5. Integration

EXERCISE 5.1

(i) ***Evaluate*** $\int \frac{-2}{\sqrt{5x-4}-\sqrt{5x-2}}dx$

Solution:

$$\int \frac{-2}{\sqrt{5x-4}-\sqrt{5x-2}}\,dx$$

$$=\int \frac{-2}{\sqrt{5x-4}-\sqrt{5x-2}} \times \frac{\sqrt{5x-4}+\sqrt{5x-2}}{\sqrt{5x-4}+\sqrt{5x-2}}\,dx$$

$$=\int \frac{-2(\sqrt{5x-4}+\sqrt{5x-2})}{(5x-4)-(5x-2)}\,dx$$

$$=\int (\sqrt{5x-4}+\sqrt{5x-2})\,dx$$

$$=\int (\sqrt{5x-4})^{\frac{1}{2}}dx+\int (\sqrt{5x-2})^{\frac{1}{2}}\,dx$$

$$=\frac{(5x-4)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\times\frac{1}{5}+\frac{(5x-2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\times\frac{1}{5}+c$$

$$=\frac{2}{15}[(5x-4)^{\frac{3}{2}}+(5x-2)^{\frac{3}{2}}]+c.$$

(ii) ***Evaluate*** $\int\left(1+x+\frac{x^2}{2!}\right)dx$

Solution:

$$\int\left(1+x+\frac{x^2}{2!}\right)dx$$

$$=\int 1dx+\int xdx+\frac{1}{2!}\int x^2dx$$

$$=x+\frac{x^2}{2}+\frac{1}{2!}\times\frac{x^3}{3}+c$$

$$=x+\frac{x^2}{2}+\frac{x^3}{6}+c.$$

(iii) ***Evaluate*** $\int \frac{3x^3-2\sqrt{x}}{x}dx$

Solution:

$$\int \frac{3x^3-2\sqrt{x}}{x}\,dx$$

$$=\int\left(\frac{3x^3}{x}-\frac{2\sqrt{x}}{x}\right)dx$$

$$=\int\left(3x^2-\frac{2}{\sqrt{x}}\right)dx$$

$$=3\int x^2dx-2\int x^{-\frac{1}{2}}\,dx$$

$$=3\cdot\left(\frac{x^3}{3}\right)-2\cdot\left[\frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right]+c$$

$$=x^3-4\sqrt{x}+c.$$

(iv) ***Evaluate*** $\int(3x^2-5)^2dx$

Solution:

$$\int(3x^2-5)^2dx=\int(9x^4-30x^2+25)\,dx$$

$$=9\int x^4dx-30\int x^2dx+25\int 1\,dx$$

$$=9\left(\frac{x^5}{5}\right)-30\left(\frac{x^3}{3}\right)+25x+c$$

$$=\frac{9x^5}{5}-10x^3+25x+c.$$

(iv) ***Evaluate*** $\int\frac{1}{x(x-1)}dx$

Solution:

$$\int\frac{1}{x(x-1)}dx=\int\frac{x-(x-1)}{x(x-1)}\,dx$$

$$=\int\left(\frac{1}{x-1}-\frac{1}{x}\right)dx$$

$$=\int\frac{1}{x-1}\,dx-\int\frac{1}{x}\,dx$$

$$=\log|x-1|-\log|x|+c$$

$$=\log\left|\frac{x-1}{x}\right|+c.$$

(vi) If $f'(x) = x^2 + 5$ and $f(0) = -1$, then find the value of $f(x)$.

Solution:

By the definition of integral

$f(x) = \int f'(x)\,dx = \int (x^2+5)\,dx$

$= \int x^2 dx + 5\int 1\,dx$

$= \frac{x^3}{3} + 5x + c \quad \ldots (1)$

Now, $f(0) = -1$ gives

$f(0) = 0 + 0 + c = -1$

$\therefore\ c = -1$

$\therefore$ from (1), $f(x) = \frac{x^3}{3} + 5x - 1.$

(vii) If $f'(x) = 4x^3 - 3x^2 + 2x + k$, $f(0) = 1$ and $f(1) = 4$, find $f(x)$

Solution:

By the definition of integral

$f(x) = \int f'(x)\,dx = \int (4x^3 - 3x^2 + 2x + k)\,dx$

$= 4\int x^3\,dx - 3\int x^2\,dx + 2\int x\,dx + k\int 1\,dx$

$= 4\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right) + kx + c$

$\therefore\ f(x) = x^4 - x^3 + x^2 + kx + c \quad \ldots (1)$

Now, $f(0) = 1$ gives

$f(0) = 0 - 0 + 0 + 0 + c = 1 \qquad \therefore\ c = 1$

$\therefore$ from (1), $f(x) = x^4 - x^3 + x^2 + kx + 1 \quad \ldots (2)$

Further $f(1) = 4$ gives

$f(1) = 1 - 1 + 1 + k + 1 = 4 \qquad \therefore\ k = 2$

$\therefore$ from (2), $f(x) = x^4 - x^3 + x^2 + 2x + 1.$

(viii) If $f'(x) = \frac{x^2}{2} - kx + 1$, $f(0) = 2$ and $f(3) = 5$, find $f(x)$

Solution:

By the definition of integral

$f(x) = \int f'(x)dx = \int\left(\frac{x^2}{2} - kx + 1\right)dx$

$= \frac{1}{2}\int x^2 dx - k\int x dx + \int 1\,dx$

$= \frac{1}{2}\left(\frac{x^3}{3}\right) - k\left(\frac{x^2}{2}\right) + x + c$

$\therefore\ f(x) = \frac{x^3}{6} - \frac{kx^2}{2} + x + c \quad \ldots (1)$

Now, $f(0) = 2$ gives

$f(0) = 0 - 0 + 0 + c = 2 \qquad \therefore\ c = 2$

$\therefore$ from (1), $f(x) = \frac{x^3}{6} - \frac{kx^2}{2} + x + 2 \quad \ldots (2)$

Further $f(3) = 5$ gives

$f(3) = \frac{27}{6} - \frac{9k}{2} + 3 + 2 = 5$

$\therefore\ \frac{9k}{2} = \frac{9}{2} \qquad \therefore\ k = 1$

$\therefore$ from (2), $f(x) = \frac{x^3}{6} - \frac{x^2}{2} + x + 2.$

EXERCISE 5.2

Evaluate the following.

