{n\to\infty} b_n = L$ and the function g is continuous at L, then

$$\lim_{n\to\infty} g(b_n) = g(L)$$

Since $\lim_{n\to\infty} b_n = 2$, $g(x) = \ln x$ is continuous on $(0,\infty)$, in particular it is continuous at $x = 2$, by the above theorem we confirm that

$\lim_{n\to\infty} \ln\left(\frac{2n^2+1}{n^2+1}\right) = \lim_{n\to\infty} g(b_n)$ Use $b_n = \frac{2n^2+1}{n^2+1}$

$= g(2)$ Use $\lim_{n\to\infty} b_n = 2$

$= \ln 2$ Replace x by 2 in $g(x) = \ln x$

Thus $\lim_{n\to\infty} \ln\left(\frac{2n^2+1}{n^2+1}\right) = \boxed{\ln 2}$

Q50E

Consider the following sequence:

$$a_n = \frac{(\ln n)^2}{n}$$

The objective is to determine whether the sequence is convergent or divergent.

If it is converges, need to find its limit.

Use the following result:

Let $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$, where n is an integer.

Let $f(x) = \frac{(\ln x)^2}{x}$

In this fraction, notice that the numerator and denominator both approaches to ∞ as $x \to \infty$.

Use L'Hospital's rule, to find the limit.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Find the limit.

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{(\ln x)^2}{x} \\ &= \lim_{x\to\infty} \frac{2\ln x \frac{d}{dx}(\ln x)}{1} \\ &= \lim_{x\to\infty} \frac{2\ln x \frac{1}{x}}{1} \\ &= \lim_{x\to\infty} \frac{2\ln x}{x}\end{aligned}$$

Again apply the L'Hospital's Rule to find the limit, because the numerator and denominator both approaches to ∞.

$$\begin{aligned}\lim_{x\to\infty} \frac{2\ln x}{x} &= \lim_{x\to\infty} \frac{2\cdot\frac{1}{x}}{1} \\ &= \lim_{x\to\infty} \frac{2}{x} \\ &= 0\end{aligned}$$

Therefore, the given sequence is convergent and the limit of the sequence is, $\boxed{\lim_{n\to\infty} \frac{(\ln n)^2}{n} = 0}$.

Q51E

Given that $a_n = \arctan(\ln n)$

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan^{-1}(\ln n)$$

As $n \to \infty$ we have $\ln n \to \infty$

and as $\ln n \to \infty$ we have $\tan^{-1}(\ln n) \to \pi/2$

Hence $\lim_{n\to\infty} a_n = \dfrac{\pi}{2}$

The sequence $\{a_n\}$ convergent to $\dfrac{\pi}{2}$.

Q52E

Consider the sequence,

$$a_n = n - \sqrt{n+1}\sqrt{n+3}$$

Need to determine whether the sequence is convergent or divergent.

Note that the sequence $\{a_n\}$ is **convergent** if the limit $\lim_{n\to\infty} a_n$ **exists**, **divergent** if the limit **does not exist**.

Rewrite the sequence by rationalizing the denominator as,

$$a_n = n - \sqrt{n+1}\sqrt{n+3}$$

$$= \left(n - \sqrt{n+1}\sqrt{n+3}\right) \cdot \frac{\left(n + \sqrt{n+1}\sqrt{n+3}\right)}{\left(n + \sqrt{n+1}\sqrt{n+3}\right)}$$

$$= \frac{\left(n^2 - (n+1)(n+3)\right)}{\left(n + \sqrt{n+1}\sqrt{n+3}\right)} \quad \text{Since}\,(a-b)(a+b) = a^2 - b^2$$

$$= \frac{n^2 - \left(n^2 + 3n + n + 3\right)}{\left(n + \sqrt{n+1}\sqrt{n+3}\right)}$$

$$= \frac{\cancel{n^2} - \cancel{n^2} - 3n - n - 3}{\left(n + \sqrt{n+1}\sqrt{n+3}\right)}$$

$$= \frac{-(4n+3)}{n + \sqrt{n+1}\sqrt{n+3}}$$

Now evaluate the limit of the sequence to test for the convergence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-(4n+3)}{\left(n+\sqrt{n+1}\sqrt{n+3}\right)}$$

$$= \lim_{n\to\infty} \frac{-(4n+3)}{\left[n+\left(\sqrt{n}\cdot\sqrt{1+\frac{1}{n}}\right)\left(\sqrt{n}\sqrt{1+\frac{3}{n}}\right)\right]}$$

$$= \lim_{n\to\infty} \frac{-(4n+3)}{n+n\cdot\sqrt{1+\frac{1}{n}}\cdot\sqrt{1+\frac{3}{n}}} \qquad \text{Since } \sqrt{n}\sqrt{n}=n$$

$$= \lim_{n\to\infty} \frac{-n\left(4+\frac{3}{n}\right)}{n\left(1+\sqrt{1+\frac{1}{n}}\sqrt{1+\frac{3}{n}}\right)}$$

$$= \frac{-(4+3\cdot 0)}{1+\sqrt{1+0}\cdot\sqrt{1+0}} \qquad \text{Since } \frac{1}{n}\to 0 \text{ as } n\to\infty$$

$$= \frac{-4}{1+(1\cdot 1)}$$

$$= \frac{-4}{2}$$

$$= -2$$

Thus,

$\lim_{n\to\infty} a_n = -2$, a finite value.

Hence the limit of the sequence exists.

Therefore, the sequence $\{a_n\}$ converges to -2

Q53E

We have $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\ldots\ldots\ldots\}$

Since the sequence takes only two values 0 and 1

Therefore the sequence a_n does not approach any single number as the number of terms increases.

So it is $\boxed{\text{divergent}}$.

Q54E

We have $a_n = \left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \ldots\ldots\right\}$

Since $a_{2n-1} = \frac{1}{n}$ and $a_{2n} = \frac{1}{n+2}$ for all n

So given sequence is the combination of these two subsequences

Since $\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} \frac{1}{n} = 0$

And $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{1}{n+2} = 0$

So $\boxed{\lim_{n\to\infty} a_n = 0}$, therefore the given sequence converges

Q55E

We have $a_n = \frac{n!}{2^n}$ where $n! = 1\cdot 2\cdot 3\cdot 4\cdots n$

Then $a_n = \frac{1\cdot 2\cdot 3\cdot 4\cdots n}{2\cdot 2\cdot 2\cdot 2\cdots \text{upto } n \text{ times}}$

Since numerator > denominator for large value of n

[For example if we take n = 5 then numerator = 120 and denominator = 32

or if we take n = 10 then numerator = 3628800 and denominator = 1024]

So $a_n \to \infty$ as $n \to \infty$

Therefore sequence $a_n = \frac{n!}{2^n}$ $\boxed{\text{diverges}}$

Q56E

We have $a_n = \frac{(-3)^n}{n!}$

Then $|a_n| = \frac{3^n}{n!}$

Since Numerator < Denominator for large value of n (n >7)

For example if we take n= 7 then numerator = 3^7 = 2187, denominator = 7!=5040

And if we take n = 15 then Numerator = 3^{15} = 14348907,

Denominator = 15! = 1307674368000

So $\boxed{\lim_{n\to\infty} a_n = 0}$

Thus given sequence is convergent and converges to 0

Q57E

Consider the sequence

$$a_n = 1 + (-2/e)^n$$

Its need to determine whether the sequence is convergent or divergent by the graph of the sequence. If the sequence is convergent, guess the value of the limit from the graph and then prove this guess.

To graph the sequence $a_n = 1 + (-2/e)^n$, use Maple software.

First enter the sequence by using the expression command.

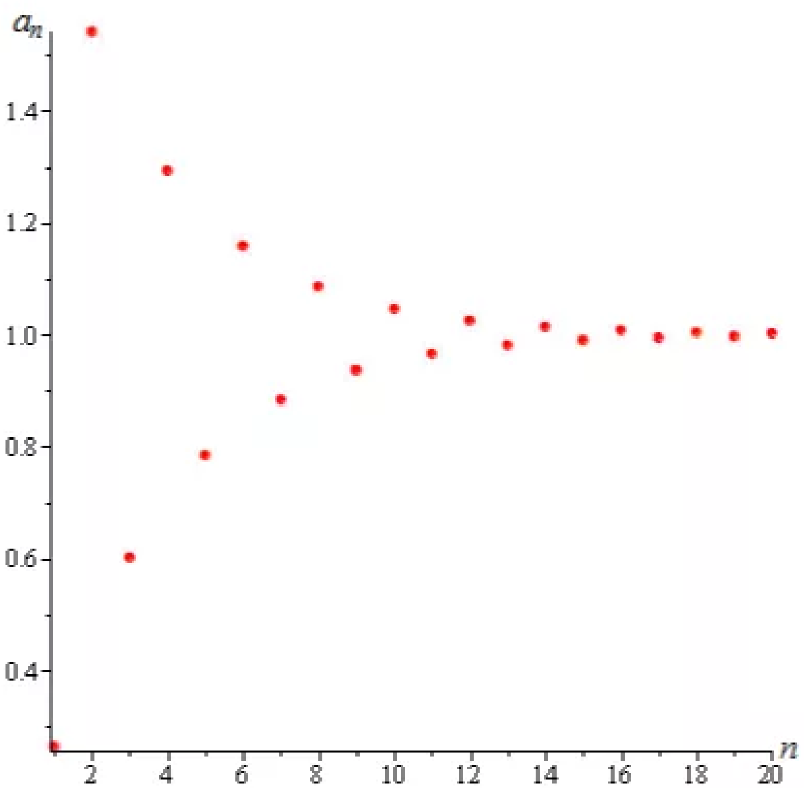
$$> \ expr := 1 + \left(\frac{-2}{\exp(1)}\right)^n$$

$$1 + \left(-\frac{2}{e}\right)^n$$

Use the following Maple command to get the plot of the sequence.

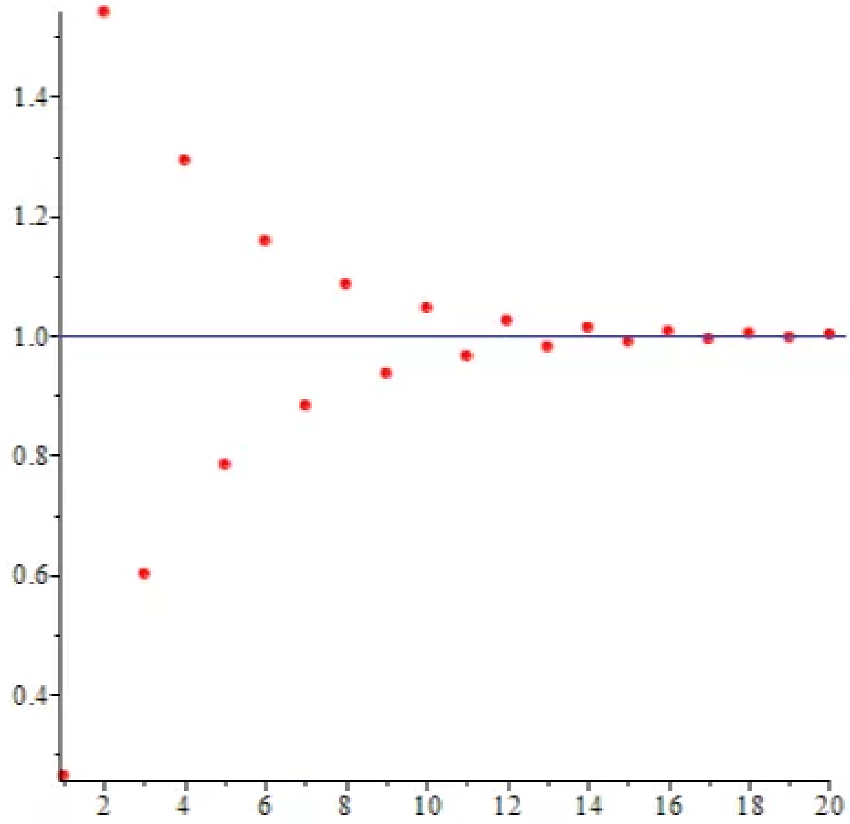
Plot([seq([n,expr],n=1..20)],style=point,symbol=solidcircle,color=red)

$> \ plot([seq([n, expr], n = 1..20)], style = point, symbol = solidcircle, color = red)$



From the graph observe that, terms of the sequence $a_n = 1 + (-2/e)^n$ are approaches to single value as n increases. So, the given sequence is convergent.

Draw a horizontal line on the graph at $a_n = 1$ as shown below.



From the graph observe that, terms of the sequence $a_n = 1 + (-2/e)^n$ as n increases are in the neighborhood of 1.

So, the given sequence has limit 1. ...... (1)

We have that

$$-2 < e(\approx 2.3) \Rightarrow -\frac{2}{e} < 1$$

So

$$\lim_{n\to\infty}\left[1+(-2/e)^n\right] = \lim_{n\to\infty} 1 + \lim_{n\to\infty}(-2/e)^n$$

$= 1 + 0$ Use the fact that, $a^n \to 0$ as $n \to \infty$ if $a < 1$

$= 1$

Thus $\lim_{n\to\infty}\left[1+(-2/e)^n\right] = 1$ ...... (2)

From arguments (1) and (2), it can be observed that given sequence has limit 1 by graphing the sequence and by evaluating limit of it by analytically.

Q58E

Consider the sequence

$$a_n = \sqrt{n}\sin\left(\pi/\sqrt{n}\right)$$

Its need to determine whether the sequence is convergent or divergent by the graph of the sequence. If the sequence is convergent, guess the value of the limit from the graph and then prove this guess.

To graph the sequence $a_n = \sqrt{n}\sin\left(\pi/\sqrt{n}\right)$, use Maple software.

First enter the sequence by using the expression command.

