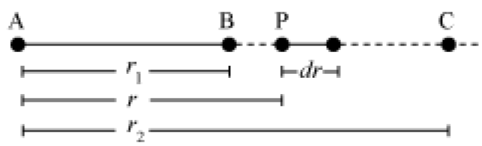
Electrostatic Potential And Capacitance

Electrostatic Potential Energy & Electrostatic Potential

- Change in electric potential energy of a system of charges is defined as the negative of the work done by the electric forces as the configuration of the system is changed.

- Let us consider a system of two charges q_1 & q_2, separated by a certain distance. Suppose that q_1 is fixed at A and q_2 is gradually moved away from its initial position B towards C; as shown in the diagram below:



On its way towards C, let us consider an arbitrary point P when the charge q_2 is at a distance *r* from A, and then a small displacement *dr* from P.

Force on q_2 when it is at P is,

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$$, along AB.

The work done by this force in the small displacement *dr* is,

$$dW = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} dr$$

Hence the total work done as the charge q_2 moves from B to C is,

$$W = \int dW = \int_{r_1}^{r_2} \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} dr = \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Thus the change in potential energy of this system of charge is,

$$U_C - U_B = -W = \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

Now consider the charge q_2 to be brought from infinity to the position P, the potential energy stored in the system of these two charges will be,

$$U_r - U_\alpha = \frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{\alpha}\right) = \frac{q_1 q_2}{4\pi\varepsilon_0 r} = U_r$$

The above equation gives the potential energy of a pair of charges.

- Electric potential at a point P is defined as the change in electric potential energy per unit positive charge when it is brought from a reference point to the point P, in presence of an electric field.

  Suppose a unit positive charge is brought from a reference point Q to the point P, and let V_P & V_Q be the potentials at P and Q respectively, the change in electric potential of the system is,

$$V_P - V_Q = \frac{U_P - U_Q}{q}$$

$$\text{Or, } \Delta V = \frac{\Delta U}{q}$$

  If the reference point Q is at infinity, we can write,

$$V_P = \frac{U_P}{q}$$

$$\because U_\alpha = 0$$

- Electric potential is also defined as the work done by an external agent on a unit positive charge in moving the test charge form a reference point to a point P, without changing its kinetic energy.

  Let W_{ext} be the work done by the external agent in bringing the unit positive charge from the reference point Q to the point P without changing its kinetic energy, and W_E be the work done by the electric field when the charge was moved. Then from the work-energy theorem the net work done on the charge is zero. Hence, we can write,

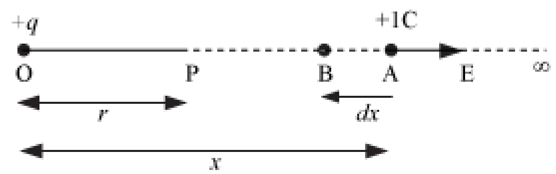
$$W_{ext} + W_E = 0$$

$$\Rightarrow W_{ext} = -W_E = \Delta U$$

$$\text{Or, } \Delta V = \frac{-W_E}{q}$$

- **Electric Potential Due to a Point Charge**

  Let P be the point at a distance 'r' from the origin O at which the electric potential due to charge $+q$ is required.



The electric potential at point *P* is the amount of work done in carrying a unit positive charge from ∞ to *P*.

Let 'A' be an intermediate point on this path, where OA = *x*. The electrostatic force on unit positive charge is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q}{x^2}$$, along OA

Small work done in moving through a distance '*dx*',

$$dW = \vec{F}\cdot d\vec{x} = Fdx\cos 180° = -Fdx$$

∴Total work done in moving unit positive charge from ∞ to point *P* is

$$W = \int_{\infty}^{r} -Fdx = \int_{\infty}^{r} -\frac{1}{4\pi\varepsilon_0}\frac{q}{x^2}dx$$

$$W = -\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{x}\right]_{\infty}^{r}$$

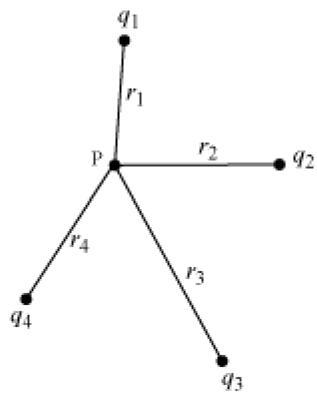
$$W = -\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}+\frac{1}{\infty}\right] = \frac{q}{4\pi\varepsilon_0 r}$$

