3. Integration (Integral Calculus)

Integration Rules

Integral of a Function

A function ?(x) is called a primitive or an antiderivative of a function f(x), if **?'(x) = f(x)**.

Let f(x) be a function. Then the collection of all its primitives is called the indefinite integral of f(x) and is denoted by ∫f(x) dx.
Thus,

$$\frac{d}{dx}(\phi(x)+c)=f(x) \Rightarrow \int f(x)\,dx=\phi(x)+c$$

where ?(x) is primitive of f(x) and c is an arbitrary constant known as the constant of integration.

Common Functions	Function	Integral
Constant	$\int a\,dx$	$ax + C$
Variable	$\int x\,dx$	$x^2/2 + C$
Square	$\int x^2\,dx$	$x^3/3 + C$
Reciprocal	$\int (1/x)\,dx$	$\ln\lvert x\rvert + C$
Exponential	$\int e^x\,dx$	$e^x + C$
	$\int a^x\,dx$	$a^x/\ln(a) + C$
	$\int \ln(x)\,dx$	$x \ln(x) - x + C$
Trigonometry (x in radians)	$\int \cos(x)\,dx$	$\sin(x) + C$
	$\int \sin(x)\,dx$	$-\cos(x) + C$
	$\int \sec^2(x)\,dx$	$\tan(x) + C$

Integration Rules

1. **Chain rule :**
   $\int u.v\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots\ldots + (-1)^{n-1} u^{n-1}v_n + (-1)^n \int u^n.v_n\,dx$
   Where stands for n^{th} differential coefficient of u and stands for n^{th} integral of v.
2. **Sum Rule**
   $\int(f + g)\,dx = \int f\,dx + \int g\,dx$
3. **Difference Rule**
   $\int(f - g)\,dx = \int f\,dx - \int g\,dx$
4. **Multiplication by constant**
   $\int cf(x)\,dx = c\int f(x)\,dx$
5. **Power Rule (n≠-1)**
   $\int x^n\,dx = x^{n+1}/(n+1) + C$

Fundamental Integration Formulae

(1) (i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

(ii) $\int (ax+b)^n dx = \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1} + c\ ,\ n \neq -1$

(2) (i) $\int \frac{1}{x} dx = \log|\ x|\ + c$

(ii) $\int \frac{1}{ax+b} dx = \frac{1}{a} (\log|\ ax+b|) \ + c$

(3) $\int e^x dx = e^x + c$

(4) $\int a^x dx = \frac{a^x}{\log_e a} + c$

(5) $\int \sin x \, dx = -\cos x + c$

(6) $\int \cos x \, dx = \sin x + c$

(7) $\int \sec^2 x \, dx = \tan x + c$

(8) $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

(9) $\int \sec x \tan x \, dx = \sec x + c$

(10) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

(11) $\int \tan x \, dx = -\log|\cos x| + c = \log|\sec x| + c$

(12) $\int \cot x \, dx = \log|\sin x| + c = -\log|\operatorname{cosec} x| + c$

(13) $\int \sec x \, dx = \log|\sec x + \tan x| + c = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(14) $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + c = \log \tan \frac{x}{2} + c$

(15) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c = -\cos^{-1} x + c$

(16) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c = -\cos^{-1} \frac{x}{a} + c$

(17) $\int \frac{dx}{1 + x^2} = \tan^{-1} x + c = -\cot^{-1} x + c$

(18) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c = \frac{-1}{a} \cot^{-1} \frac{x}{a} + c$

(19) $\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x + c = -\operatorname{cosec}^{-1} x + c$

(20) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + c = \frac{-1}{a} \operatorname{cosec}^{-1} \frac{x}{a} + c$

