

24. Scalar or Dot Products

Exercise 24.1

1 A. Question

Find $\vec{a} \cdot \vec{b}$, when

$$\vec{a} = \hat{i} - 2\hat{j} + \hat{k} \text{ and } \vec{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$$

Answer

For any vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$$\vec{a} \cdot \vec{b} = x_1x_2 + y_1y_2 + z_1z_2$$

Given Vectors:

$$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$$

$$\vec{a} \cdot \vec{b} = 1 \times 4 + (-2) \times (-4) + 1 \times 7$$

$$\vec{a} \cdot \vec{b} = 19.$$

1 B. Question

Find $\vec{a} \cdot \vec{b}$, when

$$\vec{a} = \hat{j} + 2\hat{k} \text{ and } \vec{b} = 2\hat{i} + \hat{k}$$

Answer

For any vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$$\vec{a} \cdot \vec{b} = x_1x_2 + y_1y_2 + z_1z_2$$

$$\vec{a} = \hat{j} + 2\hat{k}$$

$$\vec{b} = 2\hat{i} + \hat{k}$$

$$\vec{a} \cdot \vec{b} = 0 \times 2 + 0 \times 2 + 1 \times 2$$

$$\vec{a} \cdot \vec{b} = 2$$

1 C. Question

Find $\vec{a} \cdot \vec{b}$, when

$$\vec{a} = \hat{j} - \hat{k} \text{ and } \vec{b} = 2\hat{i} + 3\hat{j} - 2\hat{k}$$

Answer

For any vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$$\vec{a} \cdot \vec{b} = x_1x_2 + y_1y_2 + z_1z_2$$

$$\vec{a} = \hat{j} - \hat{k}$$

$$\vec{b} = 2\hat{i} + 3\hat{j} - 2\hat{k}$$

$$\vec{a} \cdot \vec{b} = 0 \times 2 + 1 \times 3 + (-1) \times (-2)$$

$$\vec{a} \cdot \vec{b} = 5.$$

2 A. Question

For what value of λ are the vector $\vec{a}$ and $\vec{b}$ perpendicular to each other? Where :

$$\vec{a} = \lambda\hat{i} + 2\hat{j} + \hat{k} \text{ and } \vec{b} = 4\hat{i} - 9\hat{j} + 2\hat{k}$$

Answer

For any vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

If $\vec{a}$ and $\vec{b}$ are $\perp$ to each other then $\vec{a} \cdot \vec{b} = 0$

$$\vec{a} = \lambda\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b} = 4\hat{i} - 9\hat{j} + 2\hat{k}$$

Now

$$\vec{a} \cdot \vec{b} = 0$$

$$\lambda \times 4 + 2 \times (-9) + 1 \times 2 = 0$$

$$\lambda \times 4 = 16$$

$$\lambda = \frac{16}{4}$$

$$\lambda = 4$$

2 B. Question

For what value of λ are the vector $\vec{a}$ and $\vec{b}$ perpendicular to each other? Where :

$$\vec{a} = \lambda\hat{i} + 2\hat{j} + \hat{k} \text{ and } \vec{b} = 5\hat{i} - 9\hat{j} + 2\hat{k}$$

Answer

For any vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

If $\vec{a}$ and $\vec{b}$ are $\perp$ to each other then $\vec{a} \cdot \vec{b} = 0$

$$\vec{a} = \lambda\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b} = 5\hat{i} - 9\hat{j} + 2\hat{k}$$

Now

$$\vec{a} \cdot \vec{b} = 0$$

$$\lambda \times 5 + 2 \times (-9) + 1 \times 2 = 0$$

$$\lambda \times 5 = 16$$

$$\lambda = \frac{16}{5}$$

2 C. Question

For what value of λ are the vector $\vec{a}$ and $\vec{b}$ perpendicular to each other? Where :

$$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k} \text{ and } \vec{b} = 3\hat{i} + 2\hat{j} - \lambda\hat{k}$$

Answer

For any vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

If $\vec{a}$ and $\vec{b}$ are $\perp$ to each other then $\vec{a} \cdot \vec{b} = 0$

$$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{b} = 3\hat{i} + 2\hat{j} - \lambda\hat{k}$$

Now

$$\vec{a} \cdot \vec{b} = 0$$

$$2 \times 3 + 3 \times 2 + 4 \times (-\lambda) = 0$$

$$-4\lambda = -12$$

$$\lambda = \frac{12}{4}$$

$$\lambda = 3$$

2 D. Question

For what value of λ are the vector $\vec{a}$ and $\vec{b}$ perpendicular to each other? Where :

$$\vec{a} = \lambda\hat{i} + 3\hat{j} + 2\hat{k} \text{ and } \vec{b} = \hat{i} - \hat{j} + 3\hat{k}$$

Answer

For any vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

If $\vec{a}$ and $\vec{b}$ are $\perp$ to each other then $\vec{a} \cdot \vec{b} = 0$

$$\vec{a} = \lambda\hat{i} + 3\hat{j} + 2\hat{k}$$

$$\vec{b} = \hat{i} - \hat{j} + 3\hat{k}$$

Now

$$\vec{a} \cdot \vec{b} = 0$$

$$\lambda \times 1 + 3 \times (-1) + 2 \times 3 = 0$$

$$\lambda - 3 + 6 = 0$$

$$\lambda = -3$$

3. Question

If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = 4$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 6$. Find the angle between $\vec{a}$ and $\vec{b}$.

Answer

Given Data:

$$|\vec{a}| = 4, |\vec{b}| = 3 \text{ and } \vec{a} \cdot \vec{b} = 6$$

Calculation:

Using formula $\vec{a} \cdot \vec{b} = |\vec{a}| \times |\vec{b}| \times \cos\theta$

$$|\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a} \cdot \vec{b}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{6}{4 \times 3}$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \theta = \frac{\pi}{3}$$

Therefore angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$.

4. Question

If $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = -\hat{j} + 2\hat{k}$, find $(\vec{a} - 2\vec{b}) \cdot (\vec{a} + \vec{b})$.

Answer

Given data:

$$\vec{a} = \hat{i} - \hat{j}$$

$$\vec{b} = -\hat{j} + 2\hat{k}$$

Now

$$\Rightarrow \vec{a} - 2\vec{b} = (\hat{i} - \hat{j}) - 2(-\hat{j} + 2\hat{k})$$

$$\vec{a} - 2\vec{b} = \hat{i} - \hat{j} + 2\hat{j} - 4\hat{k}$$

$$\vec{a} - 2\vec{b} = \hat{i} + \hat{j} - 4\hat{k}$$

$$\Rightarrow \vec{a} + \vec{b} = (\hat{i} - \hat{j}) + (-\hat{j} + 2\hat{k})$$

$$\vec{a} + \vec{b} = \hat{i} - \hat{j} - \hat{j} + 2\hat{k}$$

$$\vec{a} + \vec{b} = \hat{i} - 2\hat{j} + 2\hat{k}$$

Consider

$$(\vec{a} - 2\vec{b}) \cdot (\vec{a} + \vec{b}) = (\hat{i} + \hat{j} - 4\hat{k}) \cdot (\hat{i} - 2\hat{j} + 2\hat{k})$$

$$(\vec{a} - 2\vec{b}) \cdot (\vec{a} + \vec{b}) = 1 \times 1 + 1 \times (-2) + (-4) \times 2$$

$$(\vec{a} - 2\vec{b}) \cdot (\vec{a} + \vec{b}) = 1 - 2 - 8$$

$$(\vec{a} - 2\vec{b}) \cdot (\vec{a} + \vec{b}) = -9$$

5 A. Question

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where:

$$\vec{a} = \hat{i} - \hat{j} \text{ and } \vec{b} = \hat{j} + \hat{k}$$

Answer

Using formula $\vec{a} \cdot \vec{b} = |\vec{a}| \times |\vec{b}| \times \cos\theta$

Given Data:

$$i. \vec{a} = \hat{i} - \hat{j}$$

$$\vec{b} = \hat{j} + \hat{k}$$

$$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a} \cdot \vec{b}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(\hat{i} - \hat{j})(\hat{j} + \hat{k})}{\sqrt{1^2 + 1^2} \times \sqrt{1^2 + 1^2}}$$

$$\cos\theta = \frac{1 \times 0 + (-1) \times 1 + 0 \times 1}{\sqrt{2} \times \sqrt{2}}$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\theta = \pi - \frac{\pi}{3}$$

$$\therefore \theta = \frac{2\pi}{3}$$

Therefore angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$.

5 B. Question

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where:

$$\vec{a} = 3\hat{i} - 2\hat{j} - 6\hat{k} \text{ and } \vec{b} = 4\hat{i} - \hat{j} + 8\hat{k}$$

Answer

Using formula $\vec{a} \cdot \vec{b} = |\vec{a}| \times |\vec{b}| \times \cos\theta$

$$\vec{a} = 3\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\vec{b} = 4\hat{i} - \hat{j} + 8\hat{k}$$

$$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a} \cdot \vec{b}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(3\hat{i} - 2\hat{j} - 6\hat{k})(4\hat{i} - \hat{j} + 8\hat{k})}{\sqrt{3^2 + (-2)^2 + (-6)^2} \times \sqrt{4^2 + (-1)^2 + 8^2}}$$

$$\cos\theta = \frac{3 \times 4 + (-2) \times (-1) + (-6) \times 8}{\sqrt{9 + 4 + 36} \times \sqrt{16 + 1 + 64}}$$

$$\cos\theta = -\frac{34}{\sqrt{49} \times \sqrt{81}}$$

$$\cos\theta = -\frac{34}{7 \times 9}$$

$$\theta = \cos^{-1}\left(-\frac{34}{63}\right)$$

$$\theta = 122.66^\circ$$

Therefore angle between $\vec{a}$ and $\vec{b}$ is 122.66° .

5 C. Question

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where:

$$\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k} \text{ and } \vec{b} = 4\hat{i} + 4\hat{j} - 2\hat{k}$$

Answer

Using formula $\vec{a} \cdot \vec{b} = |\vec{a}| \times |\vec{b}| \times \cos\theta$

$$\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$$

$$\vec{b} = 4\hat{i} + 4\hat{j} - 2\hat{k}$$

$$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a} \cdot \vec{b}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(2\hat{i} - \hat{j} + 2\hat{k})(4\hat{i} + 4\hat{j} - 2\hat{k})}{\sqrt{2^2 + (-1)^2 + (2)^2} \times \sqrt{4^2 + 4^2 + (-2)^2}}$$

$$\cos\theta = \frac{2 \times 4 + (-1) \times 4 + 2 \times (-2)}{\sqrt{4 + 1 + 4} \times \sqrt{16 + 16 + 4}}$$

$$\cos\theta = \frac{0}{\sqrt{9} \times \sqrt{36}}$$

$$\cos\theta = \frac{0}{3 \times 6}$$

$$\theta = \cos^{-1}(0)$$

$$\theta = \frac{\pi}{2}$$

Therefore angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{2}$.

5 D. Question

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where:

$$\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k} \text{ and } \vec{b} = \hat{i} + \hat{j} - 2\hat{k}$$

Answer

Using formula $\vec{a} \cdot \vec{b} = |\vec{a}| \times |\vec{b}| \times \cos\theta$

$$\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$$

$$\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$$

$$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a} \cdot \vec{b}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(2\hat{i} - 3\hat{j} + \hat{k})(\hat{i} + \hat{j} - 2\hat{k})}{\sqrt{2^2 + (-3)^2 + 1^2} \times \sqrt{1^2 + 1^2 + (-2)^2}}$$

$$\cos\theta = \frac{2 \times 1 + (-3) \times 1 + 1 \times (-2)}{\sqrt{4+9+1} \times \sqrt{1+1+4}}$$

$$\cos\theta = -\frac{3}{\sqrt{14} \times \sqrt{6}}$$

$$\cos\theta = -\frac{3}{\sqrt{84}}$$

$$\theta = \cos^{-1}\left(-\frac{3}{\sqrt{84}}\right)$$

Therefore angle between $\vec{a}$ and $\vec{b}$ is $\cos^{-1}\left(-\frac{3}{\sqrt{84}}\right)$.

5 E. Question

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where:

$$\vec{a} = \hat{i} + 2\hat{j} - \hat{k}, \vec{b} = \hat{i} - \hat{j} + \hat{k}$$

Answer

Using formula $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos\theta$

$$\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$$

$$\vec{b} = \hat{i} - \hat{j} + \hat{k}$$

$$\Rightarrow |\vec{a}| |\vec{b}| \cos\theta = \vec{a} \cdot \vec{b}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

$$\cos\theta = \frac{(\hat{i} + 2\hat{j} - \hat{k}) \cdot (\hat{i} - \hat{j} + \hat{k})}{\sqrt{1^2 + 2^2 + (-1)^2} \times \sqrt{1^2 + (-1)^2 + 1^2}}$$

$$\cos\theta = \frac{1 \times 1 + 2 \times (-1) + (-1) \times 1}{\sqrt{1+4+1} \times \sqrt{1+1+1}}$$

$$\cos\theta = -\frac{2}{\sqrt{2} \times 3}$$

$$\cos\theta = -\frac{\sqrt{2}}{3}$$

$$\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{3}\right)$$

Therefore angle between $\vec{a}$ and $\vec{b}$ is $\cos^{-1}\left(-\frac{\sqrt{2}}{3}\right)$

6. Question

Find the angles which the vector $\vec{a} = \hat{i} - \hat{j} + \sqrt{2}\hat{k}$ makes with the coordinate axes.

Answer

Calculation:

Angle with x-axis

$$\vec{a} = \hat{i} - \hat{j} + \sqrt{2}\hat{k}$$

unit vector along x axis is $\hat{i}$

$$\text{So, } \vec{b} = \hat{i}$$

$$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a} \cdot \vec{b}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(\hat{i} - \hat{j} + \sqrt{2}\hat{k})(\hat{i})}{\sqrt{1^2 + (-1)^2 + (\sqrt{2})^2} \times \sqrt{1^2}}$$

$$\cos\theta = \frac{1}{\sqrt{4} \times \sqrt{1}}$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{3}$$

Therefore angle between $\vec{a}$ and x axis is $\frac{\pi}{3}$

Angle with y-axis

$$\vec{a} = \hat{i} - \hat{j} + \sqrt{2}\hat{k}$$

unit vector along y axis is $\hat{j}$

$$\text{So, } \vec{b} = \hat{j}$$

$$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a} \cdot \vec{b}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(\hat{i} - \hat{j} + \sqrt{2}\hat{k})(\hat{j})}{\sqrt{1^2 + (-1)^2 + (\sqrt{2})^2} \times \sqrt{1^2}}$$

$$\cos\theta = -\frac{1}{\sqrt{4} \times \sqrt{1}}$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\theta = \pi - \frac{\pi}{3}$$

$$\theta = \frac{2\pi}{3}$$

Therefore angle between $\vec{a}$ and y axis is $\frac{2\pi}{3}$

Angle with z-axis

$$\vec{a} = \hat{i} - \hat{j} + \sqrt{2}\hat{k}$$

unit vector along z axis is $\hat{k}$

$$\text{So, } \vec{b} = \hat{k}$$

$$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a} \cdot \vec{b}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(\hat{i} - \hat{j} + \sqrt{2}\hat{k}) \cdot (\hat{k})}{\sqrt{1^2 + (-1)^2 + (\sqrt{2})^2} \times \sqrt{1^2}}$$

$$\cos\theta = \frac{\sqrt{2}}{\sqrt{4} \times \sqrt{1}}$$

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\theta = \frac{\pi}{4}$$

Therefore angle between $\vec{a}$ and z axis is $\frac{\pi}{4}$

7 A. Question

Dot product of a vector with $\hat{i} + \hat{j} - 3\hat{k}$, $\hat{i} + 3\hat{j} - 2\hat{k}$ and $2\hat{i} + \hat{j} + 4\hat{k}$ are 0, 5 and 8 respectively. Find the vector.

Answer

Given Data:

Vectors:

$$\vec{a} = \hat{i} + \hat{j} - 3\hat{k}$$

$$\vec{b} = \hat{i} + 3\hat{j} - 2\hat{k}$$

$$\vec{c} = 2\hat{i} + \hat{j} + 4\hat{k}$$

Their Dot products are 0, 5 and 8.

Calculation:

Let the required vector be,

$$\vec{h} = x\hat{i} + y\hat{j} + z\hat{k}$$

Now,

$$\vec{a} \cdot \vec{h} = 0$$

$$(\hat{i} + \hat{j} - 3\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 0$$

$$x + y - 3z = 0 \dots \text{Eq. 1}$$

Similarly

$$\Rightarrow \vec{b} \cdot \vec{h} = 5$$

$$(\hat{i} + 3\hat{j} - 2\hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = 5$$

$$x + 3y - 2z = 5 \dots \text{Eq. 2}$$

$$\Rightarrow \vec{c} \cdot \vec{h} = 8$$

$$(2\hat{i} + \hat{j} + 4\hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = 8$$

$$2x + y + 4z = 8 \dots \text{Eq. 3}$$

Subtract Eq. 1 from Eq. 2

$$(x + 3y - 2z) - (x + y - 3z) = 5 - 0$$

$$\Rightarrow 2y + z = 5 \dots \text{Eq. 4}$$

Subtract Eq. 3 from (2 × Eq. 2)

$$2(x + 3y - 2z) - 2x + y + 4z = (2 \times 5) - 8$$

$$5y - 8z = 2 \dots \text{Eq. 5}$$

Adding Eq. 5 with (8 × Eq. 4)

$$8(2y + z) + (5y - 8z) = 8 \times 5 + 2$$

$$\Rightarrow 21y = 42$$

$$\Rightarrow y = 2$$

From Eq. 5,

$$5 \times 2 - 8z = 2$$

$$\Rightarrow z = 1$$

From Eq. 1

$$x + y - 3z = 0$$

$$\Rightarrow x + 2 - 3 \times 1 = 0$$

$$\Rightarrow x = 1$$

$$\therefore \text{required vector is } \vec{h} = \hat{i} + 2\hat{j} + \hat{k}$$

7 B. Question

Dot product of a vector with vectors $\hat{i} - \hat{j} + \hat{k}$, $2\hat{i} + \hat{j} - 3\hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$ are respectively 4, 0 and 2. Find the vector.

Answer

Vectors:

$$\vec{a} = \hat{i} - \hat{j} + \hat{k}$$

$$\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$$

$$\vec{c} = \hat{i} + \hat{j} + \hat{k}$$

Their Dot products are 4, 0 and 2.

Calculation:

Let the required vector be,

$$\vec{h} = x\hat{i} + y\hat{j} + z\hat{k}$$

Now,

$$\vec{a} \cdot \vec{h} = 0$$

$$(\hat{i} - \hat{j} + \hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = 4$$

$$x - y + z = 4 \dots \text{Eq. 1}$$

Similarly

$$\Rightarrow \vec{b} \cdot \vec{h} = 0$$

$$(2\hat{i} + \hat{j} - 3\hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = 5$$

$$2x + y - 3z = 0 \dots \text{Eq. 2}$$

$$\Rightarrow \vec{c} \cdot \vec{h} = 2$$

$$(\hat{i} + \hat{j} + \hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = 2$$

$$x + y + z = 2 \dots \text{Eq. 3}$$

Subtract Eq. 1 from Eq. 3

$$(x + y + z) - (x - y + z) = 2 - 4$$

$$\Rightarrow 2y = -2$$

$$y = -1$$

Now putting the value of y in equation(2) and equation (3) we get,

$$2x - 3z = 1 \dots (\text{Eq(4)})$$

$$x + z = 3 \dots (\text{Eq(5)})$$

$$\text{Eq(4)} - 2 \times \text{Eq (5)}$$

$$-5z = -5$$

$$z = 1$$

Now putting value of z in equation (1) we get,

$$x - y + z = 4$$

$$x + 1 + 1 = 4$$

$$x = 2$$

So the vector is,

$$\therefore \text{required vector is } \vec{h} = 2\hat{i} - \hat{j} + \hat{k}$$

8 A. Question

If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ , then prove that

$$\cos \frac{\theta}{2} = \frac{1}{2} |\hat{a} + \hat{b}|$$

Answer

Given Data: Two unit vectors inclined at an angle θ

Proof:

Since vectors are unit vectors

$$\therefore |\hat{a}| = |\hat{b}| = 1$$

Now,

$$\begin{aligned}\Rightarrow |\hat{a} + \hat{b}|^2 &= (\hat{a} + \hat{b})^2 \\&= (\hat{a})^2 + (\hat{b})^2 + 2 \hat{a} \cdot \hat{b} \\&= |\hat{a}|^2 + |\hat{b}|^2 + 2 \times |\hat{a}| \times |\hat{b}| \times \cos \theta \\&= 1 + 1 + 2 \times 1 \times 1 \times \cos \theta \\&= 2 + 2 \cos \theta \\&= 2(1 + \cos \theta)\end{aligned}$$

Using the identity, $(1 + \cos \theta) = 2 \cos^2 \frac{\theta}{2}$

$$\begin{aligned}&= 2 \times 2 \cos^2 \frac{\theta}{2} \\&= 4 \cos^2 \frac{\theta}{2} \\&\Rightarrow |\hat{a} + \hat{b}|^2 = 4 \cos^2 \frac{\theta}{2} \\&\Rightarrow |\hat{a} + \hat{b}| = \sqrt{4 \cos^2 \frac{\theta}{2}} \\&\Rightarrow |\hat{a} + \hat{b}| = 2 \cos \frac{\theta}{2} \\&\Rightarrow \text{(i) } \cos \frac{\theta}{2} = \frac{1}{2} |\hat{a} + \hat{b}|\end{aligned}$$

8 B. Question

If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ , then prove that

$$\tan \frac{\theta}{2} = \frac{|\hat{a} - \hat{b}|}{|\hat{a} + \hat{b}|}$$

Answer

$$\begin{aligned}\Rightarrow |\hat{a} - \hat{b}|^2 &= (\hat{a} - \hat{b})^2 \\&= (\hat{a})^2 + (\hat{b})^2 - 2 \hat{a} \cdot \hat{b} \\&= |\hat{a}|^2 + |\hat{b}|^2 - 2 \times |\hat{a}| \times |\hat{b}| \times \cos \theta \\&= 1 + 1 - 2 \times 1 \times 1 \times \cos \theta \\&= 2 - 2 \cos \theta \\&= 2(1 - \cos \theta)\end{aligned}$$

Using the identity, $(1 - \cos \theta) = 2 \sin^2 \frac{\theta}{2}$

$$\begin{aligned}&= 2 \times 2 \sin^2 \frac{\theta}{2} \\&= 4 \sin^2 \frac{\theta}{2}\end{aligned}$$

$$\Rightarrow |\hat{a} - \hat{b}|^2 = 4 \sin^2 \frac{\theta}{2}$$

$$\Rightarrow |\hat{a} - \hat{b}| = \sqrt{4 \sin^2 \frac{\theta}{2}}$$

$$\Rightarrow |\hat{a} - \hat{b}| = 2 \sin \frac{\theta}{2}$$

$$\sin \frac{\theta}{2} = \frac{1}{2} |\hat{a} - \hat{b}|$$

Dividing above by result (i) we will get,

$$\frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \frac{\frac{1}{2} |\hat{a} - \hat{b}|}{\frac{1}{2} |\hat{a} + \hat{b}|}$$

$$(ii) \tan \frac{\theta}{2} = \frac{|\hat{a} - \hat{b}|}{|\hat{a} + \hat{b}|}$$

Proved

9. Question

If the sum of two unit vectors is a unit vector prove that the magnitude of their difference is $\sqrt{3}$.

