Chapter 4

Continuous Time Fourier Analysis

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Introduction
- Approximation of a signal by other signals
- Fourier series representation of periodic signals
- Complex fourier series
- Even and odd Signals
- Harmonic form of fourier series
- Amplitude and phase spectra of periodic signal
- Linearity
- Time shifting
- Periodic convolution
- Parseval's theorem
- Continuous time (C.T.) fourier transform
- Fourier spectra
- Convergence of fourier transform
- Relation between fourier transform and laplace transform
- Common fourier transform pairs

Introduction

There are four distinct fourier representations, each applicable to different class of signals. The Fourier series (FS) applies to continuous-time periodic signals, and the discrete-time Fourier series (DTFS) applies to discrete time periodic signals. Non-periodic signals have Fourier transform representations. The Fourier transform (FT) applies to a signal that is continuous in time and non-periodic. The discrete-time Fourier transform (DTFT) applies to a signal that is discrete in time and non-periodic.

Relationship between time properties of a signal and the appropriate Fourier representation:

Time/property	Periodic	Non periodic
Continuous (t)	Fourier series (FS)	Fourier Transform (FT)
Discrete [n]	Discrete-time Fourier series (DTFS)	discrete-time Fourier transform (DTFT)

Approximation of a Signal by Other Signals

Let us consider two signals $f_1(t)$ and $f_2(t)$ we can approximate $f_1(t)$ in terms of $f_2(t)$ over a certain interval $(t_1 < t < t_2)$ as $f_1(t) \simeq C_{12} f_2(t)$ for $(t_1 < t < t_2)$.

We must choose c_{12} such that the error between actual function and the approximated function is minimum and is given by:

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) \cdot f_2(t)dt}{\int_{t_1}^{t_2} f_2^{\,2}(t)dt}$$

$f_1(t)$ has a component of wave form $f_2(t)$, and this component has a magnitude C_{12}. If C_{12} vanishes, then the signal $f_1(t)$ contains no component of signal $f_2(t)$ and we say the functions are orthogonal to each other over interval (t_1, t_2) if $\int_{t_1}^{t_2} f_1(t) \cdot f_2(t)dt = 0$,

similarly in case of complex functions $f_1(t), f_2(t)$ $f_1(t) \simeq C_{12} f_2(t)$ is approximation of $f_1(t)$ in terms of $f_2(t)$, and C_{12} to minimize mean square error magnitude is given by:

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2^{*}(t)dt}{\int_{t_1}^{t_2} f_2(t) f_2^{*}(t)dt},$$

where $f_2^{*}(t)$ is complex conjugate of $f_2(t)$.

For a set of complex functions $\{g_r(t)\}$ $r = 1, 2 \ldots)$ mutually orthogonal over the interval (t_1, t_2)

$$\int_{t_1}^{t_2} g_m\, g_n^{*}(t)\, dt = \begin{cases} 0, \text{if } m \neq n \\ K_m, \text{if } m = n \end{cases}$$

If this set of functions is complete, then any function $f(t)$ can be expressed as

$$f(t) = C_1 g_1(t) + C_2 g_2(t) \cdots + C_r g_r(t) + \cdots$$

where

$$C_r = \frac{1}{K_r} \int_{t_1}^{t_2} f(t) g_r^*(t) dt$$

$$K_r = \int_{t_1}^{t_2} g_r(t) g_r^*(t) dt.$$

Let us consider a set of n functions $g_1(t)$, $g_2(t) \ldots g_n(t)$ which are orthogonal to one another over an interval t_1 to t_2.

$$\int_{t_1}^{t_2} g_j(t) g_k(t) dt = 0, \text{ for } j \neq k \quad \text{and} \quad \text{consider} \int_{t_1}^{t_2} g_j^2(t) dt = K_j$$

An arbitrary function $f(t)$ can be approximated over an interval (t_1, t_2) by a linear combination of these n mutually orthogonal functions

$$f(t) \simeq c_1 g_1(t) + c_2 g_2(t) + \cdots + c_k g_k(t)$$
$$+ \cdots + c_n g_n(t) = \sum_{r=1}^{n} c_r g_r(t).$$

For the best approximation, mean square error over the interval has to be minimized then

$$C_r = \frac{\int_{t_1}^{t_2} f(t) g_r(t) dt}{\int_{t_1}^{t_2} g_r^2(t) dt} = \frac{1}{K_r} \int_{t_1}^{t_2} f(t) g_r(t) dt.$$

From the above discussion we can conclude that, $f(t)$ contains a component of signal $g_r(t)$ and this component has a magnitude C_r. Representation of $f(t)$ by a set of infinite mutually orthogonal functions is called generalized Fourier series representation of $f(t)$.

The representation of square wave by sinusoids.

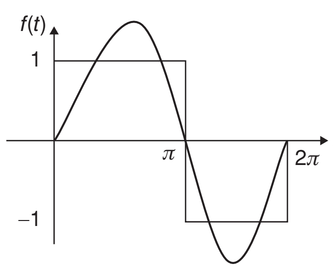


Figure 1 One term approximation

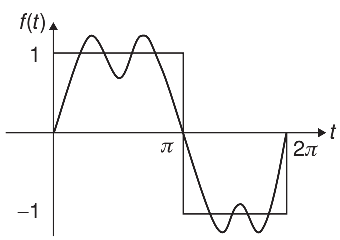


Figure 2 Two term approximation

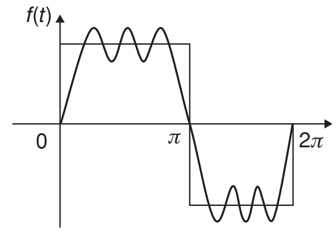


Figure 3 Three term approximation

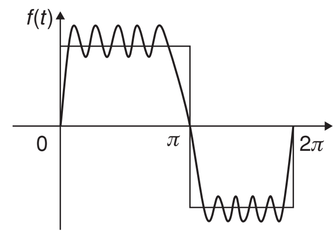


Figure 4 Five term approximation

From the approximation by sinusoids, we can observe that by increasing the number of terms we get nearly square wave.

Some examples of orthogonal functions: $\sin\omega_o t$, $\sin 2\omega_o t$, $\sin 3\omega_o t$, … etc., form orthogonal set over any interval $(t_0, t_0 + 2\pi/\omega_o)$.

Similarly $\cos\omega_o t$, $\cos 2\omega_o t$, $\cos 3\omega_o t$ … etc also form orthogonal set, in the same interval similarly $\cos n\omega_o t$ and $\sin m\omega_0 t$ also orthogonal over the same interval.

$$\int_{t_0}^{t_0+2\pi/\omega_0} \sin n\omega_o t \cdot \sin m\omega_o t \, dt = 0$$

$$\int_{t_0}^{t_0+2\pi/\omega_0} \cos n\omega_o t \cdot \cos m\omega_o t \, dt = 0 \text{ for all } m \neq n$$

$$\int_{t_0}^{t_0+2\pi/\omega_0} \sin m\omega_o t \cdot \cos n\, \omega_o t \, dt = 0$$

So composite set of functions consisting of set $\cos n\omega_o t$ and $\sin n\omega_o t$ for $(n = 0, 1, 2 \ldots)$ forms a complete orthogonal set.

$$\int_{t_0}^{t_0+2\pi/\omega_0} \sin^2 n\omega_o t \cdot dt = \int_{t_0}^{t_0+2\pi/\omega_0} \cos^2 n\omega_o t \, dt = \frac{\pi}{\omega_o} = \frac{T_0}{2}.$$

We can represent an arbitrary function $f(t)$ by linear combination of sine and cosine functions which is called as trigonometric Fourier series.

A set of exponential functions $\{e^{jn\omega_o t}\}(n = 0, \pm 1, \pm 2, \pm 3 \ldots)$ is orthogonal over an interval $(t_0, t_0 + 2\pi/\omega_o)$ for any value of

$$t_0 \int_{t_0}^{t_0+2\pi/\omega_0} (e^{jn\omega_o t})(e^{jn\omega_o t})^* dt = \frac{2\pi}{\omega_o} = T_0 \text{ for } m = n$$

So we can represent an arbitrary function $f(t)$ by a linear combination of exponential functions over an interval $(t_0, t_0 + 2\pi/\omega_o)$

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jn\omega_o t} (t_0 < t < t_0 + 2\pi/w_o).$$

This is called as complex exponential Fourier series representation

$$C_k = \frac{1}{T_0} \int_{t_0}^{t_0+2\pi/\omega_o} f(t) e^{-jk\omega_o t} dt.$$

Fourier Series Representation of Periodic Signals

If $x(t) = x(t + T_0)$, then $x(t)$ said to be periodic, the smallest positive value of T_0 is called fundamental period of $x(t)$.

Examples: $x(t) = \cos(\omega_o t + \phi)$,

$$x(t + T_0) = \cos[\omega_o(t + T_0) + \phi]$$
$$= \cos(\omega_o t + \phi + \omega_o T_0).$$

$x(t + T_0) = x(t)$ for minimum value $\omega_o T_0 = 2\pi$ or for all multiples of 2π, but we need to consider smallest value as fundamental period T_0. So $T_0 = \dfrac{2\pi}{\omega_o}$ is fundamental period, $x(t) = e^{j\omega_o t}$ are periodic where $\omega_o = \dfrac{2\pi}{T_0} = 2\pi f_0$ is the fundamental angular frequency.

Consider representing a periodic signal as a weighted super position of complex sinusoidal since the weighted super position must have the same period as the signal, each sinusoid in the superposition must have the same period as the signal, this implies that the frequency of each sinusoid must be an integer multiple of signal's fundamental frequency.

A sinusoid whose frequency is an integer multiple of a fundamental frequency is said to be a harmonic of the sinusoid at the fundamental frequency.

For example: $e^{jk\omega_o t}$ is the Kth harmonic of the $e^{j\omega_o t}$.

Complex Fourier Series

The complex Fourier series representation of a periodic signal $x(t)$ with fundamental period T_0 and fundamental frequency ω_0 is given by:

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}, \omega_0 = \frac{2\pi}{T_0}.$$

C_K: complex Fourier coefficient.

$$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

$\int_{T_0}$ denotes integral over one period of T_0.

$c_0 = \dfrac{1}{T_0}\displaystyle\int_{T_0} x(t)dt$, which indicates C_0 equals the average value of $x(t)$ over one period.

The Fourier series coefficients c_k are known as a frequency-domain representation of $x(t)$ because each Fourier series coefficient is associated with a complex sinusoid of a different frequency.

If $x(t)$ is real then $C_{-K} = C^*_K$

Trigonometric Fourier Series

The trigonometric Fourier series representation of a periodic signal $x(t)$ with fundamental period T_0, (or) fundamental angular frequency ω_0 is given by

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$a_k = \frac{2}{T_0}\int_{T_0} x(t)\cos k\omega_0 t\,dt$$

$$b_k = \frac{2}{T_0}\int_{T_0} x(t)\sin k\omega_0 dt.$$

The trigonometric Fourier coefficient a_k, b_k, and complex Fourier coefficient C_K are related as:

$$\frac{a_0}{2} = c_0, a_k = c_k + c_{-k}, b_k = j(c_k - c_{-k})$$

$$c_k = \frac{1}{2}(a_k - jb_k), c_{-k} = \frac{1}{2}(a_k + jb_k)$$

Even and Odd Signals

If a periodic signal $x(t)$ is even, then $b_k = 0$, and its Fourier series contains only cosine terms.

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t, \omega_0 = \frac{2\pi}{T_0}$$

If $x(t)$ is odd, then $a_k = 0$, and its Fourier series contains only sine terms

$$x(t) = \sum_{k=-\infty}^{\infty} b_k \sin k\omega_0 t, \omega_0 = \frac{2\pi}{T_0}$$

Harmonic Form of Fourier Series

Other form of the Fourier series representation of a real periodic signal $x(t)$ with fundamental period T_0 is given by [known as harmonic form Fourier series of $x(t)$]:

$$x(t) = D_o + \sum_{k=1}^{\infty} D_K \cos(k\omega_0 t - \theta_k), \omega_0 = \frac{2\pi}{T_0}$$

The term D_0 is known as dc component, and the term $D_k \cos(k\omega_0 t - \theta_k)$ is referred to as the kth harmonic component of $x(t)$.

The coefficients D_k and θ_k are related to trigonometric Fourier series coefficients a_k and b_k as:

$$D_o = \frac{a_0}{2}, D_K = \sqrt{{a_k}^2 + {b_k}^2}, \theta_k = \tan^{-1}\left(\frac{b_k}{a_k}\right)$$

Dirichlet Conditions

For convergence of Fourier series:

1. $x(t)$ is absolutely integrable over any period
$$\int_{T_0} |x(t)|\,dt < \infty.$$
2. $x(t)$ has a finite number of maxima and minima with in any field interval of 'T'.
3. $x(t)$ has a finite number of discontinuities with in any finite interval of 'T' and each of these discontinuities is finite.

Table 1 *Effects of symmetry on Fourier coefficients*

Symmetry	Condition	The Trigonometric Fourier Co-efficient to be Zero
Even	$x(t) = x(-t)$	$b_n = 0$, no sine terms will be present
Odd	$x(-t) = -x(t)$	$a_n = 0$, no cos terms will be present
Half wave	$x(t) = -x\left(t \pm \frac{T}{2}\right)$	dc term $a_0 = 0$; a_n, $b_n = 0$ for n even. All even coefficients will be zero

Amplitude and Phase Spectra of Periodic Signal

Complex Fourier coefficients C_k can be expressed as $C_K = |C_K|e^{j\phi_k}$. A plot $|C_k|$ versus angular frequency ω is called amplitude spectrum of $x(t)$. A plot of ϕ_K versus ω is called the phase spectrum of $x(t)$. Both are discrete in nature.

For a real, periodic signal $x(t)$ we have,

$$|C_{-k}| = |C_k|, \ \phi_{-k} = -\phi_k.$$

So amplitude spectrum is an even function of ω, where as phase spectrum is an odd function of ω, for real periodic signal.