In any of the fundamental integration formulae, if x is replaced by ax+b, then the same formulae is applicable but we must divide by coefficient of x or derivative of (ax+b) i.e., a. In general, if ∫f(x) dx = φ(x) + c, then

$$\int f(ax+b)\,dx = \frac{1}{a}\phi(ax+b)+c$$

$$\int \sin(ax+b)\,dx = \frac{-1}{a}\cos(ax+b)+c,$$

$$\int \sec(ax+b)\,dx = \frac{1}{a}\log|\sec(ax+b)+\tan(ax+b)| + c \quad etc.$$

Some more Results

(i) $$\int \frac{1}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c = \frac{-1}{a}\coth^{-1}\frac{x}{a}+c,$$

when $x > a$

(ii) $$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c = \frac{1}{a}\tanh^{-1}\frac{x}{a}+c,$$

when $x < a$

(iii) $$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\{|x+\sqrt{x^2-a^2}|\} + c = \cosh^{-1}\left(\frac{x}{a}\right)+c$$

(iv) $$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\{|x+\sqrt{x^2+a^2}|\} + c = \sinh^{-1}\left(\frac{x}{a}\right)+c$$

(v) $$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)+c$$

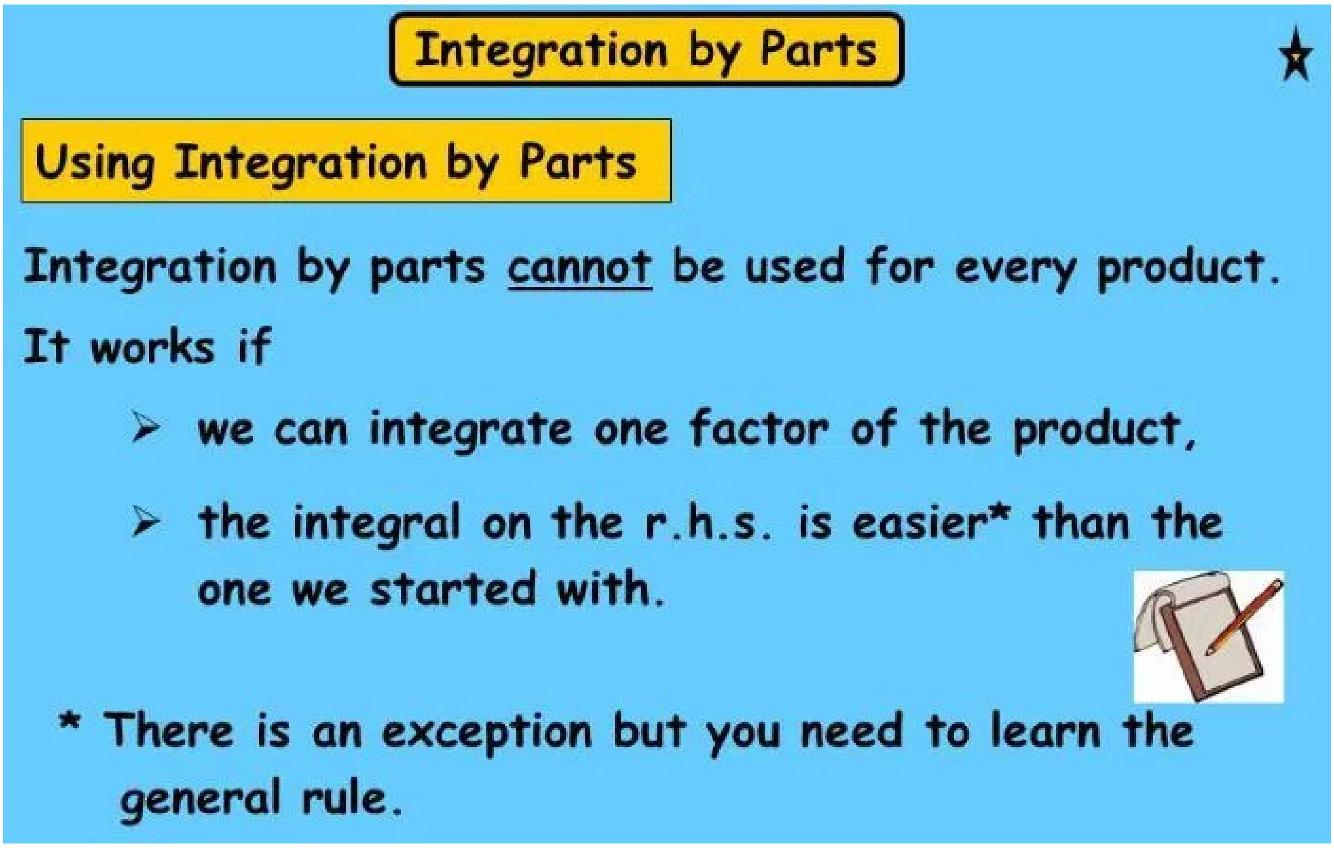
(vi) $$\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\log\{x+\sqrt{x^2-a^2}\}+c$$