(i) $\int x\sqrt{1+x^2}\,dx$

Solution:

Let $I = \int x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\cdot x\,dx$

Put $1 + x^2 = t$

$\therefore\ 2xdx = dt \qquad \therefore\ xdx = \frac{dt}{2}$

$\therefore\ I = \int \sqrt{t}\,\frac{dt}{2} = \frac{1}{2}\int t^{\frac{1}{2}}dt$

$= \frac{1}{2}\cdot\frac{t^{\frac{3}{2}}}{(3/2)} + c$

$= \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c.$

(ii) $\int \frac{x^3}{\sqrt{1+x^4}}dx$

Solution:

Let $I = \int \frac{x^3}{\sqrt{1+x^4}}\,dx$

Put $1 + x^4 = t \qquad \therefore\ 4x^3dx = dt$

$\therefore\ x^3dx = \frac{dt}{4}$

$\therefore\ I = \int \frac{1}{\sqrt{t}}\cdot\frac{dt}{4} = \frac{1}{4}\int t^{-\frac{1}{2}}dt$

$= \frac{1}{4}\cdot\frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c$

$= \frac{1}{2}\sqrt{1+x^4} + c.$

(iii) $\int (e^x + e^{-x})^2 (e^x - e^{-x}) dx$

Solution:

$$\text{Let } I = \int (e^x + e^{-x})^2 (e^x - e^{-x})\, dx$$

Put $e^x + e^{-x} = t$

$\therefore\ (e^x - e^{-x})\, dx = dt$

$\therefore\ I = \int t^2 dt = \frac{t^3}{3} + c$

$= \frac{(e^x + e^{-x})^3}{3} + c.$

(iv) $\int \frac{1+x}{x + e^{-x}} dx$

Solution:

$$\text{Let } I = \int \frac{1+x}{x+e^{-x}}\, dx$$

$$= \int \frac{(1+x)e^x}{(x+e^{-x})e^x}\, dx$$

$$= \int \frac{(1+x)e^x}{xe^x + 1}\, dx$$

Put $xe^x + 1 = t$

$\therefore\ (xe^x + e^x \times 1)\, dx = dt$

$\therefore\ (1+x)e^x\, dx = dt$

$\therefore\ I = \int \frac{1}{t}\, dt = \log |t| + c$

$= \log |xe^x + 1| + c.$

(v) $\int (x+1)(x+2)^7(x+3) dx$

Solution:

$$\text{Let } I = \int (x+1)(x+2)^7(x+3)\, dx$$

$= \int (x+2)^7 (x+1)(x+3)\, dx$

$= \int (x+2)^7 [(x+2)-1][(x+2)+1]\, dx$

$= \int (x+2)^7 [(x+2)^2 - 1]\, dx$

$= \int [(x+2)^9 - (x+2)^7]\, dx$

$= \int (x+2)^9 dx - \int (x+2)^7 dx$

$= \frac{(x+2)^{10}}{10} - \frac{(x+2)^8}{8} + c.$

(vi) $\int \frac{1}{x \log x} dx$

Solution:

Put $\log x = t \quad \therefore\ \frac{1}{x}\, dx = dt$

$\therefore\ \int \frac{dx}{x \cdot \log x} = \int \frac{1}{\log x} \cdot \frac{1}{x}\, dx$

$= \int \frac{1}{t}\, dt = \log |t| + c$

$= \log |\log x| + c.$

(vii) $\int \frac{x^5}{x^2+1} dx$

Solution:

$$\text{Let } I = \int \frac{x^5}{x^2+1}\, dx = \int \frac{x^4}{x^2+1} \cdot x\, dx$$

$$= \int \frac{(x^2)^2}{x^2+1} \cdot x\, dx$$

Put $x^2 + 1 = t \qquad \therefore\ 2x\, dx = dt$

$\therefore\ x dx = \frac{dt}{2}$ and $x^2 = t - 1$

$\therefore\ I = \int \frac{(t-1)^2}{t} \cdot \frac{dt}{2} = \frac{1}{2} \int \left(\frac{t^2 - 2t + 1}{t} \right) dt$

$= \frac{1}{2} \int \left(t - 2 + \frac{1}{t} \right) dt$

$= \frac{1}{2} \int t dt - \int 1 dt + \frac{1}{2} \int \frac{1}{t}\, dt$

$= \frac{1}{2} \cdot \frac{t^2}{2} - t + \frac{1}{2} \log |t| + c$

$= \frac{1}{4}(x^2+1)^2 - (x^2+1) + \frac{1}{2} \log |x^2+1| + c.$

(viii) $\int \frac{2x+6}{\sqrt{x^2+6x+3}} dx$

Solution:

$$\text{Let } I = \int \frac{2x+6}{\sqrt{x^2+6x+3}}\, dx$$

Put $x^2 + 6x + 3 = t$

$\therefore\ (2x+6) dx = dt$

$\therefore\ I = \int \frac{1}{\sqrt{t}}\, dt = \int t^{-\frac{1}{2}}\, dt$

$= \frac{t^{\frac{1}{2}}}{1/2} + c = 2\sqrt{x^2+6x+3} + c.$

$(ix) \int \frac{1}{\sqrt{x}+x} dx$

Solution:

$$\text{Let } I = \int \frac{1}{\sqrt{x}+x} dx = \int \frac{1}{\sqrt{x}(1+\sqrt{x})} dx$$

$$= \int \frac{1}{1+\sqrt{x}} \cdot \frac{1}{\sqrt{x}} dx$$

Put $1+\sqrt{x} = t \qquad \therefore \frac{1}{2\sqrt{x}} dx = dt$

$\therefore \frac{1}{\sqrt{x}} dx = 2\,dt$

$$\therefore I = \int \frac{1}{t} \cdot 2dt = 2\int \frac{1}{t}\,dt$$

$$= 2\log|t| + c = 2\log|1+\sqrt{x}| + c.$$

$(x) \int \frac{1}{x(x^6+1)} dx$

Solution:

$$\text{Let } I = \int \frac{1}{x(x^6+1)} dx$$

$$= \int \frac{x^5}{x^6(x^6+1)} dx$$

Put $x^6 = t \qquad \therefore 6x^5 dx = dt$

$\therefore x^5 dx = \frac{1}{6} dt$

$$\therefore I = \int \frac{1}{t(t+1)} \cdot \frac{dt}{6}$$

$$= \frac{1}{6}\int \frac{(t+1)-t}{t(t+1)} dt = \frac{1}{6}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$$

$$= \frac{1}{6}\left[\int \frac{1}{t}\,dt - \int \frac{1}{t+1}\,dt\right]$$

$$= \frac{1}{6}[\log(t) - \log|t+1|] + c$$

$$= \frac{1}{6}\log\left|\frac{t}{t+1}\right| + c = \frac{1}{6}\log\left|\frac{x^6}{x^6+1}\right| + c.$$

EXERCISE 5.3

Evaluate the following.

1) $\int \frac{3e^{2t}+5}{4e^{2t}-5} dt$

Solution:

$$\text{Let } I = \int \frac{3e^{2t}+5}{4e^{2t}-5} dt$$

Put, Numerator = A(Denominator) + $B\left[\frac{d}{dx}\text{(Denominator)}\right]$

$$\therefore 3e^{2t}+5 = A(4e^{2t}-5) + B\left[\frac{d}{dt}(4e^{2t}-5)\right]$$

$$= A(4e^{2t}-5) + B[4e^{2t} \times 2 - 0]$$

$$\therefore 3e^{2t}+5 = (4A+8B)e^{2t} - 5A$$

Equating the coefficient of e^{2t} and constant on both sides, we get

$4A + 8B = 3 \qquad \ldots (1)$

and $-5A = 5 \qquad \therefore A = -1$

$\therefore$ from (1), $4(-1) + 8B = 3$

$\therefore 8B = 7 \qquad \therefore B = \frac{7}{8}$

$$\therefore 3e^{2t}+5 = -(4e^{2t}-5) + \frac{7}{8}(8e^{2t})$$

$$\therefore I = \int \left[\frac{-(4e^{2t}-5) + \frac{7}{8}(8e^{2t})}{4e^{2t}-5}\right] dt$$

$$= \int \left[-1 + \frac{\frac{7}{8}(8e^{2t})}{4e^{2t}-5}\right] dt$$

$$= -\int 1dt + \frac{7}{8}\int \frac{8e^{2t}}{4e^{2t}-5} dt$$

$$= -t + \frac{7}{8}\log|4e^{2t}-5| + c$$

$$\ldots \left[\because \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c\right].$$

2) $\int \frac{20-12e^x}{3e^x-4} dx$

Solution:

$$\text{Let } I = \int \frac{20 - 12e^x}{3e^x - 4} dx$$

Put, Numerator = A(Denominator) + $B\left[\frac{d}{dx}\text{(Denominator)}\right]$

$$\therefore\ 20 - 12e^x = A(3e^x - 4) + B\left[\frac{d}{dx}(3e^x - 4)\right]$$

$$= A(3e^x - 4) + B(3e^x - 0)$$

$\therefore\ 20 - 12e^x = (3A + 3B)e^x - 4A$

Equating the coefficient of e^x and constant on both sides, we get

$3A + 3B = -12$ ... (1)

and $-4A = 20$ $\therefore\ A = -5$

$\therefore$ from (1), $3(-5) + 3B = -12$

$\therefore\ 3B = 3$ $\therefore\ B = 1$

$\therefore\ 20 - 12e^x = -5(3e^x - 4) + (3e^x)$

$$\therefore\ I = \int\left[\frac{-5(3e^x - 4) + (3e^x)}{3e^x - 4}\right] dx$$

$$= \int\left(-5 + \frac{3e^x}{3e^x - 4}\right) dx$$

$$= -5\int 1\,dx + \int \frac{3e^x}{3e^x - 4} dx$$

$$= -5x + \log|3e^x - 4| + c$$

$$\ldots \left[\because \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c\right]$$

[**Note** : *Answer in the textbook is incorrect.*]

3) $\displaystyle\int \frac{3e^x + 4}{2e^x - 8} dt$

Solution:

$$\text{Let } I = \int \frac{3e^x + 4}{2e^x - 8} dx$$

Put, Numerator = A(Denominator) + $B\left[\frac{d}{dx}\text{(Denominator)}\right]$

$$\therefore\ 3e^x + 4 = A(2e^x - 8) + B\left[\frac{d}{dx}(2e^x - 8)\right]$$

$$= A(2e^x - 8) + B(2e^x - 0)$$

$\therefore\ 3e^x + 4 = (2A + 2B)e^x - 8A$

Equating the coefficient of e^x and constant on both sides, we get

$2A + 2B = 3$ ... (1)

and $-8A = 4$ $\therefore\ A = -\frac{1}{2}$

$\therefore$ from (1), $2\left(-\frac{1}{2}\right) + 2B = 3$

$\therefore\ 2B = 4$ $\therefore\ B = 2$

$$\therefore\ 3e^x + 4 = -\frac{1}{2}(2e^x - 8) + 2(2e^x)$$

$$\therefore\ I = \int\left(\frac{-\frac{1}{2}(2e^x - 8) + 2(2e^x)}{2e^x - 8}\right) dx$$

$$= \int\left[-\frac{1}{2} + \frac{2(2e^x)}{2e^x - 8}\right] dx$$

$$= -\frac{1}{2}\int 1\,dx + 2\int \frac{2e^x}{2e^x - 8} dx$$

$$= -\frac{1}{2}x + 2\log|2e^x - 8| + c$$

$$\ldots \left[\because \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c\right]$$