Properties of Continuous Time Fourier Series

$x(t)$ is a periodic signal with period T_0 and fundamental frequency $\omega_0 = 2\pi/T_0$, then the Fourier series coefficients of $x(t)$ are denoted by C_k, then we can use the notation $x(t) \xleftrightarrow{FS} C_k$.

Linearity

If $x(t)$, and $y(t)$ denote two periodic signals with period T_0 and which have Fourier coefficients denoted by a_k and b_k, respectively,

$$x(t) \xleftrightarrow{FS} a_k, \ y(t) \xleftrightarrow{FS} b_k$$

as $x(t)$, and $y(t)$ have the same period T_0, it easily follows that any linear combination of the two signals will also be periodic with period T_0. $z(t) = Ax(t) + By(t)$, then Fourier series co-efficients of $z(t)$ are $C_k = A\,a_k + B\,b_k$

$$z(t) = Ax(t) + By(t) \xleftrightarrow{FS} C_k = A\,a_k + Bb_k$$

Time Shifting

If $x(t) \xleftrightarrow{FS} a_k$ then

$$x(t - t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} a_k = e^{-jk}(2\pi/T_0)t_0 a_k$$

When a periodic signal is shifted in time, the magnitude of its Fourier series coefficients remains unaltered, only phase will be changed.

Time Reversal

If $x(t) \xleftrightarrow{FS} C_k$ than $x(-t) \xleftrightarrow{FS} C_{-k}$, in other words time reversal applied to a continuous time signal results in a time reversal of the corresponding sequence of Fourier series co-efficients if $x(t)$ is even, then $x(-t) = x(t)$ then its Fourier series coefficients are also even $C_{-k} = C_k$ similarly if $x(t)$ is odd, then $x(-t) = -x(t)$, then so are its Fourier series co-efficients $C_{-k} = -C_k$.

Frequency Shifting

If $x(t) \xleftrightarrow{FS} C_k$ then

$$e^{jm\omega_0 t} x(t) \xleftrightarrow{FS} C_{k-m}$$

Time Scaling

If $x(t)$ is periodic with period T_0, and fundamental frequency $\omega_0 = 2\pi/T_0$ then $x(at)$, $a > 0$, is periodic with period $\frac{T_0}{a}$ and fundamental frequency $a\omega_0$ then Fourier series representation of

$$x(at) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

The Fourier coefficients C_k have not changed, but the Fourier series representation has changed because of the change in fundamental frequency.

Multiplication

If $x(t) \xleftrightarrow{FS} a_k$, $y(t) \xleftrightarrow{FS} b_k$, and $x(t)$, $y(t)$ are periodic with period T_0, then $x(t)$. $y(t)$ is also periodic with period T_0 then

$$x(t) \cdot y(t) \xleftrightarrow{FS} C_k = \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$$

C_K can be interpreted as the discrete time convolution of the sequence representing Fourier series co-efficients of $x(t)$ and $y(t)$.

Conjugation and Conjugate Symmetry

If $x(t) \xleftrightarrow{FS} C_k$, then $x^*(t) \xleftrightarrow{FS} C_{-k^*}$

If $x(t)$ is real signal then $x(t) = x^*(t)$

so $C_k = C_{-k^*}$ (or) $C_k^* = C_{-k}$

so we can write $|C_k| = |C_{-k}|, \angle C_k = -\angle C_k$

$\text{Re}\{C_k\} = \text{Re}\{C_{-k}\}$, $\text{Im}\{C_k\} = -\text{Im}\{C_{-k}\}$

When $x(t)$ is real $C_k^* = C_{-k}$,

When $x(t)$ is even, then $C_{-k} = C_k$

So when $x(t)$ is real and even $C_k = C_k^*$

And $C_k = C_{-k}$, i.e., the Fourier series coefficients are also real and even

When $x(t)$ is real $C_k^* = C_{-k}$,

And when $x(t)$ is odd $C_{-k} = -C_k$

By combining both, $C_{k^*} = -C_k$

So when $x(t)$ is real and odd, the Fourier series coefficients are also odd and purely imaginary ($\because\ C_k^* = -C_k$)

Periodic Convolution

If $x(t)$ and $y(t)$ are periodic with period T_0, then $x(t) \overset{FS}{\longleftrightarrow} a_k$, $y(t) \overset{FS}{\longleftrightarrow} b_k$

$$z(t) = \int_{T_0} x(\tau)y(t-\tau)d\tau \overset{FS}{\longleftrightarrow} C_k = T_0 a_k b_k$$

Differentiation

If $x(t) \overset{FS}{\longleftrightarrow} C_k$ is periodic with angular frequency ω_0, then $\frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} jk\omega_0 c_k$

Integration

If $x(t)$ is finite valued, periodic (ω_0), and if $C_0 = 0$ then $\int_{-\infty}^{t} x(t)\,dt \overset{FS}{\longleftrightarrow} \frac{1}{jk\omega_0} C_k$

Even Odd Decomposition of Real Signals

If $x(t)$ can be written as $x(t) = x_e(t) + x_0(t)$ as sum of even and odd components of $x(t)$.

$$x(-t) = x_e(t) - x_0(t)$$

$$x_e(t) = \frac{x(t)+x(-t)}{2}, x_0(t) = \frac{x(t)-x(-t)}{2}$$

when $x(t)$ is real, $x(t) \overset{FS}{\longleftrightarrow} C_k$, $x(-t) \overset{FS}{\longleftrightarrow} C_{-k}$

Similarly $C_k = C_{-k}^*$ (or) $C_{-k} = C_k^*$

So for $x_e(t) = \frac{1}{2}\{x(t)+x(-t)\} \overset{FS}{\longleftrightarrow} \frac{1}{2}\{C_k + C_k^*\}$

$$= \text{Re}\{C_k\}$$

For $x_0(t) = \frac{1}{2}\{x(t)-x(-t)\} \overset{FS}{\longleftrightarrow} \frac{1}{2}\{C_k - C_k^*\}$

$$= j\text{Im}\{C_k\}$$

$$x_e(t) \overset{FS}{\longleftrightarrow} \text{Re}\{C_k\}$$

$$x_0(t) \overset{FS}{\longleftrightarrow} j\,\text{Im}\{C_k\}$$

Parseval's Theorem

The average power of periodic signal $x(t)$ over any period is

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |C_K|^2$$

Parseval's theorem states that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Example 1: Which of the following signal will not have Fourier series representation.

(A) $\sin 4t + \cos 9t$ (B) $1 + \cos 2\pi t$
(C) e^{-j5t} (D) $\cos 4\pi t + \sin 3t$

Solution: Only periodic signals will have Fourier series representation.

For $\sin 4t + \cos 9t \to$ period is LCM of $\left(\frac{2\pi}{4}, \frac{2\pi}{9}\right) = 2\pi$.

Cos $2\pi t$ is periodic with period $T_0 = \frac{2\pi}{2\pi} = 1$, and DC component 1 is present in the signal $1 + \cos 2\pi t$ is also periodic.

For e^{-j5t} periodic with period $\frac{2\pi}{5}$. But $\cos 4\pi t + \sin 3t$ is not periodic, $T_1 = \frac{2\pi}{4\pi} = \frac{1}{2}$, $T_2 = \frac{2\pi}{3}, \frac{T_1}{T_2}$ is not rational, so option (D) will not have Fourier series expansion.

Continuous Time (CT) Fourier Transform

The function $X(\omega)$, and $x(t)$ are said to be Fourier transform pairs defined as below

$$x(t) \leftrightarrow X(\omega)$$

The Fourier transform of $x(t)$ is

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

The inverse Fourier transform of $X(\omega)$ is

$$x(t) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Fourier Spectra

Fourier transform of $x(t)$ can be expressed as $X(\omega) = |X(\omega)|e^{j\phi(\omega)}$. $X(\omega)$, Fourier transform of non periodic signal $x(t)$ is the frequency domain specification of $x(t)$, and is referred as Fourier spectrum of $x(t)$,

$|X(\omega)|$ is called magnitude spectrum of $x(t)$

$\phi(\omega)$ is called the phase spectrum of $x(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

For real signal $x(t)$

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt$$

So $X(-\omega) = X^*(\omega)$, $|X(\omega)| = |X(-\omega)|$, $\varphi(-\omega) = -\varphi(\omega)$.

Here also, the amplitude spectrum $|X(\omega)|$ is an even function and the phase spectrum $\varphi(\omega)$ is an odd function of ω.

Convergence of Fourier Transform

Dirichlet conditions

1. $x(t)$ is absolutely integrable $\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$.
2. $x(t)$ has finite number of maxima and minima with in any finite interval.
3. $x(t)$ has a finite number of discontinuities with in any finite interval, and each of these discontinuities is finite.

Relation Between Fourier Transform and Laplace Transform

Fourier transform of $x(t)$ is $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$. The bilateral Laplace transform of $x(t)$ is $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$. We can observe from these equations, Fourier transform is special case of Laplace transform in which $s = j\omega$. That is, $|X(s)|_{s=j\omega} = F\{x(t)\}$, if we substitute $s = \sigma + j\omega$ in Laplace transform equation:

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma + j\omega)t}\,dt$$

$$= \int_{-\infty}^{\infty} (x(t)e^{-\sigma t})e^{-j\omega t}\,dt = F\{x(t)e^{-\sigma t}\}.$$

Bilateral Laplace transform of $x(t)$ can be interpreted as the Fourier transform of $x(t)e^{-\sigma t}$. If $x(t)$ is absolutely integrable, the Fourier transform of $x(t)$ can be obtained from the laplace transform of $x(t)$ with $S = j\omega$.

Examples 2: Find the Fourier transform of the function $f(t) = e^{-at}\,u(t)$.

Solution: The Fourier transform does not converge for $a \le 0$, since $\int_{-\infty}^{\infty} e^{-at}u(t) = \infty\, a \le 0$. $f(t)$ is not absolutely integrable for $a \le 0$.

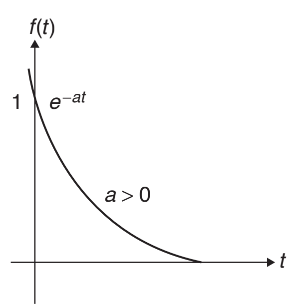
For $a > 0$,

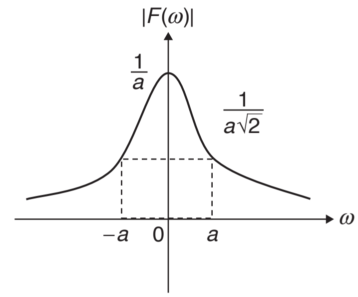
$$F(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt$$

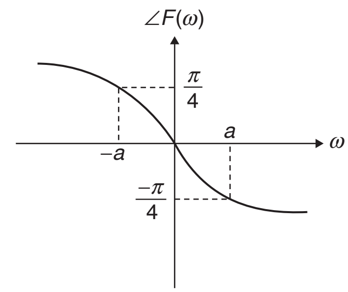
$$= \int_{-\infty}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \frac{-1}{a+j\omega}e^{-a\ +j\omega)t}\Big|_{0}^{\infty}$$

$$= \frac{1}{a+j\omega} = \frac{-1}{\sqrt{a^2+\omega^2}}\angle -\tan^{-1}(\omega/a)$$







Examples 3: Find the Fourier transform of the rectangular pulse?

$$x(t) = \begin{cases} 1 & |t| < T_0 \\ 0 & |t| > T_0 \end{cases}$$

Solution: $x(t)$ is absolutely integrable for $T_0 < \infty$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= \int_{-T_0}^{T_0} 1\cdot e^{-j\omega t}\,dt$$

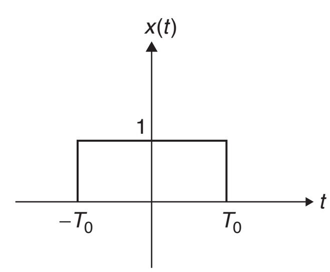
$$= \frac{-1}{j\omega}e^{-j\omega t}\Big|_{-T_o}^{T_o}$$

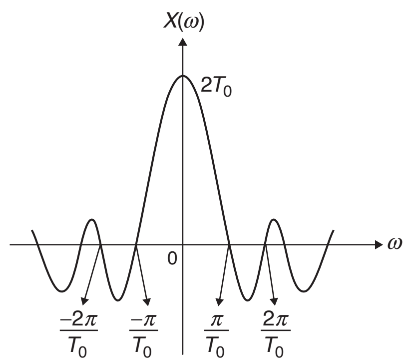
$$= \frac{-1}{j\omega}(e^{-j\omega T_0} - e^{+j\omega T_0}) = 2\frac{\sin \omega T_0}{\omega}$$

$\sin c$ function can be defined as $\sin c(x) = \dfrac{\sin \pi x}{\pi x}$

$$X(\omega) = 2T_0\frac{\sin \omega T_0}{\omega T_0} = 2T_0 \sin c(\omega T_0/\pi)$$

when $\omega \to 0, X(\omega) = \lim_{\omega \to 0}\dfrac{2T_0 \sin \omega T_0}{\omega T_0} = 2T_0$ (by using L Hospital's rule)





The magnitude spectrum is $|X(\omega)| = 2\left|\dfrac{\sin(\omega T_0)}{\omega}\right|$.

Phase spectrum is $\angle X(\omega) = \begin{cases} 0, \sin(\omega T_0)/\omega > 0 \\ \pi, \sin \omega T_0/\omega < 0. \end{cases}$

From the above example, we can see that If T_0 increases, then nonzero extent of $x(t)$ increases, and $X(\omega)$ becomes more concentrated about the frequency origin. Similarly when T_0 decreases $x(t)$ width reduces, $X(\omega)$ becomes less concentrated about the frequency origin. The duration of $x(t)$ is inversely related to the width or bandwidth of $X(\omega)$.