$$= \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\cosh^{-1}\left(\frac{x}{a}\right)+c$$

(vii) $$\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2\log\{x+\sqrt{x^2+a^2}\}+c$$

$$= \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2\sinh^{-1}\left(\frac{x}{a}\right)$$

Integration by Parts



(1) When integrand involves more than one type of functions:

We may solve such integrals by a rule which is known as **integration by parts**. We know that,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow d(uv) = udv + vdu \Rightarrow \int d(uv) = \int udv + \int vdu$$

If u and v are two functions of x, then

$$\int \underset{I\ II}{uv}\, dx = u \int v\, dx - \int \{\frac{du}{dx}.\int v dx\} dx$$

i.e., the integral of the product of two functions = (First function) × (Integral of second function) – Integral of {(Differentiation of first function) × (Integral of second function)}
Before applying this rule proper choice of first and second function is necessary. Normally we use the following methods :

1. In the product of two functions, one of the function is not directly integrable (i.e., log|x|, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$……etc), then we take it as the first function and the remaining function is taken as the second function.
2. If there is no other function, then unity is taken as the second function e.g. in the integration of $\int \sin^{-1}x\, dx$, $\int \log x\, dx$, 1 is taken as the second function.
3. If both of the function are directly integrable then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable. Usually, we use the following preference order for the first function.
(Inverse, Logarithmic, Algebraic, Trigonometric, Exponential). This rule is simply called as "**ILATE**".

(2) Integral is of the form $\int e^x \{f(x)+f'(x)\}\, dx$:

If the integral is of the form $\int e^x \{f(x)+f'(x)\}\, dx$ then by breaking this integral into two integrals integrate one integral by parts and keeping other integral as it is, by doing so, we get

(i) $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$

(ii) $\int e^{mx} [mf(x) + f'(x)] dx = e^{mx} f(x) + c$

(iii) $\int e^{mx} \left[f(x) + \frac{f'(x)}{m} \right] dx = \frac{e^{mx} f(x)}{m} + c$

(3) Integral is of the form ∫[x f'(x)+f(x)] dx:

If the integral is of the form ∫[x f'(x)+f(x)] dx then by breaking this integral into two integrals, integrate one integral by parts and keeping other integral as it is, by doing so, we get,
∫[x f'(x)+f(x)] dx = x f(x) + c

(4) Integrals of the form ∫e^{ax} sin bx dx ∫e^{ax} cos bx dx:

$$\int e^{ax} \sin bx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin(bx - \tan^{-1} \frac{b}{a}) + c$$

$$\int e^{ax} . \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + c$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos\left(bx - \tan^{-1} \frac{b}{a}\right) + c$$

$$\int e^{ax} . \sin(bx + c) dx = \frac{e^{ax}}{a^2 + b^2}[a \sin(bx + c) - b \cos(bx + c)] + k$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin\left[(bx + c) - \tan^{-1}\left(\frac{b}{a}\right)\right] + k$$

$$\int e^{ax} . \cos(bx + c) dx = \frac{e^{ax}}{a^2 + b^2}[a \cos(bx + c) - b \sin(bx + c)] + k$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos\left[(bx + c) - \tan^{-1}\left(\frac{b}{a}\right)\right] + k$$

Integration by Substitution

(1) When integrand is a function i.e.,∫f[?(x)] ?'(x) dx:

Here, we put ?(x)=t, so that ?'(x) dx=dt and in that case the integrand is reduced to ∫f(t) dt.