Answer

The sum of two unit vectors is a unit vector

Calculation:

Since $|\hat{a}| = |\hat{b}| = 1$

Also,

$$|\hat{a} + \hat{b}| = 1$$

Now squaring both sides we get

$$\Rightarrow |\hat{a} + \hat{b}|^2 = 1^2$$

$$|\hat{a}|^2 + |\hat{b}|^2 + 2 \hat{a} \cdot \hat{b} = 1$$

$$1^2 + 1^2 + 2 \hat{a} \cdot \hat{b} = 1$$

$$\Rightarrow \hat{a} \cdot \hat{b} = -\frac{1}{2}$$

Now,

$$\Rightarrow |\hat{a} - \hat{b}|^2 = (\hat{a} - \hat{b})^2$$

$$= (\hat{a})^2 + (\hat{b})^2 - 2 \hat{a} \cdot \hat{b}$$

$$= |\hat{a}|^2 + |\hat{b}|^2 - 2 \hat{a} \cdot \hat{b}$$

Using the above value,

$$= 1^2 + 1^2 - 2 \left(-\frac{1}{2}\right)$$

$$= 3$$

$$\therefore |\hat{a} - \hat{b}|^2 = 3$$

$$\Rightarrow |\hat{a} - \hat{b}| = \sqrt{3}$$

Hence, the magnitude of their difference is $\sqrt{3}$.

10. Question

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three mutually perpendicular unit vectors, then prove that $|\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}$.

Answer

Given Data:

Three mutually perpendicular unit vectors

$$\hat{a} \cdot \hat{b} = \hat{b} \cdot \hat{c} = \hat{c} \cdot \hat{a} = 0$$

$$\text{Since } |\hat{a}| = |\hat{b}| = |\hat{c}| = 1$$

Calculation:

$$|\hat{a} + \hat{b} + \hat{c}|^2 = (\hat{a} + \hat{b} + \hat{c})^2$$

$$= (\hat{a})^2 + (\hat{b})^2 + (\hat{c})^2 + 2\hat{a} \cdot \hat{b} + 2\hat{b} \cdot \hat{c} + 2\hat{c} \cdot \hat{a}$$

$$= |\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2 + 0 + 0 + 0$$

$$= 1 + 1 + 1$$

$$= 3$$

$$\Rightarrow |\hat{a} + \hat{b} + \hat{c}|^2 = 3$$

$$\therefore |\hat{a} + \hat{b} + \hat{c}| = \sqrt{3}$$

11. Question

If $|\vec{a} + \vec{b}| = 60$, $|\vec{a} - \vec{b}| = 40$ and $|\vec{b}| = 46$, find $|\vec{a}|$

Answer

Given Data:

$$|\vec{a} + \vec{b}| = 60$$

$$|\vec{a} - \vec{b}| = 40$$

$$|\vec{b}| = 46$$

Calculation:

$$\Rightarrow |\vec{a} + \vec{b}|^2 = 60^2$$

$$(\vec{a})^2 + (\vec{b})^2 + 2\vec{a} \cdot \vec{b} = 3600$$

$$|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} = 3600 \dots \text{Eq. 1}$$

Now,

$$\Rightarrow |\vec{a} - \vec{b}|^2 = 40^2$$

$$(\vec{a} - \vec{b})^2 = 1600$$

$$(\vec{a})^2 + (\vec{b})^2 - 2 \vec{a} \cdot \vec{b} = 1600$$

$$|\vec{a}|^2 + |\vec{b}|^2 - 2 \vec{a} \cdot \vec{b} = 1600 \dots \text{Eq. 2}$$

Adding Eq. 1 and Eq. 2

$$2(|\vec{a}|^2 + |\vec{b}|^2) + 2 \vec{a} \cdot \vec{b} - 2 \vec{a} \cdot \vec{b} = 3600 + 1600$$

$$2(|\vec{a}|^2 + |\vec{b}|^2) = 5200$$

$$(|\vec{a}|^2 + 46^2) = \frac{5200}{2}$$

$$(|\vec{a}|^2 + 2116) = 2600$$

$$|\vec{a}|^2 = 2600 - 2116$$

$$|\vec{a}|^2 = 484$$

$$|\vec{a}| = \sqrt{484}$$

$$|\vec{a}| = 22$$

12. Question

Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined with the coordinate axes

Answer

Calculation:

Angle with x-axis

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

unit vector along x axis is $\hat{i}$

$$\text{So, } \vec{b} = \hat{i}$$

$$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\alpha = \vec{a} \cdot \vec{b}$$

$$\cos\alpha = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\alpha = \frac{(\hat{i} + \hat{j} + \hat{k}) \cdot (\hat{i})}{\sqrt{1^2 + 1^2 + 1^2} \times \sqrt{1^2}}$$

$$\cos\alpha = \frac{1}{\sqrt{3} \times \sqrt{1}}$$

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

Angle with y-axis

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

unit vector along y axis is $\hat{j}$

$$\text{So, } \vec{b} = \hat{j}$$

$$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\beta = \vec{a} \cdot \vec{b}$$

$$\cos\beta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\beta = \frac{(\hat{i} + \hat{j} + \hat{k})(\hat{j})}{\sqrt{1^2 + 1^2 + 1^2} \times \sqrt{1^2}}$$

$$\cos\beta = \frac{1}{\sqrt{3} \times \sqrt{1}}$$

$$\cos\beta = \frac{1}{\sqrt{3}}$$

Angle with z-axis

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

unit vector along z axis is $\hat{k}$

$$\text{So, } \vec{b} = \hat{k}$$

$$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\gamma = \vec{a} \cdot \vec{b}$$

$$\cos\gamma = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\gamma = \frac{(\hat{i} + \hat{j} + \hat{k})(\hat{j})}{\sqrt{1^2 + 1^2 + 1^2} \times \sqrt{1^2}}$$

$$\cos\gamma = \frac{1}{\sqrt{3} \times \sqrt{1}}$$

$$\cos\gamma = \frac{1}{\sqrt{3}}$$

Hence $\alpha = \beta = \gamma$.

13. Question

Show that the vectors $\vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$, $\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$, $\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$ are mutually perpendicular unit vectors.

Answer

Given Data:

$$\vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$$

$$\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

$$\vec{a} \cdot \vec{b} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}) \cdot \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$$

$$\vec{a} \cdot \vec{b} = \frac{1}{49}(2 \times 3 + 3 \times (-6) + 6 \times 2)$$

$$\vec{a} \cdot \vec{b} = \frac{1}{49}(6 + -18 + 12)$$

$$\vec{a} \cdot \vec{b} = 0$$

Similarly,

$$\vec{b} \cdot \vec{c} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) \cdot \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

$$\vec{b} \cdot \vec{c} = \frac{1}{49}(3 \times 6 + (-6) \times 2 + 2 \times (-3))$$

$$\vec{b} \cdot \vec{c} = \frac{1}{49}(18 - 12 - 6)$$

$$\vec{b} \cdot \vec{c} = 0$$

$$\vec{c} \cdot \vec{a} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k}) \cdot \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$\vec{c} \cdot \vec{a} = \frac{1}{49}(6 \times 2 + 2 \times 3 + (-3) \times 6)$$

$$\vec{c} \cdot \vec{a} = \frac{1}{49}(12 + 6 - 18)$$

$$\vec{c} \cdot \vec{a} = 0$$

$$\therefore \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = 0$$

Hence these vectors are mutually perpendicular.

14. Question

For any two vectors $\vec{a}$ and $\vec{b}$, show that : $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0 \Leftrightarrow |\vec{a}| = |\vec{b}|$.

Answer

$$\text{Let } (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

$$|\vec{a}|^2 - |\vec{b}|^2 = 0$$

$$|\vec{a}|^2 = |\vec{b}|^2$$

$$\Rightarrow |\vec{a}| = |\vec{b}|$$

$$\text{Let } \Rightarrow |\vec{a}| = |\vec{b}|$$

Squaring both sides

$$|\vec{a}|^2 = |\vec{b}|^2$$

$$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 0$$

$$\Rightarrow (\vec{a})^2 - (\vec{b})^2 = 0$$

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

$$\text{Hence, } (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0 \Leftrightarrow |\vec{a}| = |\vec{b}|$$

15. Question

If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{c} = \hat{i} + 3\hat{j} - \hat{k}$, find λ such that $\vec{a}$ is perpendicular to $\lambda\vec{b} + \vec{c}$.

Answer

Given Data:

$$\vec{a} = (2\hat{i} - \hat{j} + \hat{k})$$

$$\vec{b} = (\hat{i} + \hat{j} - 2\hat{k})$$

$$\vec{c} = (\hat{i} + 3\hat{j} - \hat{k})$$

$$\vec{d} = \lambda\vec{b} + \vec{c}$$

$$\vec{d} = \lambda(\hat{i} + \hat{j} - 2\hat{k}) + (\hat{i} + 3\hat{j} - \hat{k})$$

$$\vec{d} = (\lambda + 1)\hat{i} + (\lambda + 3)\hat{j} - (2\lambda + 1)\hat{k}$$

For this vector to be $\perp$

$$\vec{a} \cdot \vec{d} = 0$$

$$(2\hat{i} - \hat{j} + \hat{k}) \cdot ((\lambda + 1)\hat{i} + (\lambda + 3)\hat{j} - (2\lambda + 1)\hat{k}) = 0$$

$$2(\lambda + 1) - 1(\lambda + 3) - 1 \cdot (2\lambda + 1) = 0$$

$$2(\lambda + 1) - 1(\lambda + 3) - 1 \cdot (2\lambda + 1) = 0$$

$$-\lambda - 2 = 0$$

$$\therefore \lambda = -2$$

16. Question

If $\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$ and $\vec{q} = \hat{i} + 3\hat{j} - 5\hat{k}$, then find the value of λ , so that $\vec{p} + \vec{q}$ and $\vec{p} - \vec{q}$ are perpendicular vectors.

Answer

Given Data:

$$\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$$

$$\vec{q} = \hat{i} + 3\hat{j} - 5\hat{k}$$

$$\vec{p} + \vec{q} = (5\hat{i} + \lambda\hat{j} - 3\hat{k}) + (\hat{i} + 3\hat{j} - 5\hat{k})$$

$$\vec{p} + \vec{q} = (6\hat{i} + (\lambda + 3)\hat{j} - 8\hat{k})$$

Also,

$$\vec{p} - \vec{q} = (5\hat{i} + \lambda\hat{j} - 3\hat{k}) - (\hat{i} + 3\hat{j} - 5\hat{k})$$

$$\vec{p} - \vec{q} = (4\hat{i} + (\lambda - 3)\hat{j} + 2\hat{k})$$

For this vector to be $\perp$

$$(\vec{p} + \vec{q}) \cdot (\vec{p} - \vec{q}) = 0$$

$$(6\hat{i} + (\lambda + 3)\hat{j} - 8\hat{k}) \cdot (4\hat{i} + (\lambda - 3)\hat{j} + 2\hat{k}) = 0$$

$$6 \times 4 + (\lambda + 3)(\lambda - 3) - 16 = 0$$

$$24 + \lambda^2 - 9 - 16 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

17. Question

If $\vec{\alpha} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{\beta} = 2\hat{i} + \hat{j} - 4\hat{k}$, then express $\vec{\beta}$ in the form of $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to

$\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

Answer

Given Data:

$$\vec{\alpha} = 3\hat{i} + 4\hat{j} + 5\hat{k}$$

$$\vec{\beta} = 2\hat{i} + \hat{j} - 4\hat{k}$$

Now

$$\vec{\beta}_1 \parallel \vec{\alpha}$$

$$\vec{\beta}_1 = \lambda \vec{\alpha}$$

$$\Rightarrow \vec{\beta}_1 = \lambda(3\hat{i} + 4\hat{j} + 5\hat{k})$$

Also,

$$\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$$

$$\Rightarrow \vec{\beta}_2 = \vec{\beta} - \vec{\beta}_1$$

$$\vec{\beta}_2 = (2\hat{i} + \hat{j} - 4\hat{k}) - \lambda(3\hat{i} + 4\hat{j} + 5\hat{k})$$

$$\vec{\beta}_2 = (2 - 3\lambda)\hat{i} + (1 - 4\lambda)\hat{j} - (4 + 5\lambda)\hat{k}$$

$$\vec{\beta}_2 \perp \vec{\alpha}$$

$$\vec{\beta}_2 \cdot \vec{\alpha} = 0$$

$$((2 - 3\lambda)\hat{i} + (1 - 4\lambda)\hat{j} - (4 + 5\lambda)\hat{k}) \cdot (3\hat{i} + 4\hat{j} + 5\hat{k}) = 0$$

$$3(2 - 3\lambda) + 4(1 - 4\lambda) - 5(4 + 5\lambda) = 0$$

$$-50\lambda = 10$$

$$\therefore \lambda = -\frac{1}{5}$$

$$\Rightarrow \vec{\beta}_1 = -\frac{1}{5}(3\hat{i} + 4\hat{j} + 5\hat{k})$$

Using the above value,

$$\Rightarrow \vec{\beta}_2 = \vec{\beta} - \vec{\beta}_1$$

$$\vec{\beta}_2 = (2 - 3\lambda)\hat{i} + (1 - 4\lambda)\hat{j} - (4 + 5\lambda)\hat{k}$$

$$\vec{\beta}_2 = (2 - 3(-\frac{1}{5}))\hat{i} + (1 - 4(-\frac{1}{5}))\hat{j} - (4 + 5(-\frac{1}{5}))\hat{k}$$

$$\vec{\beta}_2 = \frac{1}{5}(13\hat{i} + 9\hat{j} - 15\hat{k})$$

$$\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$$

18. Question

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \cdot \vec{b} = 0$. But, the converse need not be true. Justify your answer with an example.

Answer

$$\vec{a} = (2\hat{i} - \hat{j} + \hat{k})$$

$$\vec{b} = -\hat{i} + \hat{j} + 3\hat{k}$$

$$\vec{a} \cdot \vec{b} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + \hat{j} + 3\hat{k})$$

$$\vec{a} \cdot \vec{b} = -2 - 1 + 3$$

$$\vec{a} \cdot \vec{b} = 0$$

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 1^2}$$

$$|\vec{a}| = \sqrt{6}$$

$$|\vec{a}| \neq 0$$

Similarly,

$$|\vec{b}| = \sqrt{(-1)^2 + 1^2 + 3^2}$$

$$|\vec{b}| = \sqrt{11}$$

$$|\vec{b}| \neq 0$$

19. Question

Show that the vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$, $\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right angled triangle.

Answer

Given Vectors:

$$\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$$

$$\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$$

First show that the vectors form a triangle, so we use the addition of vector

$$\vec{b} + \vec{c} = (\hat{i} - 3\hat{j} + 5\hat{k}) + (2\hat{i} + \hat{j} - 4\hat{k})$$

$$\therefore \vec{b} + \vec{c} = 3\hat{i} - 2\hat{j} + \hat{k}$$

$$\therefore \vec{b} + \vec{c} = \vec{a}$$

Hence these vectors form a triangle

Now we will use Pythagoras theorem to prove this is a right angle triangle.

$$|\vec{a}| = \sqrt{3^2 + (-2)^2 + 1^2}$$

$$|\vec{a}| = \sqrt{14}$$

$$|\vec{b}| = \sqrt{1^2 + (-3)^2 + 5^2}$$

$$|\vec{a}| = \sqrt{35}$$

$$|\vec{c}| = \sqrt{2^2 + 1^2 + (-4)^2}$$

$$|\vec{c}| = \sqrt{21}$$

$$\Rightarrow |\vec{a}|^2 + |\vec{c}|^2 = 14 + 21$$

$$\Rightarrow |\vec{a}|^2 + |\vec{c}|^2 = 35$$

$$\Rightarrow |\vec{a}|^2 + |\vec{c}|^2 = |\vec{b}|^2$$

Therefore these vectors form a right angled triangle.

20. Question

If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ .

Answer

Given Data:

$$\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{c} = 3\hat{i} + \hat{j}$$

$$\vec{d} = \vec{a} + \lambda\vec{b}$$

$$\vec{d} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k})$$

$$\vec{d} = (-\lambda + 2)\hat{i} + (2\lambda + 2)\hat{j} + (\lambda + 3)\hat{k}$$

For this vector to be $\perp$

$$\vec{c} \cdot \vec{d} = 0$$

$$(3\hat{i} + \hat{j}) \cdot ((-\lambda + 2)\hat{i} + (2\lambda + 2)\hat{j} + (\lambda + 3)\hat{k}) = 0$$

$$3(-\lambda + 2) + 1(2\lambda + 2) = 0$$

$$-\lambda + 8 = 0$$

$$\lambda = 8$$

The value of λ is 8.

21. Question

Find the angles of a triangle whose vertices are A(0, -1, -2), B(3, 1, 4) and C(5, 7, 1).

Answer

Given Data:

$$\vec{A} = -1\hat{j} - 2\hat{k}$$

$$\vec{B} = 3\hat{i} + \hat{j} + 4\hat{k}$$

$$\vec{C} = 5\hat{i} + 7\hat{j} + \hat{k}$$

$$\vec{AB} = \vec{B} - \vec{A}$$

$$= (3\hat{i} + \hat{j} + 4\hat{k}) - (-1\hat{j} - 2\hat{k})$$

$$= 3\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\vec{BC} = \vec{C} - \vec{B}$$

$$= (5\hat{i} + 7\hat{j} + \hat{k}) - (3\hat{i} + \hat{j} + 4\hat{k})$$

$$= 2\hat{i} + 6\hat{j} - 3\hat{k}$$

$$\vec{AC} = \vec{C} - \vec{A}$$

$$= (5\hat{i} + 7\hat{j} + \hat{k}) - (-1\hat{j} - 2\hat{k})$$

$$= 5\hat{i} + 8\hat{j} + 3\hat{k}$$

Now the angle A

$$\cos A = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| \times |\vec{AC}|}$$

$$\cos A = \frac{(3\hat{i} + 2\hat{j} + 6\hat{k})(5\hat{i} + 8\hat{j} + 3\hat{k})}{\sqrt{3^2 + 2^2 + 6^2} \times \sqrt{5^2 + 8^2 + 3^2}}$$

$$\cos A = \frac{15 + 16 + 18}{\sqrt{49} \times \sqrt{98}}$$

$$\cos A = \frac{49}{49\sqrt{2}}$$

$$\cos A = \frac{1}{\sqrt{2}}$$

$$A = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$A = \frac{\pi}{4}$$

Now the angle B

$$\cos B = \frac{\vec{BC} \cdot \vec{BA}}{|\vec{BC}| \times |\vec{BA}|}$$

$$\cos B = \frac{(2\hat{i} + 6\hat{j} - 3\hat{k})(-3\hat{i} - 2\hat{j} - 6\hat{k})}{\sqrt{2^2 + 6^2 + (-3)^2} \times \sqrt{(-3)^2 + (-2)^2 + (-6)^2}}$$

$$\cos B = \frac{-6 - 12 + 18}{\sqrt{49} \times \sqrt{49}}$$

$$\cos B = \frac{0}{49}$$

$$\cos B = 0$$

$$B = \cos^{-1}(0)$$

$$B = \frac{\pi}{2}$$

Now the sum of angles of a triangle is π

$$\therefore A + B + C = \pi$$

$$\frac{\pi}{4} + \frac{\pi}{2} + C = \pi$$

$$\therefore C = \pi - \frac{3\pi}{4}$$

$$\therefore C = \frac{\pi}{4}$$

22. Question

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $1/2$.