Examples 4: Find the Fourier transform of the $f(t) = \delta(t)$.

Solution: $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$

$$= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = e^{j\omega.0} = 1$$

$$\delta(t)\,\underline{\text{FT}}\,1.$$

Time domain signal defined at origin, but frequency components are present from $\omega = -\infty$ to ∞.

Examples 5: What is the inverse Fourier transform of $F(\omega) = 2\pi\delta(\omega)$

Solution: $f(t) = s\dfrac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}2\pi \cdot e^{j0.t} = 1$$

So inverse Fourier transform of $2\pi\delta(\omega)$ is 1.

$$1 \leftrightarrow 2\pi\delta(\omega)$$

This implies that the frequency content of a DC signal is concentrated entirely at $\omega = 0$.

Properties of the Continuous-time Fourier Transform

Linearity:

$$a_1x_1(t) + a_2 x_2 (t) \leftrightarrow a_1 X_1 (\omega) + a_2 X_2 (\omega)$$

Time shifting: $x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$

By delaying a signal by t_0 seconds $x(t - t_0)$, the magnitude spectrum $|x(\omega)|$ will not be changed, but the phase spectrum $\angle X(\omega)$ will be changed by $-\omega t_0$. So time delay in a signal causes a linear phase shift in its spectrum.

Frequency shifting: $e^{j\omega_0 t} x(t) \leftrightarrow X(\omega - \omega_0)$

The multiplication of a signal by a factor of $e^{j\omega_0 t}$ shifts the spectrum of that signal by $\omega = \omega_0$. We can observe the duality between the time shifting and frequency shifting properties.

The multiplication of a signal $x(t)$ by a sinusoid of frequency ω_0 shifts the spectrum $X(\omega)$ by $\pm\,\omega_0$

$$x(t)\cos\omega_0 t = \frac{1}{2}[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}] \leftrightarrow$$

$$\frac{1}{2}[X(\omega - \omega_0) + X(\omega + \omega_0)].$$

Time scaling: $x(at) \leftrightarrow \dfrac{1}{|a|}X\left(\dfrac{\omega}{a}\right)$.

The scaling property states that time compression of a signal results in its spectral expansion, and time expansion of the signal results in its spectral compression.

The scaling property implies that if $x(t)$ is wider, its spectrum is narrower, and vice versa, doubling the signal duration halves its bandwidth, and vice versa. This suggests that the band width of a signal is inversely proportional to the signal duration or width (in seconds).

Time reversall: $x(-t) \leftrightarrow X(-\omega)$.

For time scaling property if we consider $a = -1$, we can obtain the inversion property of time and frequency.

Duality $X(t) \leftrightarrow 2\pi x(-\omega)$

Differentiation in the time domain:

$$\frac{d}{dt}x(t) \leftrightarrow j\omega X(\omega),$$

$$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega).$$

Differentiation in the frequency domain:

$$(-jt)x(t) \leftrightarrow \frac{d}{d\omega}X(\omega)$$

Differentiation in time domain corresponds to multiplication by $j\omega$ in the frequency domain.

Integration in time domain:

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \pi X(0)\delta(\omega) + \frac{1}{j\omega}X(\omega).$$

Integration in the time domain corresponds to division by $j\omega$ in the frequency domain and the impulse term reflects the dc or average value that can result from integration.

Convolution:

$$x_1(t) * x_2(t) \leftrightarrow X_1(\omega)\, X_2(\omega).$$

The Fourier transform maps the convolution of two signals into the product of their Fourier transforms.

Multiplication:

$$x_1(t) \cdot x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega).$$

Multiplication in the time domain corresponds to convolution in frequency domain. We can observe the duality property between time and frequency domains from above two properties.

Parseval's relations:

$$\int_{-\infty}^{\infty} x_1(\lambda) X_2(\lambda) d\lambda = \int_{-\infty}^{\infty} X_1(\lambda) x_2(\lambda) d\lambda$$

$$\int_{-\infty}^{\infty} x_1(t) x_2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) X_2(-\omega) d\omega,$$

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega.$$

Parseval's identity says that energy content E of $x(t)$, can be computed by integrating $|X(\omega)|^2$ over all frequencies, for this reason $|X(\omega)|^2$ is often called as energy density spectrum of $x(t)$.

Real signal: If $x(t)$ is real, and let $x(t) = x_e(t) + x_0(t)$, where $x_e(t)$, $x_0(t)$ are even and odd components of $x(t)$, Fourier transform of $x(t)$ is $X(\omega) = A(\omega) + jB(\omega)$ then, $X(-\omega) = X^*(\omega)$. So we can conclude that

$$x_e(t) \xleftrightarrow{\text{FT}} \text{Re}\{X(\omega)\} = A(\omega),$$

$$x_o(t) \xleftrightarrow{\text{FT}} j\text{Im}\{X(\omega)\} = jB(\omega).$$

This shows that the Fourier transform of an even signal is a real function of ω, and Fourier transform of an odd signal is a pure imaginary function of ω. The $X(\omega)$ is complex conjugate symmetric for real signal $x(t)$. When $X^*(\omega) = X(-\omega)$.

By taking the real and imaginary parts of this expression gives Re $\{X(\omega)\}$ = Re$\{X(-\omega)\}$ and Im $\{X(\omega)\}$ = $-$Im$\{X(-\omega)\}$. In other words, if $x(t)$ is real valued, then the real part of the transform is an even function of frequency, while the imaginary part is an odd function of frequency.

This also implies that the magnitude spectrum is an even function of frequency, while the phase spectrum is an odd function.

Table 2 *Common Fourier Transform pairs*

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos \omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin \omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\frac{1}{j\omega}$
$e^{-at}u(t),\ a>0$	$\frac{1}{j\omega+a}$
$te^{-at}u(t),\ a>0$	$\frac{1}{(j\omega+a)^2}$
$e^{-a\lvert t\rvert}, a>0$	$\frac{2a}{a^2+\omega^2}$
$\frac{1}{a^2+t^2}$	$e^{-a\lvert\omega\rvert}$
$e^{-at^2},\ a>0$	$\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}$
$p_a(t)=\begin{cases}1 & \lvert t\rvert<a\\ 0 & \lvert t\rvert>a\end{cases}$	$2\frac{\sin\omega a}{\omega}$
$\frac{\sin at}{\pi t}$	$p_a(\omega)=\begin{cases}1 & \lvert\omega\rvert<a\\ 0 & \lvert\omega\rvert>a\end{cases}$
$sgn(t)$	$\frac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0), \omega_0=\frac{2\pi}{T}$

Fourier Transform Representation of Periodic Signals

If $x(t)$ is a periodic signal with fundamental frequency ω_0, then the Fourier series representation of $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} dt,\ \omega_0 = \frac{2\pi}{T_0}.$$

Where C_k—complex Fourier coefficient

$$C_k = \frac{1}{T_o}\int_{T_o} x(t)e^{-jk\omega_o t}\,dt$$

we have $1 \xrightarrow{\text{FT}} 2\pi\delta(\omega)$. By using frequency shifting property $e^{jk\omega_0 t} \xrightarrow{\text{FT}} 2\pi\,\delta(\omega - k\omega_0)$.

We can apply linearity property of the Fourier transform to obtain:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \xrightarrow{\text{FT}} X(\omega) = 2\pi,\ \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k\omega_0).$$

Thus the Fourier transform of a periodic signal is a series of impulses spaced by the fundamental frequency ω_0. The kth impulse has strength of $2\pi C_k$, C_k is kth Fourier series co-efficient.

Fourier Transform in Terms of Frequency (f)

We can change the variable ω to f (in Hertz), by writing $\omega = 2\pi f$ and $d\omega = 2\pi df$. So $X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$, $x(t) = \int_{-\infty}^{\infty} X(2f)e^{j2\pi ft}\,df$, we can get $X(f)$ by changing ω to $2\pi f$.

Similarly $\delta(\omega) = \delta(2\pi f) = \frac{1}{2\pi}\delta(f)$,

$$x(t) = e^{-at}u(t) \to X(\omega) = \frac{1}{a + j\omega} \to X(f) = \frac{1}{a + j2\pi f},$$

$$x(t) = e^{-|t|} \to X(\omega) = \frac{2a}{a^2 + \omega^2} \to X(f) = \frac{2a}{a^2 + 4\pi^2 f^2},$$

$$x(t) = u(t) \to X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega} \to X(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f},$$

$$x(t) = e^{-\pi t^2} \to X(\omega) = e^{-\omega^2/4\pi} \to x(f) = e^{-\pi f^2},$$

$$\cos(2\pi at) \to \pi\,[\delta(\omega + 2\pi a) + \delta(\omega - 2\pi a)]$$

$$\to \frac{1}{2}[d(f + a) + \delta(f - a)]$$

$$\sin(2\pi at) \to j\pi\,[\delta(\omega + 2\pi a) - \delta(\omega - 2\pi a)]$$

$$\to \frac{j}{2}[\delta(f + a) + \delta(f - a)]$$

Examples 6: Find the Fourier transform of $x(t) = e^{3t}\,u(-t)$

Solution: $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$

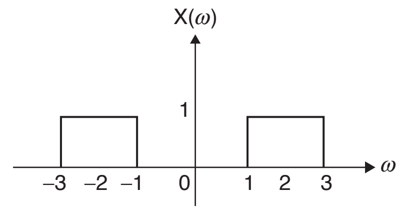
$$= \int_{-\infty}^{0} e^{3t}\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{(3-j\omega)}\,dt \text{ replace } t \text{ with } -t$$

$$= -\int_{0}^{\infty} e^{-(3-j\omega)t}\,dt$$

$$= \frac{-e^{-(3-j\omega)t}}{(3-j\omega)}\Bigg|_0^{\infty} = \frac{1}{3 - j\omega}$$

Examples 7: Find the inverse Fourier transform for the spectra depicted in figure.



Solution: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$

$$\text{So } x(t) = \frac{1}{2\pi}\left[\int_{-3}^{-1} 1e^{j\omega t}\,d\omega + \int_{1}^{3} 1e^{j\omega t}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left\{\frac{e^{j\omega t}}{jt}\Bigg|_{-3}^{-1} + \frac{e^{j\omega t}}{jt}\Bigg|_{1}^{3}\right\}$$

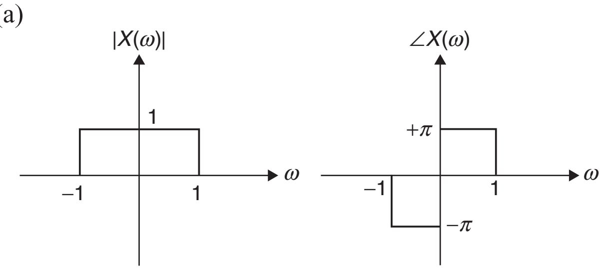
$$= \frac{1}{2\pi\,jt}(e^{-jt} - e^{-j3t} + e^{j3t} - e^{jt})$$

$$= \frac{1}{\pi t}\left\{\left(\frac{e^{j3t} - e^{-j3t}}{2j}\right) - \left(\frac{e^{jt} - e^{-jt}}{2j}\right)\right\}$$

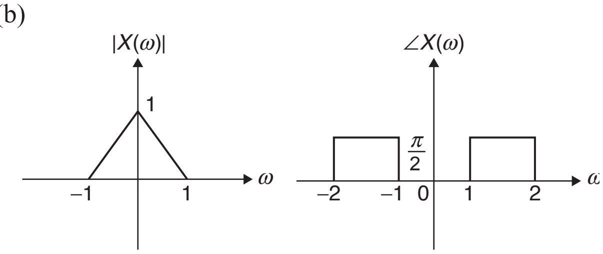
$$= \frac{1}{\pi t}(\sin 3t - \sin t)$$

Examples 8: For the frequency domain signal depicted in figure, find the nature of time domain signal?

(a)



(b)



Solution: For (a) $|x(\omega)|$ is an even function and $\angle X(\omega)$ is an odd function.

So we can say that $x(t)$ is real and odd.

For (b) $|x(\omega)|$ is an even function as well as $\angle X(\omega)$ is also an even function so $x(t)$ is real and even.

Examples 9: Find $x(t)$ for the spectra;

$$X(\omega) = \begin{cases} j\omega \; |\omega| < 1 \\ 0 \; |\omega| > 1. \end{cases}$$

Solution: $X(\omega) = \begin{cases} j\omega |\omega| < 1 \\ 0 |\omega| > 1. \end{cases}$

We can see $X(\omega) = j\omega\, z(\omega)$

Where $Z(\omega) = \begin{cases} 1 \,|\, \omega \,| < 1 \\ 0 \,|\, \omega \,| > 1. \end{cases}$

Then we can have $z(t) = \dfrac{1}{\pi t}\sin t$

$z(t) \underleftrightarrow{\text{FT}} Z(\omega)$, then $\dfrac{dz(t)}{dt} \leftrightarrow j\omega Z(\omega) = X(\omega)$.

The inverse Fourier transform of $X(\omega)$ is

$$x(t) = \frac{d}{dt} z(t) = \frac{d}{dt}\left(\frac{1}{\pi t}\sin t\right),$$

$$x(t) = \frac{1}{\pi t}\cos t - \frac{1}{\pi t^2}\sin t.$$

Examples 10: Find the time domain signal corresponding to the Fourier transform:

$$X(\omega) = \frac{5j\omega + 12}{-\omega^2 + 5j\omega + 6}$$

Solution: $X(\omega) = \dfrac{5(j\omega) + 12}{(j\omega)^2 + 5(j\omega) + 6}$

$$= \frac{5j\omega + 12}{(j\omega + 2)(j\omega + 3)} = \frac{A}{j\omega + 2} + \frac{B}{j\omega + 3}$$

$$= \frac{5\times -2 + 12}{-2 + 3}\cdot\frac{1}{j\omega + 2} + \frac{5\times -3 + 12}{-3 + 2}\,\frac{1}{j\omega + 3}$$

$$= \frac{2}{j\omega + 2} + \frac{3}{j\omega + 3}.$$

By taking inverse Fourier transform.

$$x(t) = 2e^{-2t}\,u(t) + 3e^{-3t}\,u(t)$$

Examples 11: What is inverse Fourier transform of the step function in frequency? ($X(\omega) = u(\omega)$)

Solution: $X(\omega) = u(\omega)$ from the duality property $x(t) \underleftrightarrow{\text{FT}} X(\omega)$, $X(t) \underleftrightarrow{\text{FT}} 2\pi x(-\omega)$.