(2) When integrand is the product of two factors such that one is the derivative of the others i.e., I = ∫f(x)f'(x) dx:

In this case we put f(x)=t and convert it into a standard integral.

(3) Integral of a function of the form f(ax + b):
Here we put ax + b = t and convert it into standard integral. Obviously if ∫f(x) dx = ?(x) then ∫f(ax +

b) dx = $\frac{1}{a}$?(ax + b) + c.

(4) If integral of a function of the form $\frac{f'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}dx = \log[f(x)] + c$$

(5) If integral of a function of the form $[f(x)]^n f'(x)$

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c, \quad [n \neq -1]$$

(6) If the integral of a function of the form $\frac{f'(x)}{\sqrt{f(x)}}$

$$\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$$

(7) Standard substitutions

	Integrand form	Substitution
(i)	$\sqrt{a^2 - x^2}$, $\frac{1}{\sqrt{a^2 - x^2}}$, $a^2 - x^2$	$x = a\sin\theta$, or $x = a\cos\theta$
(ii)	$\sqrt{x^2 + a^2}$, $\frac{1}{\sqrt{x^2 + a^2}}$, $x^2 + a^2$	$x = a\tan\theta$ or $x = a\sinh\theta$
(iii)	$\sqrt{x^2 - a^2}$, $\frac{1}{\sqrt{x^2 - a^2}}$, $x^2 - a^2$	$x = a\sec\theta$ or $x = a\cosh\theta$
(iv)	$\sqrt{\frac{x}{a+x}}$, $\sqrt{\frac{a+x}{x}}$, $\sqrt{x(a+x)}$, $\frac{1}{\sqrt{x(a+x)}}$	$x = a\tan^2\theta$
(v)	$\sqrt{\frac{x}{a-x}}$, $\sqrt{\frac{a-x}{x}}$, $\sqrt{x(a-x)}$, $\frac{1}{\sqrt{x(a-x)}}$	$x = a\sin^2\theta$
(vi)	$\sqrt{\frac{x}{x-a}}$, $\sqrt{\frac{x-a}{x}}$, $\sqrt{x(x-a)}$, $\frac{1}{\sqrt{x(x-a)}}$	$x = a\sec^2\theta$
(vii)	$\sqrt{\frac{a-x}{a+x}}$, $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
(viii)	$\sqrt{\frac{x-\alpha}{\beta-x}}$, $\sqrt{(x-\alpha)(\beta-x)}$, $(\beta > \alpha)$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

Integration by Substitution Problems with Solutions

1.

$$\int \frac{dx}{1+e^x} =$$

(a) $\log(1+e^x)$ (b) $-\log(1+e^{-x})$

(c) $-\log(1-e^{-x})$ (d) $\log(e^{-x}+e^{-2x})$

Solution:

$$\int \frac{dx}{1+e^x} = \int \frac{e^{-x}}{1+e^{-x}}\,dx$$

Put $1+e^{-x} = t \Rightarrow e^{-x}dx = -dt$, then it reduces to

$$-\int \frac{dt}{t} = -\log t = -\log(1+e^{-x}).$$

2.

$$\int \frac{dx}{e^x + e^{-x}} =$$

(a) $\tan^{-1}(e^{-x})$ (b) $\tan^{-1}(e^x)$

(c) $\log(e^x - e^{-x})$ (d) $\log(e^x + e^{-x})$

Solution:

$$\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{dt}{t^2+1} = \tan^{-1}(t)$$

$= \tan^{-1}(e^x)+c$, $\{$Putting $e^x = t \Rightarrow e^x dx = dt\}$.