Answer

Given Data:

$$|\vec{a}| = |\vec{b}|$$

$$\vec{a} \cdot \vec{b} = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

$$|\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a} \cdot \vec{b}$$

$$|\vec{a}| \times |\vec{a}| \times \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$|\vec{a}|^2 \times \frac{1}{2} = \frac{1}{2}$$

$$|\vec{a}|^2 = \frac{1 \times 2}{2}$$

$$|\vec{a}|^2 = 1$$

$$\therefore |\vec{a}| = |\vec{b}| = 1$$

Magnitude of vectors is unity.

23. Question

Show that the points whose position vectors are $\vec{a} = 4\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$, $\vec{c} = \hat{i} - \hat{j}$ form a right triangle.

Answer

Given Data:

$$\vec{a} = 4\hat{i} - 3\hat{j} + \hat{k}$$

$$\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$$

$$\vec{c} = \hat{i} - \hat{j}$$

$$\vec{AB} = \vec{B} - \vec{A}$$

$$= (2\hat{i} - 4\hat{j} + 5\hat{k}) - (4\hat{i} - 3\hat{j} + \hat{k})$$

$$= -2\hat{i} - \hat{j} + 4\hat{k}$$

$$\vec{BC} = \vec{C} - \vec{B}$$

$$= (\hat{i} - \hat{j}) - (2\hat{i} - 4\hat{j} + 5\hat{k})$$

$$= -\hat{i} + 3\hat{j} - 5\hat{k}$$

$$\vec{CA} = \vec{A} - \vec{C}$$

$$= (4\hat{i} - 3\hat{j} + \hat{k}) - (\hat{i} - \hat{j})$$

$$= 3\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec{AB} \cdot \vec{CA} = (-2\hat{i} - \hat{j} + 4\hat{k}) \cdot (3\hat{i} - 2\hat{j} + \hat{k})$$

$$= -2 \times 3 + (-1) \times (-2) + 4 \times 1$$

$$= -6 + 2 + 4$$

$$\vec{AB} \cdot \vec{CA} = 0$$

$$\overrightarrow{AB} \perp \overrightarrow{CA}$$

Angle A right angle, ABC is right angle triangle.

24. Question

If the vertices A, B, C of ΔABC have position vectors $(1, 2, 3), (-1, 0, 0), (0, 1, 2)$ respectively, what is the magnitude of $\angle ABC$?

Answer

Given Data:

$$\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{B} = -\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\vec{C} = 0\hat{i} + \hat{j} + 2\hat{k}$$

$$\overrightarrow{AB} = \vec{B} - \vec{A}$$

$$= (-\hat{i} + 0\hat{j} + 0\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= -2\hat{i} - 2\hat{j} - 3\hat{k}$$

$$\overrightarrow{BC} = \vec{C} - \vec{B}$$

$$= (0\hat{i} + \hat{j} + 2\hat{k}) - (-\hat{i} + 0\hat{j} + 0\hat{k})$$

$$= \hat{i} + \hat{j} + 2\hat{k}$$

$$\overrightarrow{AC} = \vec{C} - \vec{A}$$

$$= (0\hat{i} + \hat{j} + 2\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= -\hat{i} - \hat{j} - \hat{k}$$

Now the angle B

$$\cos B = \frac{\overrightarrow{BC} \cdot \overrightarrow{BA}}{|\overrightarrow{BC}| \times |\overrightarrow{BA}|}$$

$$\cos B = \frac{(\hat{i} + \hat{j} + 2\hat{k})(-2\hat{i} - 2\hat{j} - 3\hat{k})}{\sqrt{1^2 + 1^2 + (2)^2} \times \sqrt{2^2 + 2^2 + 3^2}}$$

$$\cos B = \frac{2 + 2 + 6}{\sqrt{6} \times \sqrt{17}}$$

$$\cos B = \frac{10}{\sqrt{102}}$$

$$B = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

25. Question

If A, B, C have position vectors $(0, 1, 1), (3, 1, 5), (0, 3, 3)$ respectively, show that is right angled at C.

Answer

Given Data:

$$\vec{A} = 0\hat{i} + \hat{j} + \hat{k}$$

$$\vec{B} = 3\hat{i} + \hat{j} + 5\hat{k}$$

$$\vec{C} = 0\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\overrightarrow{AB} = \vec{B} - \vec{A}$$

$$= (3\hat{i} + 1\hat{j} + 5\hat{k}) - (0\hat{i} + \hat{j} + \hat{k})$$

$$= 3\hat{i} + 4\hat{k}$$

$$\overrightarrow{BC} = \vec{C} - \vec{B}$$

$$= (0\hat{i} + 3\hat{j} + 3\hat{k}) - (3\hat{i} + 1\hat{j} + 5\hat{k})$$

$$= -3\hat{i} + 2\hat{j} - 2\hat{k}$$

$$\overrightarrow{AC} = \vec{C} - \vec{A}$$

$$= (0\hat{i} + 3\hat{j} + 3\hat{k}) - (0\hat{i} + \hat{j} + \hat{k})$$

$$= 2\hat{j} + 2\hat{k}$$

Now the angle C

$$\cos C = \frac{\overrightarrow{BC} \cdot \overrightarrow{AC}}{|\overrightarrow{BC}| \times |\overrightarrow{AC}|}$$

$$\cos C = \frac{(-3\hat{i} + 2\hat{j} - 2\hat{k}) \cdot (2\hat{j} + 2\hat{k})}{\sqrt{(-3)^2 + 2^2 + (-2)^2} \times \sqrt{2^2 + 2^2}}$$

$$\cos C = \frac{0 + 4 - 4}{\sqrt{6} \times \sqrt{17}}$$

$$\cos C = 0$$

$$C = \frac{\pi}{2}$$

So angle C is a right angle triangle.

26. Question

Find the projection of $\vec{b} + \vec{c}$ on $\vec{a}$, where $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

Answer

we know that $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos x$ where x is the angle between two vectors, so $\frac{(\vec{a} \cdot \vec{b})}{|\vec{a}|}$ gives the projection of vector b on a

Now applying the formula for projection of $\vec{b} + \vec{c}$ on $\vec{a}$

$$\vec{b} + \vec{c} = \hat{i} + 2\hat{j} - 2\hat{k} + 2\hat{i} - \hat{j} + 4\hat{k}$$

$$\vec{b} + \vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$$

$$|\vec{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = (2\hat{i} - 2\hat{j} + \hat{k}) \cdot (3\hat{i} + \hat{j} + 2\hat{k})$$

$$\hat{i} \cdot \hat{i} = 1; \hat{j} \cdot \hat{j} = 1; \hat{k} \cdot \hat{k} = 1$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = 6 - 2 + 2 = 6$$

Substituting these values in above formula, we get

$$\frac{[\vec{a} \cdot (\vec{b} + \vec{c})]}{|\vec{a}|} = \frac{6}{3} = 2$$

27. Question

If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are orthogonal.

Answer

meaning of orthogonal is that two vectors are perpendicular to each other, so their dot product is zero.

$$\vec{a} + \vec{b} = (5\hat{i} - \hat{j} - 3\hat{k}) + (\hat{i} + 3\hat{j} - 5\hat{k})$$

$$\vec{a} + \vec{b} = 6\hat{i} + 2\hat{j} - 8\hat{k}$$

Similarly,

$$\vec{a} - \vec{b} = (5\hat{i} - \hat{j} - 3\hat{k}) - (\hat{i} + 3\hat{j} - 5\hat{k})$$

$$\vec{a} - \vec{b} = 4\hat{i} - 4\hat{j} + 2\hat{k}$$

So, to satisfy the orthogonal condition $\vec{a} \cdot \vec{b} = 0$

$$\hat{i} \cdot \hat{i} = 1; \hat{j} \cdot \hat{j} = 1; \hat{k} \cdot \hat{k} = 1$$

$$(6\hat{i} + 2\hat{j} - 8\hat{k}) \cdot (4\hat{i} - 4\hat{j} + 2\hat{k}) = (24 - 8 - 16) = 0$$

Hence proved

28. Question

A unit vector $\vec{a}$ makes angles $\frac{\pi}{4}$ and $\frac{\pi}{3}$ with $\hat{i}$ and $\hat{j}$ respectively and an acute angle θ with $\hat{k}$. Find the angle θ and components of $\vec{a}$.

Answer

Assume, $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

Using formula: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos x$

$$|\vec{a}| = 1$$

since it is a unit vector

First taking dot product with $\hat{i}$

$$\vec{a} \cdot \hat{i} = |\vec{a}||\hat{i}|\cos x$$

$$x = \cos\left(\frac{\pi}{4}\right)$$

$$x = \frac{1}{\sqrt{2}}$$

Taking dot product with $\hat{j}$

$$\vec{a} \cdot \hat{j} = |\vec{a}||\hat{j}|\cos x$$

$$y = \cos\left(\frac{\pi}{3}\right)$$

$$y = \frac{1}{2}$$

Now we have $\vec{a}$ as $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + z\hat{k}$

Since the magnitude of $\vec{a}$ is 1

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + z^2 = 1$$

$$z^2 = 1 - \frac{1}{2} - \frac{1}{4}$$

$$z^2 = \frac{1}{4}$$

$$z = \frac{1}{2} \text{ or } z = -\frac{1}{2}$$

Considering, $z = \frac{1}{2}$

$$\vec{a} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$$

Therefore, angle with $\hat{k}$ is

$$\vec{a} \cdot \hat{k} = |\vec{a}| |\hat{k}| \cos x$$

$$\frac{1}{2} = \cos x$$

$$x = \frac{\pi}{3}$$

29. Question

If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = 1$, then find the value of $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$.

Answer

Expanding the given equation $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$, we get,

$$6|\vec{a}|^2 + 21(\vec{a} \cdot \vec{b}) - 10(\vec{a} \cdot \vec{b}) - 35|\vec{b}|^2$$

$$6(2)^2 + 11(1) - 35(1)^2$$

$$24 + 11 - 35 = 0$$

$$\text{Hence, } (3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b}) = 0.$$

30. Question

If $\vec{a}$ is a unit vector, then find $|\vec{x}|$ in each of the following

$$(i) (\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$$

$$(ii) (\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$$

Answer

(i) Expanding the given equation

$$|\vec{x}|^2 - |\vec{a}|^2 = 8$$

$$|\vec{a}| = 1 \text{ as given}$$

$$|\vec{x}|^2 = 9$$

$$|\vec{x}| = 3 \text{ or } |\vec{x}| = -3$$

(ii) expanding the given equation

$$|\vec{x}|^2 - |\vec{a}|^2 = 12$$

$$|\vec{a}| = 1 \text{ as given}$$

$$|\vec{x}|^2 = 13$$

$$|\vec{x}| = \sqrt{13} \text{ or } |\vec{x}| = -\sqrt{13}$$

31. Question

Find $|\vec{a}|$ and $|\vec{b}|$, if

$$(i) (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 12 \text{ and } |\vec{a}| = 2|\vec{b}|$$

$$(ii) (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8 \text{ and } |\vec{a}| = 8|\vec{b}|$$

$$(iii) (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 3 \text{ and } |\vec{a}| = 2|\vec{b}|$$

Answer

(i) expanding the given equation

$$|\vec{a}|^2 - |\vec{b}|^2 = 12$$

$$\text{Substituting } |\vec{a}| = 2|\vec{b}|$$

$$4|\vec{b}|^2 - |\vec{b}|^2 = 12$$

$$3|\vec{b}|^2 = 12$$

$$|\vec{b}| = 2 \text{ or } -2$$

$$|\vec{a}| = 4 \text{ or } -4$$

(ii) expanding the given equation

$$|\vec{a}|^2 - |\vec{b}|^2 = 8$$

$$\text{Substituting, } |\vec{a}| = 8|\vec{b}|$$

$$64|\vec{b}|^2 - |\vec{b}|^2 = 8$$

$$63|\vec{b}|^2 = 8$$

$$|\vec{b}|^2 = \frac{8}{63}$$

$$|\vec{b}| = \frac{2\sqrt{2}}{3\sqrt{7}} \text{ or } -\frac{2\sqrt{2}}{3\sqrt{7}}$$

$$|\vec{a}| = \frac{16\sqrt{2}}{3\sqrt{7}} \text{ or } -\frac{16\sqrt{2}}{3\sqrt{7}}$$

(iii) expanding the given equation

$$|\vec{a}|^2 - |\vec{b}|^2 = 3$$

Substituting, $|\vec{a}| = 2|\vec{b}|$

$$4|\vec{b}|^2 - |\vec{b}|^2 = 3$$

$$3|\vec{b}|^2 = 3$$

$$|\vec{b}| = 1 \text{ or } -1$$

$$|\vec{a}| = 2 \text{ or } -2$$

32. Question

Find $|\vec{a} - \vec{b}|$, if

(i) $|\vec{a}| = 2, |\vec{b}| = 5$ and $\vec{a} \cdot \vec{b} = 8$

(ii) $|\vec{a}| = 3, |\vec{b}| = 4$ and $\vec{a} \cdot \vec{b} = 1$

(iii) $|\vec{a}| = \sqrt{3}, |\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 4$

Answer

(i) using formula,

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a} \cdot \vec{b})}$$

Substituting the given values in above equation we get,

$$|\vec{a} - \vec{b}| = \sqrt{2^2 + 5^2 - 2(8)}$$

$$|\vec{a} - \vec{b}| = \sqrt{13}$$

(ii) using formula,

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a} \cdot \vec{b})}$$

Substituting the given values in above equation we get,

$$|\vec{a} - \vec{b}| = \sqrt{3^2 + 4^2 - 2(1)}$$

$$|\vec{a} - \vec{b}| = \sqrt{23}$$

(iii) using formula,

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a} \cdot \vec{b})}$$

Substituting the given values in above equation we get,

$$|\vec{a} - \vec{b}| = \sqrt{\sqrt{3}^2 + 2^2 - 2(4)}$$

$$|\vec{a} - \vec{b}| = \sqrt{-1}$$

Now this will yield imaginary value.

We know that, $\sqrt{-1} = i$ (iota)

Therefore, $|\vec{a} - \vec{b}| = i$

33. Question

Find the angle between two vectors $\vec{a}$ and $\vec{b}$, if

(i) $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{6}$

(ii) $|\vec{a}| = 3$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 1$

Answer

(i) we know that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos x$ where x is the angle between two vectors

$$\cos x = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos x = \frac{(\sqrt{6})}{2\sqrt{3}}$$

$$\cos x = \frac{1}{\sqrt{2}}$$

$$x = 45^\circ$$

(ii) we know that,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos x$$

Where, x is the angle between two vectors.

$$\cos x = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos x = \frac{(1)}{3 \times 3}$$

$$\cos x = \frac{1}{9}$$

$$x = \cos^{-1}\left(\frac{1}{9}\right)$$

34. Question

Express the vector $\vec{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as the sum of two vectors such that one is parallel to the vector $\vec{b} = 3\hat{i} + \hat{k}$ and other is perpendicular to $\vec{b}$.

Answer

let $\vec{a} = \vec{u} + \vec{v}$ where u is vector parallel to b and v is vector perpendicular to b, as given in the question.

$$5\hat{i} - 2\hat{j} + 5\hat{k} = \vec{u} + \vec{v}$$

So, $\vec{u} = p\vec{b}$; where p is some constant

$$\vec{u} = 3p\hat{i} + p\hat{k}$$

Substituting this value in above equation

$$\vec{v} = (5 - 3p)\hat{i} - 2\hat{j} + (5 - p)\hat{k}$$

Now according to conditions since vector v and b are perpendicular to each other $\vec{v} \cdot \vec{b} = 0$

$$\hat{i}.\hat{i} = 1; \hat{j}.\hat{j} = 1; \hat{k}.\hat{k} = 1$$

$$(5 - 3p)(3) + (5 - p) = 0$$

$$15 - 9p + 5 - p = 0$$

$$20 = 10p$$

$$p = 2$$

$$\text{So, } \vec{u} = 6\hat{i} + 2\hat{k}$$

substituting this value in above equation, we will get $\vec{v}$

$$\vec{v} = (5\hat{i} - 2\hat{j} + 5\hat{k}) - (6\hat{i} + 2\hat{k})$$

$$\vec{v} = -\hat{i} - 2\hat{j} + 3\hat{k}$$

35. Question

If $\vec{a}$ and $\vec{b}$ are two vectors of the same magnitude inclined at an angle of 30° such that $\vec{a}.\vec{b} = 3$, find $|\vec{a}|, |\vec{b}|$.

Answer

$$\text{Let } |\vec{a}| = |\vec{b}| = x$$

The angle between these vectors is 30°

So, applying the formula,

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos x$$

$$3 = x^2 \cos 30$$

$$x^2 = \frac{6}{\sqrt{3}}$$

$$\text{So, the magnitude of } |\vec{a}| = |\vec{b}| = \frac{6}{\sqrt{3}}$$

36. Question

Express $2\hat{i} - \hat{j} + 3\hat{k}$ as the sum of a vector parallel and a vector perpendicular to $2\hat{i} + 4\hat{j} - 2\hat{k}$.

Answer

$$\text{Let } \vec{a} = 2\hat{i} - \hat{j} + 3\hat{k} \text{ and } \vec{b} = 2\hat{i} + 4\hat{j} - 2\hat{k}$$

let $\vec{a} = \vec{u} + \vec{v}$ where u is vector parallel to b and v is vector perpendicular to b .

$$2\hat{i} - \hat{j} + 3\hat{k} = \vec{u} + \vec{v}$$

So, $\vec{u} = p\vec{b}$; where p is some constant

$$\vec{u} = p(2\hat{i} + 4\hat{j} - 2\hat{k})$$

Substituting this value in above equation

$$\vec{v} = (2 - 2p)\hat{i} + (-1 - 4p)\hat{j} + (3 + 2p)\hat{k}$$

Now according to conditions since vector v and b are perpendicular to each other $\vec{v}.\vec{b} = 0$

$$\hat{i}.\hat{i} = 1; \hat{j}.\hat{j} = 1; \hat{k}.\hat{k} = 1$$

$$2(2 - 2p) - 4(1 + 4p) - 2(3 + 2p) = 0$$

$$4 - 4p - 4 - 16p - 6 - 4p = 0$$

$$-24p = 6$$

$$p = -\frac{1}{4}$$

$$\vec{u} = -\left(\frac{1}{2}\hat{i} + \hat{j} - \frac{1}{2}\hat{k}\right)$$

Substituting this value of u vector in above equation

$$2\hat{i} - \hat{j} + 3\hat{k} = \left(-\frac{1}{2}\hat{i} - \hat{j} + \frac{1}{2}\hat{k}\right) + \vec{v}$$

$$\vec{v} = \frac{5}{2}\hat{i} + \frac{5}{2}\hat{k}$$

$$2\hat{i} - \hat{j} + 3\hat{k} = \left(-\frac{1}{2}\hat{i} - \hat{j} + \frac{1}{2}\hat{k}\right) + \left(\frac{5}{2}\hat{i} + \frac{5}{2}\hat{k}\right)$$

37. Question

Decompose the vector $6\hat{i} - 3\hat{j} - 6\hat{k}$ into vectors which are parallel and perpendicular to the vector $\hat{i} + \hat{j} + \hat{k}$.