We have time reversal property $g(t) \leftrightarrow G(\omega)$ then $g(-t) \leftrightarrow G(-\omega)$. So we can write $X(-t) \leftrightarrow 2\pi x(\omega)$

$$\frac{1}{2\pi} X(-t) \leftrightarrow x(\omega)$$

If we consider $x(t) = u(t)$, then $X(\omega) = \pi\delta(\omega) + \dfrac{1}{j\omega}$, then inverse Fourier transform of $u(\omega)$ is

$$\frac{1}{2\pi} X(-t) = \frac{1}{2\pi}\left\{\pi\delta(-t) - \frac{1}{jt}\right\} = \frac{\delta(t)}{2} - \frac{1}{2\pi jt} = \frac{\delta(t)}{2} + \frac{j}{2\pi t}$$

As $\delta(-t) = \delta(t)$

Examples 12: Find $x(t)$ if $X(\omega) = j\dfrac{d}{d\omega}\left\{\dfrac{e^{j3\omega}}{1 + j\omega/2}\right\}$.

Solution: Consider $p(t) = e^{-t}u(t) \leftrightarrow P(\omega) = \dfrac{1}{1 + j\omega}$,

then $X(\omega) = j\dfrac{d}{d\omega}\{e^{j3\omega} P(\omega/2)\}$.

If $Q(\omega) = P(\omega/2)$ then $q(t) = 2p(2t) = 2e^{-2t}\,u(2t) = 2e^{-2t}u(t)$. {time scaling $|a|\,x(at) \leftrightarrow X(\omega/a)$}

So Now $X(\omega) = j\dfrac{d}{d\omega}\{e^{j3\omega} Q(\omega)\}$.

If $R(\omega) = e^{j3}\,Q(\omega)$, then $r(t) = q(t + 3) = 2e^{-2(t+3)}$ $u(t + 3)$ {time shifting $x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$}

Now $X(\omega) = j\dfrac{d}{d\omega}\{R(\omega)\}$ then $x(t) = tr(t)$

$= 2t\,e^{-2(t+3)}u(t + 3)$. [differentiation in frequency domain $tx(t) \leftrightarrow j\dfrac{d}{d\omega}\{x(\omega)\}$].

Examples 13: Find the Fourier transform of the impulse train $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$

Solution: The fundamental period of $x(t)$ is T_0, so $\omega_0 = \dfrac{2\pi}{T_0}$ is fundamental frequency.

The Fourier coefficients are:

$$C_k = \frac{1}{T_0}\int_{-T_{0/2}}^{T_{0/2}} \delta(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}$$

The Fourier transform of a periodic signal is $X(\omega)$

$$= 2\pi\sum_{k=-\infty}^{\infty} c_k\delta(\omega - k\omega_0)$$

$$= 2\pi\sum_{k=-\infty}^{\infty} \frac{1}{T_0}\delta(\omega - k\omega_0)$$

$$= \frac{2\pi}{T_0}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

Hence Fourier transform of an impulse train is also an impulse train.

Frequency Response of Continuous-time LTI Systems

Consider a continuous time LTI system with impulse response $h(t)$, and the input is $x(t) = e^{j\omega t}$, then the convolution integral gives the output as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) e^{j\omega t} \cdot e^{-j\omega\tau}\, d\tau$$

$$= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\, d\tau$$

$$= H(\omega)\, e^{j\omega t}$$

Where we can define $H(\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega t}\, d\tau.$

The output of the system is thus a complex sinusoid of the same frequency as the input, multiplied by the complex number $H(\omega)$.

$H(\omega)$ is a function of only the frequency ω, and not the time t, and is termed the frequency response of the continuous-time system.

Output $y(t)$ of a continuous-time LTI system. Equals the convolution of the input $x(t)$, with the Impulse response $h(t)$ $y(t) = x(t) * h(t)$, taking Fourier transform $Y(\omega) = X(\omega)$ $H(\omega)$, $H(\omega) = \dfrac{Y(\omega)}{X(\omega)}$.

$H(\omega)$ is called frequency Response of the system. $H(\omega) = |H(\omega)|\, e^{j\theta_H(\omega)}$, $|H(\omega)|$ is called magnitude response of the system $\theta_H(\omega)$ is phase response of the system.

The behavior of a continuous-time LTI system in the frequency domain is completely characterized by its frequency response $H(\omega)$

Input $X(\omega) = |X(\omega)|\, e^{j\theta_H(\omega)}$

$H(\omega) = |H(\omega)|\, e^{j\theta_H(\omega)}$

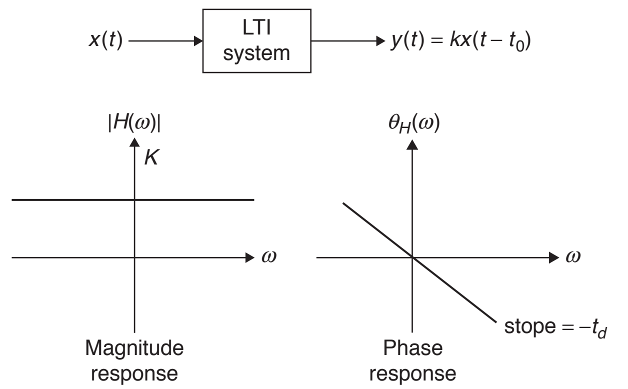
Output $Y(\omega) = |Y(\omega)| e^{j\theta_y(\omega)}$

Then $Y(\omega) = X(\omega) \cdot H(\omega)$. So $|Y(\omega)| = |X(\omega)| \cdot |H(\omega)|$

$$\theta_y(\omega) = \theta_x(\omega) + \theta_H(\omega).$$

For an LTI system, with real valued impulse response $h(t)$ and frequency response $H(\omega)$, [i.e., Fourier transform of $h(t)$] the output $y(t)$ for an input $x(t) = A\cos(\omega t - \phi)$ is $y(t)$ $= |H(\omega)| A.\ \cos(\omega t - \phi + \angle\{H(\omega)\})$ the system modifies the amplitude of input sinusoid by $H(\omega)$ and the phase by $\angle\{H(\omega)\}$.

Distortion Less Transmission



For distortion less transmission through LTI system we require, the exact input signal to be reproduced at the output, even its Amplitude is different, or it may be time delayed.

If $x(t)$ is input signal then output, $y(t) = k\, x(t - t_d)$.

By taking Fourier transform $Y(\omega) = k\, e^{-j\omega t_d} X(\omega)$. So we can write for distortion less transmission system must have the frequency response.

$$H(\omega) = |H(\omega)|\, e^{j\theta_H(\omega)} = ke^{-j\omega t_d},$$

$$|H(\omega)| = k,\ \theta_H(\omega) = -j\omega t_d.$$

That is amplitude of $H(\omega)$ must be constant over the entire frequency range and the phase of $H(\omega)$ must be linear with frequency.

Amplitude Distortion and Phase Distortion

When the amplitude spectrum $|H(\omega)|$ of the system is not constant within the frequency band of interest, the frequency components of the input signal are transmitted with different amount of gain or attenuation. This effect is called amplitude distortion. When the phase spectrum $\theta_H(\omega)$ of system is not linear with frequency, the output signal has a different wave form than the input signal because of different delays in passing through the system for different frequency components of the input signals. This form of distortion is called as phase distortion.

Equalization

To compensate for linear distortion we may use a network known as equalizer, connected is cascade with the system in question. The equalizer is designed in such a way that, inside the frequency band of interest the over all magnitude and phase responses of this cascade connection approximate the conditions for distortions less transmission to with in prescribed limits.

Consider for example a communication channel with frequency response ($H_C(\omega)$, let an equalizer of frequency response $H_{eq}(\omega)$ be connected in cascade with the channel.

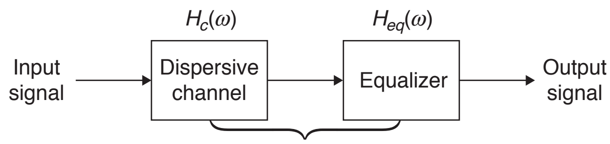


Figure 5 Distortion less transmission system

For overall transmission through the cascade connection to be distortion less, we require $H_C(\omega)\varphi \cdot H_{eq}(\omega) = e^{-j\omega t}d$. Where t_d is constant time delay, k is unity here. Therefore the frequency response of the equalizer is inversely related to that of the channel, $H_{eq}(\omega) = \dfrac{e^{-j\omega t_d}}{H_c(\omega)}$.

Example 14: Let $x(t) = \dfrac{1}{\pi t}\sin(2\pi t)$ be the input to a system with impulse response $h(t) = \dfrac{1}{\pi t}\sin(3\pi t)$, then the output of the system $y(t)$ is

Solution: $y(t) = x(t) * h(t)$, We can convert to frequency domain by taking Fourier transform $Y(\omega) = X(\omega)\,H(\omega)$. We have the Fourier transform pair

$$\frac{\sin at}{\pi t} \overset{FT}{\longleftrightarrow} P_a(\omega) = \begin{cases} 1 & |\omega| < a \\ 0 & |\omega| > a \end{cases},$$

$$x(t) \overset{FT}{\longleftrightarrow} X(\omega) = \begin{cases} 1 & |\omega| < 2\pi \\ 0 & |\omega| > 2\pi \end{cases},$$

$$h(t) \overset{FT}{\longleftrightarrow} H(\omega) = \begin{cases} 1 & |\omega| < 3\pi \\ 0 & |\omega| > 3\pi \end{cases},$$

then

$$Y(\omega) = X(\omega)\cdot H(\omega) \begin{cases} 1 & |\omega| < 2\pi \\ 0 & |\omega| > 2\pi \end{cases},$$

So, we can say $y(t) = \dfrac{1}{\pi t}\sin(2\pi t)$.

Example 15: The output of an LTI system in response to an input $x(t) = e^{-t}u(t)$ is $y(t) = e^{-2t}u(t)$. Find the frequency response and the impulse response of the system?

Solution: $x(t) = e^{-t}u(t) \overset{FT}{\longleftrightarrow} X(\omega) = \dfrac{1}{j\omega + 1}$,

$$y(t) = e^{-2t}u(t) \overset{FT}{\longleftrightarrow} Y(\omega) = \frac{1}{j\omega + 2},$$

$$y(t) = x(t) * h(t) \overset{FT}{\longleftrightarrow} Y(\omega) = X(\omega)\cdot H(\omega),$$

so $H(\omega) = \dfrac{Y(\omega)}{X(\omega)}$.

The frequency response $H(\omega) = \dfrac{j\omega + 1}{j\omega + 2}$ to find impulse response we need to take inverse Fourier transform for $H(\omega)$

$$H(\omega) = \frac{j\omega + 2 - 1}{j\omega + 2} = 1 - \frac{1}{j\omega + 2},$$

so $h(t) = \delta(t) - e^{-2t}u(t)$,

Example 16: Find the zero state response of a stable LTI system with frequency response $H(\omega) = \dfrac{1}{j\omega + 3}$ and the input is $x(t) = e^{-2t}u(t)$.

Solution: Here $X(\omega) = \dfrac{1}{j\omega + 2}, H(\omega) = \dfrac{1}{j\omega + 3}$,

then $Y(\omega) = H(\omega)\,X(\omega)$

$$= \frac{1}{(j\omega + 2)(j\omega + 3)} = \frac{1}{j\omega + 2} - \frac{1}{j\omega + 3}$$

By taking inverse Fourier transform

$$y(t) = (e^{-2t} - e^{-3t})\,u(t).$$

LTI Systems Characterized by Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t),$$

By taking Fourier transforms on both sides:

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$$

They $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$

Which is similar as in laplace transform if $s = j\omega$.

Phase and Group Delays

Whenever a signal is transmitted through a dispersive (i,e., frequency selective) system, some delay is introduced into the output signal in relation to the input signal. The delay is determined by the phase response of the system $\phi = \angle\{H(j\omega)\} = \arg\{H(j\omega)\}$.

Where $H(j\omega)$ is the frequency response of the channel. Suppose that a sinusoidal signal is transmitted through the channel at a frequency ω_c. The signal received at the channel output lags the transmitted signals by $\phi(\omega_c)$ radians. The time delay corresponding to this phase lag is simply equal to $\tau_p = \dfrac{-\phi(\omega_c)}{\omega_c}$.

Where − sign accounts for the lag. However this phase delay is not necessarily the true signals delay, this follows form the fact that a sinusoidal signal has infinite duration, with each cycle exactly like the preceding one. Such a signal does not convey information. When a modulated signals is transmitted through a communication channel, there are two different delays to be considered.

(1) The carrier or phase delay $\tau_p = \frac{-\phi(\omega_c)}{\omega_c}$.

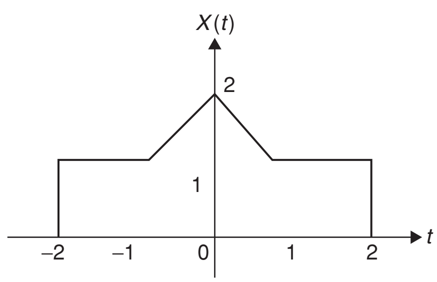
(2) The envelope or group delay

$$\tau_g = \left.\frac{-d\phi(\omega)}{d\omega}\right|_{\omega=\omega_c}.$$

The group delay is the true signal delay.