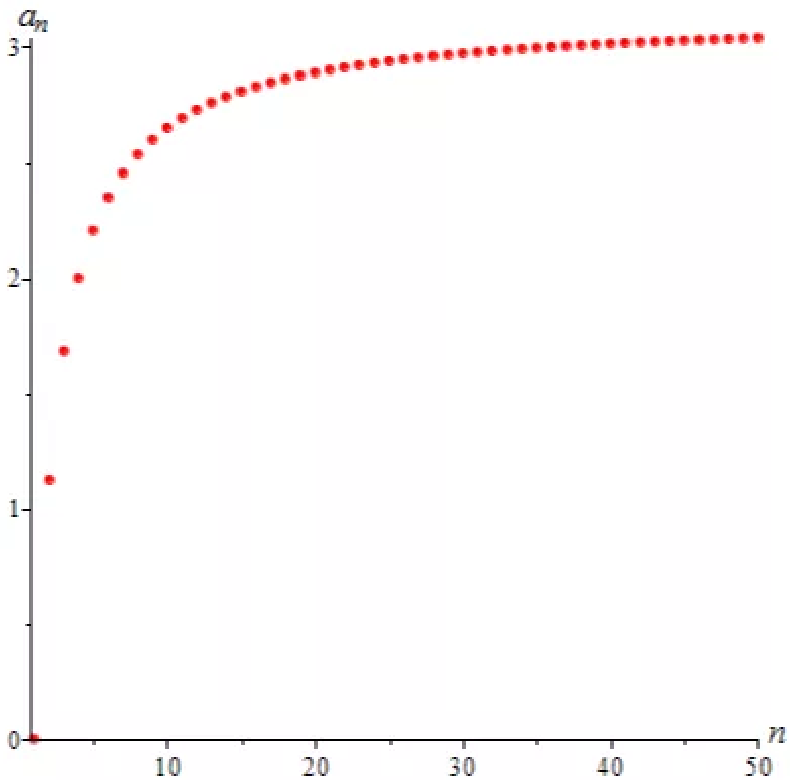
$$> expr := \sqrt{n}\sin\left(\frac{\text{Pi}}{\sqrt{n}}\right)$$

$$\sqrt{n}\sin\left(\frac{\pi}{\sqrt{n}}\right)$$

Use the following Maple command to get the plot of the sequence.

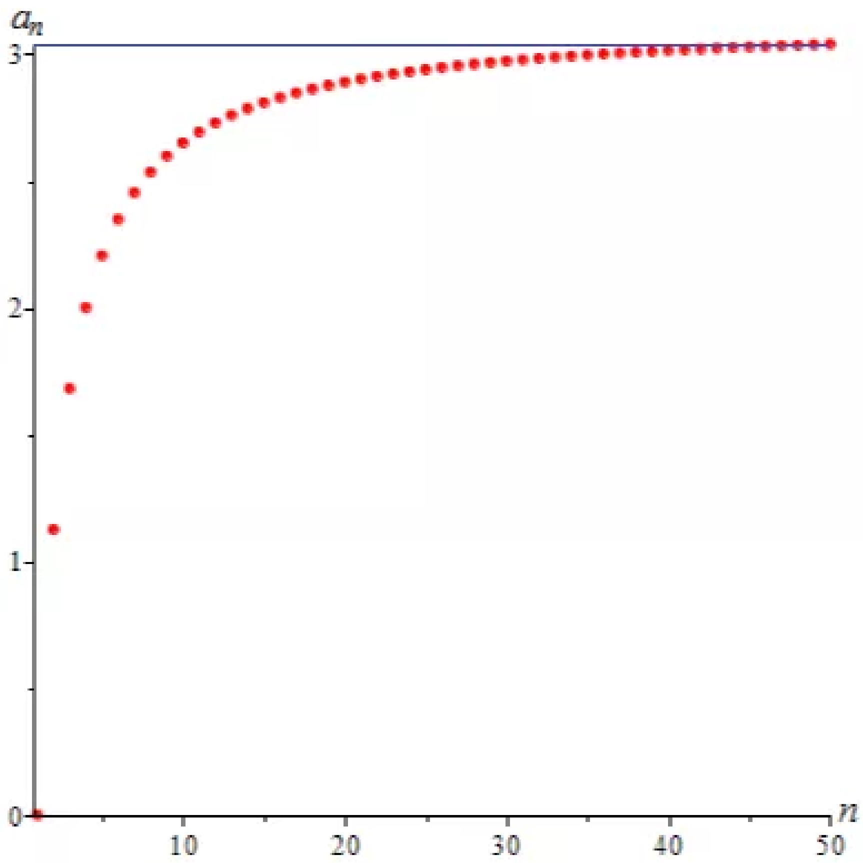
Plot([seq([n,expr],n=1..70)],style=point,symbol=solidcircle,color=red)

$$> plot([seq([n, expr], n = 1..70)], style = point, symbol = solidcircle, color = red)$$



From the graph observe that, terms of the sequence $a_n = \sqrt{n}\sin\left(\pi/\sqrt{n}\right)$ are approaches to single value as n increases. So, the given sequence is convergent.

Draw a horizontal line on the graph at $a_n = 3.1$ as shown below.



From the graph observe that, terms of the sequence $a_n = \sqrt{n}\sin\left(\pi/\sqrt{n}\right)$ as n increases are in the neighborhood of 3.1.

So, the given sequence has limit 3.1. ...... (1)

Consider the limit

$$\lim_{n\to\infty}\sqrt{n}\sin\left(\pi/\sqrt{n}\right)=\lim_{n\to\infty}\frac{\sin\left(\pi/\sqrt{n}\right)}{\left(1/\sqrt{n}\right)}$$

$$=\lim_{n\to\infty}\pi\cdot\frac{\sin\left(\pi/\sqrt{n}\right)}{\left(\pi/\sqrt{n}\right)}$$

$$=\pi\lim_{n\to\infty}\frac{\sin\left(\pi/\sqrt{n}\right)}{\left(\pi/\sqrt{n}\right)}$$

As $n\to\infty$, $\sqrt{n}\to\infty$ and $1/\sqrt{n}\to 0$, also $\pi/\sqrt{n}\to 0$

$$=\pi\lim_{\left(\pi/\sqrt{n}\right)\to 0}\frac{\sin\left(\pi/\sqrt{n}\right)}{\left(\pi/\sqrt{n}\right)}$$

$=\pi\cdot 1$ Use $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

$=\pi \quad (\approx 3.14)$

Thus $\lim_{n\to\infty}\sqrt{n}\sin\left(\pi/\sqrt{n}\right)=\pi$ ...... (2)

From arguments (1) and (2), it can be observed that given sequence has limit $\pi \quad (\approx 3.14)$ by graphing the sequence and by evaluating limit of it by analytically.

Q59E

Consider the sequence

$$a_n=\sqrt{\frac{3+2n^2}{8n^2+n}}$$

Its need to determine whether the sequence is convergent or divergent by the graph of the sequence. If the sequence is convergent, guess the value of the limit from the graph and then prove this guess.

To graph the sequence $a_n=\sqrt{\frac{3+2n^2}{8n^2+n}}$, use Maple software.

First enter the sequence by using the expression command.

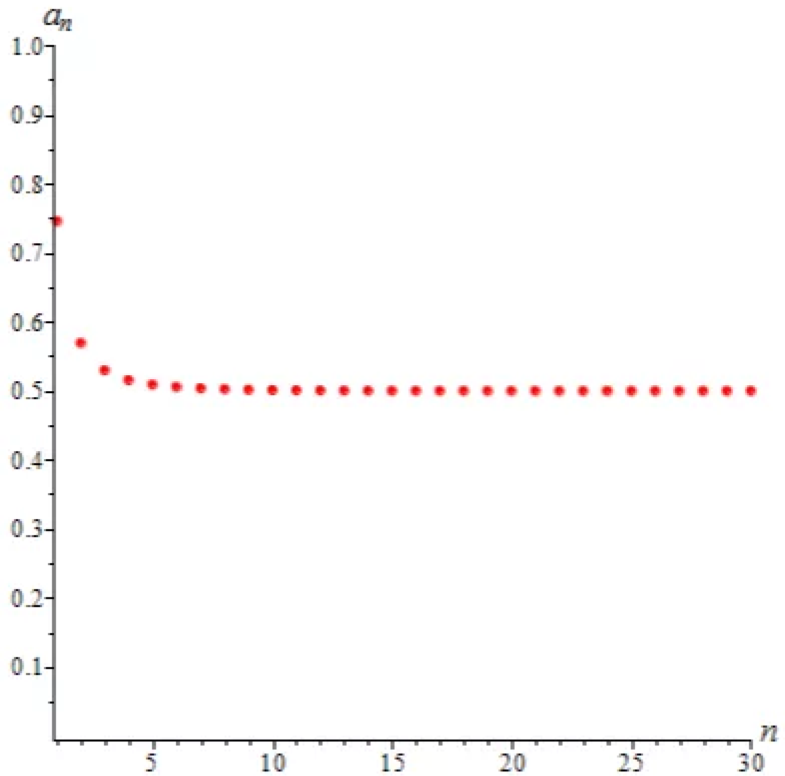
> $expr := \sqrt{\frac{3+2n^2}{8n^2+n}}$

$$\sqrt{\frac{2n^2+3}{8n^2+n}}$$

Use the following Maple command to get the plot of the sequence.

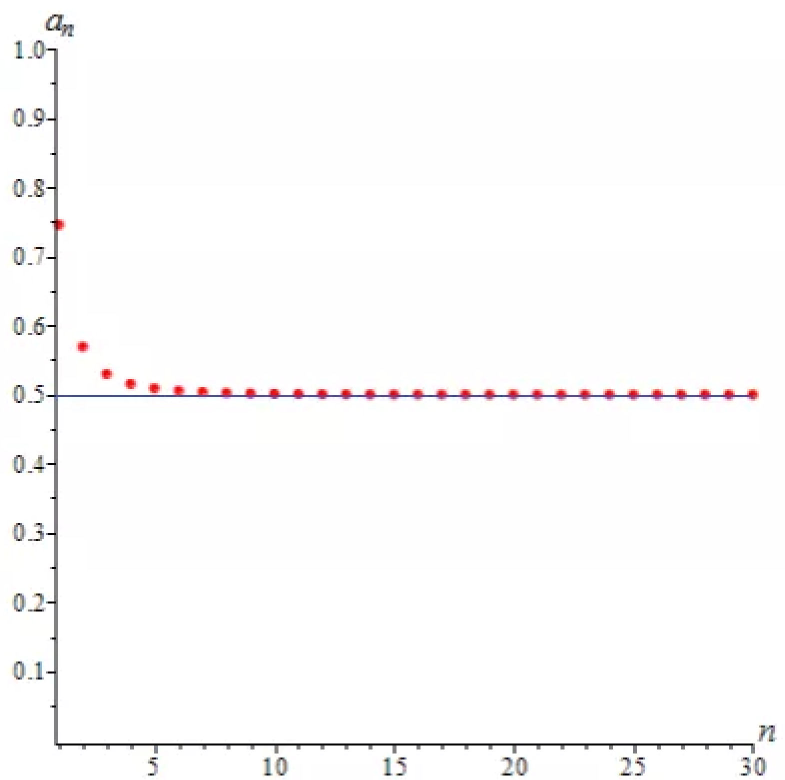
Plot([seq([n,expr],n=1..30)],style=point,symbol=solidcircle,color=red)

```
> plot([seq([n, expr], n = 1 ..30)], style = point, symbol = solidcircle, color = red)
```



From the graph observe that, terms of the sequence $a_n = \sqrt{\frac{3+2n^2}{8n^2+n}}$ are approaches to single value as n increases. So, the given sequence is convergent.

Draw a horizontal line on the graph at $a_n = 0.5$ as shown below.



From the graph observe that, terms of the sequence $a_n = \sqrt{\frac{3+2n^2}{8n^2+n}}$ as n increases are in the neighborhood of 0.5.

So, the given sequence has limit $1/2$. ...... (1)

Consider the limit

$$\lim_{n\to\infty}\sqrt{\frac{3+2n^2}{8n^2+n}} = \sqrt{\lim_{n\to\infty}\frac{3+2n^2}{8n^2+n}}$$

Factor out the highest degree term from the numerator and denominator

$$= \sqrt{\lim_{n\to\infty}\frac{n^2\left(\frac{3}{n^2}+2\right)}{n^2\left(8+\frac{1}{n}\right)}}$$

$$= \sqrt{\lim_{n\to\infty}\frac{\frac{3}{n^2}+2}{8+\frac{1}{n}}}$$

As $n\to\infty$, $1/n\to 0$ and $8/n^2\to 0$

$$\sqrt{\lim_{n\to\infty}\frac{\frac{3}{n^2}+2}{8+\frac{1}{n}}} = \sqrt{\frac{0+2}{8+0}}$$

$$= \sqrt{\frac{1}{4}}$$

$$= \frac{1}{2}$$

Thus $\lim_{n\to\infty}\sqrt{\frac{3+2n^2}{8n^2+n}} = \frac{1}{2}$ ...... (2)

From arguments (1) and (2), it can be observed that given sequence has limit $1/2$ by graphing the sequence and by evaluating limit of it by analytically.

Q60E

Consider the sequence

$$a_n = \sqrt[n]{3^n + 5^n}$$

Its need to determine whether the sequence is convergent or divergent by the graph of the sequence. If the sequence is convergent, guess the value of the limit from the graph and then prove this guess.

To graph the sequence $a_n = \sqrt[n]{3^n + 5^n}$, use Maple software.

First enter the sequence by using the expression command.

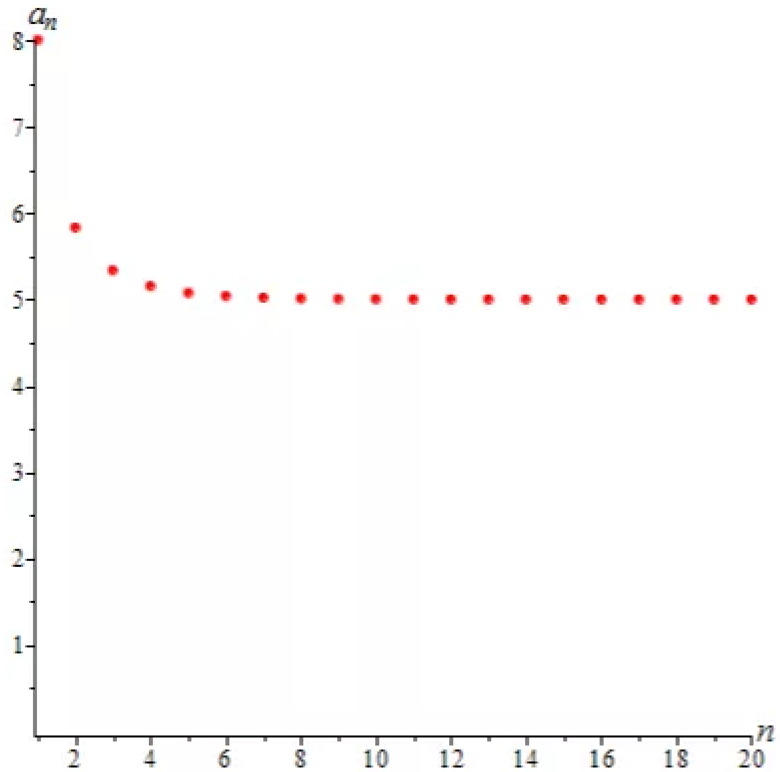
> $expr := \sqrt[n]{3^n + 5^n}$

$$(3^n + 5^n)^{\frac{1}{n}}$$

Use the following Maple command to get the plot of the sequence.