Answer

$$\text{let } \vec{a} = 6\hat{i} - 3\hat{j} - 6\hat{k} \text{ and } \vec{b} = \hat{i} + \hat{j} + \hat{k}$$

let $\vec{a} = \vec{u} + \vec{v}$ where u is vector parallel to b and v is vector perpendicular to b

$$6\hat{i} - 3\hat{j} - 6\hat{k} = \vec{u} + \vec{v}$$

So, $\vec{u} = p\vec{b}$; where p is some constant

$$\vec{u} = p(\hat{i} + \hat{j} + \hat{k})$$

Substituting this value in above equation

$$\vec{v} = (6-p)\hat{i} + (-3-p)\hat{j} + (-6-p)\hat{k}$$

Now according to conditions since vector v and b are perpendicular to each other $\vec{v} \cdot \vec{b} = 0$

$$\hat{i} \cdot \hat{i} = 1; \hat{j} \cdot \hat{j} = 1; \hat{k} \cdot \hat{k} = 1$$

$$6 - p - 3 - p - 6 - p = 0$$

$$p = -1$$

$$\text{So, } \vec{u} = -(\hat{i} + \hat{j} + \hat{k})$$

Substituting this value of $\vec{u}$ in above equation

$$6\hat{i} - 3\hat{j} - 6\hat{k} = -(\hat{i} + \hat{j} + \hat{k}) + \vec{v}$$

$$\vec{v} = 7\hat{i} - 2\hat{j} - 5\hat{k}$$

$$6\hat{i} - 3\hat{j} - 6\hat{k} = -(\hat{i} + \hat{j} + \hat{k}) + 7\hat{i} - 2\hat{j} - 5\hat{k}$$

38. Question

Let $\vec{a} = 5\hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \lambda\hat{k}$. Find such that $\vec{a} + \vec{b}$ is orthogonal to $\vec{a} - \vec{b}$.

Answer

Meaning of orthogonal is that two vectors are perpendicular to each other, so their dot product is zero.

$$\vec{a} + \vec{b} = (5\hat{i} - \hat{j} + 7\hat{k}) + (\hat{i} - \hat{j} + \beta\hat{k})$$

$$\vec{a} + \vec{b} = 6\hat{i} - 2\hat{j} + (7 + \beta)\hat{k}$$

Similarly

$$\vec{a} - \vec{b} = (5\hat{i} - \hat{j} + 7\hat{k}) - (\hat{i} - \hat{j} + \beta\hat{k})$$

$$\vec{a} - \vec{b} = 4\hat{i} + (7 - \beta)\hat{k}$$

So, to satisfy the orthogonal condition $\vec{a} \cdot \vec{b} = 0$

$$\hat{i} \cdot \hat{i} = 1; \hat{j} \cdot \hat{j} = 1; \hat{k} \cdot \hat{k} = 1$$

$$[6\hat{i} - 2\hat{j} + (7 + \beta)\hat{k}] \cdot [4\hat{i} + (7 - \beta)\hat{k}] = 24 + 49 - \beta^2 = 0$$

$$\beta = \sqrt{73}$$

39. Question

If $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$, what can you conclude about the vector $\vec{b}$?

Answer

it is given that $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$

From this, we can say that $|\vec{a}|^2 = 0$

So $\vec{a}$ is a zero vector

And from the second part $\vec{a} \cdot \vec{b} = 0$ we can say that $\vec{b}$ can be any vector perpendicular to zero vector $\vec{a}$.

40. Question

If $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, then prove that it is perpendicular to both $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

Answer

It is given that $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

So, $\vec{c} \cdot \vec{a} = 0$ and $\vec{c} \cdot \vec{b} = 0$

For $\vec{c}$ to be perpendicular to $(\vec{a} + \vec{b})$, $\vec{c} \cdot (\vec{a} + \vec{b}) = 0$

$$\vec{c} \cdot (\vec{a} + \vec{b})$$

$$\vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} = 0$$

For the second part.

For $\vec{c}$ to be perpendicular to $(\vec{a} - \vec{b})$, $\vec{c} \cdot (\vec{a} - \vec{b}) = 0$

$$\vec{c} \cdot (\vec{a} - \vec{b})$$

$$\vec{c} \cdot \vec{a} - \vec{c} \cdot \vec{b} = 0$$

Hence, proved

41. Question

If $|\vec{a}| = a$ and $|\vec{b}| = b$, prove that $\left(\frac{\vec{a}}{a^2} - \frac{\vec{b}}{b^2}\right)^2 = \left(\frac{\vec{a} - \vec{b}}{ab}\right)^2$.

Answer

we know that $|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a} \cdot \vec{b})}$

Now expanding LHS of given equation we get,

$$= \left[\frac{a^2}{a^4} + \frac{b^2}{b^4} - \frac{2\vec{a}\vec{b}}{a^2b^2} \right]$$

$$= \left[\frac{1}{a^2} + \frac{1}{b^2} - \frac{2\vec{a}\vec{b}}{a^2b^2} \right]$$

Taking LCM we get,

$$= \left[\frac{b^2 + a^2 - 2\vec{a}\vec{b}}{a^2b^2} \right]$$

Using $|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a} \cdot \vec{b})}$ re-writing the above equation

$$\left[\frac{(\vec{a} - \vec{b})^2}{ab} \right]$$

Hence, proved.

42. Question

If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $\vec{d} \cdot \vec{a} = \vec{d} \cdot \vec{b} = \vec{d} \cdot \vec{c} = 0$, then show that $\vec{d}$ is the null vector

Answer

Given that $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar and $\vec{a} \cdot \vec{d} = 0, \vec{b} \cdot \vec{d} = 0$ and $\vec{c} \cdot \vec{d} = 0$

From above given conditions we can say that either

(i) $\vec{d} = 0$ or

(ii) $\vec{d}$ is perpendicular to $\vec{a}, \vec{b}$ and $\vec{c}$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar, $\vec{d}$ cannot be simultaneously perpendicular to all three, as only three axes exist that is x, y, z

So $\vec{d}$ must be a null vector which is equal to 0

43. Question

If a vector $\vec{a}$ is perpendicular to two non-collinear vectors $\vec{b}$ and $\vec{c}$, then $\vec{a}$ is perpendicular to every vector in the plane of $\vec{b}$ and $\vec{c}$.

Answer

Given $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, so $\vec{c} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$

Let a random vector $\vec{r} = p\vec{b} + k\vec{c}$ in the plane of $\vec{b}$ and $\vec{c}$ where p and k are some arbitrary constant

Taking dot product of $\vec{r}$ with $\vec{a}$

$$\vec{r} \cdot \vec{a} = (p\vec{b} + k\vec{c}) \cdot \vec{a}$$

$$\vec{r} \cdot \vec{a} = (p\vec{b} \cdot \vec{a} + k\vec{c} \cdot \vec{a})$$

Using $\vec{c} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$

$$\vec{r} \cdot \vec{a} = 0$$

Hence, proved.....

44. Question

If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, show that the angle between the vectors $\vec{b}$ and $\vec{c}$ is given by $\cos \theta = \frac{|\vec{a}|^2 - |\vec{b}|^2 - |\vec{c}|^2}{2|\vec{b}||\vec{c}|}$.

Answer

$$\text{Given } \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$-\vec{a} = \vec{b} + \vec{c}$$

Now squaring both sides, using,

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a} \cdot \vec{b})} \text{ we get,}$$

$$|\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}| \cos x$$

$$\frac{[|\vec{a}|^2 - |\vec{b}|^2 - |\vec{c}|^2]}{2|\vec{b}||\vec{c}|} = \cos x$$

Hence, proved.

45. Question

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vector such $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. If $|\vec{u}| = 3$, $|\vec{v}| = 4$ and $|\vec{w}| = 5$, then find $\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}$.

Answer

$$\text{Given } \vec{u} + \vec{v} + \vec{w} = \vec{0}$$

Now squaring both sides using:

$$(\vec{u} + \vec{v} + \vec{w})^2 = |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 + 2\vec{u} \cdot \vec{v} + 2\vec{w} \cdot \vec{v} + 2\vec{w} \cdot \vec{u}$$

$$0 = 3^2 + 4^2 + 5^2 + 2\vec{u} \cdot \vec{v} + 2\vec{w} \cdot \vec{v} + 2\vec{w} \cdot \vec{u}$$

$$2\vec{u} \cdot \vec{v} + 2\vec{w} \cdot \vec{v} + 2\vec{w} \cdot \vec{u} = -50$$

$$\vec{u} \cdot \vec{v} + \vec{w} \cdot \vec{v} + \vec{w} \cdot \vec{u} = -25$$

46. Question

Let $\vec{a} = x^2\hat{i} + 2\hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = x^2\hat{i} + 5\hat{j} - 4\hat{k}$ be three vectors. Find the values of x for which the angle between $\vec{a}$ and $\vec{b}$ is acute and the angle between $\vec{a}$ and $\vec{b}$ is obtuse

Answer

We know that,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos x$$

Where, x is the angle between two vectors

Applying for $\vec{a}$ and $\vec{b}$

$$(x^2\hat{i} + 2\hat{j} - 2\hat{k}) \cdot (\hat{i} - \hat{j} + \hat{k}) = \sqrt{x^4 + 4 + 4} \sqrt{1 + 1 + 1} \cos x$$

$$\frac{[x^2 - 2 - 2]}{\sqrt{x^4 + 4 + 4} \sqrt{1 + 1 + 1}} = \cos x$$

$$\frac{x^2 - 4}{\sqrt{x^4 + 8} \sqrt{3}} = \cos x$$

Since angle between $\vec{a}$ and $\vec{b}$ is acute $\cos x$ should be greater than 0

$$\frac{x^2 - 4}{\sqrt{x^4 + 8} \sqrt{3}} > 0$$

$$x^2 - 4 > 0$$

$$x > 2 \text{ and } x < -2$$

applying for $\vec{b}$ and $\vec{c}$

$$(x^2\hat{i} + 5\hat{j} - 4\hat{k}) \cdot (\hat{i} - \hat{j} + \hat{k}) = \sqrt{x^4 + 25 + 16} \sqrt{1 + 1 + 1} \cos x$$

$$\frac{[x^2 - 9]}{\sqrt{x^4 + 25 + 16} \sqrt{1 + 1 + 1}} = \cos x$$

Since angle between $\vec{c}$ and $\vec{b}$ is obtuse $\cos x$ should be less than

0

$$\frac{[x^2 - 9]}{\sqrt{x^4 + 41} \sqrt{3}} < 0$$

$$x^2 - 9 < 0$$

$$x > -3 \text{ and } x < 3$$

47. Question

Find the values of x and y if the vectors $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ are mutually perpendicular vectors of equal magnitude.

Answer

given $\vec{a}$ is perpendicular to $\vec{b}$ so $\vec{b} \cdot \vec{a} = 0$

$$\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$$

$$\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$$

Applying, $\vec{b} \cdot \vec{a} = 0$

$$6 + x - y = 0$$

$$x - y = -6 \dots (i)$$

Since the magnitude of both vectors are equal

$$\sqrt{3^2 + x^2 + 1^2} = \sqrt{2^2 + 1^2 + y^2}$$

$$\sqrt{10 + x^2} = \sqrt{5 + y^2}$$

$$y^2 - x^2 = 5$$

$$(y-x)(y+x) = 5$$

$$6x + 6y = 5 \dots (ii)$$

Solving equation (i) and (ii) we get

$$x = -\frac{31}{12}; y = \frac{41}{12}$$

48. Question

If $\vec{a}$ and $\vec{b}$ are two non-collinear unit vectors such that $|\vec{a} + \vec{b}| = \sqrt{3}$, find $(2\vec{a} - 5\vec{b}) \cdot (3\vec{a} + \vec{b})$.

Answer

Given $|\vec{a}| = |\vec{b}| = 1$ and $|\vec{a} + \vec{b}| = \sqrt{3}$

$$|\vec{a} + \vec{b}| = \sqrt{3}$$

Squaring both sides

$$|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} = 3$$

$$1 + 1 + 2\vec{a} \cdot \vec{b} = 3$$

$$2\vec{a} \cdot \vec{b} = 1$$

$$\vec{a} \cdot \vec{b} = \frac{1}{2}$$

Now expanding the equation $(2\vec{a} - 5\vec{b}) \cdot (3\vec{a} + \vec{b})$

$$6|\vec{a}|^2 - 5|\vec{b}|^2 - 13\vec{a} \cdot \vec{b}$$

$$1 - \frac{13}{2} = -\frac{11}{2}$$

49. Question

If $\vec{a}, \vec{b}$ are two vectors such that $|\vec{a} + \vec{b}| = |\vec{b}|$, then prove that $\vec{a} = 2\vec{b}$ is perpendicular to $\vec{a}$.

Answer

Given $|\vec{a} + \vec{b}| = |\vec{b}|$

Squaring both sides we get,

$$|\vec{a} + \vec{b}|^2 = |\vec{b}|^2$$

$$|\vec{a} + \vec{b}| \cdot |\vec{a} + \vec{b}| = |\vec{b}| \cdot |\vec{b}|$$

$$\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + 2\vec{a} \cdot \vec{b} = |\vec{b}| \cdot |\vec{b}|$$

$$\vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} = 0$$

$$\vec{a} \cdot (\vec{a} + 2\vec{b}) = 0$$

Hence, proved.

Exercise 24.2

1. Question

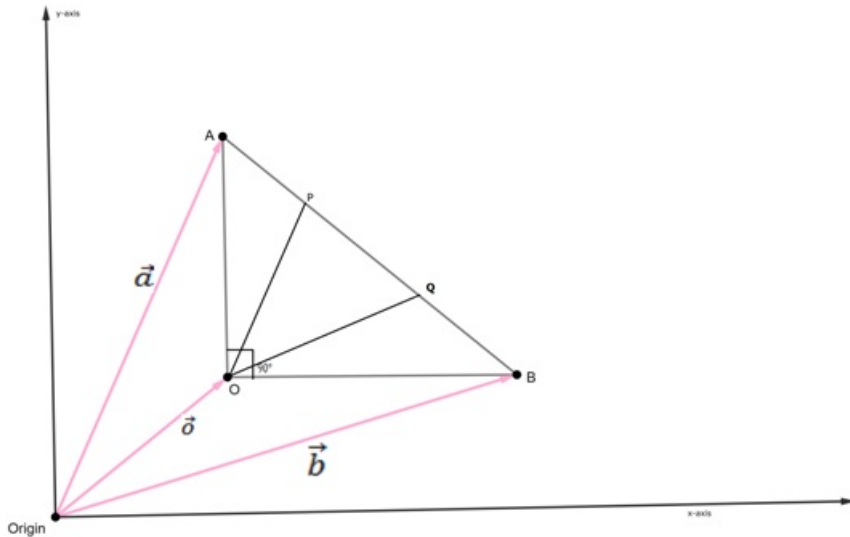
In a triangle ΔOAB , $\angle AOB = 90^\circ$. If P and Q are points of trisection of AB, prove that $OP^2 + OQ^2 = \frac{5}{9} AB^2$

Answer

Given:- $\angle AOB = 90^\circ$, P and Q are trisection of AB

i.e. $AP = PQ = QB$ or 1:1:1 division of line AB

To Prove:- $OP^2 + OQ^2 = \frac{5}{9}AB^2$



Proof:- Let $\vec{o}, \vec{a},$ and $\vec{b}$ be position vector of O, A and B respectively

Now, Find position vector of P, we use section formulae of internal division: Theorem given below

“Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$

respectively, and c be a point dividing AB internally in the ration m:n. Then the position vector of c is given by $\vec{OC} = \frac{m\vec{b} + n\vec{a}}{m + n}$

By above theorem, here P point divides AB in 1:2, so we get

$$\Rightarrow \text{Position vector of P} = \frac{\vec{b} + 2\vec{a}}{1 + 2}$$

$$\Rightarrow \text{Position vector of P} = \frac{2\vec{a} + \vec{b}}{3}$$

Similarly, Position vector of Q is calculated

By above theorem, here Q point divides AB in 2:1, so we get

$$\Rightarrow \text{Position vector of Q} = \frac{2\vec{b} + \vec{a}}{2 + 1}$$

$$\Rightarrow \text{Position vector of Q} = \frac{\vec{a} + 2\vec{b}}{3}$$

Length OA and OB in vector form

$$\Rightarrow \vec{OA} = \text{Position vector of A} - \text{Position vector of O}$$

$$\Rightarrow \vec{OA} = \vec{a} - \vec{o}$$

$$\Rightarrow \vec{OB} = \text{Position vector of B} - \text{Position vector of O}$$

$$\Rightarrow \vec{OB} = \vec{b} - \vec{o}$$

Now length/distance OP in vector form

$$\vec{OP} = \text{Position vector of P} - \text{Position vector of O}$$

$$\Rightarrow \vec{OP} = \frac{2\vec{a} + \vec{b}}{3} - \vec{o}$$

$$\Rightarrow \vec{OP} = \frac{2\vec{a} + \vec{b} - 3\vec{o}}{3}$$

$$\Rightarrow \vec{OP} = \frac{2\vec{a} + \vec{b} - 2\vec{d} - \vec{d}}{3}$$

$$\Rightarrow \vec{OP} = \frac{2\vec{a} - 2\vec{d} + \vec{b} - \vec{d}}{3}$$

Putting $\vec{OA}$ and $\vec{OB}$ values

$$\Rightarrow \vec{OP} = \frac{2\vec{OA} + \vec{OB}}{3}$$

length/distance OQ in vector form

$\vec{OQ}$ = Position vector of Q – Position vector of O

$$\Rightarrow \vec{OQ} = \frac{\vec{a} + 2\vec{b}}{3} - \vec{d}$$

$$\Rightarrow \vec{OQ} = \frac{\vec{a} + 2\vec{b} - 3\vec{d}}{3}$$

$$\Rightarrow \vec{OQ} = \frac{\vec{a} + 2\vec{b} - 2\vec{d} - \vec{d}}{3}$$

$$\Rightarrow \vec{OQ} = \frac{\vec{a} - \vec{d} + 2\vec{b} - 2\vec{d}}{3}$$

Putting $\vec{OA}$ and $\vec{OB}$ values

$$\Rightarrow \vec{OQ} = \frac{\vec{OA} + 2\vec{OB}}{3}$$

Taking LHS

$$OP^2 + OQ^2$$

$$= \left(\frac{2\vec{OA} + \vec{OB}}{3} \right)^2 + \left(\frac{\vec{OA} + 2\vec{OB}}{3} \right)^2$$

$$= \frac{4(\vec{OA})^2 + (\vec{OB})^2 + 4(\vec{OA}) \cdot (\vec{OB}) + (\vec{OA})^2 + 4(\vec{OB})^2 + 4(\vec{OA}) \cdot (\vec{OB})}{9}$$

as we know in case of dot product

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

Angle between OA and OB is 90° ,

$$\Rightarrow \vec{OA} \cdot \vec{OB} = |\vec{OA}| |\vec{OB}| \cos 90^\circ$$

$$\Rightarrow \vec{OA} \cdot \vec{OB} = 0$$

Therefore, $OP^2 + OQ^2$

$$= \frac{4(\vec{OA})^2 + (\vec{OB})^2 + 0 + (\vec{OA})^2 + 4(\vec{OB})^2 + 0}{9}$$

$$= \frac{4(\vec{OA})^2 + (\vec{OB})^2 + (\vec{OA})^2 + 4(\vec{OB})^2}{9}$$

$$= \frac{5(\vec{OA})^2 + 5(\vec{OB})^2}{9}$$

$$= \frac{5(OA^2 + OB^2)}{9}$$

As from figure $OA^2 + OB^2 = AB^2$

$$= \frac{5(AB)^2}{9}$$

= RHS

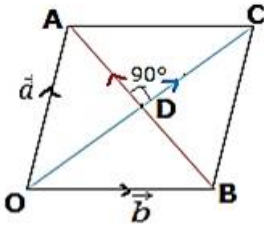
Hence, Proved.

2. Question

Prove that: If the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

Answer

Given:- Quadrilateral OACB with diagonals bisect each other at 90° .



Proof:- It is given diagonal of a quadrilateral bisect each other

Therefore, by property of parallelogram (i.e. diagonal bisect each other) this quadrilateral must be a parallelogram.

Now as Quadrilateral OACB is parallelogram, its opposite sides must be equal and parallel.

$\Rightarrow OA = BC$ and $AC = OB$

Let, O is at origin.

$\vec{a}$ and $\vec{b}$ are position vector of A and B

Therefore from figure, by parallelogram law of vector addition

$$\vec{OC} = \vec{a} + \vec{b}$$

And, by triangular law of vector addition

$$\vec{AB} = \vec{a} - \vec{b}$$

As given diagonal bisect each other at 90°

Therefore AB and OC make 90° at their bisecting point D

$$\Rightarrow \angle ADC = \angle CDB = \angle BDO = \angle ODA = 90^\circ$$

Or, their dot product is zero

$$\Rightarrow (\vec{OC}) \cdot (\vec{AB}) = 0$$

$$\Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

$$\Rightarrow |\vec{a}|^2 + \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b} - |\vec{b}|^2 = 0$$

$$\Rightarrow |\vec{a}|^2 = |\vec{b}|^2$$

$$\Rightarrow OA = OB$$

Hence we get

$$OA = AC = CB = OB$$

i.e. all sides are equal

Therefore by property of rhombus i.e

Diagonal bisect each other at 90°

And all sides are equal

Quadrilateral OACB is a rhombus

Hence, proved.