Solved Examples

Example 1: For the signal $x(t)$ Shown in figure, find $X(0)$ and $\int_{-\infty}^{\infty} X(\omega)d\omega$



Solution: We know the Fourier transform of $x(t)$ is $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$, when $\omega = 0, X(0) = \int_{-\infty}^{\infty} x(t)e^{0}dt$ $= \int_{-\infty}^{\infty} x(t)\,dt$ $\int_{-\infty}^{\infty} x(t)\,dt$ is the area under the curve $x(t)$

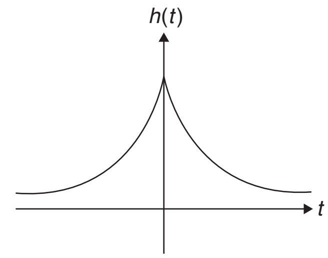
$$X(0) = \int_{-\infty}^{\infty} x(t)dt = 4\times 1 + \frac{1}{2}\times 1\times 1\times 2 = 5.$$

Inverse Fourier transform is $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$, When $t = 0, \int_{-\infty}^{\infty} X(\omega)d\omega = 2\pi x(0) = 2\pi\times 5 = 10\pi$.

Example 2: Fourier transform of a signal $x(t)$ is given by $X(\omega) = \frac{\omega^2+28}{\omega^2+16}$, then $x(t)$, is:

Solution: $x(\omega) = \frac{\omega^2+28}{\omega^2+16} = \frac{\omega^2+16+12}{\omega^2+16} = 1 + \frac{12}{\omega^2+16}$

consider $h(t) = \{e^{-a}\,|t| = \begin{cases} e^{-at}; t > 0 \\ e^{at}; t < 0 \end{cases}$



$h(t) = e^{-at}\,u(t) + e^{at}\,u(-t)'$ by taking Fourier transform on both sides

$$H(\omega) = \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{a^2+\omega^2},\quad e^{-a}|t| \leftrightarrow \frac{2a}{\omega^2+a^2},$$

here $X(\omega) = 1 + \frac{12}{\omega^2+16} = 1 + \frac{3}{2}\frac{8}{\omega^2+16}$,

By taking inverse Fourier transform

$$x(t) = \delta(t) + \frac{3}{2}e^{-4|t|}.$$

Example 3: Determine the inverse Fourier Transform of $X(\omega) = 4\pi\,\delta(\omega) - 2\pi\,\delta(\omega - 2\pi) + 2\pi\,\delta(\omega + 2\pi)$

(A) $2[1 - \sin 2\pi t]$ (B) $2j[1 - \sin 2\pi t]$

(C) $2[1 - j\sin 2\pi t]$ (D) $2j[1 - j\sin 2\pi t]$

Solution: (C)

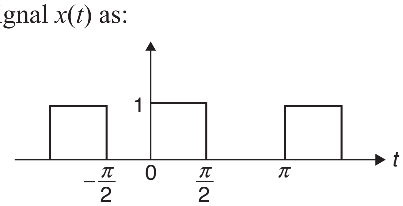
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(4\pi\delta(\omega) - 2\pi\delta(\omega-2\pi) + 2\pi\delta(\omega+2\pi)e^{j\omega t}d\omega$$

$$= 2 - e^{j2\pi t} + e^{-j2\pi t}$$

$$= 2 - 2j\left(\frac{e^{j2\pi t} - e^{-j2\pi t}}{2j}\right)$$

$$= 2 - 2j\sin 2\pi t = 2(1 - j\sin 2\pi t).$$

Example 4: Determine the Fourier series co-efficients for periodic signal $x(t)$ as:



(A) $\frac{j}{k\pi}[1 - e^{-jk\pi}]$ (B) $\frac{1}{jk\pi}[1 + e^{-jk\pi}]$.

(C) $\frac{1}{jk\pi}[1 - e^{-jk\pi}]$ (D) $\frac{j}{jk\pi}[1 + e^{-jk\pi}]$.

Solution: (C)

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt, T = \pi, w_0 = 2,$$

$$c_k = \frac{2}{T}\frac{(e^{-jk\omega_0 t})_0^{\frac{\pi}{2}}}{-jk\omega_0} = \frac{1}{jk\pi}\left(1 - e^{-jk\omega_0\frac{\pi}{2}}\right),$$

$$= \frac{1}{jk\pi}(1 - e^{-jk\pi}).$$

Example 5: Find the Fourier transform of the signal $x(t) = te^{-2|t-3|}$.

(A) $e^{-3j\omega}\left(\dfrac{12}{4+\omega^2} - \dfrac{j8\omega}{(4+\omega^2)^2}\right)$.

(B) $e^{3j\omega}\left(\dfrac{12}{4+\omega^2} + \dfrac{j8\omega}{(4+\omega^2)^2}\right)$.

(C) $e^{-3j\omega}\left(\dfrac{12}{4+\omega^2} + \dfrac{j8\omega}{(4+\omega^2)^2}\right)$.

(D) $e^{3j\omega}\left(\dfrac{12}{4+\omega^2} - \dfrac{j8\omega}{(4+\omega^2)^2}\right)$.

Solution: (A)

$$e^{-2|t|} \to \frac{4}{4+\omega^2},\ x_1(t-3) \to e^{-3j\omega}x_1(\omega),\ tx_2(t) \to \frac{jd}{d\omega}x_2(\omega),$$

$$X(\omega) = j\frac{d}{d\omega}\left|e^{-3j\omega}\frac{4}{4+\omega^2}\right|$$

$$= j\frac{d}{d\omega}\left|e^{-3j\omega}\frac{d}{d\omega}\left(\frac{4}{4+\omega^2}\right) + \left(\frac{4}{4+\omega^2}\right)\frac{d}{d\omega}(e^{-3j\omega})\right)$$

$$= j\left|\frac{-e^{-3j\omega}8\omega}{(4+\omega^2)^2} - 3j\left(\frac{4}{4+\omega^2}\right)e^{-3j\omega}\right)$$

$$= e^{-3j\omega}\left|\frac{12}{4+\omega^2} - \frac{j8\omega}{(4+\omega^2)^2}\right).$$

Example 6: Determine the Inverse Fourier transform of $X(\omega) = \dfrac{2j\omega+5}{(j\omega+2)^2}$:

(A) $(2e^{-2t} - te^{-2t})\,u(t)$.
(B) $(2e^{-2t} + te^{-2t})\,u(t)$.
(C) $(2e^{-2t} - te^{2t})\,u(t)$.
(D) $(2e^{-2t} - te^{-2t})\,u(t)$.

Solution: (B)

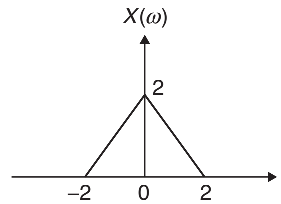
$$X(\omega) = \frac{2j\omega+5}{(j\omega+2)^2} = \frac{2j\omega+4}{(j\omega+2)^2} + \frac{1}{(j\omega+2)^2}$$

$$= \frac{2(j\omega+2)}{(j\omega+2)^2} + \frac{1}{(j\omega+2)^2}$$

$$= \frac{2}{j\omega+2} + \frac{1}{(j\omega+2)^2}$$

Hence, $x(t) = 2e^{-2t}\,u(t) + te^{-2t}\,u(t)$
$x(t) = (2e^{-2t} + te^{-2t})\,u(t) = (2 + t)\,e^{-2t}\,u(t)$.

Example 7: Consider the signal $X(\omega)$:



Find $\int_{-\infty}^{\infty}[x(t)]^2\,dt$?

(A) 0 (B) −1
(C) $2/\pi$ (D) $8/3\pi$

Solution: (D)
According to parseavall

$$\int_{-\infty}^{\infty} x(t)^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X^2(\omega)d\omega = \frac{1}{2\pi}\cdot\frac{16}{3} = \frac{8}{3\pi}$$

Example 8: A real valued continuous-time signal $x(t)$ has a fundamental period $T = 16$, the non zero Fourier series coefficients for $x(t)$ are $x(2) = x(-2) = 1$, $x(4) = x(-4) = j$, the signal $x(t)$ would be?

(A) $\cos\dfrac{\pi}{4}t + \cos\dfrac{\pi}{2}t$

(B) $\cos\dfrac{\pi}{4}t + \cos\left(\dfrac{\pi}{2}t + \dfrac{\pi}{2}\right)t$

(C) $2\left[\cos\dfrac{\pi}{4}t + \cos\left(\dfrac{\pi}{2}t + \dfrac{\pi}{2}\right)\right]$

(D) $2\left[\cos\dfrac{\pi}{2}t + \cos\left(\dfrac{\pi}{4}t + \dfrac{\pi}{2}\right)\right]$

Solution: (C)

$$x(t) = \sum_{k=-\infty}^{\infty} x(k)e^{jk\omega_0 t}$$

$$= x(-2)e^{-j2\omega_0 t} + x(2)e^{j2\omega_0 t} + x(-4)e^{-j4\omega_0 t} + x(4)e^{j4\omega_0 t}$$

$$= e^{-j2\left(\frac{\pi}{8}\right)t} + e^{j2\left(\frac{\pi}{8}\right)t} - e^{+j4\left(\frac{\pi}{8}\right)t} + je^{-j4\left(\frac{\pi}{8}t\right)}$$

$$= 2\left[\frac{e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{4}t}}{2}\right] - 2\left[\frac{e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}}{2j}\right]$$

$$= 2\left[\cos\frac{\pi}{4}t - \sin\frac{\pi}{2}t\right]$$

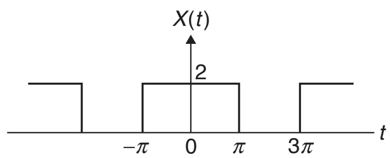
$$= 2\left[\cos\frac{\pi}{4}t + \cos\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)\right]$$

Exercises

Practice Problems I

Directions for questions 1 to 25: Select the correct alternative from the given choices.

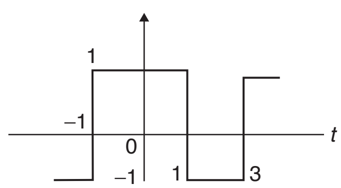
1. Determine the Fourier series coefficient for periodic signal $x(t)$ as:



(A) $\frac{2}{k\pi}\sin\frac{k\pi}{4}$. (B) $\frac{2}{k\pi}\sin\frac{k\pi}{2}$.

(C) $\frac{4}{k\pi}\sin\frac{k\pi}{4}$. (D) $\frac{4}{k\pi}\sin\frac{k\pi}{2}$.

2. Determine the Fourier series coefficient for periodic signal $x(t)$ as:



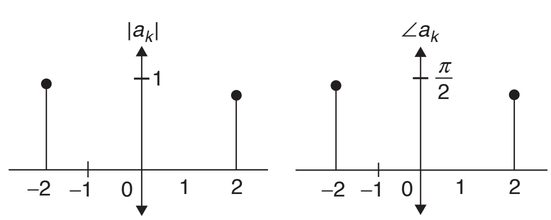
(A) $\frac{1}{2\pi k}[2e^{-jk\pi/2} - e^{-3jk\pi/2} + e^{jk\pi/2}]$.

(B) $\frac{j}{2\pi k}[2e^{jk\pi/2} + e^{-3jk\pi/2} + e^{jk\pi/2}]$.

(C) $\frac{j}{2\pi k}[2e^{-jk\pi/2} - e^{-3jk\pi/2} + e^{jk\pi/2}]$.

(D) $\frac{1}{2\pi k}[2e^{-jk\pi/2} + e^{-3jk\pi/2} - e^{jk\pi/2}]$.

3. Fourier series coefficient of the time domain signal as $\omega_0 = 2\pi$



Determine the corresponding time domain signal.

(A) $e^{j\pi/2}\cos 2\pi t$. (B) $2e^{j\pi/2}\cos 4\pi t$.

(C) $2e^{-j\pi/2}\cos 2\pi t$. (D) $2e^{-j\pi/2}\cos 4\pi t$.

4. Suppose the periodic signal $x(t)$ has fundamental period 'T' and Fourier coefficients a_k, let b_k be the Fourier coefficients of $y(t)$, where $y(t) = \frac{d^2x(t)}{dt^2}$, the Fourier coefficient a_k in terms of b_k will be:

(A) $\frac{-T^2 b_k}{4\pi^2 k^2}$ (B) $\frac{T^2 b_k}{2\pi^2 k^2}$

(C) $\frac{T^2 b_k}{\pi^2 k^2}$ (D) $\frac{2T^2 b_k}{\pi^2 k^2}$

5. Find the Fourier transform of signal $x(t) = e^{-3t}\,u(t-2)$.

(A) $\frac{e^{-2(3+j\omega)}}{3+j\omega}$. (B) $\frac{e^{-3(2+j\omega)}}{2+j\omega}$.

(C) $\frac{e^{-2(3-j\omega)}}{3-j\omega}$. (D) $\frac{e^{-3(2-j\omega)}}{2-j\omega}$.

6. Determine the Inverse Fourier transform of $X(\omega) = \frac{-j\omega+6}{\omega^2 - 13j\omega - 36}$.

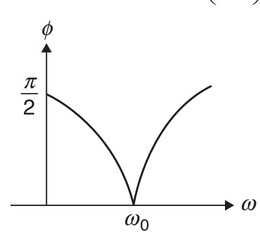
(A) $(3e^{-6t} + 2e^{-4t})u(t)$. (B) $(3e^{-6t} - 2e^{-4t})u(t)$.

(C) $(3e^{-9t} - 2e^{-4t})u(t)$. (D) $(3e^{-9t} + 2e^{-4t})u(t)$.