By the definition of potential, this is the potential at P due to the charge at O.

$$\boxed{V = \frac{q}{4\pi\varepsilon_0 r}}$$

Equipotential Surfaces & Potential due to a System of Charges

Potential due to a System of Charges



Let there be a number of point charges q_1, q_2, q_3, … q_n at distances r_1, r_2, r_3, … r_n respectively from the point P, where electric potential is to be calculated.

Now, potential at P due to charge q_1,

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1}$$

$$V_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2}$$

$$V_3 = \frac{1}{4\pi\varepsilon_0}\frac{q_3}{r_3}$$

.

.

.

$$V_n = \frac{1}{4\pi\varepsilon_0}\frac{q_n}{r_n}$$

Using superposition principle, we obtain the resultant potential at P due to total charge configuration as the algebraic sum of the potentials due to individual charges.

∴ $V = V_1 + V_3 + V_3 + \ldots + V_n$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2} + \frac{1}{4\pi\varepsilon_0}\frac{q_3}{r_3} + \ldots + \frac{1}{4\pi\varepsilon_0}\frac{q_n}{r_n}$$

Or, $$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \ldots + \frac{q_n}{r_n}\right)$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}}$$

Equipotential Surfaces

An equipotential surface is that surface at every point of which, the electric potential is the same.

We know that,

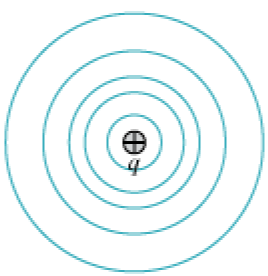
$q(V_B - V_A) = W_{AB}$, q being a test charge.

If points A and B lie on an equipotential surface, then

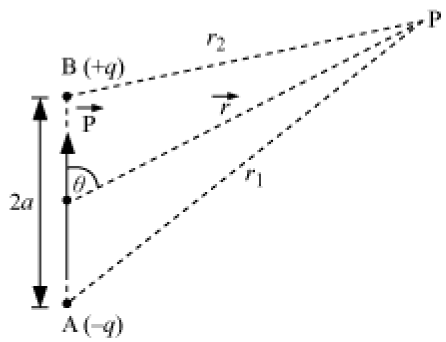
$V_B - V_A = 0$

$\therefore W_{AB} = q(V_B - V_A) = 0$

- No work is done in moving the test charge from one point of equipotential surface to the other.
- For any charge configuration, equipotential surface through a point is normal to the electric field at that point.
- Equipotential surfaces of a single point charge are concentric spherical surfaces centred at the charge.



Potential Due to an Electric Dipole



Let *P* be the point at which electric potential is required.

Potential at *P* due to − *q* charge,

$$V_1 = \frac{-q}{4\pi\varepsilon_0 r_1}$$

Potential at *P* due to + *q* charge,

$$V_2 = \frac{q}{4\pi\varepsilon_0 r_2}$$

Potential at *P* due to the dipole,

$$V = V_1 + V_2$$
$$V = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_2} - \frac{1}{r_1}\right] \quad \text{...(i)}$$

Now, by geometry

$$r_1^2 = r^2 + a^2 + 2ar\cos\theta$$
$$r_2^2 = r^2 + a^2 + 2ar\cos(180° - \theta)$$
$$= r^2 + a^2 - 2ar\cos\theta$$

$$r_1^2 = r^2\left(1 + \frac{a^2}{r^2} + \frac{2a}{r}\cos\theta\right)$$

If *r* >> *a*, $\frac{a}{r}$ is small. Therefore, $\frac{a^2}{r^2}$ can be neglected.

$$\therefore r_1^2 = r^2\left(1+\frac{2a}{r}\cos\theta\right)$$

$$r_1 = r\left(1+\frac{2a}{r}\cos\theta\right)^{\frac{1}{2}}$$

$$\frac{1}{r_1} = \frac{1}{r}\left(1+\frac{2a}{r}\cos\theta\right)^{-\frac{1}{2}}$$