3. Question

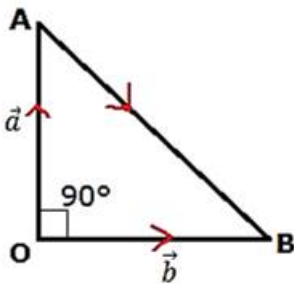
(Pythagoras's Theorem) Prove by vector method that in a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Answer

Given:- Right angle Triangle

To Prove:- Square of the hypotenuse is equal to the sum of the squares of the other two sides

Let $\triangle AOB$ be right angle triangle with right angle at O



Thus we have to prove

$$AB^2 = OA^2 + OB^2$$

Proof: - Let, O at Origin, then

$\vec{a}$ and $\vec{b}$ be position vector of A and B respectively

Since OB is perpendicular at OA, their dot product equals to zero

We know that,

$$(\text{Formula: } \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta)$$

Therefore,

$$\Rightarrow (\vec{OA}) \cdot (\vec{OB}) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} = 0 \dots\dots(i)$$

Now, We can see that, by triangle law of vector addition, $\vec{AB} = \vec{b} - \vec{a}$ Therefore,

$$(\vec{AB})^2 = (\vec{b} - \vec{a})^2$$

$$\Rightarrow (\vec{AB})^2 = a^2 + b^2 - 2\vec{a} \cdot \vec{b}$$

From equation (i)

$$\Rightarrow (\vec{AB})^2 = a^2 + b^2 - 0$$

$$\Rightarrow AB^2 = OA^2 + OB^2 \text{ (Pythagoras theorem)}$$

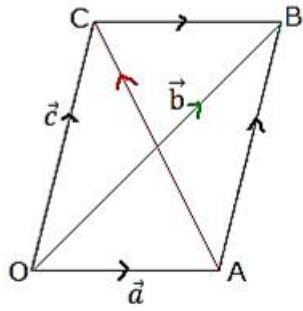
Hence, proved.

4. Question

Prove by vector method that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

Answer

Given:- Parallelogram OABC



To Prove:- $AC^2 + OB^2 = OA^2 + AB^2 + BC^2 + CO^2$

Proof:- Let, O at origin

$\vec{a}, \vec{b}$ and $\vec{c}$ be position vector of A, B and C respectively

Therefore,

$$\vec{OA} = \vec{a}, \vec{OB} = \vec{b} \text{ and } \vec{OC} = \vec{c}$$

Distance/length of AC

$$\vec{AC} = \vec{AB} + \vec{BC}$$

By triangular law:-

$$\vec{a} + \vec{b} = -\vec{c} \text{ or } \vec{a} + \vec{b} + \vec{c} = 0 \text{ the the vectors form sides of triangle}$$

$$\Rightarrow (\vec{AC})^2 = (\vec{AB} + \vec{BC})^2$$

As $AB = OC$ and $BC = OA$

From figure

$$\Rightarrow (\vec{AC})^2 = (\vec{OC} - \vec{OA})^2$$

$$\Rightarrow (\vec{AC})^2 = (\vec{c})^2 + (\vec{a})^2 - 2(\vec{c}).(\vec{a}) \dots\dots(i)$$

Similarly, again from figure

$$\Rightarrow (\vec{OB})^2 = (\vec{OA} + \vec{AB})^2$$

$$\Rightarrow (\vec{OB})^2 = (\vec{OA} + \vec{OC})^2$$

$$\Rightarrow (\vec{OB})^2 = (\vec{a} + \vec{c})^2$$

$$\Rightarrow (\vec{OB})^2 = (\vec{a})^2 + (\vec{c})^2 + 2(\vec{a}).(\vec{c}) \dots\dots(ii)$$

Now,

Adding equation (i) and (ii)

$$\Rightarrow (\vec{AC})^2 + (\vec{OB})^2 = 2|\vec{a}|^2 + 2|\vec{c}|^2 \dots\dots(iii)$$

Take RHS

$$OA^2 + AB^2 + BC^2 + CO^2$$

$$= (\vec{a})^2 + (\vec{OC})^2 + (\vec{OA})^2 + (\vec{c})^2$$

$$= (\vec{a})^2 + (\vec{c})^2 + (\vec{a})^2 + (\vec{c})^2$$

$$= 2|\vec{a}|^2 + 2|\vec{c}|^2 \dots\dots(iv)$$

Thus from equation (iii) and (iv), we get

$$LHS = RHS$$

Hence proved

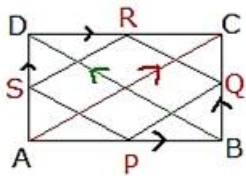
5. Question

Prove using vectors: The quadrilateral obtained by joining mid-points of adjacent sides of a rectangle is a rhombus.

Answer

Given:- ABCD is a rectangle

To prove:- PQRS is rhombus thus finding its properties in PQRS



i.e. All sides equal and parallel

Let, P, Q, R and S are midpoints of sides AB, BC, CD and DA respectively

Therefore

$$\vec{PB} = \frac{\vec{AB}}{2} = \vec{AP}$$

$$\vec{BQ} = \frac{\vec{BC}}{2} = \vec{QC}$$

$$\vec{CR} = \frac{\vec{CD}}{2} = \vec{RD}$$

$$\vec{DS} = \frac{\vec{DA}}{2} = \vec{SA}$$

also $AB = CD$, $BC = AD$ (ABCD is rectangle opposite sides are equal)

Therefore

$$AP = PB = DR = RC \text{ and } BQ = QC = AS = SD \dots\dots(i)$$

IMP:- Direction/arrow head of vector should be placed correctly

Now, considering in vector notion and applying triangular law of vector addition, we get

$$\Rightarrow \vec{PQ} = \vec{PB} + \vec{BQ}$$

$$\Rightarrow \vec{PQ} = \frac{\vec{AB}}{2} + \frac{\vec{BC}}{2}$$

$$\Rightarrow \vec{PQ} = \frac{\vec{AB} + \vec{BC}}{2}$$

$$\Rightarrow \vec{PQ} = \frac{\vec{AC}}{2}$$

Magnitude $PQ = AC$

$$\text{and } \vec{SR} = \vec{RD} + \vec{DS}$$

$$\Rightarrow \vec{SR} = \frac{\vec{CD}}{2} + \frac{\vec{DA}}{2}$$

$$\Rightarrow \overrightarrow{SR} = \frac{\overrightarrow{CD} + \overrightarrow{DA}}{2}$$

$$\Rightarrow \overrightarrow{SR} = \frac{\overrightarrow{CA}}{2}$$

Magnitude SR = AC

Thus sides PQ and SR are equal and parallel

It shows PQRS is a parallelogram

Now,

$$\Rightarrow (\overrightarrow{PQ})^2 = (\overrightarrow{PQ}).(\overrightarrow{PQ})$$

$$\Rightarrow (\overrightarrow{PQ})^2 = (\overrightarrow{PB} + \overrightarrow{BQ}).(\overrightarrow{PB} + \overrightarrow{BQ})$$

$$\Rightarrow (\overrightarrow{PQ})^2 = (\overrightarrow{PB}).(\overrightarrow{PB}) + (\overrightarrow{PB}).(\overrightarrow{BQ}) + (\overrightarrow{PB}).(\overrightarrow{BQ}) + (\overrightarrow{BQ}).(\overrightarrow{BQ})$$

By Dot product, we know

$$\vec{a}.\vec{a} = |\vec{a}|^2$$

$$\vec{a}.\vec{b} = 0 ; \text{if angle between them is } 90^\circ$$

Here ABCD is rectangle and have 90° at A, B, C, D

$$\Rightarrow (\overrightarrow{PQ})^2 = |\overrightarrow{PB}|^2 + |\overrightarrow{BQ}|^2$$

And

$$\Rightarrow (\overrightarrow{PS})^2 = (\overrightarrow{PS}).(\overrightarrow{PS})$$

again by triangular law

$$\Rightarrow (\overrightarrow{PS})^2 = (\overrightarrow{PA} + \overrightarrow{AS}).(\overrightarrow{PA} + \overrightarrow{AS})$$

$$\Rightarrow (\overrightarrow{PS})^2 = ((\overrightarrow{PA}).(\overrightarrow{PA}) + (\overrightarrow{PA}).(\overrightarrow{AS}) + (\overrightarrow{PA}).(\overrightarrow{AS}) + (\overrightarrow{AS}).(\overrightarrow{AS}))$$

By Dot product, we know

$$\vec{a}.\vec{a} = |\vec{a}|^2$$

$$\vec{a}.\vec{b} = 0 ; \text{if angle between them is } 90^\circ$$

Here ABCD is rectangle and have 90° at A, B, C, D

$$\Rightarrow (\overrightarrow{PS})^2 = |\overrightarrow{PA}|^2 + |\overrightarrow{AS}|^2$$

From above similarities of sides of rectangle in eq (i), we have

$$\Rightarrow (\overrightarrow{PS})^2 = |\overrightarrow{PB}|^2 + |\overrightarrow{BQ}|^2$$

Hence PQ = PS

And from above results we have

All sides of parallelogram are equal

$$PQ = QR = RS = SP$$

Hence proved by property of rhombus (all sides are equal and opposite sides are parallel), PQRS is rhombus

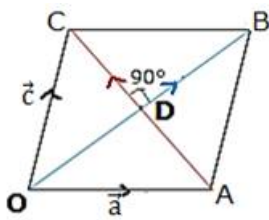
6. Question

Prove that the diagonals of a rhombus are perpendicular bisectors of each other.

Answer

Given:- Rhombus OABC i.e all sides are equal

To Prove:- Diagonals are perpendicular bisector of each other



Proof:- Let, O at the origin

D is the point of intersection of both diagonals

$\vec{a}$ and $\vec{c}$ be position vector of A and C respectively

Then,

$$\vec{OA} = \vec{a}$$

$$\vec{OC} = \vec{c}$$

Now,

$$\vec{OB} = \vec{OA} + \vec{AB}$$

$$\Rightarrow \vec{OB} = \vec{OA} + \vec{OC}$$

as $AB = OC$

$$\Rightarrow \vec{OB} = \vec{a} + \vec{c} \dots\dots(i)$$

Similarly

$$\Rightarrow \vec{AC} = \vec{AO} + \vec{OC}$$

$$\Rightarrow \vec{AC} = -\vec{a} + \vec{c} \dots\dots(ii)$$

Tip:- Directions are important as sign of vector get changed

Magnitude are same $AC = OB = \sqrt{a^2 + c^2}$

Hence from two equations, diagonals are equal

Now let's find position vector of mid-point of OB and AC

$$\Rightarrow \vec{OD} = \vec{DB} = \frac{\vec{OA} + \vec{OB}}{2}$$

$$\Rightarrow \vec{OD} = \vec{DB} = \frac{\vec{a} + \vec{c}}{2}$$

and

$$\Rightarrow \vec{AD} = \vec{DC} = \frac{\vec{AO} + \vec{OC}}{2}$$

$$\Rightarrow \vec{AD} = \vec{DC} = \frac{-\vec{a} + \vec{c}}{2}$$

Magnitude is same $AD = DC = OD = DB = 0.5(\sqrt{a^2 + c^2})$

Thus the position of mid-point is same, and it is the bisecting point D

By Dot Product of OB and AC vectors we get,

$$\Rightarrow (\vec{OB}) \cdot (\vec{AC}) = (\vec{a} + \vec{c}) \cdot (\vec{c} - \vec{a})$$

$$\Rightarrow (\overrightarrow{OB}) \cdot (\overrightarrow{AC}) = (\vec{c} + \vec{a}) \cdot (\vec{c} - \vec{a})$$

$$\Rightarrow (\overrightarrow{OB}) \cdot (\overrightarrow{AC}) = |\vec{c}|^2 - |\vec{a}|^2$$

$$\Rightarrow (\overrightarrow{OB}) \cdot (\overrightarrow{AC}) = (\overrightarrow{OC})^2 - (\overrightarrow{OA})^2$$

As the side of a rhombus are equal $OA = OC$

$$\Rightarrow (\overrightarrow{OB}) \cdot (\overrightarrow{AC}) = OC^2 - OC^2$$

$$\Rightarrow (\overrightarrow{OB}) \cdot (\overrightarrow{AC}) = 0$$

Hence OB is perpendicular on AC

Thus diagonals of rhombus bisect each other at 90°

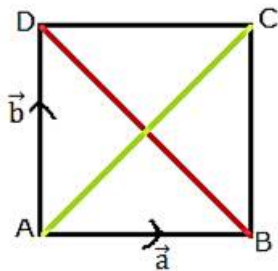
7. Question

Prove that the diagonals of a rectangle are perpendicular if and only if the rectangle is a square.

Answer

Given:- ABCD is a rectangle i.e $AB = CD$ and $AD = BC$

To Prove:- ABCD is a square only if its diagonal are perpendicular



Proof:- Let A be at the origin

$\vec{a}$ and $\vec{b}$ be position vector of B and D respectively

Now,

By parallelogram law of vector addition,

$$\Rightarrow \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

Since in rectangle opposite sides are equal $BC = AD$

$$\Rightarrow \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$$

$$\Rightarrow \overrightarrow{AC} = \vec{a} + \vec{b}$$

and

$$\Rightarrow \overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$$

Negative sign as vector is opposite

$$\Rightarrow \overrightarrow{BD} = \vec{a} - \vec{b}$$

$$\Rightarrow \overrightarrow{BD} = \vec{a} - \vec{b}$$

Diagonals are perpendicular to each other only

$$\Rightarrow (\overrightarrow{AC}) \cdot (\overrightarrow{BD}) = 0$$

$$\Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

$$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 0$$

$$\Rightarrow |\vec{a}|^2 = |\vec{b}|^2$$

$$\Rightarrow AB^2 = AD^2$$

$$\Rightarrow AB = AD$$

Hence all sides are equal if diagonals are perpendicular to each other

ABCD is a square

Hence proved

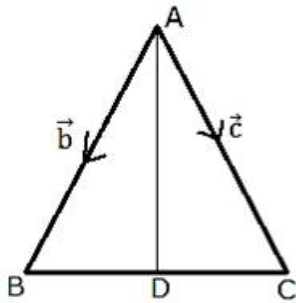
8. Question

If AD is the median of $\triangle ABC$, using vectors, prove that $AB^2 + AC^2 = 2(AD^2 + CD^2)$.

Answer

Given:- $\triangle ABC$ and AD is median

To Prove:- $AB^2 + AC^2 = 2(AD^2 + CD^2)$



Proof:- Let, A at origin

$\vec{b}$ and $\vec{c}$ be position vector of B and C respectively

Therefore,

$$\vec{AB} = \vec{b} \text{ and } \vec{AC} = \vec{c}$$

Now position vector of D, mid-point of BC i.e divides BC in 1:1.

Section formula of internal division: Theorem given below

“Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$

respectively, and c be a point dividing AB internally in the ration m:n. Then the position vector of c is given by $\vec{OC} = \frac{m\vec{b} + n\vec{a}}{m + n}$

Position vector of D is given by

$$\Rightarrow \vec{AD} = \frac{\vec{b} + \vec{c}}{2}$$

Now distance/length of CD

$\vec{CD}$ = position vector of D-position vector of C

$$\Rightarrow \vec{CD} = \frac{\vec{b} + \vec{c}}{2} - \vec{c}$$

$$\Rightarrow \vec{CD} = \frac{\vec{b} - \vec{c}}{2}$$

Now taking RHS

$$= 2(AD^2 + CD^2)$$

$$= 2 \left[\left(\frac{\vec{b} + \vec{c}}{2} \right)^2 + \left(\frac{\vec{b} - \vec{c}}{2} \right)^2 \right]$$

$$= \frac{2}{4} [(\vec{b} + \vec{c})^2 + (\vec{b} - \vec{c})^2]$$

$$= \frac{1}{2} [(\vec{b})^2 + (\vec{c})^2 + 2(\vec{b}) \cdot (\vec{c}) + (\vec{b})^2 + (\vec{c})^2 - 2(\vec{b}) \cdot (\vec{c})]$$

$$= \frac{1}{2} [2(\vec{b})^2 + 2(\vec{c})^2]$$

$$= (\vec{b})^2 + (\vec{c})^2$$

$$= AB^2 + AC^2$$

= LHS

Hence proved

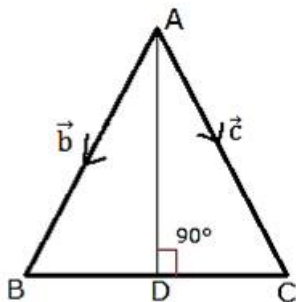
9. Question

If the median to the base of a triangle is perpendicular to the base, then the triangle is isosceles.

Answer

Given:- ΔABC , AD is median

To Prove:- If AD is perpendicular on base BC then ΔABC is isosceles



Proof:- Let, A at Origin

$\vec{b}$ and $\vec{c}$ be position vector of B and C respectively

Therefore,

$$\overrightarrow{AB} = \vec{b} \text{ and } \overrightarrow{AC} = \vec{c}$$

Now position vector of D, mid-point of BC i.e divides BC in 1:1

Section formula of internal division: Theorem given below

“Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$

respectively, and c be a point dividing AB internally in the ration m:n. Then the position vector of c is given by $\overrightarrow{OC} = \frac{m\vec{b} + n\vec{a}}{m+n}$

Position vector of D is given by

$$\Rightarrow \overrightarrow{AD} = \frac{\vec{b} + \vec{c}}{2}$$

Now distance/length of BC

$\overrightarrow{BC}$ = position vector of C-position vector of B

$$\Rightarrow \overrightarrow{BC} = \vec{c} - \vec{b}$$

Now, assume median AD is perpendicular at BC

Then by Dot Product

$$\Rightarrow (\overrightarrow{AD}) \cdot (\overrightarrow{BC}) = 0$$

$$\Rightarrow \left(\frac{\vec{b} + \vec{c}}{2} \right) \cdot (\vec{c} - \vec{b}) = 0$$

$$\Rightarrow (\vec{c} + \vec{b}) \cdot (\vec{c} - \vec{b}) = 0$$

$$\Rightarrow |\vec{c}|^2 - |\vec{b}|^2 = 0$$

$$\Rightarrow |\vec{c}|^2 = |\vec{b}|^2$$

$$\Rightarrow AC = AB$$

Thus two sides of $\triangle ABC$ are equal

Hence $\triangle ABC$ is isosceles triangle

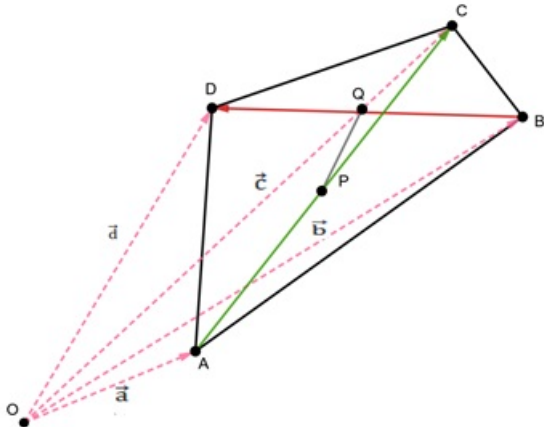
10. Question

In a quadrilateral ABCD, prove that $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2$ where P and Q are middle points of diagonals AC and BD.

Answer

Given:- Quadrilateral ABCD with AC and BD are diagonals. P and Q are mid-point of AC and BD respectively

To Prove:- $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2$



Proof:- Let, O at Origin

$\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be position vector of A, B, C and D respectively

As P and Q are mid-point of AC and BD,

Then, position vector of P, mid-point of AC i.e divides AC in 1:1

and position vector of Q, mid-point of BD i.e divides BD in 1:1

Section formula of internal division: Theorem given below

"Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$

respectively, and c be a point dividing AB internally in the ration m:n. Then the position vector of c is given by $\overrightarrow{OC} = \frac{m\vec{b} + n\vec{a}}{m + n}$ "

Hence

Position vector of P is given by

$$= \frac{\vec{a} + \vec{c}}{2}$$

Position vector of Q is given by

$$= \frac{\vec{b} + \vec{d}}{2}$$

Distance/length of PQ

$$\Rightarrow \overrightarrow{PQ} = \text{position vector of Q} - \text{position vector of P}$$

$$\Rightarrow \overrightarrow{PQ} = \frac{\vec{b} + \vec{d}}{2} - \frac{\vec{a} + \vec{c}}{2}$$

Distance/length of AC

$$\Rightarrow \overrightarrow{AC} = \text{position vector of C} - \text{position vector of A}$$

$$\Rightarrow \overrightarrow{AC} = \vec{c} - \vec{a}$$

Distance/length of BD

$$\Rightarrow \overrightarrow{BD} = \text{position vector of D} - \text{position vector of B}$$

$$\Rightarrow \overrightarrow{BD} = \vec{d} - \vec{b}$$

Distance/length of AB

$$\Rightarrow \overrightarrow{AB} = \text{position vector of B} - \text{position vector of A}$$

$$\Rightarrow \overrightarrow{AB} = \vec{b} - \vec{a}$$

Distance/length of BC

$$\Rightarrow \overrightarrow{BC} = \text{position vector of C} - \text{position vector of B}$$

$$\Rightarrow \overrightarrow{BC} = \vec{c} - \vec{b}$$

Distance/length of CD

$$\Rightarrow \overrightarrow{CD} = \text{position vector of D} - \text{position vector of C}$$

$$\Rightarrow \overrightarrow{CD} = \vec{d} - \vec{c}$$

Distance/length of DA

$$\Rightarrow \overrightarrow{DA} = \text{position vector of A} - \text{position vector of D}$$

$$\Rightarrow \overrightarrow{DA} = \vec{a} - \vec{d}$$

Now, by LHS

$$= AB^2 + BC^2 + CD^2 + DA^2$$

$$= (\vec{b} - \vec{a})^2 + (\vec{c} - \vec{b})^2 + (\vec{d} - \vec{c})^2 + (\vec{a} - \vec{d})^2$$

$$= 2 \left[|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{d}|^2 - \vec{a}\vec{b} \cos \theta_1 - \vec{c}\vec{b} \cos \theta_2 - \vec{c}\vec{d} \cos \theta_3 - \vec{a}\vec{d} \cos \theta_4 \right]$$

Where $\theta_1, \theta_2, \theta_3, \theta_4$ are angle between vectors

Take RHS

$$AC^2 + BD^2 + 4PQ^2$$

$$= (\vec{c} - \vec{a})^2 + (\vec{d} - \vec{b})^2 + 4\left(\frac{\vec{b} + \vec{d}}{2} - \frac{\vec{a} + \vec{c}}{2}\right)^2$$

$$= (\vec{c} - \vec{a})^2 + (\vec{d} - \vec{b})^2 + ((\vec{b} + \vec{d}) - (\vec{a} + \vec{c}))^2$$

$$= (\vec{c} - \vec{a})^2 + (\vec{c} + \vec{a})^2 + (\vec{d} - \vec{b})^2 + (\vec{d} + \vec{b})^2 + 2(\vec{b} + \vec{d}) \cdot (\vec{a} + \vec{c})$$

$$= 2\left[|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{d}|^2 - \vec{a}\vec{b}\cos\theta_1 - \vec{c}\vec{b}\cos\theta_2 - \vec{c}\vec{d}\cos\theta_3 - \vec{a}\vec{d}\cos\theta_4\right]$$

Thus LHS = RHS

Hence proved

Very short answer

1. Question

What $\vec{a}$ and $\vec{b}$ is the angle between vectors and with magnitudes 2 and $\sqrt{3}$ respectively? Given $\vec{a} \cdot \vec{b} = \sqrt{3}$.