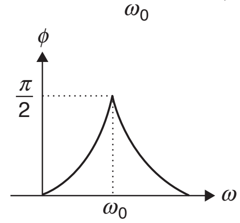
7. Which one of the following represents the phase response of the function?

$$H(s) = \frac{s^2 + \omega_0{}^2}{s^2 + \left(\frac{\omega_0}{\theta}\right)s + \omega_0{}^2}$$

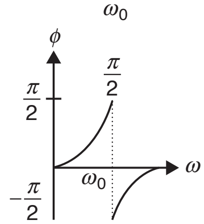
(A)



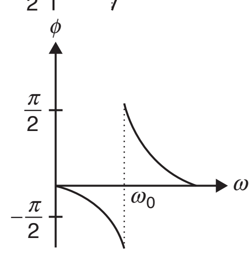
(B)



(C)



(D)



8. Which of the following statements are true?

(1) A Fourier series for an even periodic function will consist entirely of cosine terms.
(2) A Fourier series for an odd periodic function will consist entirely of sine terms.
(3) A Fourier series for an odd periodic function will consist entirely of cosine terms.
(4) A Fourier series for an even periodic function will consist entirely of sine terms.

(A) 3, 4 (B) 1, 2 (C) 1, 3 (D) 2, 4

9. The effect of time displacement τ in a periodic function ($n\omega_0$ frequency):

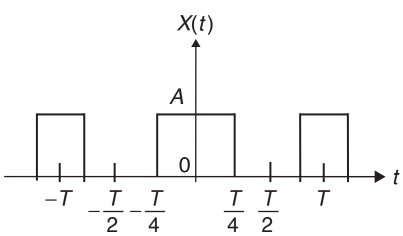
(A) Frequency spectrum remains same.
(B) Magnetic spectrum remains constant.
(C) Magnetic spectrum shifts by $n\omega_0 t$.
(D) Phase spectrum remains constant.

10. Which of the following is true?

(1) $x(t)$ is real and even function, then the Fourier series coefficients are real and even
(2) $x(t)$ is real and odd function then the Fourier series coefficients are imaginary and odd.

(A) Both (B) Neither (1) nor (2)
(C) Only (1) (D) Only (2)

11. Fourier series of the wave form shown:



(A) $A + \frac{A}{\pi}\left[\cos\omega_0 t - \frac{\cos 3\omega_0 t}{3} + \frac{\cos 5\omega_0 t}{5} \cdots\right]$.

(B) $\frac{A}{2} + \frac{2A}{\pi}\left[\cos\omega_0 t - \frac{\cos 3\omega_0 t}{3} + \frac{\cos 5\omega_0 t}{5} \cdots\right]$.

(C) $\frac{A}{2} + \frac{2A}{\pi}\left[\cos\omega_0 t - \frac{\cos 4\omega_0 t}{4} + \frac{\cos 6\omega_0 t}{6} \cdots\right]$.

(D) $\frac{A}{2} - \frac{2A}{\pi}\left[\cos\omega_0 t - \frac{\cos 3\omega_0 t}{3} + \frac{\cos 5\omega_0 t}{5} \cdots\right]$.

12. Match the following Fourier transform pairs

Group (1)	Group (2)
P. $e^{-j\omega t_o}$	**1.** $\delta(t)$
Q. $e^{-a\lvert w\rvert}$	**2.** $\delta(t - t_0)$
R. 1	**3.** $\frac{1}{a^2 + t^2}$
S. $2\pi\delta(\omega)$	**4.** 1

(A) P – 4, Q – 3, R – 2, S – 1.
(B) P – 3, Q – 2, R – 1, S – 4.
(C) P – 2, Q – 3, R – 1, S – 4.
(D) P – 2, Q – 3, R – 4, S – 1.

13. A rectangular function defined as:

$f(t) = 1/2,\ 0 < t < \pi,\ -1/2,\ (\pi < t < 2\pi)$, then find approximated function by a wave form sin t over the interval $(0, 2\pi)$ such that the mean square error is minimum.

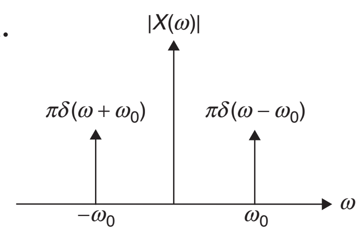
(A) $\frac{4}{\pi}\sin t$. (B) $\frac{8}{\pi}\sin t$.

(C) $\frac{2}{\pi}\sin t$. (D) $\frac{1}{\pi}\sin t$.

14. Match the following with their magnitude spectrum.

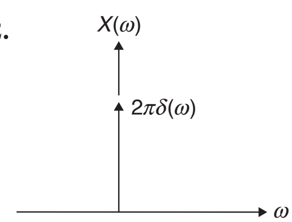
P. $x(t) = 1$

1.



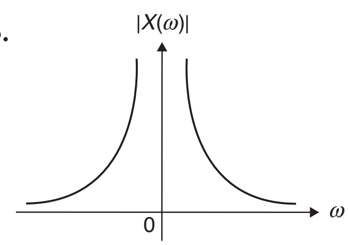
Q. Signum function.

2.



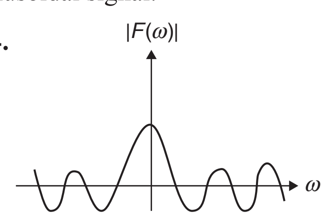
R. Rectangular Pulse.

3.



S. Cosinusoidal signal.

4.



(A) P – 2, Q – 1, R – 4, S – 3.
(B) P – 1, Q – 2, R – 3, S – 4.
(C) P – 2, Q – 3, R – 1, S – 4.
(D) P – 2, Q – 3, R – 4, S – 1.

15. Find the inverse Fourier transform of $\dfrac{7+j\omega}{(3+j\omega)^2}$

(A) $(4t-1)\,e^{-3t}\,u(t)$ (B) $(4t+1)\,e^{3t}\,u(t)$

(C) $(4+4)\,e^{-3t}\,u(t)$ (D) $(4t+1)\,e^{-3t}\,u(t)$

16. The impulse response of an LTI system is $h(t) = e^{-2t}$ $u(t)$, find the response of the system for $x(t) = e^{-6t}\,u(t)$

(A) $\dfrac{1}{4}(e^{-2t} + e^{-6t})u(t)$

(B) $\dfrac{1}{4}(e^{-2t} - e^{-6t})u(t)$

(C) $\dfrac{-1}{4}(e^{-2t} - e^{-6t})u(t)$

(D) $\dfrac{-1}{4}(e^{-2t} + e^{-6t})u(t)$

Common Data for Questions 17 and 18:

17. Fourier transform of the signal $x(t)$ is $X(\omega) = \dfrac{1}{\omega^2}(e^{j\omega} - j\omega e^{-j\omega} - 1)$. Find the Fourier transfom of $x(-t)$

(A) $\dfrac{1}{\omega^2}(e^{-j\omega} + j\omega e^{j\omega} - 1)$.

(B) $\dfrac{1}{\omega^2}(e^{-j\omega} - j\omega e^{j\omega} - 1)$.

(C) $\dfrac{1}{\omega^2}(e^{j\omega} + j\omega e^{-j\omega} - 1)$.

(D) $\dfrac{1}{\omega^2}(e^{-j\omega} - j\omega e^{-j\omega} - 1)$.

18. Find the Fourier transform of $x(-t-1)$:

(A) $\dfrac{e^{j\omega}}{\omega^2}(e^{-j\omega} + j\omega e^{j\omega} - 1)$;

(B) $\dfrac{e^{-j\omega}}{\omega^2}(e^{-j\omega} + j\omega e^{-j\omega} - 1)$;

(C) $\dfrac{e^{-j\omega}}{\omega^2}(e^{j\omega} + j\omega e^{j\omega} - 1)$;

(D) $\dfrac{e^{-j\omega}}{\omega^2}(e^{-j\omega} - j\omega e^{j\omega} - 1)$;

19. The Fourier transform of $e^{-t}u(t)$ is $\dfrac{1}{1+j\omega}$, then Fourier transform of $\dfrac{1}{1+t}$ is:

(A) $2\pi e^{-j\omega}\,u(j\omega)$ (B) $2\pi e^{j\omega}\,u(j\omega)$

(C) $2\pi e^{-j\omega}\,u(-j\omega)$ (D) $2\pi e^{j\omega}\,u(-j\omega)$

20. Fourier coefficients of $x[n]$ are: $C_k = \{3, 2+j, 1, 2-j\}$. The value of $x[6]$ is:

(A) 0 (B) 3

(C) $2-j$ (D) $2+j$

21. Frequency response $H(j\omega)$ of causal LTI system of difference equation:

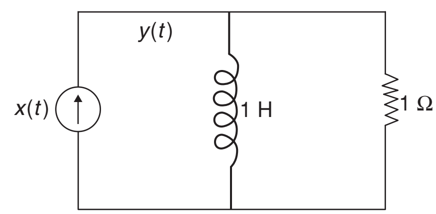
$$y[n] + \frac{1}{3}y[n-1] = x[n]$$

(A) $\dfrac{1}{1+\frac{1}{3}e^{-j\omega}}$ (B) $\dfrac{1}{1-\frac{1}{3}e^{-j\omega}}$

(C) $\dfrac{1}{1-\frac{1}{6}e^{-j\omega}}$ (D) $\dfrac{1}{1+\frac{1}{4}e^{-j\omega}}$

Common Data for Questions 22 and 23:

22. Consider a causal LTI system implemented as the RL circuit shown—a current source produces an input current $x(t)$, and system output is considered to find the differential equation relating to $x(t)$ and $y(t)$:



(A) $\dfrac{dy(t)}{dt} - y(t) = x(t)$ (B) $\dfrac{dy(t)}{dt} = x(t)$

(C) $y(t) = 0$ (D) $\dfrac{dy(t)}{dt} + y(t) = x(t)$

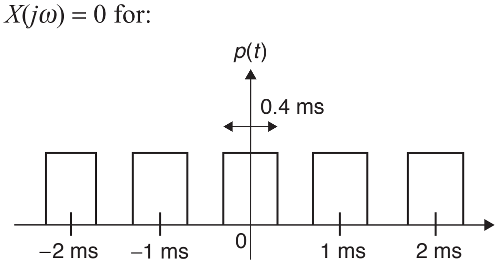
23. The frequency response of this system for input $x(t) = e^{j\omega t}$

(A) $H(j\omega) = 1$ (B) $H(j\omega) = \dfrac{1}{1+j\omega}$

(C) $H(j\omega) = \dfrac{1}{j\omega}$ (D) $H(j\omega) = j\omega$

Common Data for Questions 24 and 25:

24. A signal $x(t)$ is multiplied with a rectangular pulse train $p(t)$ as shown. $x(t)$ can be recovered from product $x(t)$. $p(t)$ by using an ideal low pass filter, if $X(j\omega) = 0$ for:



(A) $\omega < 2000\pi$ (B) $\omega < 1000\pi$

(C) $\omega > 1000\pi$ (D) $\omega > 2000\pi$

25. From the above question, the pass band gain of the ideal low pass filter needed to recover $x(t)$ from $x(t)$. $p(t)$ is:

(A) 4 (C) 10

(B) 2 (D) 2.5

Practice Problems 2

Directions for questions 1 to 20: Select the correct alternative from the given choices.

1. Which of the following is not a Dirichlet condition?

(A) $\frac{1}{T}\int_0^T |x(t)|^2\, dt < \infty$

(B) $\frac{1}{T}\int_0^T |x(t)|^2\, dt > 0$

(C) $x(t)$ has finite number of maxima and minima.

(D) $x(t)$ has a finite number of discontinuities.

2. If $f(t)$ is an even function of time period $T/2$, then,

(A) $\int_{-T/2}^{T/2} f(t)dt = 2\int_0^{T/2} f(t)dt$

(B) $\int_0^T f(t)dt = \frac{1}{2}\int_0^{T/2} f(t)dt$

(C) $\int_{-T/2}^{T/2} f(t)dt = 0$

(D) A, B

3. $\int_{-T}^{T} f(t)dt = 0, f(t)$ is function of time period $2T$, then $f(t)$ is,

(A) Even function
(B) Odd function
(C) Symmetric function
(D) Can't say

4. The phase spectrum of a function is:

(A) Even function
(B) Symmetric function
(C) Anti symmetric function
(D) Continuous function

5. Power spectrum is symmetrical about.

(A) Horizontal axis
(B) Horizontal axis passing through origin
(C) Vertical axis passing through origin
(D) Vertical axis.