Similarly, $\frac{1}{r_2} = \frac{1}{r}\left(1-\frac{2a}{r}\cos\theta\right)^{-\frac{1}{2}}$

Putting these values in (i), we obtain

$$V = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\left(1-\frac{2a}{r}\cos\theta\right)^{-\frac{1}{2}} - \frac{1}{r}\left(1+\frac{2a}{r}\cos\theta\right)^{-\frac{1}{2}}\right]$$

Using Binomial theorem and retaining terms up to the first order in $\frac{a}{r}$, we obtain

$$V = \frac{q}{4\pi\varepsilon_0 r}\left[1+\frac{a}{r}\cos\theta-\left(1-\frac{a}{r}\cos\theta\right)\right]$$

$$V = \frac{q}{4\pi\varepsilon_0 r}\left[1+\frac{a}{r}\cos\theta-1+\frac{a}{r}\cos\theta\right]$$

$$V = \frac{q\times 2a\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\boxed{V = \frac{P\cos\theta}{4\pi\varepsilon_0 r^2}}$$

As $P\cos\theta = \vec{p}\cdot\vec{r}$,

$$\therefore \boxed{V = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}} \qquad \text{for } (r >> a)$$

Potential Energy in an External Field

Potential Energy of a Single Charge

Work done in bringing a charge q from infinity to a point P, in an external field $= q \cdot V(\vec{r})$

Let,

$\vec{E}$ = Strength of the external electric field

$V(\vec{r})$ = External potential at any point P, of position vector $\vec{r}$

This work done is stored in the charged particle in the form of its potential energy.

∴Potential energy of a single charge q, at distance $\vec{r}$, in an external field $= q \cdot V(\vec{r})$

Potential Energy of a System of Two Charges in an External Field

The work done in bringing a charge q_1 from infinity to position $\vec{r}_1$ is

$W_1 = q_1 . V(\vec{r}_1)$

Now, the work done in bringing a charge q_2 from infinity to position $\vec{r}_2$, against an external field is

$W_2 = q \cdot V(\vec{r}_2)$

Let,

q_1, q_2 = Two point charges at position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively

$\vec{E}$ = Intensity of the external electric field

$V(\vec{r}_1)$ = Potential at $\vec{r}_1$ due to the external field

$V(\vec{r}_2)$ = Potential at $\vec{r}_2$ due to the external field

For bringing q_2 from infinity to position $\vec{r}_2$, work has to be done against the field due to q_1.

$$\therefore W_3 = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}}$$

Where, r_{12} = Distance between q_1 and q_2

Total work done in assembling the charge configuration = Potential energy of the system

$$U = W_1 + W_2 + W_3$$

$$\boxed{U = q_1 \cdot V(\vec{r}_1) + q_2 \cdot V(\vec{r}_2) + \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}}}$$

Potential Energy of an Electric Dipole, When Placed in a Uniform Electric Field

Suppose an electric dipole of dipole moment p is placed along a direction, making an angle $\theta$with the direction of an external uniform electric field E. Then, the torque acting on the dipole is given by

$$\tau = pE\sin\theta$$

If the dipole is rotated through an infinitesimally small angle $d\theta$, against the torque acting on it, then the small work done is given by

$$dW = \rho d\theta = pE\sin d\theta$$

If the dipole is oriented, making an angle θ_1 to θ_2 with the electric filed, then the total work done is given by

$$W = \int_{\theta_1}^{\theta_2} pE\sin\theta d\theta = pE\left|-\cos\theta\right|_{\theta_1}^{\theta_2}$$

$W = pE\ (\cos\theta_1 - \cos\theta_2)$

This work done is stored in the dipole in the form of its potential energy.

$\therefore U = pE\ (\cos 90° - \cos\theta)$

$U = -\ pE\cos\theta$

$U = -\ \vec{p}\cdot\vec{E}$

Electrostatics of Conductors & Dielectrics and Polarisation

- Inside a conductor, electrostatic field is zero. In static solution, the free charges have so distributed themselves that the electric field is zero everywhere inside.

- At the surface of a charged conductor, electrostatic field must be normal to the surface at every point.

- The interior of a conductor can have no excess charge in the static situation.