4) $\displaystyle\int \frac{2e^x + 5}{2e^x + 1} dt$

Solution:

$$\text{Let I} = \int \frac{2e^x + 5}{2e^x + 1} dx$$

$$\text{Let } 2e^x + 5 = A(2e^x + 1) + B\frac{d}{dx}(2e^x + 1)$$

$$= 2Ae^x + A + B(2e^x)$$

$$\therefore\ 2e^x + 5 = (2A + 2B)e^x + A$$

Comparing the coefficients of e^x and constant term on both sides, we get

2A + 2B = 2 and A = 5

Solving these equations, we get

B = - 4

$$\therefore\ I = \int \frac{5\left(2e^x + 1\right) - 4\left(2e^x\right)}{2e^x + 1} dx$$

$$= 5\int dx - 4\int \frac{2e^x}{2e^x + 1} dx$$

$$\therefore\ I = 5x - 4\log\left|2e^x + 1\right| + c \quad \ldots$$

$$\left[\int \frac{f\prime(x)}{f(x)} dx = \log|f(x)| + c\right]$$

EXERCISE 5.4

Evaluate the following.

1) $\int \frac{1}{4x^2-1}dx$

Solution:

$$\int \frac{1}{4x^2-1}\,dx = \frac{1}{4}\int \frac{1}{x^2-(1/4)}\,dx$$

$$=\frac{1}{4}\int \frac{1}{x^2-\left(\frac{1}{2}\right)^2}\,dx$$

$$=\frac{1}{4}\times\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{x-\frac{1}{2}}{x+\frac{1}{2}}\right|+c$$

$$=\frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right|+c.$$

2) $\int \frac{1}{x^2+4x-5}dx$

Solution:

$$\int \frac{1}{x^2+4x-5}\,dx$$

$$=\int \frac{1}{(x^2+4x+4)-4-5}\,dx$$

$$=\int \frac{1}{(x+2)^2-(3)^2}\,dx$$

$$=\frac{1}{2\times 3}\log\left|\frac{x+2-3}{x+2+3}\right|+c$$

$$=\frac{1}{6}\log\left|\frac{x-1}{x+5}\right|+c.$$

3) $\int \frac{1}{4x^2-20x+17}dx$

Solution:

$$\int \frac{1}{4x^2-20x+17}\,dx$$

$$=\frac{1}{4}\int \frac{1}{x^2-5x+\frac{17}{4}}\,dx$$

$$=\frac{1}{4}\int \frac{1}{\left(x^2-5x+\frac{25}{4}\right)-\frac{25}{4}+\frac{17}{4}}\,dx$$

$$=\frac{1}{4}\int \frac{1}{\left(x-\frac{5}{2}\right)^2-(\sqrt{2})^2}\,dx$$

$$=\frac{1}{4}\times\frac{1}{2\sqrt{2}}\log\left|\frac{x-\frac{5}{2}-\sqrt{2}}{x-\frac{5}{2}+\sqrt{2}}\right|+c$$

$$=\frac{1}{8\sqrt{2}}\log\left|\frac{2x-5-2\sqrt{2}}{2x-5+2\sqrt{2}}\right|+c.$$

4) $\int \frac{x}{4x^4-20x^2-3}dx$

Solution:

$$\text{Let } I=\int \frac{x}{4x^4-20x^2-3}\,dx$$

Put $x^2=t$ $\quad\therefore\ 2x\,dx=dt$

$$\therefore\ x\,dx=\frac{dt}{2}$$

$$\therefore\ I=\int \frac{1}{4t^2-20t-3}\cdot\frac{dt}{2}$$

$$=\frac{1}{2}\times\frac{1}{4}\int \frac{1}{t^2-5t-\frac{3}{4}}\,dt$$

$$=\frac{1}{8}\int \frac{1}{\left(t^2-5t+\frac{25}{4}\right)-\frac{25}{4}-\frac{3}{4}}\,dt$$

$$=\frac{1}{8}\int\frac{1}{\left(t-\frac{5}{2}\right)^2-(\sqrt{7})^2}\,dt$$

$$=\frac{1}{8}\times\frac{1}{2\sqrt{7}}\log\left|\frac{t-\frac{5}{2}-\sqrt{7}}{t-\frac{5}{2}+\sqrt{7}}\right|+c$$

$$=\frac{1}{16\sqrt{7}}\log\left|\frac{2t-5-2\sqrt{7}}{2t-5+2\sqrt{7}}\right|+c$$

$$=\frac{1}{16\sqrt{7}}\log\left|\frac{2x^2-5-2\sqrt{7}}{2x^2-5+2\sqrt{7}}\right|+c.$$

[**Note :** *Answer in the textbook is incorrect.*]