6. The Fourier transform of $u(t)$, unit step function

(A) $\pi\delta(\omega)+\frac{j}{\omega}$ (B) $\pi\delta(\omega)-\frac{j}{\omega}$

(C) $\pi\delta(\omega)-\frac{1}{j\omega}$ (D) $-\pi\delta(\omega)+\frac{1}{j\omega}$

7. Match the Fourier transforms of group (1), to group (2) where $F\{x(t)\} = X(w)$:

Group (1)		Group (2)	
P.	$x(t-t_0)$	**1.**	$\frac{dX(\omega)}{d\omega}$
Q.	$e^{j\omega_0 t}x(t)$	**2.**	$X(\omega-\omega_0)$
R.	$\frac{dx(t)}{dt}$	**3.**	$e^{-j\omega_0 t}X(\omega)$
S.	$(-jt)\, x(t)$	**4.**	$j\omega\, X(\omega)$

(A) P – 2, Q – 3, R – 1, S – 4
(B) P – 2, Q – 3, R – 1, S – 4
(C) P – 3, Q – 2, R – 1, S – 4
(D) P – 3, Q – 2, R – 4, S – 1

8. Determine the complex exponential Fourier series coefficients of $\sin^2 t$.

(A) $\frac{-1}{4},\frac{1}{2},\frac{-1}{4}$ (B) $\frac{1}{2},\frac{1}{4},\frac{-1}{2}$

(C) $1,\frac{1}{2},-1$ (D) None

9. Fourier series of full wave rectified sine wave, $\sin\omega_0 t$ for $t = 0$ to $T/2$ is:

(A) $\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos 2\omega_0 t}{3}+\frac{\cos 4\omega_0 t}{15}+\frac{\cos 6\omega_0 t}{35}+\cdots\right]$

(B) $\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos 3\omega_0 t}{3}-\frac{\cos 5\omega_0 t}{5}+\frac{\cos 7\omega_0 t}{7}+\cdots\right]$

(C) $\frac{2}{\pi}+\frac{4}{\pi}\left[\frac{\cos 2\omega_0 t}{3}-\frac{\cos 4\omega_0 t}{15}+\frac{\cos 6\omega_0 t}{35}+\cdots\right]$

(D) $\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\sin 2\omega_0 t}{3}+\frac{\sin 4\omega_0 t}{15}+\frac{\sin 6\omega_0 t}{35}+\cdots\right]$

10. Fourier transform of sinusoidal signal $\sin\omega_0 t$ is:

(A) $\pi\,[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$

(B) $\frac{\omega}{j}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

(C) $j\pi\,[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$

(D) $j\pi\,[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)$

11. Which of the following signal will not have sine coefficients in Fourier series expansion?

(A) $x(t) = 1-t$ (B) $x(t) = \cos t$

(C) $x(t) = \frac{1}{1+t}$ (D) $x(t) = (1-t)^2$

12. The Fourier transform of the signal $e^{-3|t|}$ is:

(A) $\frac{6}{9-\omega^2}$ (B) $\frac{6}{9+\omega^2}$

(C) $\frac{6}{\omega^2-9}$ (D) $\frac{-6}{\omega^2+9}$

13. Find the Fourier transform of $e^{-3|t-t_0|}+e^{3|t+t_0|}$

(A) $\dfrac{12\cos\omega t_0}{9+\omega^2}$ (B) $\dfrac{12\sin\omega t_0}{9+\omega^2}$

(C) $\dfrac{j12\sin\omega t_0}{9-\omega^2}$ (D) $\dfrac{12\cos\omega t_0}{9-\omega^2}$

14. Fourier transform of a Gaussian pulse is

(A) sinc pulse (B) Sin function

(C) Gaussian pulse (D) Impulse train

15. The average power of $x(n)$ in terms of Fourier series coefficients C_k is

(A) $\sum_{k=0}^{\infty}|C_k|^2$ (B) $\dfrac{1}{N}\sum_{k=0}^{\infty}C_k^{\ 2}$

(C) $\dfrac{1}{N}\sum_{k=0}^{N-1}|C_k|^2$ (D) $\sum_{k=0}^{N-1}|C_k|^2$

16. If we replace 'z' in z transform of $x[n]$ with $e^{j\omega}$ we will get:

(A) Continuous-time Fourier transform
(B) Discrete-time Fourier transform
(C) Hilbert transform
(D) Complex z-transform

17. The function $x(t)$ has the Fourier transform $X(\omega)$ then,

$\int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt = ?$

(A) $\dfrac{1}{2\pi}x(\omega)$ (B) $2\pi X(-\omega)$

(C) $2\pi x(-\omega)$ (D) $\dfrac{1}{2\pi}x(-\omega)$

18. If $F\{x(t)\}=\dfrac{\omega}{s^2+\omega^2}$, then the value of $\lim\limits_{x\to\infty} x(t)$ is:

(A) Cannot be determined
(B) Zero
(C) Unity
(D) Infinite

19. Match the time domain signals and their Fourier transforms:

List I		List II	
P.	$1/2\pi$	**1.**	$\delta(\omega-\omega_0)$
Q.	$e^{j\omega_0 t}/2\pi$	**2.**	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)$
R.	$\text{Cos }\omega_0 t$	**3.**	$\delta(\omega)$
S.	$\text{Sin }\omega_0 t$	**4.**	$j\pi(\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$

(A) P – 1, Q – 3, R – 4, S – 2;
(B) P – 3, Q – 1, R – 4, S – 2;
(C) P – 3, Q – 1, R – 2, S – 4;
(D) P – 1, Q – 3, R – 2, S – 4;

20. Delaying a signal by t_0 seconds in time domain does:

(A) Not change its amplitude and phase spectrum
(B) Not change its amplitude spectrum but the phase spectrum is changed by $-\omega t_0$
(C) Not change its phase spectrum but the amplitude spectrum is scaled by t_0
(D) Change phase spectrum by ωt_0 and amplitude spectrum by t_0.

PREVIOUS YEARS' QUESTIONS

1. The z-transform of a signal $x[n]$ is given by $4z^{-3}+3z^{-1}+2-6z^2+2z^3$. It is applied to a system, with a transfer function $H(z)=3z^{-1}-2$. Let the output be $y(n)$. Which of the following is true? **[2009]**

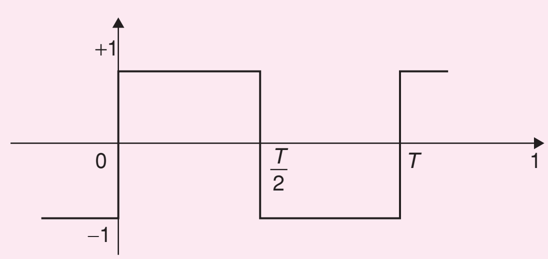
(A) $y(n)$ is non-causal with finite support
(B) $y(n)$ is causal with infinite support
(C) $y(n)=0;\ |n|>3$
(D) $\text{Re}[Y(z)]_{z=e^{j\theta}}=-\text{Re}[Y(z)]_{z=e^{-j\theta}};\ \text{Im}[Y(z)]z=e^{j\theta}.$

$\text{Im}[Y(z)]_{z=e^{-j\theta}};\ -\pi\le\theta<\pi$

2. The Fourier series coefficient, of a periodic signal $x(t)$, expressed as $x(t)=\sum_{k=-\infty}^{\infty}a_k e^{j2\pi kt/T}$ are given by $a_{-2}=2-j1$; $a_{-1}=0.5+j0.2$; $a_0=j2$; $a_1=0.5-j0.2$; $a_2=2+j1$; and $a_k=0$; For $|k|>2$. Which of the following is true? **[2009]**

(A) $x(t)$ has finite energy because only finitely many coefficients are non-zero.
(B) $x(t)$ has zero average value because it is periodic.
(C) The imaginary part of $x(t)$ is constant
(D) The real part of $x(t)$ is even.

3. The second harmonic component of the periodic waveform given in the figure has an amplitude of: **[2010]**



(A) 0 (B) 1

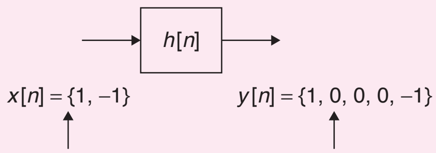
(C) $2/\pi$ (D) $\sqrt{5}$

4. $x(t)$ is a positive rectangular pulse from $t=-1$ to $t=+1$ with unit height as shown in the figure. The value of $\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega$ {where $X(\omega)$ is the Fourier transform of $x(t)$} is: **[2010]**

(A) 2 (B) 2π

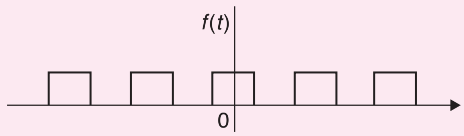
(C) 4 (D) 4π

5. Given the finite length input $x[n]$ and the corresponding finite length output $y[n]$ of an LTI system as shown below, the impulse response $h[n]$ of the system is: **[2010]**



(A) $h[n] = \{\underset{\uparrow}{1},0,0,1\}$ (B) $h[n] = \{\underset{\uparrow}{1},0,1\}$
(C) $h[n] = \{\underset{\uparrow}{1},1,1,1\}$ (D) $h[n] = \{\underset{\uparrow}{1},1,1\}$

6. The Fourier series expansion $f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$ of the periodic signal shown below will contain the following nonzero terms. **[2011]**



(A) a_0 and b_n, $n = 1, 3, 5, \ldots\infty$
(B) a_0 and a_n, $n = 1, 2, 3, \ldots\infty$
(C) a_0, a_n and b_n, $n = 1, 2, 3, \ldots\infty$
(D) a_0 and a_n, $n = 1, 3, 5, \ldots\infty$

7. If $x[n] = (1/3)^{|n|} - (1/2)^n u[n]$, then the region of convergence (ROC) of its z-transforms in the z-plane will be: **[2012]**

(A) $\frac{1}{3} < |z| < 3$ (B) $\frac{1}{3} < |z| < \frac{1}{2}$
(C) $\frac{1}{2} < |z| < 3$ (D) $\frac{1}{3} < |z|$

8. Given $F(z) = \frac{1}{z+1} - \frac{2}{z+3}$. If C is a counterclockwise path in z-plane such that $|z + 1| = 1$, the value of $\frac{1}{2\pi j}\oint_c f(z)dz$ is: **[2012]**

(A) –2   (B) –1
(C) 1   (D) 2

9. Let $y[n]$ denotes the convolution of $h[n]$ and $g[n]$, where $h[n] = (1/2)^n u[n]$ and $g[n]$ is a Causal sequence. If $y[0] = 1$ and $y[1] = \frac{1}{2}$, then g[1] equals: **[2012]**

(A) 0   (B) $\frac{1}{2}$
(C) 1   (D) 3/2

10. The Fourier transforms of a signal $h(t)$ is $H(j\omega) = (2\cos\omega)(\sin 2\omega)/\omega$. The value of $h(0)$ is: **[2012]**

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$
(C) 1   (D) 2

11. For a periodic signal $v(t) = 30\sin 100t + 10\cos 300t + 6\sin(500t + \pi/4)$, the fundamental frequency in rad/s is: **[2013]**

(A) 100   (B) 300
(C) 500   (D) 1500

12. For a periodic square wave, which one of the following statements is TRUE? **[2014]**

(A) The Fourier series coefficients do not exist.
(B) The Fourier series coefficients exist but the reconstruction converges at no point.
(C) The Fourier series coefficients exist and the reconstruction converges at most points.
(D) The Fourier series coefficients exist and the reconstruction converges at every point.

13. Let $f(t)$ be a continuous-time signal and let $F(\omega)$ be its Fourier transform and $g(t)$ defined by: **[2014]**

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$g(t) = \int_{-\infty}^{\infty} F(u)e^{-jut}\,du$$

What is the relationship between $f(t)$ and $g(t)$?

(A) $g(t)$ would always be proportional to $f(t)$.
(B) $g(t)$ would be proportional to $f(t)$ if $f(t)$ is an even function.
(C) $g(t)$ would be proportional to $f(t)$ only if $f(t)$ is a sinusoidal function.
(D) $g(t)$ would never be proportional to $f(t)$.

14. The following discrete-time equations result from the numerical integration of the differential equations of an un-damped simple harmonic oscillator with state variable x and y. The integration time step is h.

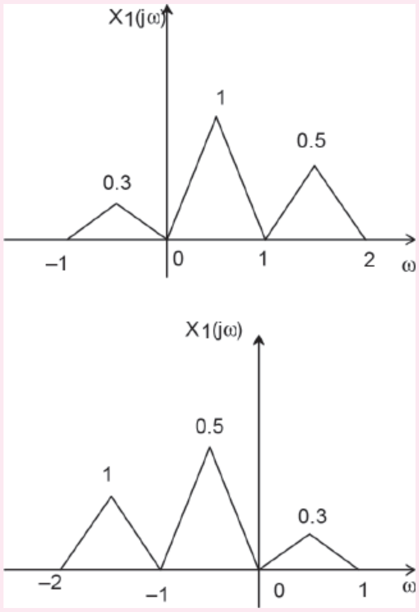
$$\frac{x_{k+1} - x_k}{h} = y_k$$

$$\frac{y_{k+1} - y_k}{h} = -x_k$$

For this discrete-time system, which one of the following statements is TRUE?

(A) The system is not stable for $h > 0$
(B) The system is stable for $h > \frac{1}{\pi}$
(C) The system is stable for $0 < h < \frac{1}{2\pi}$
(D) The system is stable for $\frac{1}{2\pi} < h < \frac{1}{\pi}$

15. Suppose $x_1(t)$ and $x_2(t)$ have the Fourier transforms as shown below.



Which one of the following statements is TRUE? **[2016]**

(A) $x_1(t)$ and $x_2(t)$ are complex and $x_1(t)\, x_2(t)$ is also complex with nonzero imaginary part
(B) $x_1(t)$ and $x_2(t)$ are real and $x_1(t)\, x_2(t)$ is also real
(C) $x_1(t)$ and $x_2(t)$ are complex but $x_1(t)\, x_2(t)$ is real
(D) $x_1(t)$ and $x_2(t)$ are imaginary but $x_1(t)\, x_2(t)$ is real

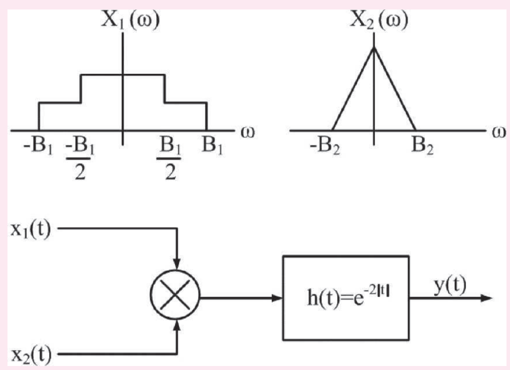
16. The output of a continuous-time, linear time-invariant system is denoted by $\boldsymbol{T}\{x(t)\}$ where $x(t)$ is the input signal. A signal $z(t)$ is called eigen-signal of the system $\boldsymbol{T}$, when $\boldsymbol{T}\{x(t)\} = xz(t)$, where $x(t)$ is a complex number, in general, and is called an eigenvalue of $\boldsymbol{T}$. Suppose the impulse response of the system $\boldsymbol{T}$ is real and even. Which of the following statements is TRUE? **[2016]**

(A) cos (t) is an eigen-signal but sin (t) is not
(B) cos(t) and sin(t) are both eigen-signals but with different eigenvalues
(C) sin(t) is an eigen-signal but cos (t) is not
(D) cos(t) and sin(t) are both eigen-signals but with identical eigenvalues

17. Suppose the maximum frequency in a band-limited signal $x(t)$ is 5kHz. Then, the maximum frequency in $x(t)$ COS$(2000\pi t)$, in kHz, is ________. **[2016]**

18. Let $x_1(t) \leftrightarrow X_1(\omega)$ and $x_2(t) \leftrightarrow X_2(\omega)$ be two signals whose Fourier Transforms are as shown in the figure below. In the figure $h(t) = e^{-2|t|}$ denotes the impulse response.