- Electrostatic potential is constant throughout the volume of the conductor and has the same value (as inside) on its surface.

- Electric field at the surface of a charged conductor is given by,

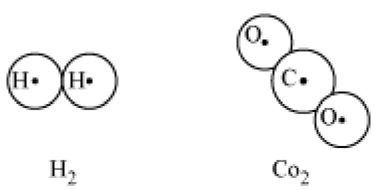
$$E = \frac{\sigma}{\varepsilon_0}\hat{n}$$

- Electrostatic shielding – It is the phenomenon of protecting a certain region of space from external electric field.

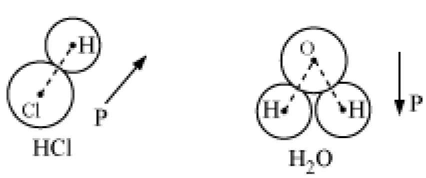
Dielectrics and polarisation

- Non-polar dielectrics – The centre of positive charge coincides with centre of negative charge in the molecule.

Example:



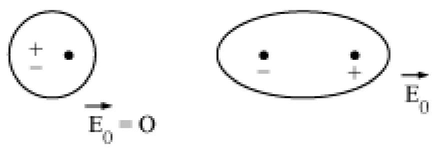
- Polar dielectrics – The centres of positive and negative charges do not coincide because of the asymmetric shape of the molecules.



Polar molecules

Non-Polar Dielectrics

- When a non-polar dielectric is held in an external electric field $\vec{E}_0$, the centre of positive charge in each molecule is pulled in the direction of $\vec{E}_0$ and the negative charge centre is pulled in a direction opposite to $\vec{E}_0$.
- The two centres of positive and negative charges in the molecule are separated. The molecules get distorted. The non-polar molecule gets polarised or a tiny dipole moment is imparted to each molecule.

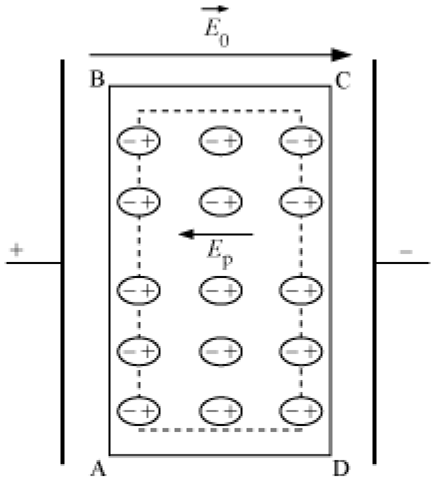


Polar Dielectrics

- When no external field is applied, the different permanent dipoles of such a dielectric are oriented randomly. Therefore, the total dipole moment is zero.
- When an external electric field is applied, the individual dipole moments tend to align with the field.
- A net dipole moment in the direction of the external field is developed i.e., the dielectric is polarised.
- Thus, each molecule becomes a tiny electric dipole, with a dipole moment parallel to the external field and proportional to it. Induced dipole moment P acquired by the molecule may be written as

  $P = \alpha\varepsilon_0 E_0$

  Where, α is a constant of proportionality and is called atomic/molecular polarisibility



Consider any small volume element in the interior of the slab, shown in dotted in the above figure. Inside the dotted portion, there is no net charge density. The negative ends of the dipoles remain unneutralised at the surface AB and positive ends of the dipole remain unneutralised at the surface CD.

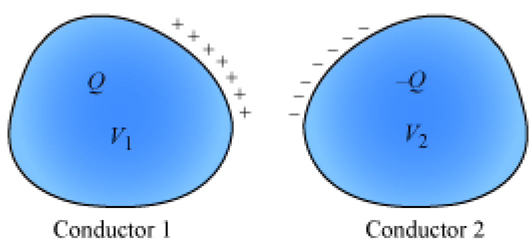
They set up an electric field $\vec{E}_P$ opposite to $\vec{E}_0$.

∴ Effective electric field in a polarised dielectric = $E = E_0 - E_P$

Here, E is called the reduced value of the electric field.

Capacitors and Capacitance

- A capacitor is a system of two conductors separated by an insulator. The conductors have charges Q and $-Q$, with potential difference $V = V_1 - V_2$ between them.