Answer

We know,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta \text{ where } \theta \text{ is the angle between } \vec{a} \text{ and } \vec{b}.$$

$$\text{Given, } |\vec{a}|=2 \quad |\vec{b}|=\sqrt{3}$$

$$\vec{a} \cdot \vec{b} = 2 \cdot \sqrt{3} \cos\theta$$

$$\text{So, } \cos\theta = \frac{1}{2}$$

$$\theta = 60^\circ$$

2. Question

If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a} \cdot \vec{b} = 6$, $|\vec{a}| = 3$ and $|\vec{b}| = 4$. Write the projection of on

Answer

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = 6$$

Given,

$$|\vec{a}|=3, |\vec{b}|=4$$

$$6 = 3 \times 4 \cos\theta$$

$$6 = 12\cos\theta$$

$$\cos\theta = \frac{1}{2}$$

3. Question

Find the cosine of the angle between the vectors $4\hat{i} - 3\hat{j} + 3\hat{k}$ and $2\hat{i} - \hat{j} - \hat{k}$.

Answer

We know,

$$\text{If } \vec{A} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}, \vec{B} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

$$(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) \cdot (a_2\hat{i} + b_2\hat{j} + c_2\hat{k}) = a_1 \cdot a_2 + b_1 \cdot b_2 + c_1 \cdot c_2$$

$$\text{And } \vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

$$\text{So, } \vec{A} \cdot \vec{B} = (a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) \cdot (a_2\hat{i} + b_2\hat{j} + c_2\hat{k})$$

$$= a_1 \cdot a_2 + b_1 \cdot b_2 + c_1 \cdot c_2$$

$$= |\vec{A}||\vec{B}|\cos\theta$$

$$|\vec{A}| = \sqrt{34}$$

$$|\vec{B}| = \sqrt{6}$$

$$\text{Here, } 4 \times 2 + (-3) \times (-1) + 3 \times (-1) = 8$$

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

$$= \sqrt{34} \times \sqrt{6} \cos\theta$$

$$= \sqrt{204} \cos\theta$$

$$= 8$$

$$\cos\theta = \frac{8}{\sqrt{204}} = 0.56$$

4. Question

If the vectors $3\hat{i} + m\hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} - 8\hat{k}$ are orthogonal, find m.

Answer

Orthogonal vectors are perpendicular to each other so their dot product is always 0 as $\cos 90^\circ = 0$

$$\text{If } \vec{A} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}, \vec{B} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

$$(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) \cdot (a_2\hat{i} + b_2\hat{j} + c_2\hat{k}) = a_1 \cdot a_2 + b_1 \cdot b_2 + c_1 \cdot c_2$$

$$\text{And } \vec{A} \cdot \vec{B} = 3 \times 2 + m \times (-1) + 1 \times (-8) = 0$$

$$6 - m - 8 = 0$$

$$-m - 2 = 0$$

$$m = -2$$

5. Question

If the vectors $3\hat{i} - 2\hat{j} - 4\hat{k}$ and $18\hat{i} - 12\hat{j} - m\hat{k}$ are parallel, find the value of m.

Answer

$$\text{If } \vec{A} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}, \vec{B} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

And A is parallel to B

Then $\vec{A} = k\vec{B}$, where k is some constant

$$\text{So, } k = \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{3}{18} = \frac{4}{m}$$

$$k = \frac{4}{m} = \frac{1}{6}$$

$$m = 6 \times 4$$

$$= 24$$

6. Question

If $\vec{a}$ and $\vec{b}$ are vectors of equal magnitude, write the value of $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})$.

Answer

We know that dot product is distributive.

So

$$\begin{aligned} (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) &= |\vec{a}||\vec{a}| - |\vec{a}||\vec{b}| + |\vec{a}||\vec{b}| - |\vec{b}||\vec{b}| \\ &= |\vec{a}|^2 - |\vec{b}|^2 \end{aligned}$$

We know

$$\begin{aligned} |\vec{a}| &= |\vec{b}| \\ (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) &= |\vec{a}||\vec{a}| - |\vec{a}||\vec{b}| + |\vec{a}||\vec{b}| - |\vec{b}||\vec{b}| \\ &= |\vec{a}|^2 - |\vec{b}|^2 \\ &= |\vec{a}|^2 - |\vec{a}|^2 \\ &= 0 \end{aligned}$$

7. Question

If $\vec{a}$ and $\vec{b}$ are two vectors such that $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$, find the relation between the magnitudes of $\vec{a}$ and $\vec{b}$.

Answer

We know that dot product is distributive.

So

$$\begin{aligned} (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) &= |\vec{a}||\vec{a}| - |\vec{a}||\vec{b}| + |\vec{a}||\vec{b}| - |\vec{b}||\vec{b}| \\ &= |\vec{a}|^2 - |\vec{b}|^2 \end{aligned}$$

Given that,

$$\begin{aligned} (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) &= |\vec{a}||\vec{a}| - |\vec{a}||\vec{b}| + |\vec{a}||\vec{b}| - |\vec{b}||\vec{b}| \\ &= |\vec{a}|^2 - |\vec{b}|^2 \\ &= 0 \end{aligned}$$

$$|\vec{a}|^2 - |\vec{b}|^2 = 0$$

$$|\vec{a}|^2 = |\vec{b}|^2$$

Therefore, both the vectors have equal magnitude

8. Question

For any two vectors $\vec{a}$ and $\vec{b}$ write when $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$ holds.

Answer

We know,

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta}$$

$$|\vec{a}| + |\vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta}$$

$$(|\vec{a}| + |\vec{b}|)^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$$

$$|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}| = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$$

Comparing LHS and RHS we can conclude that

$$2|\vec{a}||\vec{b}| = 2|\vec{a}||\vec{b}|\cos\theta$$

$$\cos\theta = 1 \text{ or } \theta = 0^\circ$$

9. Question

For any two vectors $\vec{a}$ and $\vec{b}$ write when $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ holds.

Answer

We know,

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta}$$

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta}$$

$$\text{If } |\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$$

$$\text{Then, } \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta} = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta}$$

$$2|\vec{a}||\vec{b}|\cos\theta = -2|\vec{a}||\vec{b}|\cos\theta$$

Comparing LHS and RHS we can conclude that

$$\cos\theta = 0 \text{ or } \theta = 90^\circ$$

10. Question

If $\vec{a}$ and $\vec{b}$ are two vectors of the same magnitude inclined at an angle of 60° such that $\vec{a} \cdot \vec{b} = 8$, write the value of their magnitude.

Answer

Given,

$$\theta = 60^\circ \text{ and } |\vec{a}| = |\vec{b}|$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$= |\vec{a}|^2 \cos 60^\circ$$

$$= 8$$

$$\vec{a} \cdot \vec{b} = |\vec{a}|^2 \times \frac{1}{2}$$

$$= 8$$

$$|\vec{a}|^2 = 16$$

$$|\vec{a}| = 4 \text{ (as magnitude cannot be negative)}$$

11. Question

If $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$, what can you conclude about the vector $\vec{b}$?

Answer

$$\vec{a} \cdot \vec{a} = \vec{a} \cdot \vec{b} = 0$$

$$|\vec{a}||\vec{a}|\cos 0^\circ = |\vec{a}||\vec{b}|\cos \theta$$

$$= 0$$

Possible answers are,

$$|\vec{a}| = 0 \text{ i.e. } \vec{a} \text{ is a null vector}$$

Or

$$\cos \theta = 0 \text{ or } \theta = 90^\circ \text{ i.e. } \vec{a} \text{ and } \vec{b} \text{ are perpendicular}$$

Or

$$|\vec{b}| = 0 \text{ i.e. } \vec{b} \text{ is a null vector}$$

12. Question

If $\vec{b}$ is a unit vector such that $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$, find $|\vec{a}|$.

Answer

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}||\vec{a}| - |\vec{a}||\vec{b}| + |\vec{a}||\vec{b}| - |\vec{b}||\vec{b}|$$

$$= |\vec{a}|^2 - |\vec{b}|^2$$

$$= 8$$

$$|\vec{a}|^2 - 1^2 = 8$$

$$|\vec{a}|^2 = 9$$

$$|\vec{a}| = 3$$

13. Question

If $\hat{a}$ and $\hat{b}$ are unit vectors such that $\hat{a} + \hat{b}$ is a unit vector, write the value of $|\hat{a} - \hat{b}|$.

Answer

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos \theta} = 1 \text{ (As given as unit vector)}$$

$$\sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos \theta} = \sqrt{1^2 + 1^2 + 1 \times 1 \times \cos \theta}$$

$$= 1$$

$$\sqrt{2 + 2\cos\theta} = 1$$

$$2 + 2\cos\theta = 1$$

$$\cos\theta = -1/2$$

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta}$$

$$= \sqrt{1 + 1 - 2 \times 1 \times 1 \times \cos\theta}$$

$$= \sqrt{2 - 2(-1/2)}$$

$$= \sqrt{3}$$

14. Question

If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $\vec{a} \cdot \vec{b} = 2$, and find $|\vec{a} - \vec{b}|$.

Answer

$$|\vec{a}| = 2, |\vec{b}| = 5$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$= 2 \times 5 \times \cos\theta$$

$$= 2$$

$$\cos\theta = \frac{2}{10} = \frac{1}{5}$$

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta}$$

$$= \sqrt{2^2 + 5^2 - 2 \times 2 \times 5 \times \cos\theta}$$

$$= \sqrt{4 + 25 - 20(1/5)}$$

$$|\vec{a} - \vec{b}| = \sqrt{4 + 25 - 20(1/5)}$$

$$= \sqrt{29 - 4} = \sqrt{25}$$

$$= 5$$

15. Question

If $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = -\hat{j} + \hat{k}$, find the projection of $\vec{a}$ on $\vec{b}$.

Answer

Projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$$|\vec{b}| = \sqrt{(-1)^2 + 1^2}$$

$$= \sqrt{2}$$

$$(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) \cdot (a_2\hat{i} + b_2\hat{j} + c_2\hat{k}) = a_1 \cdot a_2 + b_1 \cdot b_2 + c_1 \cdot c_2$$

$$\vec{a} \cdot \vec{b} = 1 \times 0 + (-1) \times (-1) + 0 \times 1$$

$$= 1$$

$$\text{Therefore projection} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{2}}$$

16. Question

For any two non-zero vectors, write the value of $\frac{|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2}{|\vec{a}|^2 + |\vec{b}|^2}$.

Answer

$$\frac{|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2}{|\vec{a}|^2 + |\vec{b}|^2} = \frac{(|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}) + (|\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b})}{|\vec{a}|^2 + |\vec{b}|^2}$$

$$= \frac{2(|\vec{a}|^2 + |\vec{b}|^2)}{|\vec{a}|^2 + |\vec{b}|^2}$$

$$= 2$$

17. Question

Write the projections of $\vec{r} = 3\hat{i} - 4\hat{j} + 12\hat{k}$ on the coordinate axes.

Answer

$$\text{x-axis} = \hat{i}$$

$$\text{y-axis} = \hat{j}$$

$$\text{z-axis} = \hat{k}$$

$$\text{proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$$

$$\text{Projection along x-axis} = \frac{3}{1} \hat{i}$$

$$= 3\hat{i}$$

$$\text{Projection along y-axis} = \frac{-4}{1} \hat{j}$$

$$= -4\hat{j}$$

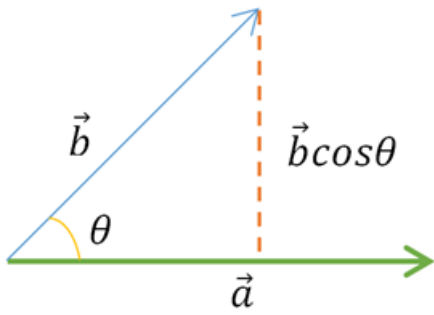
$$\text{Projection along z-axis} = \frac{12}{1} \hat{k}$$

$$= 12\hat{k}$$

18. Question

Write the component of $\vec{b}$ along $\vec{a}$.

Answer



Component of a given vector $\vec{b}$ along $\vec{a}$ is given by the length of $\vec{b}$ on $\vec{a}$.

Let θ be the angle between both the vectors.

So the length of $\vec{b}$ on $\vec{a}$ is given as: $|\vec{b}| \cos \theta$

By vector dot product, we know that:

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

$$\text{Therefore, } |\vec{b}| \cos \theta = |\vec{b}| \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$$\text{Hence, } \text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

19. Question

Write the value of $(\vec{a} \cdot \hat{i})\hat{i} + (\vec{a} \cdot \hat{j})\hat{j} + (\vec{a} \cdot \hat{k})\hat{k}$, where $\vec{a}$ is any vector.

Answer

$$\text{Let } \vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{a} \cdot \hat{i} = x \quad (1)$$

$$\vec{a} \cdot \hat{j} = y \quad (2)$$

$$\vec{a} \cdot \hat{k} = z \quad (3)$$

Put the values obtained in the given equation

We get:

$$(\vec{a} \cdot \hat{i})\hat{i} + (\vec{a} \cdot \hat{j})\hat{j} + (\vec{a} \cdot \hat{k})\hat{k} = x\hat{i} + y\hat{j} + z\hat{k}$$

i.e.

$$(\vec{a} \cdot \hat{i})\hat{i} + (\vec{a} \cdot \hat{j})\hat{j} + (\vec{a} \cdot \hat{k})\hat{k} = \vec{a}$$

20. Question

Find the value of $\theta \in (0, \pi/2)$ for which vectors $\vec{a} = (\sin \theta)\hat{i} + (\cos \theta)\hat{j}$ and $\vec{b} = \hat{i} - \sqrt{3}\hat{j} + 2\hat{k}$ are perpendicular.

Answer

Given:

$$\vec{a} = (\sin \theta)\hat{i} + (\cos \theta)\hat{j}$$

$$\vec{b} = \hat{i} - \sqrt{3}\hat{j} + 2\hat{k}$$

$$\vec{a} \cdot \vec{b} = 0 \text{ (perpendicular)}$$

So,

$$\vec{a} \cdot \vec{b} = \{(\sin\theta\hat{i} + \cos\theta\hat{j}) \cdot (\hat{i} - \sqrt{3}\hat{j} + 2\hat{k})\} = 0$$

Therefore;

$$\sin\theta - \sqrt{3}\cos\theta = 0$$

Multiply and divide the whole equation by 2:

We get

$$\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta = 0$$

By the identity:

$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$

We have:

$$\sin\left(\theta - \frac{\pi}{3}\right) = 0$$

$$\sin\left(\theta - \frac{\pi}{3}\right) = \sin n\pi$$

So

$$\left(\theta - \frac{\pi}{3}\right) = n\pi$$

$$\theta = n\pi + \frac{\pi}{3}, n \in I$$

21. Question

Write the projection of $\hat{i} + \hat{j} + \hat{k}$ along the vector $\hat{j}$.

Answer

$$\text{Let, } \vec{a} = \hat{i} + \hat{j} + \hat{k} \text{ \& } \vec{b} = \hat{j}$$

We know that, projection of $\vec{a}$ along $\vec{b}$ is given by:

$$\text{proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$$

$$\text{Also, } \vec{a} \cdot \vec{b} = 1$$

$$\text{\& } |\vec{b}| = 1$$

$$\text{So, } \text{proj}_{\vec{b}} \vec{a} = 1(\hat{j}) = \hat{j}$$

22. Question

Write a vector satisfying $\vec{a} \cdot \hat{i} = \vec{a} \cdot (\hat{i} + \hat{j}) = \vec{a} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$.

Answer

$$\text{Let } \vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{a} \cdot \hat{i} = x$$

$$\vec{a}(\hat{i} + \hat{j}) = x + y$$

$$\vec{a}(\hat{i} + \hat{j} + \hat{k}) = x + y + z$$

For all the equations to be equal to 1;

$$\text{i.e. } x = x + y$$

$$= x + y + z$$

$$= 1$$

$$\text{So, } x = 1;$$

$$\& x + y = 1$$

$$\& x + y + z = 1$$

We get: $x = 1, y = z = 0$

Therefore, $\vec{a} = \hat{i}$

23. Question

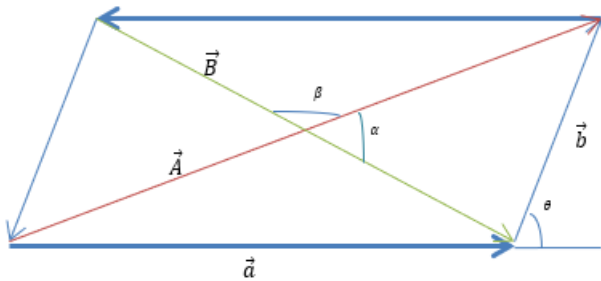
If $\vec{a}$ and $\vec{b}$ are unit vectors, find the angle between $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

Answer

Since, $|\vec{a}| = |\vec{b}| = 1$

Let $\vec{A} = \vec{a} + \vec{b}$ & $\vec{B} = \vec{a} - \vec{b}$

Angle between $\vec{a}$ & $\vec{b}$ is θ and angle between $\vec{A}$ & $\vec{B}$ is α & β



By vector addition method;

we have:

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$$

$$= 2(1 + \cos\theta)$$

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$= 2(1 - \cos\theta)$$

So,

$$|\vec{a} + \vec{b}| = 2\cos\frac{\theta}{2}$$

$$|\vec{a} - \vec{b}| = 2\sin\frac{\theta}{2}$$

Now in the parallelogram:

Area of parallelogram = (product of two sides and the sine of angle between them)

i.e. $area = |\vec{a}| \times |\vec{b}| \times \sin \theta$ (1)

Also area of parallelogram = sum of area of all four triangle

And area of each triangle = $\frac{1}{2}bh$

So, Area = $2\left\{\frac{1}{2} * \frac{|\vec{A}|}{2} * \frac{|\vec{B}|}{2} \sin \alpha\right\} + 2\left\{\frac{1}{2} * \frac{|\vec{A}|}{2} * \frac{|\vec{B}|}{2} \sin \beta\right\}$

Since α & β are supplementary

$A = 4\left\{\frac{1}{2} * \frac{|\vec{A}|}{2} * \frac{|\vec{B}|}{2} \sin \alpha\right\} = \frac{1}{2} * |\vec{A}| |\vec{B}| \sin \alpha$ (2)

From (1) & (2) we get:

$$\sin \alpha = \frac{2|\vec{a}||\vec{b}| \sin \theta}{|\vec{A}||\vec{B}|} = \frac{2|\vec{a}||\vec{b}| \sin \theta}{|\vec{a} + \vec{b}||\vec{a} - \vec{b}|}$$

$$\sin \alpha = \frac{2 * 1 * 1 * \sin \theta}{2 \sin \frac{\theta}{2} * 2 \cos \frac{\theta}{2}} = \frac{2 \sin \theta}{2 \sin \theta} = 1$$

$$\alpha = \sin^{-1} 1 = \frac{\pi}{2}$$

24. Question

If $\vec{a}$ and $\vec{b}$ are mutually perpendicular unit vectors, write the value of $|\vec{a} + \vec{b}|$.