For the system shown below, the minimum sampling rate required to sample $y(t)$, so that $y(t)$ can be uniquely reconstructed form its samples, is **[2016]**



(A) $2B_1$ (B) $2(B_1 + B_2)$
(C) $4(B_1 + B_2)$ (D) ∞

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** C	**3.** B	**4.** A	**5.** A	**6.** C	**7.** D	**8.** B	**9.** B	**10.** A
11. B	**12.** C	**13.** C	**14.** D	**15.** D	**16.** B	**17.** A	**18.** C	**19.** D	**20.** A
21. A	**22.** D	**23.** B	**24.** B	**25.** D					

Practice Problems 2

1. B	**2.** A	**3.** B	**4.** C	**5.** C	**6.** B	**7.** D	**8.** A	**9.** A	**10.** C
11. B	**12.** B	**13.** A	**14.** C	**15.** D	**16.** B	**17.** C	**18.** A	**19.** C	**20.** B

Previous Years' Questions

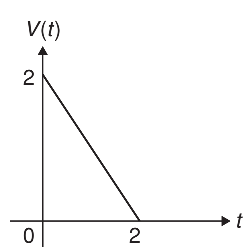
1. A	**2.** A	**3.** A	**4.** D	**5.** C	**6.** D	**7.** C	**8.** C	**9.** A	**10.** D
11. A	**12.** C	**13.** B	**14.** A	**15.** C	**16.** D	**17.** 6	**18.** B		

TEST
SIGNALS AND SYSTEMS

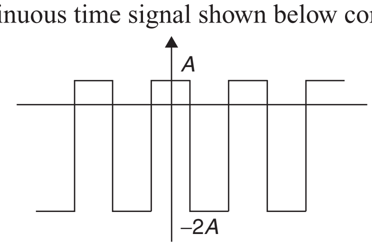
Time: 60 min.

Direction for questions 1 to 25: Select the correct alternative from the given choices.

1. The system represented by the input-output relationship $y(t) = \int_{-\infty}^{4t} x(\tau)d\tau, t > 0$ is
   (A) Linear and causal
   (B) Linear but not causal
   (C) Causal but not linear
   (D) Neither linear nor causal
2. Which of the following cannot be the Fourier series expansion of a periodic signal?
   (A) $x(t) = 2\cos t + 3\cos 3t$
   (B) $x(t) = 2\cos \pi t + 7\cos t$
   (C) $x(t) = \cos t + 0.5$
   (D) $x(t) = 2\cos t 1.5\pi t + \sin 3.5\pi t$
3. The transfer function of a system is given by $H(s) = \frac{1}{s^2(s-2)}$. The impulse response of the system is: ($*$ denotes convolution, and $u(t)$ is the unit step function).
   (A) $(t^2 * e^{-2t})\, u(t)$ (B) $(t * e^{2t})\, u(t)$
   (C) $(t^2 e^{-2t})\, u(t)$ (D) $te^{-2t}\, u(t)$
4. $y(n)$ denotes the output and $x(n)$ denotes the input of a discrete time system given by the difference equation $y[n] - 0.8\, y[n-1] = x[n] + 1.25\, x[n+1]$. Its right sided impulse response is
   (A) Anticausal (B) Unstable
   (C) Periodic (D) Non negative
5. A linear time invariant system with an impulse response $h(t)$ produces output $y(t)$, when input $x(t)$ is applied. When the input $x(t-\tau)$ is applied to a system with impulse response $h(t-\tau)$, the output will be ________.
   (A) $y(t-\tau)$ (B) $y(t)$
   (C) $y(t-2\tau)$ (D) $y(2(t-\tau))$
6. Laplace transform of $V(t)$ shown in wave form is



   (A) $\frac{-2}{s} + \frac{1}{s^2[1-e^{-2s}]}$
   (B) $\frac{2}{s} - \frac{1}{s^2}[1-e^{-2s}]$
   (C) $\frac{-2}{s} - \frac{1}{s^2[1-e^{-2s}]}$
   (D) $\frac{2}{s} - \frac{1}{s^2[1-e^{-2s}]}$
7. Let $x(t)$ be a periodic signal with time period T. Let $y(t) = x(t-t_0) + x(t+t_0)$. The Fourier series coefficients of $y(t)$ are denoted by b. If $b_k = 0$ for all odd K, then t_0 can be equal to
   (A) $\frac{T}{8}$ (B) $\frac{T}{4}$
   (C) $\frac{T}{2}$ (D) $2T$
8. A signal $x(n) = \sin(\omega_0 n + \phi)$ is the input to a linear time-invariant system having a frequency response $H(e^{j\omega})$. If the output of the system is $Ax(n-n_0)$, then the most general form of $\angle H(e^{j\omega})$ will be
   (A) $-n_0\omega_0 + \beta$ for any arbitrary real β
   (B) $-n_0\omega_0 + 2\pi k$ for any arbitrary integer k
   (C) $n_0\omega_0 + 2\pi k$ for any arbitrary integer k
   (D) $-n_0\omega_0 + \phi$
9. A system defined by $y[n] = \sum_{k=-\infty}^{n} x[k]$, where $x[n]$ is causal is an example of
   (A) Invertible system
   (B) Memory less system
   (C) Non invertible system
   (D) Averaging system
10. Two discrete time systems with impulse responses $h_1[n] = \delta[n-1]$ and $h_2[n] = \delta[n-2]$ are connected in cascade. The overall impulse response of the cascaded system is
    (A) $\delta[n-1] + \delta[n-2]$ (B) $\delta[n-4]$
    (C) $\delta[n-3]$ (D) $\delta[n-1]\,\delta[n-2]$
11. A continuous time signal shown below contains

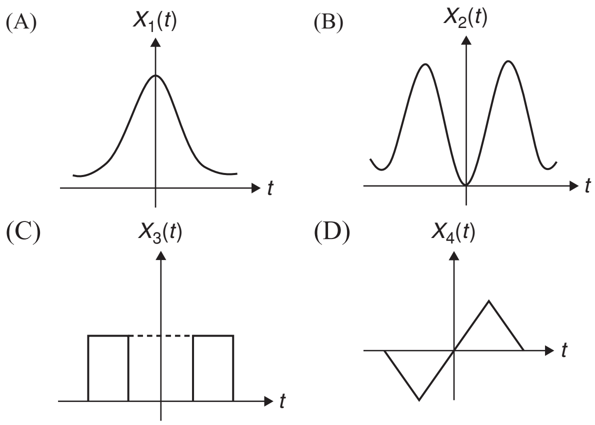


    (A) Only cosine terms and zero value for the dc component
    (B) Only cosine terms and a positive value for the dc component.
    (C) Only cosine terms and a negative value for the dc component.
    (D) Only sine terms and a negative value for the dc component.
12. $F\{\ \}$ denotes Fourier transform, $F\{e^{-t}u(t)\} = \frac{1}{1+j2\pi f}$. If so $F\left\{\frac{1}{1+j2\pi t}\right\}$ is
    (A) $e^{f}u(f)$ (B) $e^{-f}u(f)$
    (C) $e^{f}u(-f)$ (D) $e^{-f}u(-f)$
13. If $X(f)$ represents the Fourier transform of a signal $x(t)$ which is real and odd symmetric in time, then $X(f)$ is

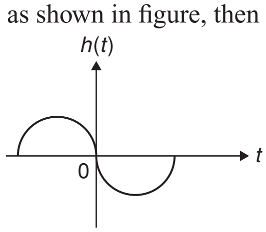
(A) Complex (B) Imaginary
(C) Real (D) Real and non negative

14. Which of the following real signals will not satisfy the condition?

$$\int_{-\infty}^{\infty} X(j\omega)\, d\omega = 0$$



15. Consider a system whose input x and output y are related by the equation $y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(2\tau)d\tau$, where $h(t)$ is as shown in figure, then the system is



(A) Causal, LTI (B) BIBO, causal, LTI
(C) Low pair, LTI (D) BIBO, LTI

16. If $u(t)$ is the unit step and $\delta(t)$ is the unit impulse function, the inverse z-transform of $F(z) = \frac{1}{z+1}$ for $k > 0$ is ______.

(A) $(-1)^k \delta(k)$ (B) $\delta(k) - (-1)^k$
(C) $(-1)^k u(k)$ (D) $u(k) - (-1)^k$

17. An LTI system having transfer function $\frac{s^2+1}{s^2+2s+1}$ and $x(t) = \sin(t+1)$ is in steady state. The output is sampled at a rate ω_s rad/s to obtain the final output $\{Y(k)\}$, which of the following is true?

(A) $Y(\cdot)$ is zero for all sampling frequencies ω_s
(B) $Y(\cdot)$ is non zero for all sampling frequencies for ω_s
(C) $Y(\cdot)$ is non zero for $\omega_s > 2$, but zero for $\omega_s < 2$
(D) $Y(\cdot)$ is zero for $\omega_s > 2$, but non zero for $\omega_s < 2$

18. Given the relationship between the input $u(t)$ and the output $y(t)$ to be $y(t) = \int_0^2 (2+t-\tau)e^{-3(t-\tau)}u(\tau)d\tau$ The transfer function $\frac{Y(s)}{U(s)}$

(A) $\frac{2e^{-2s}}{s+3}$ (B) $\frac{s+2}{(s+3)^2}$

(C) $\frac{2s+5}{s+3}$ (D) $\frac{2s+7}{(s+3)^2}$

Common Data for Questions 19 and 20: The impulse response $h(t)$ of a linear time invariant continuous time system is given by $h(t) = \exp(-2t)u(t)$, where $u(t)$ denotes the unit step function.

19. The frequency response $H(\omega)$ of this system in terms of angular frequency ω, is given by $H(\omega) =$

(A) $\frac{1}{1+j2\omega}$ (B) $\frac{1}{2+j\omega}$

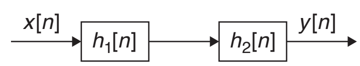
(C) $\frac{j\omega}{1+j2\omega}$ (D) $\frac{j\omega}{2+j\omega}$

20. The output of this system, to the sinusoidal input $x(t) = 2\cos(2t)$ for all time t, is __________.

(A) $2e^{-0.5}\cos(2t - 0.25\pi)$ (B) $2e^{0.5}\cos(2t + 0.25\pi)$

(C) $\frac{1}{\sqrt{2}}\cos(2t - 0.125\pi)$ (D) $\frac{1}{\sqrt{2}}\cos(2t - 0.25\pi)$

21. Consider the figure given below. The impulse response of the overall system is



$$h_1[n] = \left(\frac{1}{2}\right)^n u[n],\ h_2[n] = \left(\frac{1}{4}\right)^n u[n]$$

(A) $\left(\frac{1}{2}\right)^n\left[2-\left(\frac{1}{2}\right)^n\right]u[n]$ (B) $\left(\frac{1}{2}\right)\left[1-\left(\frac{1}{2}\right)^n\right]u[n]$

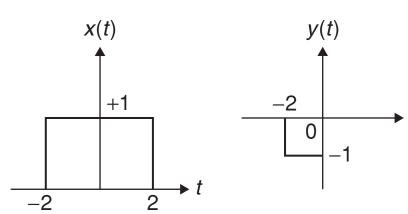
(C) $\left(\frac{1}{2}\right)^n\left[1-\left(\frac{1}{2}\right)^n\right]u[n]$ (D) $\frac{1}{2}\left[2-\left(\frac{1}{2}\right)^n\right]u[n]$

22. The ROC of the z-transform of the discrete time sequence $x(n) = \left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$ is

(A) $\frac{1}{3} < |z| < \frac{1}{2}$ (B) $|z| > \frac{1}{2}$

(C) $|z| < \frac{1}{3}$ (D) $2 < |z| < 3$

23. Let $x(t)$ and $y(t)$ (with Fourier transforms $X(f)$ and $Y(f)$ respectively) be related as shown in the given figure. Then $Y(f)$ is?



(A) $-\frac{1}{2}X\left[\frac{f}{2}\right]e^{j2\pi f}$ (B) $+\frac{1}{2}X\left[\frac{f}{2}\right]e^{j4\pi f}$

(C) $\frac{1}{4}X\left[\frac{f}{4}\right]e^{j2\pi f}$ (D) $-\frac{1}{2}X\left[\frac{f}{4}\right]e^{-4j\pi f}$

24. To which one of the following difference equations the impulse response, $h(n) = \delta(n+2) - \delta(n-2)$ corresponds to?

(A) $y(n+2) = x(n) - x(n-2)$

(B) $y(n-2) = x(n) - x(n-4)$

(C) $y(n) = x(n+2) + x(n-2)$

(D) $y(n) = -x(n+2) + x(n-2)$

25. A stable LTI system has impulse response

$h(n) = \begin{cases} a^n & n \geq 0 \\ b^n & n < 0, \end{cases}$ then the values of a and b are?

(A) $|a| < 1$ and $|b| < 1$

(B) $|a| > 1$ and $|b| > 1$

(C) $|a| < 1$ and $|b| > 1$

(D) $|a| > 1$ and $|b| < 1$

Answers Keys

1. B	**2.** B	**3.** B	**4** D	**5.** C	**6.** B	**7.** B	**8.** B	**9.** A	**10.** C
11. C	**12.** C	**13.** B	**14.** A	**15.** D	**16.** B	**17.** A	**18.** D	**19.** B	**20.** D
21. A	**22.** A	**23.** A	**24.** B	**25.** C					