- The electric field in the region between the conductors is proportional to the charge Q.
- If potential difference V is the work done per unit positive charge in taking a small test charge from the conductor 2 to 1 against the field, then V is proportional to Q and the ratio Q/V is a constant.

$C = \dfrac{Q}{V}$

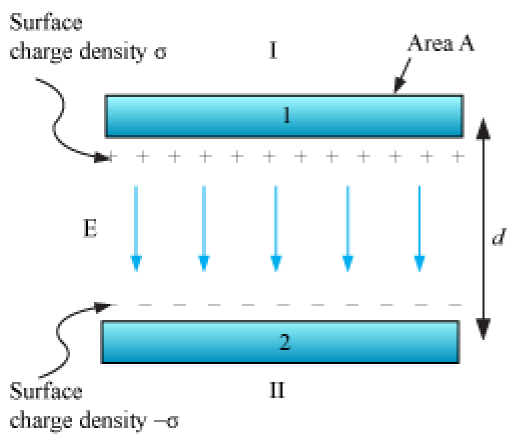
The constant C is called the capacitance of the capacitor.

- Capacitance C depends on shape, size and separation of the system of two conductors.
- The SI unit of capacitance is Farad.

  1 Farad = 1 Coulomb volt^{-1}

Parallel Plate Capacitor

- A parallel plate capacitor consists of two large plane parallel conducting plates separated by a small distance.



- Let A be the area of each plate and d the separation between them. The two plates have charges Q and $-Q$.
- Surface charge density of plate 1, $\sigma = Q/A$, and that of plate 2 is σ.
- Electric field in different regions:

  Outer region I,

  $E = \dfrac{\sigma}{2\varepsilon_0} - \dfrac{\sigma}{2\varepsilon_0} = 0$

  Outer region II,

$$E = \frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} = 0$$

In the inner region between plates 1 and 2, the electric fields due to the two charged plates add up. So,

E=σ2ε0+σ2ε0=σε0=Qε0AE=σ2ε0+σ2ε0=σε0=Qε0A

- The direction of electric field is from the positive to the negative plate.
- For uniform electric field, potential difference is simply the electric field multiplied by the distance between the plates, i.e.

$$V = E\,d = \frac{1}{\varepsilon_0}\frac{Qd}{A}$$

Capacitance *C* of the parallel plate capacitor,

$$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}$$

Effect of Dielectric on Capacitance

- Consider two large plates, each of area *A*, separated by a distance *d.*
- The charge on the plates is ± *Q*, corresponding to linear charge density ±*σ*. When there is vacuum between the plates,

$$E_0 = \frac{\sigma}{\varepsilon_0}$$

And the potential difference,

$V_0 = E_0 d$

Capacitance in this case,

$$C_0 = Q/V_0 = \varepsilon_0 \frac{A}{d}$$

- Consider a dielectric inserted between the plates, fully occupying the intervening region. The electric field in the dielectric then corresponds to the case when the net surface charge density on the plates is ± $(\sigma - \sigma_p)$. That is,

$$E = \frac{\sigma - \sigma_p}{\varepsilon_0}$$

Therefore, the potential difference across the plates,

$$V = Ed = \frac{\sigma - \sigma_p}{\varepsilon_0} d$$

For linear dielectric, we expect σ_p to be proportional to E_0, i.e. to σ. Thus, $(\sigma - \sigma_p)$ is proportional to σ and we can write

$$\sigma - \sigma_p = \frac{\sigma}{K}$$

Here, K is a constant characteristic of the dielectric.

Clearly, $K > 1$

Then,

$$V = \frac{\sigma d}{\varepsilon_0 K} = \frac{Qd}{A\varepsilon_0 K}$$

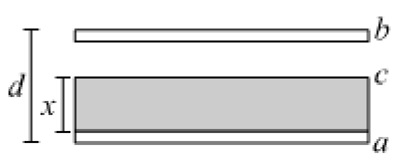
Capacitance C, with dielectric between the plates is then

$$C = \frac{Q}{V} = \frac{\varepsilon_0 KA}{d}$$

The product $\varepsilon_0 K$ is called the permittivity of the medium and is denoted by ε.