5) $\int\frac{x^3}{16x^8-25}dx$

Solution:

$$\text{Let } I=\int\frac{x^3}{16x^8-25}\,dx$$

Put $x^4=t \qquad \therefore\ 4x^3dx=dt$

$$\therefore\ x^3dx=\frac{dt}{4}$$

$$\therefore\ I=\int\frac{1}{16t^2-25}\cdot\frac{dt}{4}$$

$$=\frac{1}{4}\times\frac{1}{16}\int\frac{1}{t^2-\frac{25}{16}}\,dt$$

$$=\frac{1}{64}\int\frac{1}{t^2-\left(\frac{5}{4}\right)^2}\,dt$$

$$=\frac{1}{64}\times\frac{1}{2\times\frac{5}{4}}\log\left|\frac{t-\frac{5}{4}}{t+\frac{5}{4}}\right|+c$$

$$=\frac{1}{160}\log\left|\frac{4t-5}{4t+5}\right|+c$$

$$=\frac{1}{160}\log\left|\frac{4x^4-5}{4x^4+5}\right|+c.$$

[**Note :** *Answer in the textbook is incorrect.*]

6) $\int\frac{1}{a^2-b^2x^2}dx$

Solution:

$$\int\frac{1}{a^2-b^2x^2}\,dx=\frac{1}{b^2}\int\frac{1}{\frac{a^2}{b^2}-x^2}\,dx$$

$$=\frac{1}{b^2}\int\frac{1}{\left(\frac{a}{b}\right)^2-x^2}\,dx$$

$$=\frac{1}{b^2}\times\frac{1}{2\left(\frac{a}{b}\right)}\log\left|\frac{\frac{a}{b}+x}{\frac{a}{b}-x}\right|+c$$

$$=\frac{1}{2ab}\log\left|\frac{a+bx}{a-bx}\right|+c.$$

7) $\int\frac{1}{7+6x-x^2}dx$

Solution:

$$\int\frac{1}{7+6x-x^2}\,dx$$

$$=\int\frac{1}{7-(x^2-6x+9)+9}\,dx$$

$$=\int\frac{1}{(4)^2-(x-3)^2}\,dx$$

$$=\frac{1}{2\times4}\log\left|\frac{4+x-3}{4-x+3}\right|+c$$

$$=\frac{1}{8}\log\left|\frac{1+x}{7-x}\right|+c.$$

8) $\int\frac{1}{\sqrt{3x^2+8}}dx$

Solution:

$$\int\frac{1}{\sqrt{3x^2+8}}\,dx=\int\frac{1}{\sqrt{(\sqrt{3}x)^2+(\sqrt{8})^2}}\,dx$$

$$=\frac{\log\left|\sqrt{3}x+\sqrt{(\sqrt{3}x)^2+(\sqrt{8})^2}\right|}{\sqrt{3}}+c$$

$$=\frac{1}{\sqrt{3}}\log\left|\sqrt{3}x+\sqrt{3x^2+8}\right|+c.$$

9) $\int \frac{1}{\sqrt{x^2+4x+29}}dx$

Solution:

$$\int \frac{1}{\sqrt{x^2+4x+29}}\,dx$$

$$= \int \frac{1}{\sqrt{(x^2+4x+4)+25}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+2)^2+(5)^2}}\,dx$$

$$= \log\left|(x+2)+\sqrt{(x+2)^2+(5)^2}\right|+c$$

$$= \log\left|(x+2)+\sqrt{x^2+4x+29}\right|+c.$$

10) $\int \frac{1}{\sqrt{3x^2-5}}dx$

Solution:

$$\int \frac{1}{\sqrt{3x^2-5}}\,dx = \int \frac{1}{\sqrt{(\sqrt{3}x)^2-(\sqrt{5})^2}}\,dx$$

$$= \frac{\log\left|\sqrt{3}x+\sqrt{(\sqrt{3}x)^2-(\sqrt{5})^2}\right|}{\sqrt{3}}+c$$

$$= \frac{1}{\sqrt{3}}\log\left|\sqrt{3}x+\sqrt{3x^2-5}\right|+c.$$

11) $\int \frac{1}{\sqrt{x^2-8x-20}}dx$

Solution:

$$\int \frac{1}{\sqrt{x^2-8x-20}}\,dx$$

$$= \int \frac{1}{\sqrt{(x^2-8x+16)-16-20}}\,dx$$

$$= \int \frac{1}{\sqrt{(x-4)^2-(6)^2}}\,dx$$

$$= \log\left|(x-4)+\sqrt{(x-4)^2-(6)^2}\right|+c$$

$$= \log\left|(x-4)+\sqrt{x^2-8x-20}\right|+c.$$

EXERCISE 5.5

EXERCISE 5.5

Evaluate the following.

1) $\int x\log x$

Solution:

$$\int x\log x\,dx = \int(\log x)\cdot x\,dx$$

$$= (\log x)\int x\,dx - \int\left[\frac{d}{dx}(\log x)\int x\,dx\right]dx$$

$$= (\log x)\cdot\frac{x^2}{2} - \int\frac{1}{x}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{1}{2}x^2\log x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\cdot\frac{x^2}{2}+c = \frac{x^2}{2}\log x - \frac{x^2}{4}+c.$$

2) $\int x^2e^{4x}\,dx$

Solution:

$$\int x^2e^{4x}\,dx = x^2\int e^{4x}\,dx - \int\left[\frac{d}{dx}(x^2)\int e^{4x}\,dx\right]dx$$

$$= x^2\cdot\frac{e^{4x}}{4} - \int 2x\cdot\frac{e^{4x}}{4}\,dx$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{2}\int xe^{4x}\,dx$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{2}\left[x\int e^{4x}\,dx - \int\left\{\frac{d}{dx}(x)\int e^{4x}\,dx\right\}dx\right]$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{2}\left[x\cdot\frac{e^{4x}}{4} - \int 1\cdot\frac{e^{4x}}{4}\,dx\right]$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}x\cdot e^{4x} + \frac{1}{8}\int e^{4x}\,dx$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{8}\cdot\frac{e^{4x}}{4}+c$$