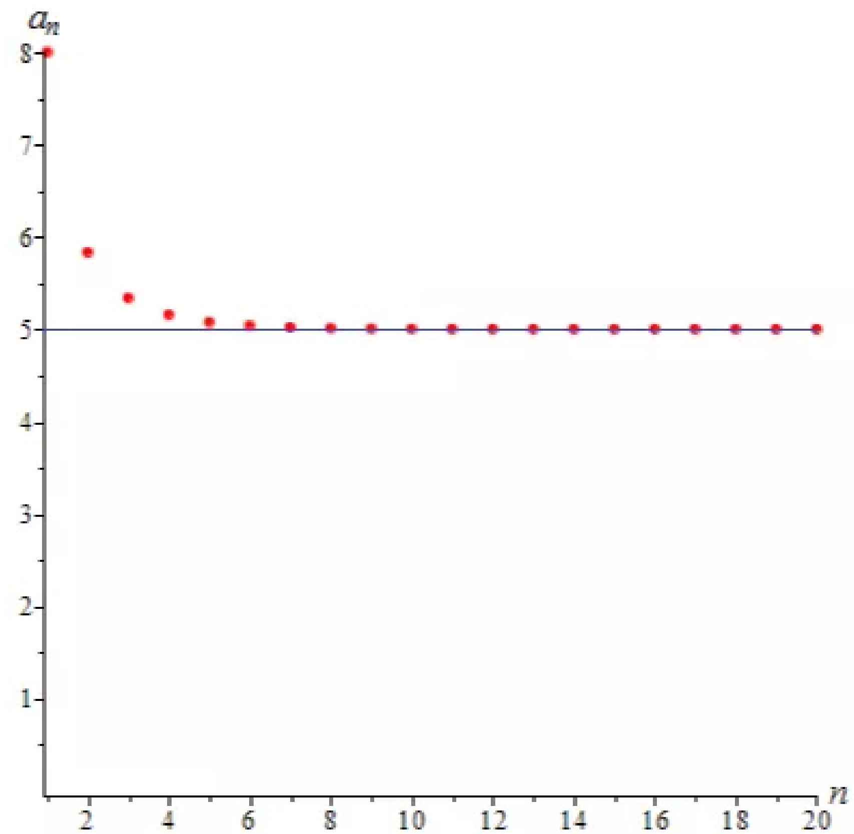
Plot([seq([n,expr],n=1..20)],style=point,symbol=solidcircle,color=red)

> $plot([seq([n, expr], n = 1..20)], style = point, symbol = solidcircle, color = red)$



From the graph observe that, terms of the sequence $a_n = \sqrt[n]{3^n + 5^n}$ are approaches to single value as n increases. So, the given sequence is convergent.

Draw a horizontal line on the graph at $a_n = 5$ as shown.



From the graph observe that, terms of the sequence $a_n = \sqrt[n]{3^n + 5^n}$ as n increases are in the neighborhood of 5.

So, the given sequence has limit 5. ...... (1)

Consider the limit

$$\lim_{n\to\infty}\sqrt[n]{3^n+5^n}=\lim_{n\to\infty}\left(3^n+5^n\right)^{\frac{1}{n}}$$

$$=\lim_{n\to\infty}\left(5^n\right)^{\frac{1}{n}}\left(\left(\frac{3}{5}\right)^n+1\right)^{\frac{1}{n}}$$

$$=\lim_{n\to\infty}5\left(\left(\frac{3}{5}\right)^n+1\right)^{\frac{1}{n}}$$

$$=5\lim_{n\to\infty}\left(\left(\frac{3}{5}\right)^n+1\right)^{\frac{1}{n}}$$

As $n\to\infty$, $a^n\to 0$ if $a<1$

Observe that, $\frac{3}{5}<1$

So, as $n\to\infty$, $\left(\frac{3}{5}\right)^n\to 0$

Thus

$$5\lim_{n\to\infty}\left(\left(\frac{3}{5}\right)^n+1\right)^{\frac{1}{n}}=5\left(0+1\right)^0$$

$$=5(1)$$

$$=5$$

Therefore $\lim_{n\to\infty}\sqrt[n]{3^n+5^n}=5$ ...... (2)

From arguments (1) and (2), it can be observed that given sequence has limit 5 by graphing the sequence and by evaluating limit of it by analytically.

Q61E

Consider the sequence

$$a_n = \frac{n^2 \cos n}{1+n^2}$$

Its need to determine whether the sequence is convergent or divergent by the graph of the sequence. If the sequence is convergent, guess the value of the limit from the graph and then prove this guess.

To graph the sequence $a_n = \frac{n^2 \cos n}{1+n^2}$, use Maple software.

First enter the sequence by using the expression command.

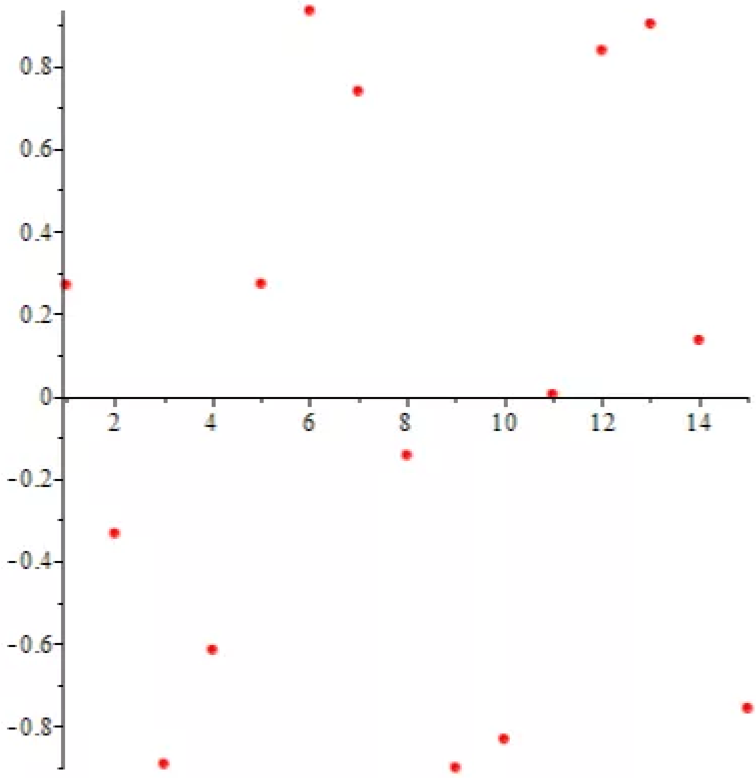
> $expr := \frac{n^2 \cdot \cos(n)}{1+n^2}$

$$\frac{n^2 \cos(n)}{n^2+1}$$

Use the following Maple command to get the plot of the sequence.

Plot([seq([n,expr],n=1..15)],style=point,symbol=solidcircle,color=red)

> *plot*([*seq*([*n*, *expr*], *n* = 1 ..15)], *style* = *point*, *symbol* = *solidcircle*, *color* = *red*)



From the graph observe that, terms of the sequence $a_n = \frac{n^2 \cos n}{1+n^2}$ are does not approaches to a single value as n increases.

So, the given sequence is divergent. ...... (1)

Consider the limit

$$\lim_{n\to\infty}\frac{n^2\cos n}{1+n^2}=\lim_{n\to\infty}\frac{n^2}{1+n^2}\times\lim_{n\to\infty}(\cos n)$$

$$=\lim_{n\to\infty}\frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)}\times\lim_{n\to\infty}(\cos n)$$

$$=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n^2}\right)}\times\lim_{n\to\infty}(\cos n)$$

$$=\frac{1}{(1+0)}\times(\text{Undefined})$$

$$=\text{Undefined}$$

Thus $\lim_{n\to\infty}\frac{n^2\cos n}{1+n^2}$ is undefined. ...... (2)

From arguments (1) and (2), it can be observed that given sequence is divergent by graphing the sequence and by evaluating limit of it by analytically.

Q62E

Given sequence $a_n=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}$

To compute the terms of this sequence, let us write the n$^{\text{th}}$ term as

$$a_n=\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)\cdot 2\cdot 4\cdot 6\cdot\ldots\cdot 2n}{2\cdot 4\cdot 6\cdot\ldots\cdot 2n\cdot n!}$$

$$=\frac{(2n)!}{2^n\cdot 1\cdot 2\cdot 3\cdot\ldots\cdot n\cdot(n!)}$$

$$=\frac{(2n)!}{2^n(n!)^2}$$

Using this term, we consider the various terms of the sequence and draw the corresponding graph.

$$a_1=\frac{2!}{2^1\cdot(1!)^2}=1$$

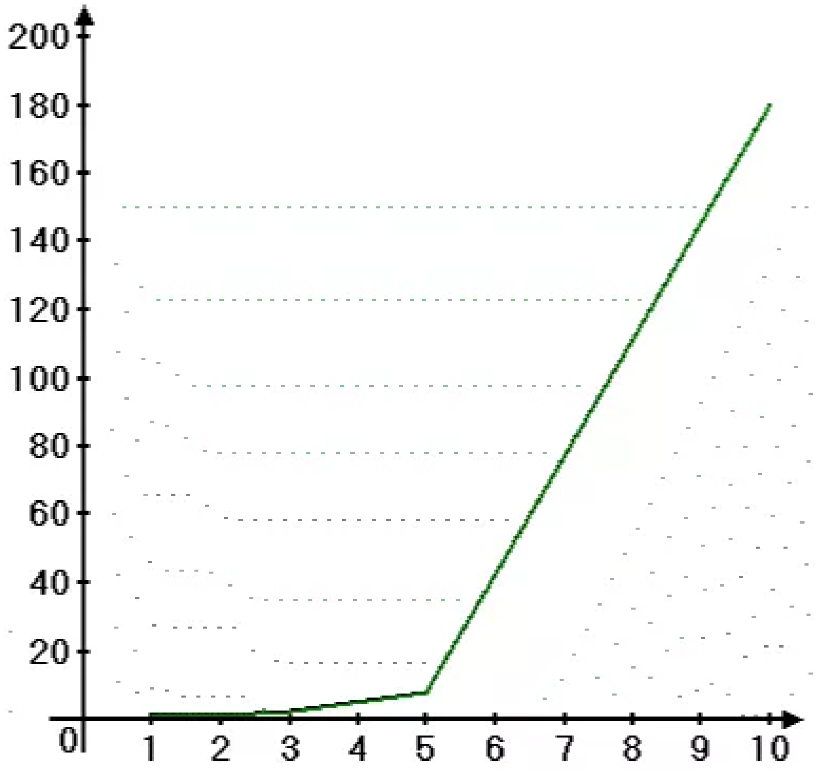
$$a_2=\frac{4!}{2^2\cdot(2!)^2}=1.5$$

$$a_3=\frac{6!}{2^3\cdot(3!)^2}=2.5$$

$$a_5=\frac{10!}{2^5\cdot(5!)^2}=7.875$$

$$a_{10}=\frac{20!}{2^{10}\cdot(10!)^2}=180.42,\ldots$$

The graph of the sequence is



From the graph of the sequence it can be observed that the slope of the curve is increasing alarmingly and so, goes ∞ and thus, the sequence diverges.

Now

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}$$
$$= \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 2\cdot 3\cdots n}$$
$$= \lim_{n\to\infty} \frac{3}{2}\cdot\frac{5}{3}\cdot\frac{7}{4}\cdots(2-\frac{1}{n})$$

we know that $\lim_{n\to\infty} \frac{1}{n} = 0$ and each of the fraction is greater than 1, there are infinite such fractions in the product.

So, the product is $= \infty$

Thus, the sequence is divergent

Q63E

Consider the sequence

$$a_n = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n)^n}$$

Its need to determine whether the sequence is convergent or divergent by the graph of the sequence. If the sequence is convergent, guess the value of the limit from the graph and then prove this guess.

Observe that, numerator of the given sequence is the product of the first $2n-1$ odd terms.

Recollect that, product of the first n natural numbers is $\frac{n(n+1)}{2}$

So, product of the first $2n-1$ odd terms is

$$1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1) = \frac{(2n-1)(2n-1+1)}{2}$$

$$= n(2n-1)$$

Thus, given sequence can be written as

$$a_n = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n)^n}$$

$$= \frac{n(2n-1)}{(2n)^n}$$

To graph the sequence $a_n = \frac{n\cdot(2n-1)}{(2n)^n}$, use Maple software.

First enter the sequence by using the expression command.

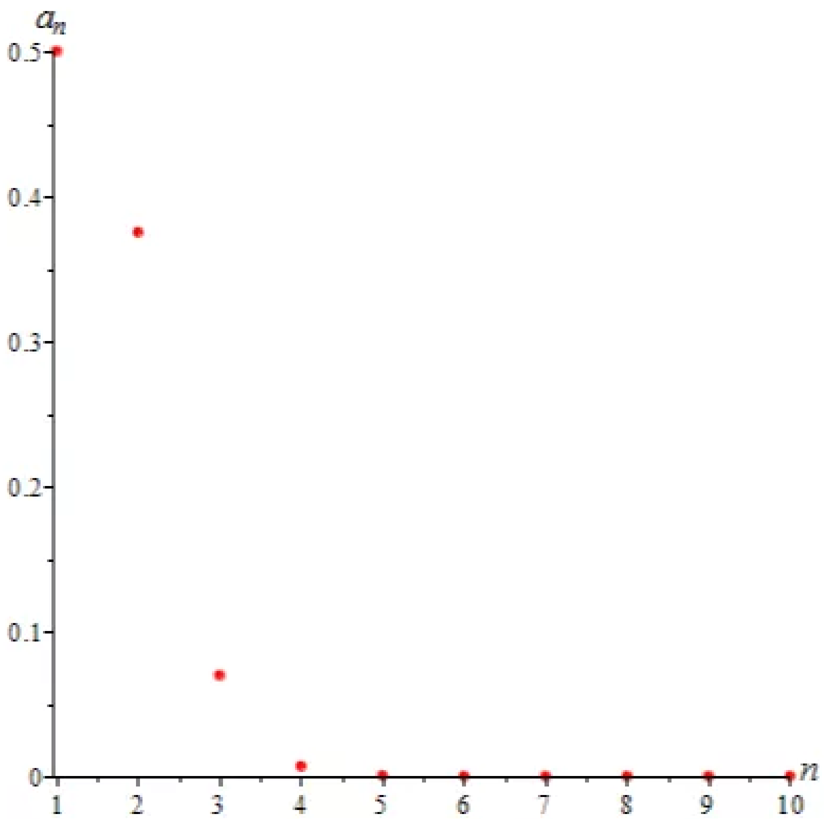
$$> expr := \frac{n\cdot(2n-1)}{(2n)^n}$$

$$\frac{n(2n-1)}{(2n)^n}$$

Use the following Maple command to get the plot of the sequence.