3.

$$\int \frac{dx}{x + x\log x} =$$

(a) $\log(1+\log x)$ (b) $\log\log(1+\log x)$

(c) $\log x + \log(\log x)$ (d) None of these

Solution:

(a) $$\int \frac{dx}{x+x\log x} = \int \frac{dx}{x(1+\log x)}$$

Now putting $1+\log x = t \Rightarrow \frac{1}{x}dx = dt$, it reduces to

$$\int \frac{dt}{t} = \log(t) = \log(1+\log x).$$

4.

$$\int \frac{\sin 2x}{1+\sin^2 x}\,dx =$$

(a) $\log \sin 2x + c$ (b) $\log(1+\sin^2 x)+c$

(c) $\frac{1}{2}\log(1+\sin^2 x)+c$ (d) $\tan^{-1}(\sin x)+c$

Solution:

(b) Put $(1+\sin^2 x)=t \Rightarrow \sin 2x\,dx = dt$

Hence $\int \frac{\sin 2x}{1+\sin^2 x}\,dx = \int \frac{1}{t}\,dt = \log(1+\sin^2 x)+c.$

5.

$$\int \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx =$$

(a) $\cot^{-1}(\tan^2 x)+c$ (b) $\tan^{-1}(\tan^2 x)+c$

(c) $\cot^{-1}(\cot^2 x)+c$ (d) $\tan^{-1}(\cot^2 x)+c$

Solution:

(b) $\int \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$

$$= \int \frac{2\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx = \int \frac{2\tan x\sec^2 x}{1+\tan^4 x}\,dx$$

Put $\tan^2 x = t \Rightarrow 2\tan x\sec^2 x\,dx = dt$, then it reduced to $\int \frac{dt}{1+t^2} = \tan^{-1} t + c = \tan^{-1}(\tan^2 x)+c$.

Trick : By inspection,

$$\frac{d}{dx}\{\cot^{-1}(\tan^2 x)\} = -\frac{1(2\tan x.\sec^2 x)}{1+\tan^4 x} = -\frac{\sin 2x}{\cos^4 x+\sin^4 x}$$

$$\Rightarrow \frac{d}{dx}\{\tan^{-1}(\tan^2 x)\} = \frac{\sin 2x}{\sin^4 x+\cos^4 x}\ .$$

Definite Integrals

Let ?(x) be the primitive or anti-derivative of a function f(x) defined on [a, b] i.e., $\frac{d}{dx}[\phi(x)] = f(x)$. Then the definite integral of f(x) over [a,b] is denoted by $\int_a^b f(x)dx$ and is defined as [?(b) − ?(a)] i.e., $\int_a^b f(x)dx = \phi(b) - \phi(a)$. This is also called *Newton Leibnitz* formula.

The numbers a and b are called the limits of integration, 'a' is called the lower limit and 'b' the upper limit. The interval [a, b] is called the interval of integration. The interval [a, b] is also known as range of integration. Every definite integral has a unique value.

Evaluation of definite integral by substitution

When the variable in a definite integral is changed, the substitutions in terms of new variable should be effected at three places.
(i) In the integrand (ii)In the differential i.e., dx (iii) In the limits

For example, if we put $\phi(x) = t$ in the integral $\int_a^b f\{\phi(x)\}\phi'(x)dx$,

then $\int_a^b f\{\phi(x)\}\phi'(x)dx = \int_{\phi(a)}^{\phi(b)} f(t)\,dt$.

Properties of definite integral

(1) $\int_a^b f(x)dx = \int_a^b f(t)dt$ *i.e.,* The value of a definite integral remains unchanged if its variable is replaced by any other symbol.

(2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ *i.e.,* by the interchange in the limits of definite integral, the sign of the integral is changed.

(3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, (where $a < c < b$)

or $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + ... + \int_{c_n}^b f(x)dx$;

(where $a < c_1 < c_2 < .... c_n < b$)

Generally this property is used when the integrand has two or more rules in the integration interval. This is useful when is not continuous in [a, b] because we can break up the integral into several integrals at the points of discontinuity so that the function is continuous in the sub-intervals.