$$= \frac{1}{4}e^{4x}\left[x^2 - \frac{x}{2} + \frac{1}{8}\right]+c.$$

3) $\int x^2e^{3x}\,dx$

Solution:

$$\int x^2 e^{4x}dx = x^2\int e^{4x}dx - \int\left[\frac{d}{dx}(x^2)\int e^{4x}dx\right]dx$$

$$= x^2\cdot\frac{e^{4x}}{4} - \int 2x\cdot\frac{e^{4x}}{4}dx$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\int x e^{4x}dx$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\left[x\int e^{4x}dx - \int\left\{\frac{d}{dx}(x)\int e^{4x}dx\right\}dx\right]$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\left[x\cdot\frac{e^{4x}}{4} - \int 1\cdot\frac{e^{4x}}{4}dx\right]$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{8}x\cdot e^{4x} + \frac{1}{8}\int e^{4x}dx$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{8}x e^{4x} + \frac{1}{8}\cdot\frac{e^{4x}}{4} + c$$

$$= \frac{1}{4}e^{4x}\left[x^2 - \frac{x}{2} + \frac{1}{8}\right] + c.$$

4) $\int x^3 e^{x^2}\,dx$

Solution:

$$\text{Let } I = \int x^3 e^{x^2}dx = \int x^2 e^{x^2}\cdot x\,dx$$

Put $x^2 = t$ $\quad\therefore\ 2x\,dx = dt$

$$\therefore\ x\,dx = \frac{dt}{2}$$

$$\therefore\ I = \int te^t\cdot\frac{dt}{2} = \frac{1}{2}\int te^t dt$$

$$= \frac{1}{2}\left[t\int e^t dt - \int\left\{\frac{d}{dt}(t)\int e^t dt\right\}dt\right]$$

$$= \frac{1}{2}[te^t - \int 1\cdot e^t dt]$$

$$= \frac{1}{2}[te^t - e^t] + c$$

$$= \frac{1}{2}(t-1)e^t + c$$

$$= \frac{1}{2}(x^2-1)e^{x^2} + c.$$

5) $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$

Solution:

$$\text{Let } I = \int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$$

Put $f(x) = \frac{1}{x}$. Then $f'(x) = -\frac{1}{x^2}$

$$\therefore\ I = \int e^x - [f(x) + f'(x)]\,dx$$

$$= e^x\cdot f(x) + c = e^x\cdot\frac{1}{x} + c.$$

6) $\int e^x\frac{x}{(x+1)^2}dx$

Solution:

$$\text{Let } I = \int e^x\cdot\frac{x}{(x+1)^2}dx$$

$$= \int e^x\left[\frac{(x+1)-1}{(x+1)^2}\right]dx$$

$$= \int e^x\left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right]dx$$

Put $f(x) = \frac{1}{x+1}$

Then $f'(x) = \frac{d}{dx}(x+1)^{-1} = -1(x+1)^{-2}\cdot\frac{d}{dx}(x+1)$

$$= \frac{-1}{(x+1)^2}\times(1+0) = \frac{-1}{(x+1)^2}$$

$$\therefore\ I = \int e^x[f(x) + f'(x)]\,dx$$

$$= e^x\cdot f(x) + c = e^x\cdot\frac{1}{x+1} + c.$$

7) $\int e^x\frac{x-1}{(x+1)^3}dx$

Solution:

$$\text{Let } I = \int e^x\cdot\frac{x-1}{(x+1)^3}dx$$

$$= \int e^x\left[\frac{(x+1)-2}{(x+1)^3}\right]dx$$

$$= \int e^x\left[\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right]dx$$

Put $f(x) = \frac{1}{(x+1)^2}$

Then $f'(x) = \frac{d}{dx}(x+1)^{-2} = -2(x+1)^{-3}\cdot\frac{d}{dx}(x+1)$

$$= \frac{-2}{(x+1)^3}\times(1+0) = \frac{-2}{(x+1)^3}$$

$$\therefore\ I = \int e^x\,[f(x) + f'(x)]\,dx$$

$$= e^x\cdot f(x) + c = e^x\cdot\frac{1}{(x+1)^2} + c.$$

8) $\int e^x\left[(logx)^2+\frac{2logx}{x}\right]dx$

Solution:

$$\text{Let } I=\int e^x\left[(\log x)^2+\frac{2\log x}{x}\right]dx$$

Put $f(x)=(\log x)^2$

Then $f'(x)=\frac{d}{dx}(\log x)^2=2\log x\cdot\frac{d}{dx}(\log x)$

$$=2\log x\times\frac{1}{x}=\frac{2\log x}{x}$$

$\therefore\ I=\int e^x[f(x)+f'(x)]dx$

$=e^x\cdot f(x)+c=e^x\cdot(\log x)^2+c.$

9) $\int\left[\frac{1}{logx}-\frac{1}{(logx)^2}\right]dx$

Solution:

$$\text{Let } I=\int\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx$$

Put $\log x=t \qquad \therefore\ x=e^t$

$\therefore\ dx=e^t\,dt$

$\therefore\ I=\int\left(\frac{1}{t}-\frac{1}{t^2}\right)e^t\,dt$

Let $f(t)=\frac{1}{t}$. Then $f'(t)=-\frac{1}{t^2}$

$\therefore\ I=\int e^t[f(t)+f'(t)]dt$

$=e^t\cdot f(t)+c=e^t\times\frac{1}{t}+c$

$=\frac{x}{\log x}+c.$

10) $\int\frac{logx}{(1+logx)^2}dx$

Solution:

$$\text{Let } I=\int\frac{\log x}{(1+\log x)^2}dx$$

Put $\log x=t \qquad \therefore\ x=e^t$

$\therefore\ dx=e^t\,dt$

$\therefore\ I=\int\frac{t}{(1+t)^2}\cdot e^t\,dt$

$=\int e^t\left[\frac{(1+t)-1}{(1+t)^2}\right]dt$

$=\int e^t\left[\frac{1}{1+t}-\frac{1}{(1+t)^2}\right]dt$

Let $f(t)=\frac{1}{1+t}$.