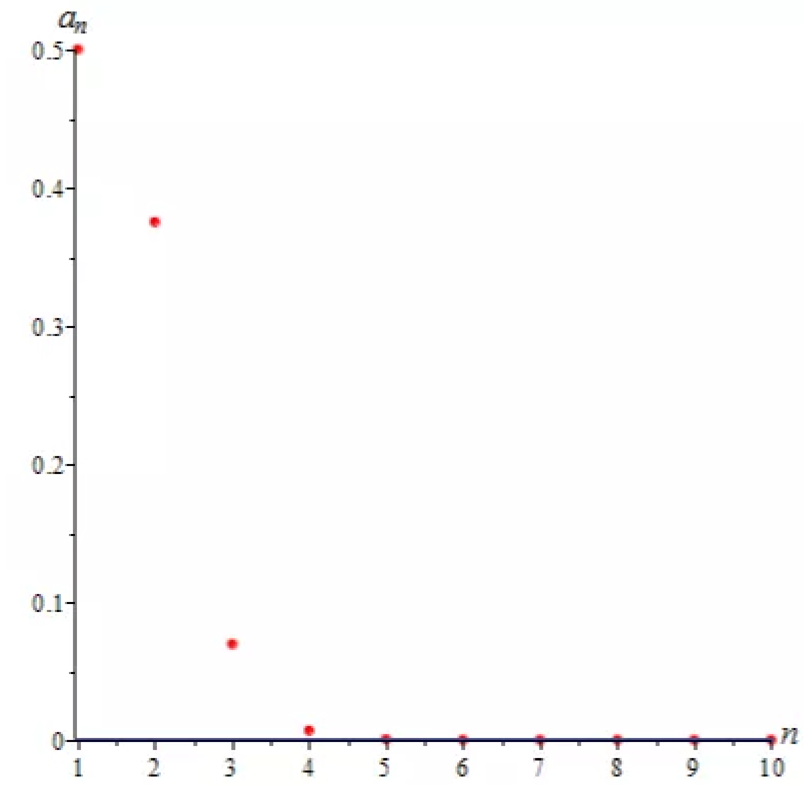
Plot([seq([n,expr],n=1..10)],style=point,symbol=solidcircle,color=red)

```
> plot([seq([n, expr], n = 1 ..10)], style = point, symbol = solidcircle, color = red)
```



From the graph observe that, terms of the sequence $a_n = \frac{n\cdot(2n-1)}{(2n)^n}$ are approaches to single value as n increases. So, the given sequence is convergent.

Draw a horizontal line on the graph at $a_n = 0.5$ as shown below.



From the graph observe that, terms of the sequence $a_n = \dfrac{n\cdot(2n-1)}{(2n)^n}$ as n increases are in the neighborhood of 0.

So, the given sequence has limit 0. ...... (1)

Consider the limit

$$\lim_{n\to\infty}\frac{n\cdot(2n-1)}{(2n)^n} = \lim_{n\to\infty}\frac{n\cdot n\left(2-\frac{1}{n}\right)}{2^n\cdot n^n}$$

$$= \lim_{n\to\infty}\frac{n^2\left(2-\frac{1}{n}\right)}{2^n\cdot n^n}$$

$$= \lim_{n\to\infty}\frac{2-\frac{1}{n}}{2^n\cdot n^{n-2}}$$

$$= \lim_{n\to\infty}\left(2-\frac{1}{n}\right)\cdot\lim_{n\to\infty}\frac{1}{2^n}\cdot\lim_{n\to\infty}\frac{1}{n^{n-2}}$$

$$=(2-0)\cdot 0\cdot 0$$

$$=0$$

Thus $\lim_{n\to\infty}\dfrac{n\cdot(2n-1)}{(2n)^n}=0$ ...... (2)

From arguments (1) and (2), it can be observed that given sequence is convergent by graphing the sequence and by evaluating limit of it by analytically.

Q64E

Consider the sequence:

$a_1 = 1 \quad a_{n+1} = 4 - a_n \quad \text{for } n \geq 1$

(a)

Find some terms a_2, a_3, a_4, a_5 of the above sequence.

$$\begin{aligned} a_2 &= 4 - a_1 \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} a_3 &= 4 - a_2 \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

$$\begin{aligned} a_4 &= 4 - a_3 \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} a_5 &= 4 - a_4 \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

From the above, the sequence is divergent because the given sequence oscillates between 1 and 3 forever.

Hence, the sequence is **divergent**.

(b)

Find some terms a_2, a_3, a_4, a_5 of the given sequence if the first term of the sequence is 2.

$$\begin{aligned} a_2 &= 4 - a_1 \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_3 &= 4 - a_2 \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_4 &= 4 - a_3 \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_5 &= 4 - a_4 \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

From the above, the sequence is convergent because the given sequence is bounded and have finite unit limit points. Hence, the sequence is **convergent**.

Q65E

We have $a_n = 1000(1.06)^n$

(A) $a_1 = 1000(1.06) = 1060$

$a_2 = 1000(1.06)^2 = 1123.60$

$a_3 = 1000(1.06)^3 = 1191.02$

$a_4 = 1000(1.06)^4 = 1262.48$

$a_5 = 1000(1.06)^5 = 1338.23$

(B)
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(1000)(1.06)^n$$
$$= 1000\lim_{n\to\infty}(1.06)^n$$
$$= \infty$$

Thus $\{a_n\}$ is $\boxed{\text{divergent}}$.

Q66E

(a)
Given

$$I_n = 100\left(\frac{(1.0025)^n - 1}{0.0025} - n\right)$$

$$I_1 = 100\left(\frac{(1.0025)^1 - 1}{0.0025} - 1\right) = 0$$

$$I_2 = 100\left(\frac{(1.0025)^2 - 1}{0.0025} - 2\right) = 0.0025$$

$$I_3 = 100\left(\frac{(1.0025)^3 - 1}{0.0025} - 3\right) = 0.0075$$

$$I_4 = 100\left(\frac{(1.0025)^4 - 1}{0.0025} - 4\right) = 0.0150$$

$$I_5 = 100\left(\frac{|(1.0025)|^5 - 1}{0.0025} - 5\right) = 0.0250$$

$$I_6 = 100\left(\frac{(1.0025)^6 - 1}{0.0025} - 6\right) = 0.0376$$

(b)
After 2 years n=24

$$\therefore I_{24} = 100\left(\frac{(1.0025)^{24} - 1}{0.0025} - 24\right) = .7028$$

Q67E

(a)
Initially, the fish fomer has 5000 fish.
Hence

$$P_o = 5000$$

Each month the number of catfish increses by 8% and 300 fish are utilized.
So after (n – 1) months the cat fish is P_{n-1}
Then

$$P_n = \left(P_{n-1} + \frac{(P_{n-1})\times 8}{100}\right) - 300$$

$$P_n = 1.08P_{n-1} - 300$$

(b)

$$P_0 = 5000$$
$$P_1 = 1.08P_0 - 300 = 5100$$
$$P_2 = 1.08P_1 - 300 = 5208$$
$$P_3 = 1.08P_2 - 300 = 5{,}324.64$$
$$P_4 = 1.08P_3 - 300 = 5{,}450.61$$
$$P_5 = 1.08P_4 - 300 = 5{,}586.66$$
$$P_6 = 1.08P_5 - 300 = 5733.59$$

After six months the number of fishes in the pond is 5734

Q68E

We start with $a_1 = 11$ which is an odd number

So for n = 1, $a_{1+1} = 3a_1 + 1$

$\Rightarrow \boxed{a_2 = 34}$ this is an even number

Then $a_3 = \frac{1}{2}a_2 = 17 \Rightarrow a_3 = 17$ this is an odd number.

Then $a_4 = 3a_3 + 1 = 52$ this is an even number

Then $a_5 = \frac{1}{2}a_4 = 26$ this is an even number

Then $a_6 = \frac{1}{2}a_5 = 13$ this is an odd number

Then $a_7 = 3a_6 + 1 = 40$ this is an even number

Then $a_8 = \frac{1}{2}a_7 = 20$ this is an even number

$\Rightarrow a_9 = \frac{1}{2}a_8 = 10$ this is an even number

$\Rightarrow a_{10} = \frac{1}{2}a_9 = 5$ this is an odd number

Similarly we find first 40 terms and make a table

a_1	11	a_{11}	16	a_{21}	1	a_{31}	4
a_2	34	a_{12}	8	a_{22}	4	a_{32}	2
a_3	17	a_{13}	4	a_{23}	2	a_{33}	1
a_4	52	a_{14}	2	a_{24}	1	a_{34}	4
a_5	26	a_{15}	1	a_{25}	4	a_{35}	2
a_6	13	a_{16}	4	a_{26}	2	a_{36}	1
a_7	40	a_{17}	2	a_{27}	1	a_{37}	4
a_8	20	a_{18}	1	a_{28}	4	a_{38}	2
a_9	10	a_{19}	4	a_{29}	2	a_{39}	1
a_{10}	5	a_{20}	2	a_{30}	1	a_{40}	4

Similarly we calculate first 40 terms with $a_1 = 25$

a_1	25	a_{11}	34	a_{21}	8	a_{31}	4
a_2	76	a_{12}	17	a_{22}	4	a_{32}	2
a_3	38	a_{13}	52	a_{23}	2	a_{33}	1
a_4	19	a_{14}	26	a_{24}	1	a_{34}	4
a_5	58	a_{15}	13	a_{25}	4	a_{35}	2
a_6	29	a_{16}	40	a_{26}	2	a_{36}	1
a_7	88	a_{17}	20	a_{27}	1	a_{37}	4
a_8	44	a_{18}	10	a_{28}	4	a_{38}	2
a_9	22	a_{19}	5	a_{29}	2	a_{39}	1
a_{10}	11	a_{20}	16	a_{30}	1	a_{40}	4

We see that (from both the tables) table 1 is contained in table 2. starts from a_{10}
In other words for a_1=11, a_1=11 and for a_1=25 , a_{10}= 11
Let first sequence be A_n and second sequence be B_n .Then relation between these sequences, that is $\boxed{B_{n+9} = A_n}$

Q69E

If $|r| \geq 1$ then $r^n \to \infty$ as $n \to \infty$

So $\{nr^n\} \to \infty$

And then $\{nr^n\}$ diverges

If $|r| < 1$ then

$$\lim_{x\to\infty} xr^x = \lim_{x\to\infty} \frac{x}{r^{-x}}$$

By L- hospital rule

$$\lim_{x\to\infty} \frac{x}{r^{-x}} = \lim_{x\to\infty} \frac{1}{(-\ln r)r^{-x}}$$

$$= \lim_{x\to\infty} \frac{r^x}{-\ln r}$$

$$= 0 \qquad \left[\begin{array}{l} \text{since } r < 1 \\ \text{then } r^x \to 0 \text{ as } x \to \infty \end{array}\right]$$

So $\lim_{n\to\infty} nr^n = 0 \quad for\ |r| < 1$

Thus $\boxed{\{nr^n\} \text{ converges when } |r| < 1}$

Q70E

(a)

Consider the convergent sequence, $\{a_n\}$.

If a sequence is convergent, then it has a limit.

Suppose the limit of the sequence $\{a_n\}$ is L.

Thus, $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$.

That is, if, for every $\varepsilon > 0$, there is a corresponding integer N such that:

If $n > N$, then $|a_n - L| < \varepsilon$.

$n > N$ so, $n+1 > N$. So, $|a_{n+1} - L| < \varepsilon$

Hence, $\lim_{n\to\infty} a_{n+1} = L$

Therefore, $\boxed{\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}}$.

(b)

Consider the sequence, $\{a_n\}$.

Suppose the sequence $\{a_n\}$ is convergent.

From the part (a)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{1+a_n}\right)$$ Use $a_n = \frac{1}{1+a_n}$

$$\lim_{n\to\infty} a_n = \frac{1}{\lim_{n\to\infty}(1+a_n)}$$

$$\lim_{n\to\infty} a_n = \frac{1}{1+\lim_{n\to\infty} a_n}$$

$L = \frac{1}{1+L}$ Use $\lim_{n\to\infty} a_n = L$

$L^2 + L = 1$ Multiply with $(1+L)$ on both sides.

$$L^2 + L - 1 = 0$$

To solve the quadratic equation, $L^2 + L - 1 = 0$, use the formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

Here, $a = 1, b = 1, c = -1$

$$L = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1}$$

$$L = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$L = \frac{-1 \pm \sqrt{5}}{2}$$

$$L = \frac{-1+\sqrt{5}}{2} \quad \text{or} \quad L = \frac{-1-\sqrt{5}}{2}$$

The sequence can be defined as $a_1 = 1$ and $a_n = \frac{1}{(1+a_n)}$ for $n \geq 1$.

The sequence is $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots$

That is the terms are reducing numerically but are all positive.

Hence, as $n \to \infty$, the limit L will be a very small positive number. It cannot be negative.

Hence, the limit of the sequence $\{a_n\}$ is, $\boxed{L = \frac{\sqrt{5}-1}{2}}$.

Q71E

Given that $\{a_n\}$ is decreasing sequence

So $a_n > a_{n+1} > a_{n+2} > a_{n+3} > \ldots\ldots$ for all $n \geq 1$

$\{a_n\}$ is bounded sequence, since all terms lie between 5 and 8.

Then by Monotonic sequence theorem $\{a_n\}$ is convergent.

So $\{a_n\}$ has a limit say L.