$\varepsilon = \varepsilon_0 K$

Capacitance of Parallel Plate Capacitor With Partially Filled Dielectric

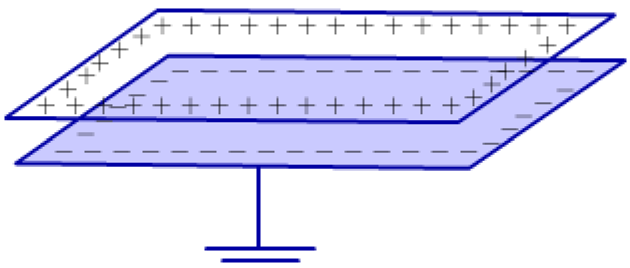


The above system is equivalent to two capacitors in series. Let the capacitance of the capacitor with dielectric be C_1 and that of without dielectric be C_2. Now,
C1=KεoAx andC2=εoA(d−x) C1=KεoAx andC2=εoA(d-x)
For series combination, equivalent capacitance, C=C1C2C1+C2C=C1C2C1+C2
Thus, C=KεoAKd−x(K−1)C=KεoAKd-x(K-1)

- **Types of Capacitors:**

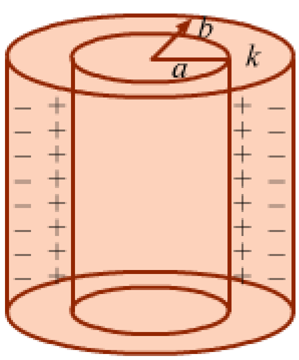
  Parallel Plate Capacitor:

- It consists of two parallel metal plates that are separated by a finite distance containing dielectric between them.
- One of the plates is positively charged and the other is negatively charged.



  Cylindrical Capacitor:

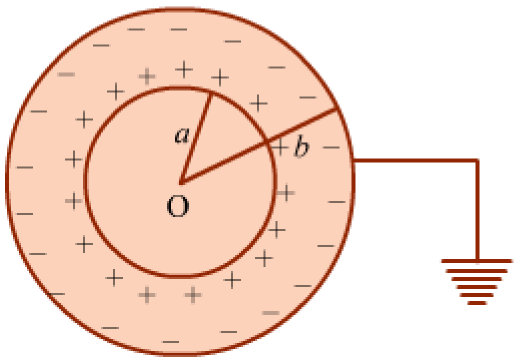
- It consists of two coaxial cylinders.
- The outer surface of the inner cylinder is positively charged and the inner surface of the outer cylinder is negatively charged.
- The space between them can be filled with the dielectric.
- Capacitance of cylindrical capacitor without dielectric is given as C=2πεollnbaC=2πεollnba, where *l* is the length of the cylinders; *a* and *b* is the radius of inner and outer cylinders, respectively.



  Spherical Capacitor:

- It consists of two concentric spheres.

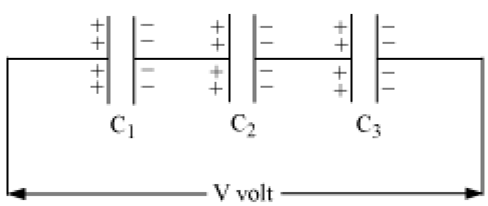
- The outer surface of the inner sphere has positively charge.
- The inner surface of outer sphere has negative charge.
- A dielectric is placed inside the space between the spheres.
- Capacitance of spherical capacitor without dielectric is given as C=4πεoabb−aC=4πεoabb-a



Combination of Capacitors & Energy Stored in a Capacitor

Combination of Capacitors

- Capacitors in series



- Capacitors are said to be connected in series when the potential difference applied across their combination is the sum of the resulting potential differences across each capacitor.

- Also in series combination the charge in all of the capacitors is same.

The potential difference across the separate capacitors are given by,

$$V_1 = \frac{Q}{C_1}, V_2 = \frac{Q}{C_2}, V_3 = \frac{Q}{C_3}$$

However,the potential difference across the series combination of capacitor is *V* volt.