Answer

Since $\vec{a}$ & $\vec{b}$ are mutually perpendicular;

Then, $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$ (1)

And $\sin \theta = \frac{(\vec{a} \times \vec{b})}{|\vec{a}||\vec{b}|}$ (2)

Squaring and adding both equations, we get;

$$(\sin \theta)^2 + (\cos \theta)^2 = \left(\frac{(\vec{a} \times \vec{b})}{|\vec{a}||\vec{b}|}\right)^2 + \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)^2$$

$$1 = \frac{(\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2}{(|\vec{a}||\vec{b}|)^2}$$

So, $(|\vec{a}||\vec{b}|)^2 = (\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2$

Hence, $|\vec{a}||\vec{b}| = \sqrt{(\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2}$

25. Question

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular unit vectors, write the value of $|\vec{a} + \vec{b} + \vec{c}|$.

Answer

Since all three vectors are mutually perpendicular, so dot product of each vector with another is zero.

i.e. $\vec{a} \cdot \vec{b} = 0, \vec{b} \cdot \vec{c} = 0, \vec{c} \cdot \vec{a} = 0$

Also, $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

$$\text{So, } |\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$$

$$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$$

$$\text{i.e. } |\vec{a} + \vec{b} + \vec{c}|^2 = 3$$

$$\text{So, } |\vec{a} + \vec{b} + \vec{c}|^2 = \sqrt{3}$$

26. Question

Find the angle between the vectors $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$.

Answer

By vector dot product, we know that:

$$\vec{a} \cdot \vec{b} = |a||b| \cos \theta$$

$$\text{So, } \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|a||b|}$$

$$\vec{a} = \hat{i} - \hat{j} + \hat{k} \text{ and } \vec{b} = \hat{i} + \hat{j} - \hat{k}.$$

$$\vec{a} \cdot \vec{b} = -1$$

$$|a| = \sqrt{3} \text{ \& } |b| = \sqrt{3}$$

Therefore,

$$\cos \theta = \frac{-1}{\sqrt{3} \cdot \sqrt{3}}$$

$$\cos \theta = \frac{-1}{3}$$

$$\text{So, } \theta = \cos^{-1}\left(\frac{-1}{3}\right)$$

27. Question

For what value of λ are the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ perpendicular to each other?

Answer

$$\text{Let } \vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k} \text{ and } \vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}.$$

$$\text{for } \vec{a} \text{ to be perpendicular to } \vec{b} \cos \theta = 0$$

$$\text{i.e. } \vec{a} \cdot \vec{b} = 0 \text{ [vector dot product]}$$

$$(2\hat{i} + \lambda\hat{j} + \hat{k}) \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 0$$

$$2 - 2\lambda + 3 = 0$$

$$5 - 2\lambda = 0$$

$$\text{Hence, } \lambda = \frac{5}{2}$$

28. Question

Find the projection of $\vec{a}$ on $\vec{b}$, if $\vec{a} \cdot \vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$.

Answer

We know that;

$$\text{proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$$

So,

$$\text{proj}_{\vec{b}} \vec{a} = \frac{2\hat{i} + 6\hat{j} + 3\hat{k}}{|\vec{b}|^2}$$

29. Question

Write the value of p for which $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel vectors.

Answer

$$\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k} \text{ and } \vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$$

for $\vec{a}$ to be parallel to $\vec{b}$ $\sin \theta = 0$

i.e. $(\vec{a} \times \vec{b}) = 0$ [vector cross product]

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 9 \\ 1 & p & 3 \end{vmatrix} = 0$$

$$\hat{i}(6 - 9p) - \hat{j}(9 - 9) + \hat{k}(3p - 2) = 0$$

$$\hat{i}(6 - 9p) + \hat{k}(3p - 2) = 0\hat{i} + 0\hat{k}$$

$$(6 - 9p) = 0 \text{ \& } (3p - 2) = 0$$

$$\text{Hence, } p = \frac{2}{3}$$

30. Question

Find the value of λ if the vectors $2\hat{i} + \lambda\hat{j} + 3\hat{k}$ and $3\hat{i} + 2\hat{j} - 4\hat{k}$ are perpendicular to each other.

Answer

$$\text{Let } \vec{a} = 2\hat{i} + \lambda\hat{j} + 3\hat{k} \text{ and } \vec{b} = 3\hat{i} + 2\hat{j} - 4\hat{k}$$

for $\vec{a}$ to be perpendicular to $\vec{b}$ $\cos \theta = 0$

i.e. $\vec{a} \cdot \vec{b} = 0$ [vector dot product]

$$(2\hat{i} + \lambda\hat{j} + 3\hat{k}) \cdot (3\hat{i} + 2\hat{j} - 4\hat{k}) = 0$$

$$6 + 2\lambda - 12 = 0$$

$$2\lambda - 6 = 0$$

$$\text{Hence, } \lambda = 3$$

31. Question

If $|\vec{a}| = 2$, $|\vec{b}| = 3$, $\vec{a} \cdot \vec{b} = 3$, find the projection of $\vec{b}$ on $\vec{a}$

Answer

$$\text{Given } |\vec{a}| = 2 \text{ and } |\vec{b}| = 3 \text{ and } |\vec{a} \cdot \vec{b}| = 3$$

The projection of $\vec{b}$ vector $\vec{a}$ on a is given by,

$$\vec{b} \cdot \hat{a} = \vec{b} \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \text{ (since scalar product is commutative)}$$

$$= \frac{3}{2}$$

32. Question

Write the angle between the two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a} \cdot \vec{b} = \sqrt{6}$

Answer

We know that the scalar product of two non-zero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is defined as,

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

$$\sqrt{6} = \sqrt{3} \times 2 \cos \theta$$

$$\cos \theta = \frac{\sqrt{3} \times \sqrt{2}}{\sqrt{3} \times 2}$$

$$= \frac{1}{\sqrt{2}}$$

$$\theta = \cos^{-1} \left(\frac{1}{\sqrt{2}} \right)$$

$$= \frac{\pi}{4}$$

33. Question

Write the projection of vector $\vec{i} + 3\vec{j} + 7\vec{k}$ on the vector $2\vec{i} - 3\vec{j} + 6\vec{k}$.

Answer

Let $\vec{a} = \vec{i} + 3\vec{j} + 7\vec{k}$ and $\vec{b} = 2\vec{i} - 3\vec{j} + 6\vec{k}$

Then the projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\vec{a} \cdot \hat{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

$$= \vec{a} \cdot \vec{b} = (\vec{i} + 3\vec{j} + 7\vec{k}) \cdot (2\vec{i} - 3\vec{j} + 6\vec{k})$$

$$= 1 \times 2 - 3 \times 3 + 7 \times 6$$

$$= 2 - 9 + 42$$

$$= 35$$

$$\text{Now, } |\vec{b}| = \sqrt{2^2 + 6^2 + 3^2}$$

$$= \sqrt{4 + 36 + 9}$$

$$= \sqrt{49}$$

$$= 7$$

$$\text{Therefore projection of } \vec{a} \text{ on } \vec{b} = \frac{35}{7}$$

$$= 5$$

34. Question

Find λ , when the projection of $\vec{a} = \lambda\vec{i} + \vec{j} + 4\vec{k}$ on $\vec{b} = 2\vec{i} + 6\vec{j} + 3\vec{k}$ is 4 units.

Answer

Given $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ is given by $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$$\vec{a} \cdot \vec{b} = (\lambda\hat{i} + \hat{j} + 4\hat{k}) \cdot (2\hat{i} + 6\hat{j} + 3\hat{k})$$

$$= 2\lambda + 6 + 12$$

$$= 2\lambda + 18$$

$$\text{Now, } |\vec{b}| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{49} = 7$$

$$\frac{2\lambda + 18}{7} = 4$$

$$2\lambda + 18 = 28$$

$$2\lambda = 10$$

$$\lambda = 5$$

35. Question

For what value of λ are the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ perpendicular to each other?

Answer

Given $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$

For two vectors to be perpendicular, the angle between them must be 90° or $\frac{\pi}{2}$

We know that $\cos 90 = 0$

$$\vec{a} \cdot \vec{b} = (2\hat{i} + \lambda\hat{j} + \hat{k}) \cdot (\hat{i} - 2\hat{j} + 3\hat{k})$$

$$= 2 - 2\lambda + 3$$

$$= 5 - 2\lambda$$

By scalar product, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$5 - 2\lambda = 0$$

$$\lambda = \frac{5}{2}$$

36. Question

Write the projection of the vector $7\hat{i} + \hat{j} - 4\hat{k}$ on the vector $2\hat{i} + 6\hat{j} + 3\hat{k}$.

Answer

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ is given by,

$$\vec{a} \cdot \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

$$\vec{a} \cdot \vec{b} = (7\hat{i} + \hat{j} - 4\hat{k}) \cdot (2\hat{i} + 6\hat{j} + 3\hat{k})$$

$$= 7 \times 2 + 1 \times 6 - 4 \times 3$$

$$= 14 + 6 - 12$$

$$=14-6$$

$$=8$$

$$|\vec{b}| = \sqrt{2^2 + 6^2 + 3^2}$$

$$= \sqrt{4 + 36 + 9}$$

$$= \sqrt{49}$$

$$= 7$$

Therefore, projection of $\vec{a}$ on $\vec{b}$ is $\frac{8}{7}$

37. Question

Write the value of λ so that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ perpendicular to each other?

Answer

Given $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$

For two vectors to be perpendicular, the angle between them must be 90° or $\frac{\pi}{2}$

We know that $\cos 90 = 0$

$$\vec{a} \cdot \vec{b} = (2\hat{i} + \lambda\hat{j} + \hat{k}) \cdot (\hat{i} - 2\hat{j} + 3\hat{k})$$

$$= 2 - 2\lambda + 3$$

$$= 5 - 2\lambda$$

By scalar product, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Therefore, $5 - 2\lambda = 0$

$$\lambda = \frac{5}{2}$$

38. Question

Write the projection of $\vec{b} + \vec{c}$ on $\vec{a}$, when $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$, and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

Answer

Given, $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$

$$\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$$

$$\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$$

$$\text{So, } \vec{b} + \vec{c} = (\hat{i} + 2\hat{j} + 2\hat{k}) + (2\hat{i} - \hat{j} + 4\hat{k})$$

$$= 3\hat{i} + \hat{j} + 6\hat{k}$$

$$= \vec{d}$$

Now, to find projection of $\vec{b} + \vec{c}$ on $\vec{a}$ i.e. $\vec{d}$ on $\vec{a}$

$$\vec{d} \cdot \vec{a} = \frac{\vec{d} \cdot \vec{a}}{|\vec{a}|}$$

$$\text{Now, } \vec{d} \cdot \vec{a} = (3\hat{i} + \hat{j} + 6\hat{k}) \cdot (2\hat{i} - 2\hat{j} + \hat{k})$$

$$= 3 \times 2 - 1 \times 2 + 6 \times 1$$

$$=6-2+6$$

$$=10$$

$$|\vec{a}| = \sqrt{2^2 + (-2)^2 + 1^2}$$

$$= \sqrt{4 + 4 + 1}$$

$$= \sqrt{9}$$

$$=3$$

$$\text{Therefore, } \frac{\vec{a} \cdot \vec{a}}{|\vec{a}|} = \frac{10}{3}$$

39. Question

If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a} + \vec{b}| = 3$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$.

Answer

Given, $|\vec{a} + \vec{b}| = 3$ and $|\vec{a}| = 5$

Also given $\vec{a}$ and $\vec{b}$ are perpendicular

$$\vec{a} \cdot \vec{b} = 0$$

$$|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})^2$$

$$3^2 = |\vec{a}|^2 + 2|\vec{a} \cdot \vec{b}| + |\vec{b}|^2$$

$$3^2 = 5^2 + |\vec{b}|^2$$

$$9 = 25 + |\vec{b}|^2$$

$$-|\vec{b}|^2 = 16$$

$$|\vec{b}| = -4$$

40. Question

If $\vec{a}$ and $\vec{b}$ vectors are such that $|\vec{a}| = 3$, $|\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$.

Answer

Given $|\vec{a}| = 3$ $|\vec{b}| = \frac{2}{3}$

Also given, $\vec{a} \times \vec{b}$ is a unit vector

$$\Rightarrow |\vec{a} \times \vec{b}| = 1$$

By vector product,

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

$$\text{Therefore, } 1 = 3 \times \frac{2}{3} \sin\theta$$

$$\Rightarrow \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

41. Question

If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + \vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.

Answer

$$\text{Given } |\vec{a}| = |\vec{b}| = 1 \text{ and } |\vec{a} + \vec{b}| = 1$$

$$\text{Now, } |\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$$

$$\Rightarrow 1 = |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2$$

$$\Rightarrow 1 = 1 + 2|\vec{a} \cdot \vec{b}| + 1$$

$$\Rightarrow -1 = 2|\vec{a} \cdot \vec{b}|$$

$$\Rightarrow -\frac{1}{2} = |\vec{a} \cdot \vec{b}|$$

$$\text{Also, } |\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}|\cos\theta$$

$$\text{Therefore, } -\frac{1}{2} = 1 \times 1 \times \cos\theta$$

$$\Rightarrow \cos\theta = -\frac{1}{2}$$

We know that $\cos 60^\circ = \frac{1}{2}$ and $\cos$ is negative in 2nd quadrant

Therefore, $\theta = 180 - 60$

$$= 120$$

$$= \frac{2\pi}{3}$$

42. Question

If $\vec{a}$ and $\vec{b}$ are unit vectors, then find the angle between $\vec{a}$ and $\vec{b}$, given that $\sqrt{3}\vec{a} - \vec{b}$ is a unit vector.

Answer

$$\text{Given, } |\vec{a}| = |\vec{b}| = 1 \text{ and } |\sqrt{3}\vec{a} - \vec{b}| = 1$$

$$\text{By scalar product, } |\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}|\cos\theta$$

By substituting the values, we get

$$\vec{a} \cdot \vec{b} = \cos\theta$$

$$|\sqrt{3}\vec{a} - \vec{b}|^2 = 1$$

$$(\sqrt{3}\vec{a})^2 - 2\sqrt{3}\vec{a} \cdot \vec{b} + \vec{b}^2 = 1$$

$$3a^2 - 2\sqrt{3}\cos\theta + b^2 = 1$$

$$\Rightarrow 3 - 2\sqrt{3}\cos\theta + 1 = 1$$

$$\Rightarrow 4 - 1 = 2\sqrt{3}\cos\theta$$

$$\Rightarrow 3 = 2\sqrt{3}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

MCQ

1. Question

Mark the correct alternative in each of the following:

The vector $\vec{a}$ and $\vec{b}$ satisfy the equation $2\vec{a} + \vec{b} = \vec{p}$ and $\vec{a} + 2\vec{b} = \vec{q}$, where $\vec{p} = \hat{i} + \hat{j}$ and $\vec{q} = \hat{i} - \hat{j}$. If θ is the angle between $\vec{a}$ and $\vec{b}$, then

A. $\cos \theta = \frac{4}{5}$

B. $\sin \theta = \frac{1}{\sqrt{2}}$

C. $\cos \theta = -\frac{4}{5}$

D. $\cos \theta = -\frac{3}{5}$

Answer

Here, $2\vec{a} + \vec{b} = \vec{p}$ and $\vec{a} + 2\vec{b} = \vec{q}$

Also, $\vec{p} = \hat{i} + \hat{j}$ and $\vec{q} = \hat{i} - \hat{j}$

$\therefore 2\vec{a} + \vec{b} = \hat{i} + \hat{j}$ and $\vec{a} + 2\vec{b} = \hat{i} - \hat{j}$

Solving above two equations for $\vec{a}$ and $\vec{b}$ we get,

$$\therefore \vec{a} = \frac{2}{6}\hat{i} + \hat{j} \text{ and } \vec{b} = \frac{2}{6}\hat{i} - \hat{j}$$

$$\therefore \vec{a} \cdot \vec{b} = \frac{2}{6} \times \frac{2}{6} + 1 \times (-1)$$

$$= \frac{4}{36} - 1$$

$$= -\frac{32}{36}$$

$$\text{Also, } |\vec{a}| = \left\{ \left(\frac{2}{6} \right)^2 + (1)^2 \right\}^{\frac{1}{2}}$$

$$= \sqrt{\frac{40}{36}}$$

$$= \frac{\sqrt{40}}{6}$$

$$\text{Ands, } |\vec{b}| = \left\{ \left(\frac{2}{6} \right)^2 + (1)^2 \right\}^{\frac{1}{2}}$$

$$= \sqrt{\frac{40}{36}}$$

$$= \frac{\sqrt{40}}{6}$$

Now, θ is the angle between $\vec{a}$ and $\vec{b}$

$$\text{So, } \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$= \frac{\left(-\frac{32}{36}\right)}{\left(\frac{\sqrt{40}}{6} \times \frac{\sqrt{40}}{6}\right)}$$

$$= -\frac{32}{40}$$

$$= -\frac{4}{5}$$

2. Question

Mark the correct alternative in each of the following:

If $\vec{a} \cdot \hat{i} = \vec{a} \cdot (\hat{i} + \hat{j}) = \vec{a} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$, then $\vec{a} =$

A. $\vec{0}$

B. $\hat{i}$

C. $\hat{j}$

D. $\hat{i} + \hat{j} + \hat{k}$

Answer

Here, $\vec{a} \cdot \hat{i} = 1$ _____ (1)

$\vec{a} \cdot (\hat{i} + \hat{j}) = 1$ _____ (2)

and $\vec{a} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ _____ (3)

From (2),

$$\vec{a} \cdot \hat{i} + \vec{a} \cdot \hat{j} = 1$$

$$\therefore \vec{a} \cdot \hat{j} = 0 \quad (\because \vec{a} \cdot \hat{i} = 1) \text{ _____ (4)}$$

From (3) and (4)

$$\vec{a} \cdot \hat{i} + \vec{a} \cdot \hat{k} = 1 \quad (\because \vec{a} \cdot \hat{j} = 0)$$

$$\therefore \vec{a} \cdot \hat{k} = 0 \quad (\because \vec{a} \cdot \hat{i} = 1)$$

So, $\vec{a} = \vec{a} \cdot \hat{i} + \vec{a} \cdot \hat{j} + \vec{a} \cdot \hat{k}$

$$= \hat{i}$$

3. Question

Mark the correct alternative in each of the following:

If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then the angle between $\vec{a}$ and $\vec{b}$ is

A. $\frac{\pi}{6}$

B. $\frac{2\pi}{3}$

C. $\frac{5\pi}{3}$

D. $\frac{\pi}{3}$

Answer

Here, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ _____(1)

$$\Rightarrow \vec{a} \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0}$$

$$\Rightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = -\vec{a} \cdot \vec{a} = -|\vec{a}|^2$$

$$\Rightarrow \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = -9 \quad (\because |\vec{a}| = 3) \text{ _____(2)}$$

From (1)

$$\Rightarrow \vec{b} \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0}$$

$$\Rightarrow \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{b} \cdot \vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} = -\vec{b} \cdot \vec{b} = -|\vec{b}|^2$$

$$\Rightarrow \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} = -25 \quad (\because |\vec{b}| = 5) \text{ _____(3)}$$

From (1)

$$\Rightarrow \vec{c} \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0}$$

$$\Rightarrow \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{c} = \vec{0}$$

$$\Rightarrow \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} = -\vec{c} \cdot \vec{c} = -|\vec{c}|^2$$

$$\Rightarrow \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} = -49 \quad (\because |\vec{c}| = 7) \text{ _____(4)}$$

From (2) and (3)

$$\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = 24 \text{ _____(5)}$$

From (2) and (5)

$$2(\vec{a} \cdot \vec{b}) = 15$$

$$\Rightarrow \vec{a} \cdot \vec{b} = \frac{15}{2}$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$

Then, $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$$= \frac{\frac{15}{2}}{3 \times 5}$$

$$= \frac{1}{2}$$

So, $\theta = \frac{\pi}{3}$

i.e. angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$.