Since 8 is an upper bound of $\{a_n\}$, so L must be less than 8

Since $\{a_n\}$ is decreasing sequence so $\boxed{5 \leq L < 8}$

Q72E

Observe that terms of sequence $a_n = (-2)^{n+1}$ are

$$(-2)^{1+1}, (-2)^{2+1}, (-2)^{3+1}, (-2)^{4+1}, (-2)^{5+1}, \cdots$$

which are equals to

$$4, -8, 16, -32, 64, \cdots$$

From the above one observe that, odd terms of the sequence $a_n = (-2)^{n+1}$ are

$4, 16, 64, \cdots$ are increasing

and even terms

$-8, -32, -128, \cdots$ are decreasing.

Since only odd terms of the given sequence are increasing and only even terms of the given sequence are decreasing, the sequence $a_n = (-2)^{n+1}$ is **not monotonic**.

Consider the sequence

$$a_n = (-2)^{n+1}$$

Its need to determine whether the sequence is increasing, decreasing or not monotonic and need to determine whether the sequence is bounded or not

Recollect the definition that, a sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$, that is $a_1 < a_2 < a_3 < \cdots$. It is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. It is called monotonic if it is either increasing or decreasing.

To determine whether the sequence is bounded or not:

The terms of the sequence $a_n = (-2)^{n+1}$ are

$4, -8, 16, -32, 64, \cdots$

Recollect the definition that, a sequence $\{a_n\}$ is bounded above if there is a number M such that

$a_n \leq M$ for all $n \geq 1$

It is bounded below if there is a number m such that

$m \leq a_n$ for all $n \geq 1$

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

From the above one observe that, odd terms of the sequence $a_n = (-2)^{n+1}$ are

$4, 16, 64, \cdots$ are bounded below by 4

and even terms

$-8, -32, -128, \cdots$ are bounded above by -8.

Since only odd terms of the given sequence are bounded below and only even terms of the given sequence are bounded above, the sequence $a_n = (-2)^{n+1}$ is **not bounded**.

Q73E

We have $a_n = \dfrac{1}{2n+3}$

Then $a_{n+1} = \dfrac{1}{2(n+1)+3} = \dfrac{1}{2n+5}$

Since $2n+5 > 2n+3$

$\dfrac{1}{2n+5} < \dfrac{1}{2n+3}$ for all $n \geq 1$

Then $a_{n+1} < a_n$ for all $n \geq 1$

Therefore $\{a_n\}$ is decreasing sequence

Since $n \geq 1$

Then $2n \geq 2$

Or $2n+3 \geq 5$

Or $\dfrac{1}{2n+3} \leq \dfrac{1}{5}$

$\Rightarrow a_n \leq \dfrac{1}{5}$,

Therefore $\{a_n\}$ is bounded sequence.

Q74E

We have $a_n = \dfrac{2n-3}{3n+4}$

Consider the function

$$f(x) = \frac{2x-3}{3x+4}$$

$$f'(x) = \frac{(3x+4)(2)-(2x-3)(3)}{(3x+4)^2} \qquad \text{[Quotient rule]}$$

$$= \frac{6x+8-6x+9}{(3x+4)^2}$$

$$= \frac{17}{(3x+4)^2} > 0 \qquad \text{for all } x$$

Thus f is an increasing function

So $a_n < a_{n+1}$ and the given sequence is increasing sequence.

Now $a_1 = \dfrac{-1}{7}, a_2 = \dfrac{1}{10}, a_3 = \dfrac{3}{13}, \ldots\ldots$

Thus $a_n \geq -\dfrac{1}{7}$ for all $n \geq 1$

And so $\{a_n\}$ is bounded below by -1/7

Now $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{2n-3}{3n+4}$

$$= \lim_{n\to\infty} \frac{2-3/n}{3+4/n} = 2/3$$

So the sequence $\{a_n\}$ is bounded above by 2/3

Therefore $\{a_n\}$ is a bounded sequence

Q75E

Consider the sequence

$$a_n = n(-1)^n$$

Its need to determine whether the sequence is increasing, decreasing or not monotonic and need to determine whether the sequence is bounded or not

Recollect the definition that, a sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$, that is $a_1 < a_2 < a_3 < \cdots$. It is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. It is called monotonic if it is either increasing or decreasing.

Observe that terms of sequence $a_n = n(-1)^n$ are

$$1(-1)^1, 2(-1)^2, 3(-1)^3, 4(-1)^4, 5(-1)^5, 6(-1)^6, \cdots$$

which are equals to

$$-1, 2, -3, 4, -5, 6, \cdots$$

From the above one observe that, odd terms of the sequence $a_n = n(-1)^n$ are

$-1, -3, -5, \cdots$ are decreasing

and even terms

$2, 4, 6, \cdots$ are increasing.

Since only odd terms of the given sequence are decreasing and only even terms of the given sequence are increasing, the sequence $a_n = n(-1)^n$ is **not monotonic**.

To determine whether the sequence is bounded or not:

The terms of the sequence $a_n = n(-1)^n$ are

$$-1, 2, -3, 4, -5, 6, \cdots$$

Recollect the definition that, a sequence $\{a_n\}$ is bounded above if there is a number M such that

$a_n \le M$ for all $n \ge 1$

It is bounded below if there is a number m such that

$m \le a_n$ for all $n \ge 1$

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

From the above one observe that, odd terms of the sequence $a_n = n(-1)^n$ are

$-1, -3, -5, \cdots$ are bounded above by -1

and even terms

$2, 4, 6, \cdots$ are bounded below by 2.

Since only odd terms of the given sequence are bounded above and only even terms of the given sequence are bounded below, the sequence $a_n = n(-1)^n$ is **not bounded**.

Q76E

We have $a_n = ne^{-n}$

We consider a function $f(x) = xe^{-x}$

Then $f'(x) = -xe^{-x} + e^{-x}$

$= (1-x)e^{-x} \le 0$ for all $x \ge 1$

So the function $f(x) = xe^{-x}$ is a decreasing function

So $a_{n+1} < a_n$

Therefore $\{a_n\}$ is decreasing sequence

For $n \ge 1$

We have $a_n \le a_1$

So $a_n \le 1/e$

Since for $n \ge 1$, $a_n \ge 0$

Therefore $\{a_n\}$ is bounded above by (1/e) and bounded below by 0.

Q77E

We have $a_n = \dfrac{n}{n^2+1}$

Consider the function $f(x) = \dfrac{x}{x^2+1}$

Then $f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$

$= \dfrac{1-x^2}{(x^2+1)^2} < 0$ *when* $x > 1$

Therefore f is a decreasing on $(1, \infty)$ and so $\{a_n\}$ is a decreasing sequence

Since $\{a_n\}$ is a decreasing for $n \ge 1$

So $a_n \le a_1$

Or $\dfrac{n}{n^2+1} \le \dfrac{1}{2}$

Thus $a_n \le \dfrac{1}{2}$

Thus sequence $\{a_n\}$ is a bounded above

Q78E

We have $a_n = n + \frac{1}{n}$

Consider the function $f = (x) = x + \frac{1}{x}$

Then $f'(x) = 1 - \frac{1}{x^2}$

$= \frac{x^2 - 1}{x^2} > 0 \qquad \text{for } x > 1$

Thus f is an increasing function on $(1, \infty)$ and therefore $\{a_n\}$ is increasing sequence

Also since given sequence is increasing so

We have $a_n > a_1$ for n > 1

Then $a_n \geq 2,$

Therefore the given sequence is bounded below

Q79E

We have a sequence $a_n = \left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\ldots\right\}$

$= \left\{2^{1/2}, 2^{1/2}.2^{1/4}, 2^{1/2}.2^{1/4}.2^{1/8}, \ldots\ldots\ldots\right\}$

$= \left\{2^{1/2}, 2^{3/4}, 2^{7/8}, \ldots\ldots\ldots\right\}$

$\Rightarrow a_n = 2^{(2^n-1)/2^n}$

Taking limit as $n \to \infty$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{(2^n-1)/2^n}$

$= 2^{\lim_{n\to\infty}(2^n-1)/2^n}$

We calculate $\lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty} \frac{1 - 1/2^n}{1}$

$= \frac{1-0}{1} = 1$

Then $\lim_{n\to\infty} a_n = 2^1$

$\Rightarrow \boxed{\lim_{n\to\infty} a_n = 2}$

Q80E

(A) We have $a_1 = \sqrt{2}$ $a_{n+1} = \sqrt{2+a_n}$

For $n = 1$,

$$a_{1+1} = \sqrt{2+a_1}$$

$$\Rightarrow a_2 = \sqrt{2+\sqrt{2}} \qquad \text{since } \sqrt{2+\sqrt{2}} > \sqrt{2}$$

Then $a_2 > a_1$

this suggest that the sequence is increasing

we assume that it is true for $n = k$

so $a_{k+1} > a_k$

$$\Rightarrow 2 + a_{k+1} > 2 + a_k$$

$$\Rightarrow \sqrt{2+a_{k+1}} > \sqrt{2+a_k}$$

$$\Rightarrow a_{k+2} > a_{k+1}$$

So sequence is increasing for $n = k+1$

By mathematical induction we have $a_{n+1} > a_n$ for all n

Therefore the sequence $\{a_n\}$ is increasing.

We calculate

$$a_1 = \sqrt{2} = 1.4142, \qquad a_2 = \sqrt{2+\sqrt{2}} = 1.84776$$

$$a_3 = \sqrt{2+a_2} = 1.96157, \qquad a_4 = 1.990$$

$$a_5 = 1.9976, \quad a_6 = 1.999, \quad a_7 = 1.9998, \quad a_8 = 1.99996$$

$$a_9 = 1.99999, \quad a_{10} = 1.9999$$

We see that terms are approaching 2

Since 2<3

So $a_n < 3$ for all $n \geq 1$

Thus 3 is an upper bound.

(B) Since $\{a_n\}$ is increasing and bounded above by 3 then $\{a_n\}$ must have limit.

Let limit be L so $\lim_{n\to\infty}\{a_n\} = L$ exists

Then

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n}$$

$$= \sqrt{2+\lim_{n\to\infty} a_n}$$

$$= \sqrt{2+L}$$

Since $a_n \to L$ then $a_{n+1} \to L$ $(as\ n\to\infty, n+1\to\infty\ also)$

$$\Rightarrow L = \sqrt{2+L}$$

$$\Rightarrow L^2 = 2+L$$

$$\Rightarrow L^2 - L - 2 = 0$$

$$\Rightarrow (L-2)(L+1) = 0$$

$$\Rightarrow L = -1 \quad or \quad L = 2$$

Since $a_1 = \sqrt{2} > 0$ and a_n is increasing

So $L \neq -1$

and thus $\boxed{\lim_{n\to\infty} a_n = 2}$

Q81E

Consider the sequence defined by

$$a_1 = 1 \quad a_{n+1} = 3 - \frac{1}{a_n}$$

Its need to show that the sequence is increasing and $a_n < 3$ for all n.

And also needed to deduce that $\{a_n\}$ is convergent and find its limit.

Begin by computing the first several terms by using the recurrence relation

$$a_1 = 1 \quad a_{n+1} = 3 - \frac{1}{a_n}$$

The terms of the sequence are given as,

$$a_1 = 1 \quad a_2 = 3 - \frac{1}{1} = 2$$

$$a_3 = 3 - \frac{1}{2} \quad a_4 = 3 - \frac{2}{5}$$

$$= \frac{5}{2} \quad = \frac{13}{5}$$

$$= 2.5 \quad = 2.6$$

Continuation to the above

$$a_5 = 3 - \frac{5}{13} \quad a_6 = 3 - \frac{13}{34}$$

$$= \frac{34}{13} \quad = \frac{89}{34}$$

$$\approx 2.615 \quad \approx 2.617$$

$$a_7 = 3 - \frac{34}{89} \quad a_8 = 3 - \frac{89}{233}$$

$$= \frac{233}{89} \quad = \frac{610}{233}$$

$$\approx 2.618 \quad = 2.618$$

These initial terms suggests that the sequence is increasing and the terms are approaching to 3.

To confirm that the sequence is increasing, we use the mathematical induction to show that $a_{n+1} > a_n$ for all $n \geq 1$.

This is true for $n = 1$ because $a_2 = 3 - \frac{1}{1} = 2 > a_1$

If we assume that it is true for $n = k$, then we have

$a_{k+1} > a_k$

So $\frac{1}{a_{k+1}} < \frac{1}{a_k}$

And $-\frac{1}{a_{k+1}} > -\frac{1}{a_k}$

$3 - \frac{1}{a_{k+1}} > 3 - \frac{1}{a_k}$

Thus $a_{k+2} > a_{k+1}$

That is, $a_{n+1} > a_n$ is true for $n = k + 1$

Therefore, the inequality is true for all n by induction.

To show that $a_n < 3$ for all n:

Since the sequence is increasing, we already know that it has lower bound:

$a_n \geq a_1 = 1$ for all n

We know that $a_1 = 1 < 3$, so the assertion is true for $n = 1$

Suppose it is true for $n = k$. Then

$a_k < 3$

So, $\frac{1}{a_k} > \frac{1}{3}$

$-\frac{1}{a_k} < -\frac{1}{3}$

$3 - \frac{1}{a_k} < 3 - \frac{1}{3} = \frac{8}{3} < 3$

Thus $a_{k+1} < 3$

This shows, by mathematical induction, that $a_n < 3$ for all n.

Since the sequence $\{a_n\}$ is increasing and bounded, by monotonic convergence theorem it follows that the sequence $\{a_n\}$ is convergent. That is, it has a limit.

Suppose that $\lim_{n\to\infty} a_n = L$, then by the recurrence relation $a_{n+1} = 3 - \frac{1}{a_n}$, we have that

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(3 - \frac{1}{a_n}\right)$$

$$= 3 - \frac{1}{\lim_{n\to\infty} a_n}$$

Since $a_n \to L$, it follows that $a_{n+1} \to L$. So we have

$$L = 3 - \frac{1}{L}$$

$$\Rightarrow L^2 = 3L - 1$$

$$\Rightarrow L^2 - 3L + 1 = 0$$

$$\Rightarrow L = \frac{-(-3) \pm \sqrt{(-3)^2 - 4\cdot 1\cdot 1}}{2\cdot 1}$$

$$= \frac{3 \pm \sqrt{9-4}}{2}$$

$$= \frac{1}{2}\left(3 \pm \sqrt{5}\right)$$

Since the sequence $\{a_n\}$ is increasing and bounded above, so it converges to supremum.