$\therefore\ f'(t)=\frac{d}{dt}(1+t)^{-1}=-1(1+t)^{-2}(0+1)$

$=\frac{-1}{(1+t)^2}$

$\therefore\ I=\int e^t[f(t)+f'(t)]dt$

$=e^t\cdot f(t)+c=e^t\times\frac{1}{1+t}+c=\frac{x}{1+\log x}+c.$

EXERCISE 5.6

Evaluate:

1) $\int\frac{2x+1}{(x+1)(x-2)}dx$

Solution:

$$\text{Let } I=\int\frac{2x+1}{(x+1)(x-2)}dx$$

Let $\frac{2x+1}{(x+1)(x-2)}=\frac{A}{x+1}+\frac{B}{x-2}$

$\therefore\ 2x+1=A(x-2)+B(x+1)$

Put $x+1=0$, i.e. $x=-1$, we get

$2(-1)+1=A(-3)+B(0) \qquad \therefore\ A=\frac{1}{3}$

Put $x-2=0$, i.e. $x=2$, we get

$2(2)+1=A(0)+B(3) \qquad \therefore\ B=\frac{5}{3}$

$\therefore\ \frac{2x+1}{(x+1)(x-2)}=\frac{(1/3)}{x+1}+\frac{(5/3)}{x-2}$

$\therefore\ I=\int\left[\frac{(1/3)}{x+1}+\frac{(5/3)}{x-2}\right]dx$

$$=\frac{1}{3}\int\frac{1}{x+1}\,dx+\frac{5}{3}\int\frac{1}{x-2}\,dx$$

$$=\frac{1}{3}\log|x+1|+\frac{5}{3}\log|x-2|+c.$$

2) $\int\frac{2x+1}{x(x-1)(x-4)}dx$

Solution:

$$\text{Let } I=\int\frac{2x+1}{x(x-1)(x-4)}\,dx$$

Let $\int\frac{2x+1}{x(x-1)(x-4)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-4}$

$\therefore\ 2x+1=A(x-1)(x-4)+Bx(x-4)+Cx(x-1)$

Put $x=0$, we get

$2(0)+1=A(-1)(-4)+B(0)(-4)+C(0)(-1)$

$\therefore\ 1=4A \qquad \therefore\ A=\frac{1}{4}$

Put $x-1=0$, i.e. $x=1$, we get

$2(1)+1=A(0)(-3)+B(1)(-3)+C(1)(0)$

$\therefore\ 3=-3B \qquad \therefore\ B=-1$

Put $x-4=0$, i.e $x=4$, we get

$2(4)+1=A(3)(0)+B(4)(0)+C(4)(3)$

$\therefore\ 9=12C \qquad \therefore\ C=\frac{3}{4}$

$$\therefore\ \frac{2x+1}{x(x-1)(x-4)}=\frac{\left(\frac{1}{4}\right)}{x}+\frac{(-1)}{x-1}+\frac{\left(\frac{3}{4}\right)}{x-4}$$

$$\therefore\ I=\int\left[\frac{\left(\frac{1}{4}\right)}{x}+\frac{(-1)}{x-1}+\frac{\left(\frac{3}{4}\right)}{x-4}\right]dx$$

$$=\frac{1}{4}\int\frac{1}{x}\,dx-\int\frac{1}{x-1}\,dx+\frac{3}{4}\int\frac{1}{x-4}\,dx$$

$$=\frac{1}{4}\log|x|-\log|x-1|+\frac{3}{4}\log|x-4|+c.$$

3) $\int\frac{x^2+x-1}{x^2+x-6}dx$

Solution:

$$\text{Let } I=\int\frac{x^2+x-1}{x^2+x-6}\,dx$$

$$=\int\frac{(x^2+x-6)+5}{x^2+x-6}\,dx$$

$$=\int\left[1+\frac{5}{x^2+x-6}\right]dx$$

$$=\int 1\,dx+5\int\frac{1}{x^2+x-6}\,dx$$

Let $\frac{1}{x^2+x-6}=\frac{1}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2}$

$\therefore\ 1=A(x-2)+B(x+3)$

Put $x+3=0$, i.e. $x=-3$, we get

$1=A(-5)+B(0) \qquad \therefore\ A=\frac{-1}{5}$

Put $x-2=0$, i.e. $x=2$, we get

$1=A(0)+B(5) \qquad \therefore\ B=\frac{1}{5}$

$$\therefore\ \frac{1}{x^2+x-6}=\frac{(-1/5)}{x+3}+\frac{(1/5)}{x-2}$$

$$\therefore\ I=\int 1\ dx+5\int\left[\frac{(-1/5)}{x+3}+\frac{(1/5)}{x-2}\right]dx$$

$$=\int 1\,dx-\int\frac{1}{x+3}\,dx+\int\frac{1}{x-2}\,dx$$

$$=x-\log|x+3|+\log|x-2|+c.$$

4) $\int\frac{x}{(x-1)^2(x+2)}dx$

Solution:

$$\text{Let } I=\int\frac{x}{(x-1)^2(x+2)}\,dx$$

Let $\frac{x}{(x-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2}$

$\therefore\ x=A(x-1)(x+2)+B(x+2)+C(x-1)^2$

Put $x-1=0$, i.e. $x=1$, we get

5) $\int\frac{3x-2}{(x+1)^2(x+3)}dx$

Solution:

$$\text{Let } I=\int \frac{3x-2}{(x+1)^2(x+3)}\,dx$$

Let $\dfrac{3x-2}{(x+1)^2(x+3)}=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{x+3}$