∴ $V = V_1 + V_2 + V_3$ (i)

Let $C_s$represent the equivalent capacitance. Then,

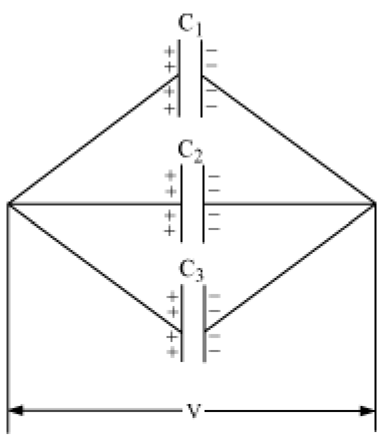
$$V = \frac{Q}{C_s} \qquad \text{(ii)}$$

Combining (i) and (ii), we obtain

$$\frac{Q}{C_s} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

$$\boxed{\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$$

- Capacitors in parallel − Capacitors are said to be connected in parallel when a potential difference that is applied across their combination results in the same across each capacitor.



- If Q is the total charge on the parallel network, then

$$Q = Q_1 + Q_2 + Q_3 \qquad \text{...(i)}$$

Let $C_P$be the equivalent capacitance of the parallel combination.

Now, $Q = C_P V$

$Q_1 = C_1 V$, $Q_2 = C_2 V$, and $Q_3 = C_3 V$ (ii)

Combining (i) and (ii), we obtain

$C_P V = C_1 V + C_2 V + C_3 V$

$$\boxed{C_P = C_1 + C_2 + C_3}$$

Energy Stored in a Charged Capacitor

The energy of a charged capacitor is measured by the total work done in charging the capacitor to a given potential.

Let us assume that initially both the plates are uncharged. Now, we have to repeatedly remove small positive charges from one plate and transfer them to the other plate.

Let

q →Total quantity of charge transferred

V →Potential difference between the two plates

Then,

$q = CV$

Now, when an additional small charge $dq$is transferred from the negative plate to the positive plate, the small work done is given by,

$$dW = Vdq = \frac{q}{C}dq$$

The total work done in transferring charge $Q$is given by,

$$W = \int_0^Q \frac{q}{C}dq = \frac{1}{C}\int_0^Q qdq = \frac{1}{C}\left[\frac{q^2}{2}\right]_0^Q$$

$$\boxed{W = \frac{Q^2}{2C}}$$

This work is stored as electrostatic potential energy Uin the capacitor.

$$U = \frac{Q^2}{2C}$$

$$U = \frac{(CV)^2}{2C} \qquad [\because Q = CV]$$

$$\boxed{U = \frac{1}{2}CV^2}$$

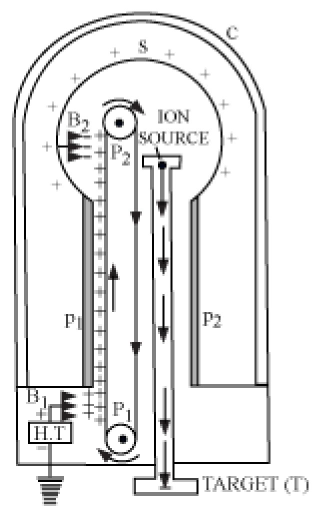
Van De Graff Generator

Van de Graff generator is a device used for building up high potential differences of the order of a few million volts.

Such high potential differences are used to accelerate charged particles such as electrons, protons, ions, etc.

Principle – It is based on the principle that charge given to a hollow conductor is transferred to outer surface and is distributed uniformly over it.

Construction:



It consists of a large spherical conducting shell (S) supported over the insulating pillars. A long narrow belt of insulating material is wound around two pulleys P_1 and P_2. B_1 and B_2 are two sharply pointed metal combs. B_1 is called the spray comb and B_2 is called the collecting comb.

Working – The spray comb is given a positive potential by high tension source. The positive charge gets sprayed on the belt.

As the belt moves and reaches the sphere, a negative charge is induced on the sharp ends of collecting comb B_2 and an equal positive charge is induced on the farther end of B_2.

This positive charge shifts immediately to the outer surface of S. Due to discharging action of sharp points of B_2, the positive charge on the belt is neutralized. The uncharged belt returns down and collects the positive charge from B_1, which in turn is collected by B_2. This is repeated. Thus, the positive charge on S goes on accumulating. In this way, voltage differences of as much as 6 or 8 million volts (with respect to the ground) can be built up.