That is, $L = \frac{1}{2}\left(3 + \sqrt{5}\right)$

Q82E

First we show that $0 < a_n \le 2$

We have $a_1 = 2,\ a_{n+1} = \frac{1}{3 - a_n}$

Then $a_2 = \frac{1}{3-2} = \frac{1}{1} = 1$

$\Rightarrow a_2 = 1$

Since a_{n+1} is a rational function of a_n

So .... $< a_3 < a_2 < a_1 = 2$

Thus this satisfies $0 < a_n \le 2$

We use mathematical induction to show that a_n is decreasing.

If a_n is decreasing then $a_n > a_{n+1} \Rightarrow a_{n+1} < a_n$ .... P_n

For n = 1

$a_2 < a_1$

$\Rightarrow 1 < 2$ which is true

So P_n is true for n = 1

Now we assume that it is true for n = k

Then $a_{k+1} < a_k$

$\Rightarrow -a_{k+1} > -a_k$

$\Rightarrow 3 - a_{k+1} > 3 - a_k$

$\Rightarrow \dfrac{1}{3 - a_{k+1}} < \dfrac{1}{3 - a_k}$

$\Rightarrow a_{k+2} < a_{k+1}$

So it is true for n = k+1

Thus by mathematical induction $a_{n+1} < a_n$ for all n. So a_n is decreasing

Since a_n is decreasing and $0 < a_n \le 2$

So $\{a_n\}$ must have a limit L

Then $\lim\limits_{n\to\infty} a_n = L$

And so $\lim\limits_{n\to\infty} a_{n+1} = L$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{3 - a_n}$$

$$= \frac{1}{3 - L}$$

$$\Rightarrow L = \frac{1}{3 - L}$$

$$\Rightarrow 3L - L^2 = 1$$

$$\Rightarrow L^2 - 3L + 1 = 0$$

$$\Rightarrow L = \frac{3 \pm \sqrt{9 - 4}}{2}$$

$$\Rightarrow L = \frac{3 \pm \sqrt{5}}{2}$$

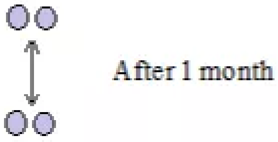
But limit $L = \dfrac{3 + \sqrt{5}}{2} > 2$ which is not possible since $a_n \le 2$

So limit $\boxed{L = \dfrac{3 - \sqrt{5}}{2}}$

Q83E

Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at the age of 2 months. Start with one new born pair.

In starting month we have new born pair, after one month it remains same.



Again after one month starting new born pair gets two months of age and its produces another pair. It can be illustrated in below figure.

Then after one month old pair produces another new pair and which new pair which one is born after previously remains same. It can be illustrated in below figure.

Again after one month previously born pair gets two months of age and its produces another pair and old pair produces another new pair.so, totally we have five pairs.

It can be illustrated in below figure.

On proceeding this manner, we get pairs as shown.

$\{1,1,2,3,5,8,13,21,\cdots\}$

Observe that, in the above set each term is the sum of two preceding terms.it van be represented recursively as show

$f_1 = 1 \quad f_2 = 1 \quad f_n = f_{n-1} + f_{n-2} \quad n \geq 3$

It represents a Fibonacci sequence $\{f_n\}$

(b)

Let $a_n = f_{n+1}/f_n$ ...... (1)

Replace n by $n-1$ in (1), then we get

$$a_{n-1} = \frac{f_{(n-1)+1}}{f_{n-1}}$$

$$= \frac{f_n}{f_{n-1}}$$

$$= \frac{f_{n-1} + f_{n-2}}{f_{n-1}}$$ Use $f_n = f_{n-1} + f_{n-2}$

$$= 1 + \frac{f_{n-2}}{f_{n-1}}$$

$$= 1 + \frac{1}{f_{n-1}/f_{n-2}}$$

$$= 1 + \frac{1}{a_{n-2}}$$

Thus $a_{n-1} = 1 + 1/a_{n-2}$

Assume that $\{a_n\}$ is convergent.

That is, $\lim_{n\to\infty} a_n = l$

Then $\lim_{n\to\infty} a_{n-1} = l$ and $\lim_{n\to\infty} a_{n-2} = l$

As $a_{n-1} = 1 + 1/a_{n-2}$, on applying limit on both sides get

$$\lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty}\left(1 + \frac{1}{a_{n-1}}\right)$$

$$= 1 + \frac{1}{\lim_{n\to\infty} a_{n-1}}$$

$$\Rightarrow \lim_{n\to\infty} a_{n-1} = 1 + \frac{1}{\lim_{n\to\infty} a_{n-1}}$$

Continuation to the above

$$\Rightarrow l = 1 + \frac{1}{l}$$

$$\Rightarrow l^2 = l + 1$$

$$\Rightarrow l^2 - l - 1 = 0$$

$$l = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Since the terms of the sequence are positive, it converges to the positive limit.

That is, $l = \frac{1+\sqrt{5}}{2}$

Q84E

(a)

Suppose that, $a_1 = a$, $a_2 = f(a)$, $a_3 = f(a_2) = f(f(a))$, $\cdots$, $a_{n+1} = f(a_n)$

Here f is continuous function.

Assume that $\lim_{n\to\infty} a_n = L$

Its need to shows that $f(L) = L$

Since $\lim_{n\to\infty} a_n = L$, by the definition of the limit of a sequence for every $\varepsilon > 0$ there is a corresponding integer N such that

"if $n > N$ then $|a_n - L| < \varepsilon$"

As $n > N$, we have that $(n+1) > n > N$ then $|a_{n+1} - L| < \varepsilon$

That is $\lim_{n\to\infty} a_{n+1} = L$ ...... (1)

Recall the theorem that, if $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$

From the data, we have that $\lim_{n\to\infty} a_n = L$ and the function f is continuous, we claim that

$\lim_{n\to\infty} f(a_n) = f(L)$ ...... (2)

From the data, we have

$a_{n+1} = f(a_n)$

$\Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n)$ ...... (3)

From (1), (2) and (3) we have that

$L = f(L)$

(b)

Consider the function $f(x) = \cos x$

By taking $a = 1$, we have to estimate the value of L up to five decimal places.

From the part (a), we observe that $f(x) = x$

That is, $\cos x = x$

So, we need to find the roots of the equation

$g(x) = x - \cos x$

Recall the, formula to find the roots of a transcendental equation $g(x)=0$ by using Newton Raphson method is

$$x_{n+1}=x_n-\frac{g(x_n)}{g'(x_n)}$$

As $g(x)=x-\cos x$, we have that

$$g'(x)=1+\sin x$$

So, Newton Raphson formula is

$$\begin{aligned}x_{n+1}&=x_n-\frac{(x_n-\cos x_n)}{1+\sin x_n}\\&=\frac{x_n+x_n\sin x_n-x_n+\cos x_n}{1+\sin x_n}\\&=\frac{x_n\sin x_n+\cos x_n}{1+\sin x_n}\end{aligned}$$

Thus

$$x_{n+1}=\frac{x_n\sin x_n+\cos x_n}{1+\sin x_n}$$

From the data, we have that

$a=1$, that is $x_0=1$

For $n=0$, we have

$$\begin{aligned}x_1&=\frac{x_0\sin x_0+\cos x_0}{1+\sin x_0}\\&=\frac{1\sin 1+\cos 1}{1+\sin 1}\\&=\frac{0.84147+0.5403}{1+0.84147}\\&=\frac{1.38177}{1.84147}\end{aligned}$$

$$=0.75036$$

Thus $x_1=0.75036$

For $n=1$, we have

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1}{1+\sin x_1}$$
$$= \frac{(0.75036)\sin(0.75036)+\cos(0.75036)}{1+\sin(0.75036)}$$
$$= \frac{(0.75036)(0.6819)+(0.73144)}{1+0.6819}$$
$$= 0.73911$$

Thus $x_2 = 0.73911$

For $n=2$, we have

$$x_3 = \frac{x_2 \sin x_2 + \cos x_2}{1+\sin x_2}$$
$$= \frac{(0.73911)\sin(0.73911)+\cos(0.73911)}{1+\sin(0.73911)}$$
$$= \frac{(0.73911)(0.67363)+(0.73907)}{1+0.67363}$$
$$= 0.73909$$

Thus $x_3 = 0.73909$

For $n=3$, we have

$$x_4 = \frac{x_3 \sin x_3 + \cos x_3}{1+\sin x_3}$$
$$= \frac{(0.73909)\sin(0.73909)+\cos(0.73909)}{1+\sin(0.73909)}$$
$$= \frac{(0.73911)(0.67362)+(0.73908)}{1+0.67362}$$
$$= 0.73909$$

Thus $x_4 = 0.73909$

Observe that, x_4 is equal to the x_3

That is, after x_3 we obtain the same limit value $L = 0.73909$

Hence, the required value is

$$L = \boxed{0.73909}$$

Q85E

(a)

Consider the limit

$$\lim_{n\to\infty}\frac{n^5}{n!}$$

It is need to guess the value of the limit by considering its graph.

To graph the limit, use Maple software.

First enter the sequence by using the expression command.

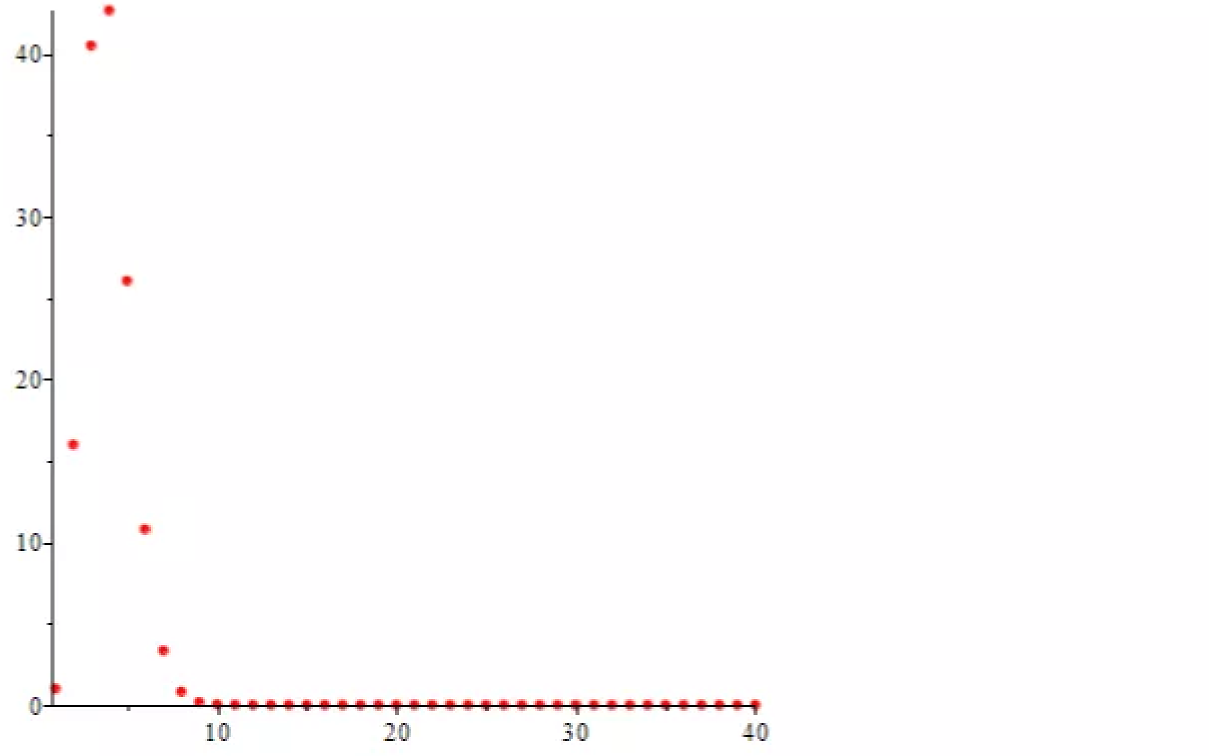
$$> expr := \frac{n^5}{n!}$$

$$\frac{n^5}{n!}$$

Use the following Maple command to get the plot of the sequence.

Plot([seq([n,expr],n=1..40)],style=point,symbol=solidcircle,color=red)

$> plot([seq([n, expr], n = 1..40)], style = point, symbol = solidcircle, color = red)$



From the graph observe that, terms of the sequence $a_n = \frac{n^5}{n!}$ are approaches to zero as n increases.

Hence $\lim_{n\to\infty}\frac{n^5}{n!} = \boxed{0}$

(b)

Its need to find the smallest values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.001$ by using the graph of the sequence in part (a).

From the graph in part (a), we observed that the sequence $a_n = \frac{n^5}{n!}$ has limit $L = 0$

Recollect the definition that, a sequence $\{a_n\}$ has the limit L and we write

$\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

if $n > N$ then $|a_n - L| < \varepsilon$

Since the sequence $a_n = \frac{n^5}{n!}$ has limit $L = 0$, by the limit definition we have that

$$\left|\frac{n^5}{n!} - 0\right| < \varepsilon \quad \text{...... (1)}$$

Suppose that $\varepsilon = 0.1$ in (1), then we need to find the smallest value of N

$$\left|\frac{n^5}{n!} - 0\right| < 0.1$$

$$\Rightarrow \frac{n^5}{n!} < 0.1$$

Observe that $n^5 > n! \;\; \forall n \le 7 \Rightarrow \frac{n^5}{n!} > 1$

So, we can't consider these n values.

For $n = 8$, $8^5 = 32768$ and $8! = 40320$

Then

$$\frac{8^5}{8!} = \frac{32768}{40320}$$

$$\approx 0.812$$

For $n=9$, $9^5=59049$ and $9!=362880$

Then

$$\frac{9^5}{9!}=\frac{59049}{362880}$$

$$\approx 0.162$$

For $n=10$, $10^5=100,000$ and $10!=3.6288\text{E}6$

Then

$$\frac{10^5}{10!}=\frac{100,000}{3.6288\text{E}6}$$

$$\approx 0.027<0.1$$

Thus

$\left|\frac{n^5}{n!}-0\right|<0.1$ for $n=10>N(=9)$

Hence, the smallest value of N for which the inequality

$\left|\frac{n^5}{n!}-0\right|<0.1 \quad \forall n>N$ is $\boxed{9}$

Suppose that $\varepsilon=0.001$ in (1), then we need to find the smallest value of N

$$\left|\frac{n^5}{n!}-0\right|<0.001$$

$$\Rightarrow \frac{n^5}{n!}<0.001$$

Observe that $n^5>n! \quad \forall n\le 7 \Rightarrow \frac{n^5}{n!}>1$

So, we can't consider these n values.