4. Question

Mark the correct alternative in each of the following:

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and α be the angle between them, then $\vec{a} + \vec{b}$ is a unit vector, if

A. $\alpha = \frac{\pi}{4}$

B. $\alpha = \frac{\pi}{3}$

C. $\alpha = \frac{2\pi}{3}$

D. $\alpha = \frac{\pi}{2}$

Answer

Here, $\vec{a}$ and $\vec{b}$ are unit vectors.

i.e. $|\vec{a}| = 1$ and $|\vec{b}| = 1$

If $\vec{a} + \vec{b}$ is unit vector then

$$|\vec{a} + \vec{b}| = 1$$

$$\Rightarrow |\vec{a} + \vec{b}|^2 = 1$$

$$\Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = 1 \quad (\because |\vec{a}|^2 = \vec{a} \cdot \vec{a})$$

$$\Rightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = 1$$

$$\Rightarrow 2(\vec{a} \cdot \vec{b}) + 2 = 1 \quad (\because |\vec{a}|^2 = \vec{a} \cdot \vec{a} = 1; |\vec{b}|^2 = \vec{b} \cdot \vec{b} = 1; \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a})$$

$$\Rightarrow (\vec{a} \cdot \vec{b}) = -\frac{1}{2}$$

Now, $\cos \alpha = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$$= -\frac{1}{2}$$

We know $\cos \frac{\pi}{3} = \frac{1}{2}$ and cosine is negative in second quadrant.

$$\therefore \alpha = \pi - \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

5. Question

Mark the correct alternative in each of the following:

The vector $(\cos \alpha + \cos \beta)\hat{i} + (\cos \alpha + \sin \beta)\hat{j} + (\sin \alpha)\hat{k}$ is a

- A. null vector
- B. unit vector
- C. constant vector
- D. none of these

Answer

$$\text{Let } \vec{a} = (\cos \alpha \cos \beta) \hat{i} + (\cos \alpha \sin \beta) \hat{j} + (\sin \alpha) \hat{k}$$

$$\text{So, } |\vec{a}|^2 = (\cos \alpha \cos \beta)^2 + (\cos \alpha \sin \beta)^2 + (\sin \alpha)^2$$

$$= \cos^2 \alpha (\cos^2 \beta + \sin^2 \beta) + \sin^2 \alpha$$

$$= \cos^2 \alpha (1) + \sin^2 \alpha$$

$$= 1$$

$$\text{i.e. } |\vec{a}| = 1$$

So, $\vec{a}$ is a unit vector.

6. Question

Mark the correct alternative in each of the following:

If the position vectors of P and Q are $\hat{i} + 3\hat{j} - 7\hat{k}$ and $5\hat{i} - 2\hat{j} + 4\hat{k}$ then the cosine of the angle between $\vec{PQ}$ and y-axis is

A. $\frac{5}{\sqrt{162}}$

B. $\frac{4}{\sqrt{162}}$

C. $-\frac{5}{\sqrt{162}}$

D. $\frac{11}{\sqrt{162}}$

Answer

Let $\vec{r}$ be the direction of $\vec{PQ}$

$$\text{Then, } \vec{r} = \vec{Q} - \vec{P} = 4\hat{i} - 5\hat{j} + 11\hat{k}$$

Let θ be the angle between $\vec{r}$ and Y-axis

$$\text{Then, } \cos \theta = \frac{\vec{r} \cdot \hat{j}}{|\vec{r}| |\hat{j}|}$$

$$= -\frac{5}{(16 + 25 + 121)^{\frac{1}{2}}}$$

$$= -\frac{5}{\sqrt{162}}$$

7. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ and $\vec{b}$ are unit vectors, then which of the following values of $\vec{a} \cdot \vec{b}$ is not possible?

- A. $\sqrt{3}$
- B. $\sqrt{3}/2$
- C. $1/\sqrt{2}$
- D. $-1/2$

Answer

Here, $\vec{a}$ and $\vec{b}$ are unit vectors.

i.e. $|\vec{a}| = 1$ and $|\vec{b}| = 1$

Now, Let θ be the angle between $\vec{a}$ and $\vec{b}$

$$\text{So, } \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\Rightarrow \vec{a} \cdot \vec{b} = \cos\theta$$

Now, we know $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$; \cos\frac{2\pi}{3} = -\frac{1}{2}$$

$$; \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

Therefore, $\vec{a} \cdot \vec{b} = \cos\theta = \sqrt{3}$ is not possible.

8. Question

Mark the correct alternative in each of the following:

If the vectors $\hat{i} - 2x\hat{j} + 2y\hat{k}$ and $\hat{i} + 2x\hat{j} - 3y\hat{k}$ are perpendicular, then the locus of (x, y) is

- A. a circle
- B. an ellipse
- C. a hyperbola
- D. none of these

Answer

Let $\vec{a} = \hat{i} - 2x\hat{j} + 2y\hat{k}$ and $\vec{b} = \hat{i} + 2x\hat{j} - 3y\hat{k}$

Given that $\vec{a}$ and $\vec{b}$ are perpendicular.

$$\text{So, } \vec{a} \cdot \vec{b} = 0$$

$$\Rightarrow 1 - 4x^2 - 6y^2 = 0$$

$$\Rightarrow 4x^2 + 6y^2 = 1$$

Here, vectors are in 3-Dimensions

$\therefore$ above equation represents an ellipse .i.e. locus of (x, y) is an ellipse.

9. Question

Mark the correct alternative in each of the following:

The vector component of $\vec{b}$ perpendicular to $\vec{a}$ is

A. $(\vec{b} \cdot \vec{c}) \vec{a}$

B. $\frac{\vec{a} \times (\vec{b} \times \vec{c})}{|\vec{a}|^2}$

C. $\vec{a} \times (\vec{b} \times \vec{c})$

D. none of these

Answer

Let $\vec{r}$ be the vector projection of $\vec{b}$ onto $\vec{a}$

Then, $\vec{r} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \cdot \vec{a}$

Now, vector component of $\vec{b}$ perpendicular to $\vec{a}$ is

$$\vec{x} = \vec{b} - \vec{r}$$

$$= \vec{b} - \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \cdot \vec{a}$$

$$= \frac{\vec{b}(\vec{a} \cdot \vec{a}) - (\vec{a} \cdot \vec{b})\vec{a}}{|\vec{a}|^2}$$

$$= \frac{\vec{a} \times (\vec{b} \times \vec{c})}{|\vec{a}|^2}$$

10. Question

Mark the correct alternative in each of the following:

The length of the longer diagonal of the parallelogram constructed on $5\vec{a} + 2\vec{b}$ and $\vec{a} - 3\vec{b}$ if its is given that $|\vec{a}| = 2\sqrt{2}$, $|\vec{b}| = 3$ and angle between $\vec{a}$ and $\vec{b}$ is $\pi/4$, is

A. 15

B. $\sqrt{113}$

C. $\sqrt{593}$

D. $\sqrt{369}$

Answer

Here, $|\vec{a}| = 2\sqrt{2}$ and $|\vec{b}| = 3$

The parallelogram is constructed on $5\vec{a} + 2\vec{b}$ and $\vec{a} - 3\vec{b}$

Then its one diagonal is $5\vec{a} + 2\vec{b} + \vec{a} - 3\vec{b} = 6\vec{a} - \vec{b}$

And other diagonal is $5\vec{a} + 2\vec{b} - \vec{a} + 3\vec{b} = 4\vec{a} + 5\vec{b}$

Length of one diagonal is $= |6\vec{a} - \vec{b}|$

$$= \{(6\vec{a} - \vec{b}) \cdot (6\vec{a} - \vec{b})\}^{\frac{1}{2}}$$

$$= \left(36\vec{a}^2 + \vec{b}^2 - 2 \times 6|\vec{a}||\vec{b}| \cos \frac{\pi}{4}\right)^{\frac{1}{2}}$$

$$= \left(36 \times 8 + 9 - 12 \times 2\sqrt{2} \times 3 \times \frac{1}{\sqrt{2}} \right)^{\frac{1}{2}}$$

$$= (288 + 9 - 12 \times 6)^{\frac{1}{2}}$$

$$= \sqrt{225}$$

$$= 15$$

Length of other diagonal is $= |4\vec{a} + 5\vec{b}|$

$$= \{(4\vec{a} + 5\vec{b}) \cdot (4\vec{a} + 5\vec{b})\}^{\frac{1}{2}}$$

$$= \left(16\vec{a}^2 + 25\vec{b}^2 + 2 \times 4 \times 5|\vec{a}||\vec{b}| \cos \frac{\pi}{4} \right)^{\frac{1}{2}}$$

$$= \left(16 \times 8 + 25 \times 9 + 40 \times 2\sqrt{2} \times 3 \times \frac{1}{\sqrt{2}} \right)^{\frac{1}{2}}$$

$$= (128 + 225 + 40 \times 6)^{\frac{1}{2}}$$

$$= \sqrt{593}$$

So, Length of the longest diagonal is $\sqrt{593}$.

11. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ is a non-zero vector of magnitude 'a' and λ is a non-zero scalar, then $\lambda\vec{a}$ is a unit vector if

A. $\lambda = 1$

B. $\lambda = -1$

C. $a = |\lambda|$

D. $a = \frac{1}{|\lambda|}$

Answer

Here, $|\vec{a}| = a$

Now, $\lambda\vec{a}$ is unit vector if $|\lambda\vec{a}| = 1$

i.e. $|\lambda||\vec{a}| = 1$

i.e. $|\lambda|a = 1$

i.e. $a = \frac{1}{|\lambda|}$

12. Question

Mark the correct alternative in each of the following:

If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a} \cdot \vec{b} \geq 0$ only when

A. $0 < \theta < \frac{\pi}{2}$

B. $0 \leq \theta \leq \frac{\pi}{2}$

C. $0 < \theta < \pi$

D. $0 \leq \theta \leq \pi$

Answer

Here, θ be the angle between $\vec{a}$ and $\vec{b}$

Then, $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$

Now, $\vec{a} \cdot \vec{b} \geq 0$

$\Rightarrow \cos\theta |\vec{a}| |\vec{b}| \geq 0$

$\Rightarrow \cos\theta \geq 0$

We know cosine is positive in first quadrant.

$\therefore 0 \leq \theta \leq \frac{\pi}{2}$

13. Question

Mark the correct alternative in each of the following:

The values of x for which the angle between $\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$, $\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse and the angle between $\vec{b}$ and the z -axis is acute and less than $\pi/6$ are

A. $x > \frac{1}{2}$ or $x < 0$

B. $0 < x < \frac{1}{2}$

C. $\frac{1}{2} < x < 15$

D. ϕ

Answer

Here, angle between $\vec{a}$ and $\vec{b}$ is obtuse

So, $\vec{a} \cdot \vec{b} \leq 0$

$\Rightarrow 14x^2 - 8x + x \leq 0$

$\Rightarrow 14x^2 - 7x \leq 0$

$\Rightarrow 2x^2 - x \leq 0$

$\Rightarrow x(2x - 1) \leq 0$

$\Rightarrow x \leq 0 \text{ and } x \geq \frac{1}{2}$

or $x \geq 0 \text{ and } x \leq \frac{1}{2}$ _____ (1)

Now, angle between $\vec{b}$ and Z -axis is acute

So, $\vec{b} \cdot \hat{k} \geq 0$

$\Rightarrow x \geq 0$ _____ (2)

∴ From (1) and (2) $0 \leq x \leq \frac{1}{2}$.

14. Question

Mark the correct alternative in each of the following:

If $\vec{a}, \vec{b}, \vec{c}$ are any three mutually perpendicular vectors of equal magnitude a , then $|\vec{a} + \vec{b} + \vec{c}|$ is equal to

- A. a
- B. $\sqrt{2}a$
- C. $\sqrt{3}a$
- D. $2a$

Answer

We know that,

$$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a} \quad (i)$$

Since, they are mutually perpendicular vectors

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0 \quad (ii)$$

And according to question

$$|\vec{a}| = |\vec{b}| = |\vec{c}|$$

Using (i) and (ii) ,

$$\begin{aligned} |\vec{a} + \vec{b} + \vec{c}| &= \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2} \\ &= \sqrt{3}|\vec{a}| \text{ Ans.} \end{aligned}$$

Smart Approach

In case of such mutually perpendicular vectors, assume vectors to be $\hat{i}, \hat{j}, \hat{k}$ and verify your answer from options.

15. Question

Mark the correct alternative in each of the following:

If the vectors $3\hat{i} + \lambda\hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} + 8\hat{k}$ are perpendicular, then λ is equal to

- A. -14
- B. 7
- C. 14
- D. $\frac{1}{7}$

Answer

We have,

$$\vec{a} = 3\hat{i} + \lambda\hat{j} + \hat{k}$$

$$\vec{b} = 2\hat{i} - \hat{j} + 8\hat{k}$$

Given that $\vec{a}$ and $\vec{b}$ are perpendicular

$$\Rightarrow \vec{a} \cdot \vec{b} = 0$$

$$\Rightarrow (3\vec{i} + \lambda\vec{j} + \vec{k}) \cdot (2\vec{i} - \vec{j} + 8\vec{k}) = 0$$

$$\Rightarrow 6 - \lambda + 8 = 0$$

$$\therefore \lambda = 14 \text{ Ans.}$$

16. Question

Mark the correct alternative in each of the following:

The projection of the vector $\hat{i} + \hat{j} + \hat{k}$ along the vector $\hat{j}$ is

- A. 1
- B. 0
- C. 2
- D. -1

Answer

Projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

Projection of $\hat{i} + \hat{j} + \hat{k}$ on $\hat{j}$ is

$$\frac{(\hat{i} + \hat{j} + \hat{k}) \cdot \hat{j}}{|\hat{j}|}$$

$$= \frac{1}{1}$$

$$= 1 \text{ Ans.}$$

17. Question

Mark the correct alternative in each of the following:

The vectors $2\hat{i} + 3\hat{j} - 4\hat{k}$ and $a\hat{i} + b\hat{j} + c\hat{k}$ are perpendicular, if

- A. $a = 2, b = 3, c = -4$
- B. $a = 4, b = 4, c = 5$
- C. $a = 4, b = 4, c = -5$
- D. $a = -4, b = 4, c = -5$

Answer

The given two vectors,

$\Rightarrow$ Their dot-product is zero

$$\Rightarrow (2\hat{i} + 3\hat{j} - 4\hat{k}) \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = 0$$

$$2a + 3b - 4c = 0$$

From the given options only option B satisfies the above equation

Hence option B is correct answer.

18. Question

Mark the correct alternative in each of the following:

If $|\vec{a}| = |\vec{b}|$, then $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) =$

- A. positive
- B. negative
- C. 0
- D. none of these

Answer

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 + (\vec{b} \cdot \vec{a}) - (\vec{a} \cdot \vec{b}) - |\vec{b}|^2$$

$$= |\vec{a}|^2 - |\vec{b}|^2 \quad (|\vec{a}| = |\vec{b}|)$$

= 0 Ans.

19. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ and $\vec{b}$ are unit vectors inclined at an angle θ , then the value of $|\vec{a} - \vec{b}|$ is

- A. $2 \sin \frac{\theta}{2}$
- B. $2 \sin \theta$
- C. $2 \cos \frac{\theta}{2}$
- D. $2 \cos \theta$

Answer

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta}$$

Given that,

$$|\vec{a}| = |\vec{b}| = 1$$

$$|\vec{a} - \vec{b}| = \sqrt{2 - 2\cos\theta} \left\{ (1 - \cos\theta) = 2\sin^2 \frac{\theta}{2} \right\}$$

$$|\vec{a} - \vec{b}| = \sqrt{(2)2\sin^2 \frac{\theta}{2}}$$

$$|\vec{a} - \vec{b}| = \left| 2\sin \frac{\theta}{2} \right| \text{ Ans.}$$

20. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ and $\vec{b}$ are unit vectors, then the greatest value of $\sqrt{3}|\vec{a} + \vec{b}| + |\vec{a} - \vec{b}|$ is

- A. 2
- B. $2\sqrt{2}$
- C. 4

D. none of these

Answer

If $\vec{a}$ and $\vec{b}$ are unit vector then

$$|\vec{a} + \vec{b}| = \left| 2\cos\frac{\theta}{2} \right|$$

$$|\vec{a} - \vec{b}| = \left| 2\sin\frac{\theta}{2} \right|$$

$$\sqrt{3}|\vec{a} + \vec{b}| + |\vec{a} - \vec{b}| = 2\sqrt{3}\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2}$$

Maximum value of $a\sin\theta + b\cos\theta$ is $\sqrt{a^2 + b^2}$

Maximum value of $2\sqrt{3}\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2}$ is 4 Ans.

21. Question

Mark the correct alternative in each of the following:

If the angle between the vectors $x\hat{i} + 3\hat{j} - 7\hat{k}$ and $x\hat{i} - x\hat{j} + 4\hat{k}$ is acute, then x lies in the interval.

A. $(-4, 7)$

B. $[-4, 7]$

C. $\mathbb{R} - [4, 7]$

D. $\mathbb{R} - (4, 7)$

Answer

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

If the angle is acute $\cos\theta > 0$

$$\Rightarrow \vec{a} \cdot \vec{b} > 0$$

$$\Rightarrow (x\hat{i} + 3\hat{j} - 7\hat{k}) \cdot (x\hat{i} - x\hat{j} + 4\hat{k}) > 0$$

$$\Rightarrow x^2 - 3x - 28 > 0$$

$$\Rightarrow (x-7)(x+4) > 0$$

$$\Rightarrow x \in \mathbb{R} - (-4, 7) \text{ Ans.}$$

22. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at an angle θ such that $|\vec{a} + \vec{b}| < 1$, then

A. $\theta < \frac{\pi}{3}$

B. $\theta > \frac{2\pi}{3}$

C. $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$

D. $\frac{2\pi}{3} < \theta < \pi$

Answer

We know that,

If $\vec{a}$ and $\vec{b}$ are two-unit vectors inclined at an angle θ

$$|\vec{a} + \vec{b}| = \left| 2\cos\frac{\theta}{2} \right|$$

According to question,

$$|\vec{a} + \vec{b}| < 1$$

$$\Rightarrow \left| 2\cos\frac{\theta}{2} \right| < 1$$

$$\Rightarrow \frac{-1}{2} < \cos\frac{\theta}{2} < \frac{1}{2}$$

$$\Rightarrow \frac{2\pi}{3} > \frac{\theta}{2} > \frac{\pi}{3}$$

$$\Rightarrow \frac{2\pi}{3} < \theta < \frac{4\pi}{3} \text{ Ans.}$$

23. Question

Mark the correct alternative in each of the following:

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three unit vectors such that $|\vec{a} + \vec{b} + \vec{c}| = 1$ and $\vec{a}$ is perpendicular to $\vec{b}$. If $\vec{c}$ makes angle α and β with $\vec{a}$ and $\vec{b}$ respectively, then $\cos \alpha + \cos \beta =$

A. $-\frac{3}{2}$

B. $\frac{3}{2}$

C. 1

D. -1

Answer

We know that,

$$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a} \quad (i)$$

Since,

$\vec{a}$ is perpendicular to $\vec{b}$

$$\Rightarrow \vec{a} \cdot \vec{b} = 0$$

And according to question

$$|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$$

We can rewrite equation (i) as

$$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 0 + 2\cos \beta + 2\cos \alpha$$

$$1 = 1 + 1 + 1 + 0 + 2(\cos \alpha + \cos \beta)$$

$$\Rightarrow \cos \alpha + \cos \beta = -1 \text{ Ans.}$$

24. Question

Mark the correct alternative in each of the following:

The orthogonal projection of $\vec{a}$ and $\vec{b}$ is

A. $\frac{(\vec{a} \cdot \vec{b})\vec{a}}{|\vec{a}|^2}$

B. $\frac{(\vec{a} \cdot \vec{b})\vec{b}}{|\vec{b}|^2}$

C. $\frac{\vec{a}}{|\vec{a}|^2}$

D. $\frac{\vec{b}}{|\vec{b}|^2}$

Answer

Key Concept/Trick: Magnitude of Projection of any vector $\vec{a}$ on $\vec{b}$

is given by $\vec{a} \cdot \hat{b}$

Now, Since it is the magnitude or length($a \cos \theta$) we have to give the length a direction in the direction of $\vec{b}$

So, we multiply the projection by unit vector of $\vec{b}$

$(\vec{a} \cdot \hat{b}) \cdot \hat{b}$ which on simplification gives option B Ans.

25. Question

Mark the correct alternative in each of the following:

If θ is an acute angle and the vector $(\sin \theta)\hat{i} + (\cos \theta)\hat{j}$ is perpendicular to the vector $\hat{i} - \sqrt{3}\hat{j}$, then $\theta =$

A. $\frac{\pi}{6}$

B. $\frac{\pi}{5}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{3}$

Answer

Since, the given two vectors are given as perpendicular their dot product must be zero

$$((\sin \theta)\hat{i} + (\cos \theta)\hat{j}) \cdot (\hat{i} - \sqrt{3}\hat{j}) = 0$$

$$\sin \theta - \sqrt{3} \cos \theta = 0$$

$$\tan \theta = \sqrt{3}$$

Since θ is acute then, $\theta = \frac{\pi}{3}$ Ans

26. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a} + \vec{b}$ is a unit vector, if $\theta =$

A. $\frac{\pi}{4}$

B. $\frac{\pi}{3}$

C. $\frac{\pi}{2}$

D. $\frac{2\pi}{3}$

Answer

We know that,

$$|\vec{a} + \vec{b}| = \left| 2\cos \frac{\theta}{2} \right|$$

According to Question,

$$|\vec{a} + \vec{b}| = 1$$

$$\Rightarrow \left| 2\cos \frac{\theta}{2} \right| = 1$$

$$\Rightarrow \cos \frac{\theta}{2} = \frac{1}{2}$$

$$\Rightarrow \frac{\theta}{2} = \frac{\pi}{3}$$

$$\theta = \frac{2\pi}{3} \text{ Ans.}$$