$\therefore\ 3x-2=A(x+1)(x+3)+B(x+3)+C(x+1)^2$

Put $x+1=0$, i.e. $x=-1$, we get

$3(-1)-2=A(0)(2)+B(2)+C(0)$

$\therefore\ -5=2B \qquad \therefore\ B=-\dfrac{5}{2}$

Put $x+3=0$, i.e. $x=-3$, we get

$3(-3)-2=A(-2)(0)+B(0)+C(4)$

$\therefore\ -11=4C \qquad \therefore\ C=-\dfrac{11}{4}$

Put $x=0$, we get

$3(0)-2=A(1)(3)+B(3)+C(1)$

$\therefore\ -2=3A+3B+C$

But $B=-\dfrac{5}{2}$ and $C=-\dfrac{11}{4}$

$\therefore\ -2=3A+3\left(-\dfrac{5}{2}\right)-\dfrac{11}{4}$

$\therefore\ 3A=-2+\dfrac{15}{2}+\dfrac{11}{4}=\dfrac{-8+30+11}{4}=\dfrac{33}{4}$

$\therefore\ A=\dfrac{11}{4}$

$$\therefore\ \frac{3x-2}{(x+1)^2(x+3)}=\frac{\left(\frac{11}{4}\right)}{x+1}+\frac{\left(-\frac{5}{2}\right)}{(x+1)^2}+\frac{\left(-\frac{11}{4}\right)}{(x+3)}$$

$$\therefore\ I=\int\left[\frac{\left(\frac{11}{4}\right)}{x+1}+\frac{\left(-\frac{5}{2}\right)}{(x+1)^2}+\frac{\left(-\frac{11}{4}\right)}{x+3}\right]dx$$

$$=\frac{11}{4}\int\frac{1}{x+1}\,dx-\frac{5}{2}\int(x+1)^{-2}\,dx-\frac{11}{4}\int\frac{1}{x+3}\,dx$$

$$=\frac{11}{4}\log|x+1|-\frac{5}{2}\cdot\frac{(x+1)^{-1}}{-1}-\frac{11}{4}\log|x+3|+c$$

$$=\frac{11}{4}\log\left|\frac{x+1}{x+3}\right|+\frac{5}{2(x+1)}+c.$$

6) $\displaystyle\int\frac{1}{x(x^5+1)}dx$

Solution:

$$\text{Let } I=\int\frac{1}{x(x^5+1)}\,dx$$

$$=\int\frac{x^4}{x^5(x^5+1)}\,dx$$

Put $x^5=t$. Then $5x^4\,dx=dt$

$\therefore\ x^4\,dx=\dfrac{dt}{5}$

$$\therefore\ I=\int\frac{1}{t(t+1)}\cdot\frac{dt}{5}$$

$$=\frac{1}{5}\int\frac{(t+1)-t}{t(t+1)}\,dt$$

$$=\frac{1}{5}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$=\frac{1}{5}\left[\int\frac{1}{t}dt-\int\frac{1}{t+1}dt\right]$$

$$=\frac{1}{5}[\log|t|-\log|t+1|]+c$$

$$=\frac{1}{5}\log\left|\frac{t}{t+1}\right|+c=\frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right|+c.$$

7) $\displaystyle\int\frac{1}{x(x^n+1)}dx$

Solution:

$$\text{Let } I=\int\frac{1}{x(x^n+1)}\,dx$$

$$=\int\frac{x^{n-1}}{x^n(x^n+1)}\,dx$$

Put $x^n=t \quad \therefore\ nx^{n-1}\,dx=dt$

$\therefore\ x^{n-1}\,dx=\dfrac{dt}{n}$

$$\therefore\ I=\int\frac{1}{t(t+1)}\cdot\frac{dt}{n}$$

$$=\frac{1}{n}\int\frac{(t+1)-t}{t(t+1)}\,dt$$

$$=\frac{1}{n}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$=\frac{1}{n}\left[\int\frac{1}{t}dt-\int\frac{1}{t+1}\,dt\right]$$

$$=\frac{1}{n}[\log|t|-\log|t+1|]+c$$

$$=\frac{1}{n}\log\left|\frac{t}{t+1}\right|+c=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+c.$$

8) $\int\frac{5x^2+20x+6}{x^3+2x^2+x}dx$

Solution:

$$\text{Let } I=\int\frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$$

$$=\int\frac{5x^2+20x+6}{x(x^2+2x+1)}\,dx$$

$$=\int\frac{5x^2+20x+6}{x(x+1)^2}\,dx$$

Let $\frac{5x^2+20x+6}{x(x+1)^2}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$

$\therefore\ 5x^2+20x+6=A(x+1)^2+Bx(x+1)+Cx$

Put $x=0$, we get

$0+0+6=A(1)+B(0)(1)+C(0)\quad\therefore\ A=6$

Put $x+1=0$, i.e. $x=-1$, we get

$5(1)+20(-1)+6=A(0)+B(-1)(0)+C(-1$

$\therefore\ -9=-C\qquad\therefore\ C=9$

Put $x=1$, we get

$5(1)+20(1)+6=A(4)+B(1)(2)+C(1)$

But $A=6$ and $C=9$

$\therefore\ 31=24+2B+9\qquad\therefore\ B=-1$

$$\therefore\ \frac{5x^2+20x+6}{x(x+1)^2}=\frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}$$

$$\therefore\ I=\int\left[\frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}\right]dx$$

$$=6\int\frac{1}{x}dx-\int\frac{1}{x+1}dx+9\int(x+1)^{-2}\,dx$$

$$=6\log|x|-\log|x+1|+9.\frac{(x+1)^{-1}}{-1}+c$$

$$=6\log|x|-\log|x+1|-\frac{9}{x+1}+c.$$