For $n=8$, $8^5=32768$ and $8!=40320$

Then

$$\frac{8^5}{8!}=\frac{32768}{40320}$$

$$\approx 0.812$$

Then

$$\frac{11^5}{11!} = \frac{161051}{3.99168E7}$$

$$\approx 0.004035$$

For $n = 12$, $12^5 = 248832$ and $12! = 4.790016E8$

Then

$$\frac{12^5}{12!} = \frac{248832}{4.790016E8}$$

$$= 0.000519 < 0.001$$

Thus

$\left|\frac{n^5}{n!} - 0\right| < 0.001$ for $n = 12 > N(=11)$

Hence, the smallest value of N for which the inequality

$\left|\frac{n^5}{n!} - 0\right| < 0.001 \quad \forall n > N$ is $\boxed{11}$

For $n = 9$, $9^5 = 59049$ and $9! = 362880$

Then

$$\frac{9^5}{9!} = \frac{59049}{362880}$$

$$\approx 0.162$$

For $n = 10$, $10^5 = 100{,}000$ and $10! = 3.6288E6$

Then

$$\frac{10^5}{10!} = \frac{100{,}000}{3.6288E6}$$

$$\approx 0.027$$

For $n = 11$, $11^5 = 161051$ and $11! = 3.99168E7$

Q86E

We have to prove that $\lim_{n\to\infty} r^n = 0$

Guessing a value for N:

Let $\in > 0$ be given, we have find an integer N such that

$|r^n - 0| < \in$ Whenever $n > N$

$\Rightarrow$ $|r^n| < \in$ Whenever $n > N$

$\Rightarrow$ $|r|^n < \in$ Whenever $n > N$

Taking *ln* of both sides.

$\Rightarrow$ $n \ln|r| < \ln \in$ Whenever $n > N$

$\Rightarrow$ $n > \ln \in / \ln|r|$ Whenever $n > N$

$[\text{Since } |r| < 1 \Rightarrow \ln|r| < 0 \Rightarrow n > \ln \in / \ln|r|]$

So, we choose $N = \ln \in / \ln|r|$

Showing that this N works:-

Given $\in > 0$, let $N = \ln \in / \ln|r|$ When $|r| < 1$

Then n > N.

$\Rightarrow$ $n > \ln \in / \ln|r|$

$\Rightarrow$ $n \ln|r| < \ln \in$ When $|r| < 1$

$\Rightarrow$ $\ln|r|^n < \ln \in$ When $|r| < 1$

$\Rightarrow$ $\ln|r^n| < \ln \in$ When $|r| < 1$

$\Rightarrow$ $|r^n| < \in$ When $|r| < 1$

$\Rightarrow$ $|r^n - 0| < \in$ When $|r| < 1$

Therefore by the definition we have

$\lim_{n\to\infty} r^n = 0$ When $|r| < 1$

Q87E

If $\lim_{n\to\infty} |a_n| = 0$

Then we have $\lim_{n\to\infty} -|a_n| = 0$

Since $-|a_n| \le a_n \le |a_n|$

Then by squeeze theorem

We have $\boxed{\lim_{n\to\infty} a_n = 0}$

Q88E

Suppose $\lim_{n\to\infty} a_n = L$ and f is continuous at L.

Need to prove that $\lim_{n\to\infty} f(a_n) = f(L)$.

Since $\lim_{n\to\infty} a_n = L$

So for $\varepsilon > 0$, there exists an integer N such that

$|a_n - L| < \varepsilon$ for all $n > N$ ...... (1)

Since f is continuous at L.

So for particular ε, there exists a $\delta > 0$ such that

$|x - L| < \delta$, then $|f(x) - f(L)| < \varepsilon$ ...... (2)

Let $\varepsilon > 0$ and an integer n_0 such that $m > n_0$

Now apply the definition of $\lim_{n\to\infty} a_n = L$.

Choose j such that $m > j$, then $|a_m - L| < \delta$.

So by (2), for $m > j$, $|f(a_m) - f(L)| < \varepsilon$.

Now take $n_0 = j$ and

Then $m > n_0$, and $|f(a_m) - f(L)| < \varepsilon$

This is true for any arbitrary $\varepsilon > 0$.

Thus, $\boxed{\lim_{n\to\infty} f(a_n) = f(L)}$.

Q89E

Suppose that

$\lim_{n\to\infty} a_n = 0$

Also suppose that $\{b_n\}$ is bounded.

Need to prove that $\lim_{n\to\infty} a_n b_n = 0$.

Since $\lim_{n\to\infty} a_n = 0$

So for $\varepsilon > 0$, there exists an integer N such that

$|a_n - 0| < \varepsilon$ for all $n > N$ ...... (1)

Since $\{b_n\}$ is bounded.

Then $|b_n| \le M$, for all $n \ge 1$ ...... (2)

Let $\varepsilon > 0$ and n be an integer such that $n > N$

Consider

$$|a_n b_n - 0| = |a_n b_n|$$

$$= |a_n||b_n|$$

$< \varepsilon M$ Since by (1) and (2)

$= \left(\frac{\varepsilon}{M}\right) M$ Choosing ε such that $\varepsilon = \frac{\varepsilon}{M}$

$= \varepsilon$

Thus, for $\varepsilon > 0$, there exists $n > N$ such that $|a_n b_n - 0| < \varepsilon$.

Hence $\boxed{\lim_{n\to\infty} a_n b_n = 0}$

Q90E

(A) We have to show that if $0 \le a < b$ then

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n+1) b^n$$

We will prove it by mathematical induction.

For n = 1 we have,

$$\frac{b^{1+1} - a^{1+1}}{b - a} < (1+1) b^1$$

$$\frac{b^2 - a^2}{b - a} < 2b$$

$$\frac{(b+a)(b-a)}{(b-a)} < 2b$$

$$(b+a) < 2b$$

$\Rightarrow$ $a < b$ Which is true as it is given that a < b.

Therefore, the given inequality is true for n = 1.

Let the given inequality be true for n = k.

So, we have,

$$\frac{b^{k+1} - a^{k+1}}{b - a} < (k+1) b^k$$

Let us check for n = k + 1.

For n = k + 1.

$$\frac{b^{(k+1)+1}-a^{(k+1)+1}}{b-a}=\frac{b\cdot b^{k+1}-a\cdot a^{k+1}}{b-a}$$

$$=\frac{bb^{k+1}-b\cdot a^{k+1}+ba^{k+1}-a\cdot a^{k+1}}{b-a}$$ Adding and subtracting $b\cdot a^{k+1}$ in numerator.

$$=\frac{b\left(b^{k+1}-a^{k+1}\right)+a^{k+1}(b-a)}{(b-a)}$$

$$=\frac{b\left(b^{k+1}-a^{k+1}\right)}{(b-a)}+\frac{a^{k+1}(b-a)}{(b-a)}$$

$$<b(k+1)b^{k}+a^{k+1}$$ Since, $\frac{b^{k+1}-a^{k+1}}{b-a}<(k+1)b^{k}$

$$<(k+1)b^{k+1}+\frac{a^{k+1}}{b^{k+1}}b^{k+1}$$

$$<(k+1)b^{k+1}+\left(\frac{a}{b}\right)^{k+1}b^{k+1}$$

$$<\left[k+1+\left(\frac{a}{b}\right)^{k+1}\right]b^{k+1}$$

$$<[k+1+1]b^{k+1}$$ Since $\left(\frac{\mathrm{a}}{\mathrm{b}}\right)^{k+1}<1$ as $a<b$

$$<(k+2)b^{k+1}$$

This tells us that the given inequality is true for $n=k+1$

Hence, $\frac{b^{n+1}-a^{n+1}}{b-a}<(n+1)b^{n}$ is true for all n.

(B) From part (a) we have proved that

$$\frac{b^{n+1}-a^{n+1}}{b-a}<(n+1)b^{n}$$

$$\Rightarrow \left(b^{n+1}-a^{n+1}\right)<(n+1)(b-a)b^{n}$$

$$\Rightarrow b^{n+1}-a^{n+1}<\left[(n+1)b-(n+1)a\right]b^{n}$$

$$\Rightarrow b^{n+1}=a^{n+1}<(n+1)b^{n+1}-(n+1)ab^{n}$$

$$\Rightarrow b^{n+1}-a^{n+1}<nb^{n+1}+b^{n+1}-(n+1)ab^{n}$$

$$\Rightarrow -a^{n+1}<nb^{n+1}-(n+1)ab^{n}$$

$$\Rightarrow (n+1)ab^{n}-nb^{n+1}<a^{n+1}$$

$$\Rightarrow b^{n}\left[(n+1)a-nb\right]<a^{n+1}$$

Hence,

$$\boxed{b^{n}\left[(n+1)a-nb\right]<a^{n+1}}$$

(C) From part (b) we have,

$$b^n\left[(n+1)a-nb\right]<a^{n+1}$$

Now putting $b=1+\frac{1}{n}$ and $a=1+\frac{1}{n+1}$

We get,

$$\left(1+\frac{1}{n}\right)^n\left[(n+1)\left(1+\frac{1}{n+1}\right)-n\left(1+\frac{1}{n}\right)\right]<\left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\Rightarrow \quad \left(1+\frac{1}{n}\right)^n\left[(n+1)+1-(n+1)\right]<\left[1+\frac{1}{n+1}\right]^{n+1}$$

$$\Rightarrow \quad \left(1+\frac{1}{n}\right)^n\left[n+1+1-n-1\right]<\left[1+\frac{1}{n+1}\right]^{n+1}$$

$$\Rightarrow \quad \left(1+\frac{1}{n}\right)^n[1]<\left[1+\frac{1}{n+1}\right]^{n+1}$$

$$\Rightarrow \quad \left(1+\frac{1}{n}\right)^n<\left(1+\frac{1}{n+1}\right)^{n+1} \qquad \text{---- (i)}$$

Given that $a_n=\left(1+\frac{1}{n}\right)^n$

So, $a_{n+1}=\left(1+\frac{1}{n+1}\right)^{n+1}$

From inequality (i)

For the sequence $\{a_n\}$, we have,

$$a_n<a_{n+1}$$

This shows that $\{a_n\}$ is increasing.

(D) From part (b) we have,

$$b^n\left[(n+1)a-nb\right]<a^{n+1}$$

Putting a = 1 and $b=1+\frac{1}{(2n)}$ we get,

$$\left(1+\frac{1}{2n}\right)^n\left[(n+1)\cdot 1-n\left(1+\frac{1}{(2n)}\right)\right]<(1)^{n+1}$$

$$\Rightarrow \quad \left(1+\frac{1}{2n}\right)^n\left[n+1-n-\frac{1}{2}\right]<1$$

$$\Rightarrow \quad \left(1+\frac{1}{2n}\right)^n\left[\frac{1}{2}\right]<1$$

$$\Rightarrow \quad \left(1+\frac{1}{2n}\right)^n<2$$

Squaring both sides we get,

$$\left[\left(1+\frac{1}{2n}\right)^n\right]^2<(2)^2$$

$$\Rightarrow \quad \left(1+\frac{1}{2n}\right)^{2n}<4 \qquad \text{---- (i)}$$

Given that $a_n=\left(1+\frac{1}{n}\right)^n$

So, $a_{2n}=\left(1+\frac{1}{2n}\right)^{2n}$

Therefore, from inequality (i) we get,

$$\boxed{a_{2n}<4}$$

(E) We have to show that $a_n<4$ for all n.

From part (c), $\{a_n\}$ is increasing. So n^th term will be less than each of it's succeeding terms of $\{a_n\}$.

So, $a_n<a_{2n}$ ---- (i)

Also from part (d) we have $a_{2n}<4$ ---- (ii)

Therefore, combining (i) and (ii) we get,

$$a_n<a_{2n}<4$$

$$\Rightarrow \quad a_n<4$$

Hence, $\boxed{a_n<4}$

(F) We have to show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists.

We know by a theorem, that every bounded Monotonic sequence is convergent.

From part (c) we have that the sequence, $\{a_n\}$ is increasing where $a_n=\left(1+\frac{1}{n}\right)^n$

Also, from part (e) we have found that $a_n<4$ for all n.

i.e. $\{a_n\}$ is bounded above by 4.

Thus we see that the sequence $\{a_n\}$ is monotonic and bounded. So $\{a_n\}$ will be convergent.

And we know that if a sequence $\{a_n\}$ is convergent then $\lim_{n\to\infty} a_n$ exists.

Therefore, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists. Since $a_n=\left(1+\frac{1}{n}\right)^n$

Hence,

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \text{ exists.}}$$

Q91E

(A) Given that a > b and $a_{n+1}=\frac{a_n+b_n}{2}, b_{n+1}=\sqrt{a_nb_n}$.

We have to show that $a_n>a_{n+1}>b_{n+1}>b_n$ by using mathematical induction.

For, n = 1 we have,

$$a_1-a_2=a_1-\frac{(a_1+b_1)}{2} \qquad \text{Since, } a_2=\frac{a_1+b_1}{2}$$

$$=\frac{2a_1-a_1-b_1}{2}$$

$$=\frac{a_1-b_1}{2}$$

Now,

$$a_1-b_1=\left[\frac{a+b}{2}-\sqrt{ab}\right]$$

$$=\left[\frac{a+b-2\sqrt{ab}}{2}\right]$$

$$=\frac{1}{2}\left[\left(\sqrt{a}\right)^2+\left(\sqrt{b}\right)^2-2\sqrt{a}\sqrt{b}\right]$$

$$=\frac{1}{2}\left[\sqrt{a}-\sqrt{b}\right]^2 = \text{always positive as } a>b$$

Therefore, $a_1-a_2=\frac{(a_1-b_1)}{2}$ = always positive.

$\Rightarrow \quad a_1>a_2$

$$a_2 - b_2 = \frac{a_1 + b_1}{2} - \sqrt{a_1 b_1}$$

$$= \frac{a_1 + b_1 - 2\sqrt{a_1 b_1}}{2}$$

$$= \frac{\left(\sqrt{a_1}\right)^2 + \left(\sqrt{b_1}\right)^2 - 2\sqrt{a_1}\sqrt{b_1}}{2}$$

$$= \frac{\left[\sqrt{a_1} - \sqrt{b_1}\right]^2}{2}$$

$\Rightarrow$ Always positive as $a_1 > b_1$ from (i)

$\Rightarrow$ $a_2 > b_2$

Now, $b_2 - b_1 = \sqrt{a_1 b_1} - b_1$

$$= \sqrt{b_1}\left(\sqrt{a_1} - \sqrt{b_1}\right)$$

$$= \frac{\sqrt{b_1}\left(\sqrt{a_1} - \sqrt{b_1}\right)\left(\sqrt{a_1} + \sqrt{b_1}\right)}{\left(\sqrt{a_1} + \sqrt{b_1}\right)}$$ (Multiplying and dividing by $\sqrt{a_1} + \sqrt{b_1}$)

$$= \frac{\sqrt{b}\left(\left(\sqrt{a_1}\right)^2 - \left(\sqrt{b_1}\right)^2\right)}{\sqrt{a_1} + \sqrt{b_1}}$$

$$= \frac{\sqrt{b}\left(a_1 - b_1\right)}{\left(\sqrt{a_1} + \sqrt{b_1}\right)}$$

> 0 as $a_1 - b_1 > 0$ and $b > 0$

Therefore, $b_2 > b_1$

Thus, we find that $a_1 > a_2 > b_2 > b_1$

i.e. the inequality $a_n > a_{n+1} > b_{n+1} > b_n$ is true for n = 1.

Let the inequality $a_n > a_{n+1} > b_{n+1} > b_n$ is true for n = k.

So, we have

$a_k > a_{k+1} > b_{k+1} > b_k$ ---- (ii)

Now, for n = k + 1. we will check whether $a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1}$ is true or not.

For this,

$$a_{k+1} - a_{k+2} = a_{k+1} - \frac{\left(a_{k+1} + b_{k+1}\right)}{2}$$ Since, $a_{k+2} = \frac{a_{k+1} + b_{k+1}}{2}$

$$= \frac{2a_{k+1} - a_{k+1} - b_{k+1}}{2}$$

$$= \frac{a_{k+1} - b_{k+1}}{2}$$

> 0 Since, from (ii) $a_{k+1} > b_{k+1}$

So, $a_{k+1} > a_{k+2}$ ---- (1)

Also, $a_{k+2} - b_{k+2} = \frac{(a_{k+1} + b_{k+1})}{2} = \sqrt{a_{k+1} b_{k+1}}$

$$= \frac{a_{k+1} + b_{k+1} - 2\sqrt{a_{k+1} b_{k+1}}}{2}$$

$$= \frac{\left(\sqrt{a_{k+1}}\right)^2 + \left(\sqrt{b_{k+1}}\right)^2 - 2\sqrt{a_{k+1}}\sqrt{b_{k+1}}}{2}$$

$$= \frac{\left(\sqrt{a_{k+1}} - \sqrt{b_{k+1}}\right)^2}{2}$$

> 0 Since, $a_{k+1} > b_{k+1}$ From (ii)

So, we have $a_{k+2} > b_{k+2}$ ---- (2)

Now, $b_{k+2} - b_{k+1} = \sqrt{a_{k+1} b_{k+1}} - b_{k+1}$

$$= \sqrt{a_{k+1}}\sqrt{b_{k+1}} - \left(\sqrt{b_{k+1}}\right)^2$$

$$= \sqrt{b_{k+1}}\left(\sqrt{a_{k+1}} - \sqrt{b_{k+1}}\right)$$

> 0 As $a_{k+1} > b_{k+1}$ and $b_{k+1} > 0$ from (ii)

So, we have $b_{k+2} > b_{k+1}$ ---- (3)

Thus from (1), (2), (3) we get,

$$a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1}$$

Therefore, the inequality $a_n > a_{n+1} > b_{n+1} > b_n$ is true for all n by mathematical induction.

(B) We have $a_1 - a = \frac{a+b}{2} - a$ Since $a_1 = \frac{a+b}{2}$

$$= \frac{a+b-2a}{2}$$

$$= \frac{b-a}{2}$$

< 0 Since, $b - a < 0$ as $a > b$

$\Rightarrow$ $a_1 < a$

Or, $a > a_1$

Also, we have proved that

$a_n > a_{n+1}$

$\Rightarrow$ $\{a_n\}$ is decreasing sequence

Also, $a > a_n$

$\Rightarrow$ $\{a_n\}$ is bounded above by a.

Thus, $\{a_n\}$ is monotonic and bound above by a.

We know that every bounded, monotonic sequence is convergent. So the sequence $\{a_n\}$ is convergent.

Also,

$$b_1 - b = \sqrt{ab} - b$$
$$= \sqrt{a}\sqrt{b} - \left(\sqrt{b}\right)^2$$
$$= \sqrt{b}\left(\sqrt{a} - \sqrt{b}\right)$$
$$> 0 \qquad \text{Since } a > b.$$

$\Rightarrow \quad b_1 > b$

Since from part (a) we have found that

$$b_{n+1} > b_n$$

$\Rightarrow$ The sequence $\{b_n\}$ is increasing

i.e. $\{b_n\}$ is monotonic.

Since $\{b_n\}$ is monotonic and $b_1 > b$

So, $b_n > b$

i.e. $\{b_n\}$ is bounded below by b.

Here we find that the sequence $\{b_n\}$ is monotonic and bounded below by b.

We know that every bounded and monotonic sequence is convergent.

Therefore, the sequence $\{b_n\}$ is convergent.

(C) We have to prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

Since $\{a_n\}$ and $\{b_n\}$ both are convergent.

So, both the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

Since $\{a_n\}$ is convergent. So we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{a_n + b_n}{2}\right) = \lim_{n\to\infty} a_n \qquad \text{Since } a_{n+1} = \frac{a_n + b_n}{2}$$

$$\Rightarrow \lim_{n\to\infty}\left(a_n + b_n\right) = 2\lim_{n\to\infty} a_n$$

$$\Rightarrow \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = 2\lim_{n\to\infty} a_n$$

$$\Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} 2a_n - \lim_{n\to\infty} a_n$$

$$\Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(2a_n - a_n\right)$$

$$\Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$$

Hence,

$$\boxed{\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n}$$

Q92E

(a)

Consider two subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ converges to L.

That is, $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$.

Use definition of limit of a sequence, to show that $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

Limit of a Sequence:

For every $\varepsilon > 0$, there exist a positive integer N such that for $n > N$,

$|a_n - L| < \varepsilon.$

As $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then

For every $\varepsilon > 0$, there exist a positive integer N_1 such that for $n > N_1$,

$|a_{2n} - L| < \varepsilon.$

Also for every $\varepsilon > 0$, there exist a positive integer N_2 such that for $n > N_2$,

$|a_{2n+1} - L| < \varepsilon.$

Let $N = \max\{N_1, 2N_2 + 1\}$.

For $n > N$,

$n = 2k$ or $n = 2k+1$, for some $k \in \mathbb{N}$.

If $n = 2k$, then $|a_{2k} - L| < \varepsilon$ as $k > N_1$.

If $n = 2k+1$, then $|a_{2k+1} - L| < \varepsilon$ as $k > N_2$.

Thus, $|a_n - L| < \varepsilon$ for all $n > N$.

Therefore, the sequence $\{a_n\}$ is convergent, and $\boxed{\lim_{n\to\infty} a_n = L}$.

(b)

Let the first term of the sequence be $a_1 = 1,$ and $a_{n+1} = 1 + \frac{1}{1+a_n}.$

Plug in $n = 1,$ to the term $a_{n+1} = 1 + \frac{1}{1+a_n},$ to get the second term of the sequence.

$$\begin{aligned} a_{1+1} &= 1 + \frac{1}{1+a_1} \\ a_2 &= 1 + \frac{1}{1+1} \\ &= 1 + \frac{1}{2} \\ &= \frac{3}{2} \end{aligned}$$

Plug in $n = 2,$ to the term $a_{n+1} = 1 + \frac{1}{1+a_n}.$

$$\begin{aligned} a_{2+1} &= 1 + \frac{1}{1+a_2} \\ a_3 &= 1 + \frac{1}{1+\frac{3}{2}} \\ &= 1 + \frac{2}{5} \\ &= \frac{7}{5} \end{aligned}$$

Plug in $n = 3$, to the term $a_{n+1} = 1 + \frac{1}{1 + a_n}$.

$$a_{3+1} = 1 + \frac{1}{1 + a_3}$$

$$a_4 = 1 + \frac{1}{1 + \frac{7}{5}}$$

$$= 1 + \frac{5}{12}$$

$$= \frac{17}{12}$$

Plug in $n = 4$, to the term $a_{n+1} = 1 + \frac{1}{1 + a_n}$.

$$a_5 = 1 + \frac{1}{1 + a_4}$$

$$= 1 + \frac{1}{1 + \frac{17}{12}}$$

$$= 1 + \frac{12}{29}$$

$$= \frac{41}{29}$$

Plug in $n=5,$ to the term $a_{n+1}=1+\frac{1}{1+a_n}$.

$$\begin{aligned} a_6 &= 1+\frac{1}{1+a_5} \\ &= 1+\frac{1}{1+\frac{41}{29}} \\ &= 1+\frac{29}{70} \\ &= \frac{99}{70} \end{aligned}$$

Plug in $n=6,$ to the term $a_{n+1}=1+\frac{1}{1+a_n}$.

$$\begin{aligned} a_7 &= 1+\frac{1}{1+a_6} \\ &= 1+\frac{1}{1+\frac{99}{70}} \\ &= 1+\frac{70}{169} \\ &= \frac{239}{169} \end{aligned}$$

Plug in $n=7,$ to the term $a_{n+1}=1+\frac{1}{1+a_n}$.

$$\begin{aligned} a_8 &= 1+\frac{1}{1+a_7} \\ &= 1+\frac{1}{1+\frac{239}{169}} \\ &= 1+\frac{169}{408} \\ &= \frac{577}{408} \end{aligned}$$

Therefore, the first eight terms of the sequence are $\boxed{1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \text{ and } \frac{577}{408}}$.

Use part **(a)** result to show that $\lim_{n\to\infty} a_n = \sqrt{2}$.

Consider the two subsequences of $\{a_n\}$ odd terms $\left\{1, \frac{7}{5}, \frac{41}{29}, \frac{239}{169}, \dots\right\}$, and the even terms $\left\{\frac{3}{2}, \frac{17}{12}, \frac{99}{70}, \frac{577}{408}, \dots\right\}$ of the sequence.

Odd terms sequence is increasing and tending to 1.41421356.

Even terms sequence are decreasing and tending to 1.41421356.

Therefore, the sequence a_n approaches to $\boxed{\sqrt{2}\left(\text{since } \sqrt{2} = 1.41421356\right)}$.

Q93E

Consider the formula $p_{n+1} = \dfrac{bp_n}{a+p_n}$

Where p_n is the fish population after *n* years and a and b are positive constants that depend on the species and its environment.

Suppose $p_n \to p$ as $n \to \infty$

Similarly $p_{n+1} \to p$ as $n \to \infty$ also

Then

$$\begin{aligned} p &= \lim_{n\to\infty} p_{n+1} \\ &= \lim_{n\to\infty} \frac{bp_n}{a+p_n} \\ &= \frac{b \lim_{n\to\infty} p_n}{a + \lim_{n\to\infty} p_n} \\ &= \frac{b_p}{a+p} \end{aligned}$$

So

$$\begin{aligned} & p = \frac{b_p}{a+p} \\ & \Rightarrow p(a+p) = b_p \\ & \Rightarrow p^2 + ap - bp = 0 \\ & \Rightarrow p(p + (a-b)) = 0 \end{aligned}$$

So $p = 0$ or $p = -(a-b) = b-a$

The only possible values for its limit are 0 and $b-a$.

Therefore $\{p_n\}$ is convergent.

(b)To prove $p_{n+1} < (b/a)\,p_n$.

$$P_{n+1} = \frac{bp_n}{a + p_n}$$
$$= \frac{b}{a}\left(\frac{p_n}{1+\frac{p_n}{a}}\right)$$

Note that $1+\frac{p_n}{a} > 1$

So $\frac{1}{1+\frac{p_n}{a}} < \frac{1}{1} = 1$

Since $p_n > 0$

$$\frac{p_n}{1+\frac{p_n}{a}} < p_n$$

With this fact,

$$p_{n+1} = \frac{bp_n}{a + p_n}$$
$$= \frac{b}{a}\left(\frac{p_n}{1+\frac{p_n}{a}}\right) < \frac{b}{a}p_n$$

(c)Using the above condition $p_{n+1} < \left(\frac{b}{a}\right)p_n$.

$$p_1 < \frac{b}{a} p_0$$

$$p_2 < \frac{b}{a} p_1 < \left(\frac{b}{a}\right)^2 p_0$$

$$p_3 < \frac{b}{a} p_2 < \left(\frac{b}{a}\right)^3 p_0$$

$$p_n < \left(\frac{b}{a}\right)^n p_0$$

Now since $b < a, \;\; 0 < \frac{b}{a} < 1$

So

$$\begin{aligned} \lim_{n\to\infty} \left(\frac{b}{a}\right)^n p_0 &= p_0 \lim_{n\to\infty} \left(\frac{b}{a}\right)^n \\ &= p_0 \cdot 0 \\ &= 0 \end{aligned}$$

Since $p_n < \left(\frac{b}{a}\right)^n p_0$ for all *n*,

By the comparison test,

$$\lim_{n\to\infty} p_n \leq \lim_{n\to\infty} \left(\frac{b}{a}\right)^n p_0 = 0$$

But $p_n \geq 0$ for all *n*, so $0 \leq \lim_{n\to\infty} p_n \leq 0,$

Then $\lim_{n\to\infty} p_n = 0.$

(d)Let us assume that $a < b$.

Suppose $\{p_n\}$ is an increasing sequence. Since $\{p_n\}$ is bounded.

Every bounded, monotonic sequence is convergent.

If $a < b$ p_n is bounded, monotonic and converges to $b - a$.

Therefore, if $p_0 < b - a$, then $\{p_n\}$ is increasing and $0 < p_n < b - a.$

Similarly if $p_0 > b - a$, then $\{p_n\}$ is decreasing and $p_n > b - a.$

Therefore our assumption is right, if $a < b$, then $\lim_{n\to\infty} p_n = b - a$.