(4) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$:

This property can be used only when lower limit is zero. It is generally used for those complicated integrals whose denominators are unchanged when x is replaced by (a – x).
Following integrals can be obtained with the help of above property.

(i) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}\,dx = \frac{\pi}{4}$

(ii) $\int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x}\,dx = \int_0^{\pi/2} \frac{\cot^n x}{1+\cot^n x}\,dx = \frac{\pi}{4}$

(iii) $\int_0^{\pi/2} \frac{1}{1+\tan^n x}\,dx = \int_0^{\pi/2} \frac{1}{1+\cot^n x}\,dx = \frac{\pi}{4}$

(iv) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \operatorname{cosec}^n x}\,dx = \int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\operatorname{cosec}^n x + \sec^n x}\,dx = \frac{\pi}{4}$

(v) $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx$

(vi) $\int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f(\cos x)\,dx$

(vii) $\int_0^{\pi/4} \log(1+\tan x)\,dx = \frac{\pi}{8}\log 2$

(viii) $\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx = \frac{-\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}$

(ix) $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} \frac{a\sec x + b\operatorname{cosec} x}{\sec x + \operatorname{cosec} x}\,dx$

$$= \int_0^{\pi/2} \frac{a\tan x + b\cot x}{\tan x + \cot x}\,dx = \frac{\pi}{4}(a+b)$$

(5) $\int_{-a}^{a} f(x)\,dx = \int_0^{a}[f(x)+f(-x)]\,dx$.

In special case :

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even function or } f(-x)=f(x) \\ 0, & \text{if } f(x) \text{ is odd function or } f(-x)=-f(x) \end{cases}$$

This property is generally used when integrand is either even or odd function of x.

(6) $\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)\,dx$

In particular, $\int_0^{2a} f(x)\,dx = \begin{cases} 0, & \text{if } f(2a-x)=-f(x) \\ 2\int_0^a f(x)\,dx, & \text{if } f(2a-x)=f(x) \end{cases}$

It is generally used to make half the upper limit.

(7) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)dx$.

(8) $\int_0^a x\,f(x)dx = \frac{1}{2}a\int_0^a f(x)dx$, if $f(a-x)=f(x)$.

(9) If $f(x)$ is a periodic function with period T, then

$$\int_0^{nT} f(x)dx = n\int_0^{T} f(x)dx$$

Deduction : If $f(x)$ is a periodic function with period T, then

$$\int_a^{a+nT} f(x)\,dx = n\int_0^{T} f(x)\,dx \quad \text{, where } n \in I$$

(a) If $a = 0$, $\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$, where $n \in I$

(b) If $n = 1$, $\int_0^{a+T} f(x)dx = \int_0^{T} f(x)dx$.

(10) $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^{T} f(x)dx$, where n, $m \in I$.

(11) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^{b} f(x)dx$, where $n \in I$.

(12) $\int_0^{2k} (x-[x])dx = k$, where k an integer, since $x-[x]$ is a periodic function with period 1.

(13) If $f(x)$ is a periodic function with period T, then $\int_a^{a+T} f(x)$ is independent of a.

(14) $\int_a^{b} f(x)dx = (b-a)\int_0^{1} f((b-a)x+a)dx$.

Summation of series by integration

We know that $\int_a^{b} f(x)dx = \lim_{n\to\infty} h\sum_{r=1}^{n} f(a+rh)$, where $nh = b-a$

Now, put $a = 0, b = 1, \therefore nh = 1$ or $h = \frac{1}{n}$.

Hence $\int_0^{1} f(x)dx = \lim_{n\to\infty} \frac{1}{n}\sum f\left(\frac{r}{n}\right)$.

Express the given series in the form $\sum \frac{1}{n} f\left(\frac{r}{h}\right)$.

Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and the limit of the sum is $\int_0^{1} f(x)dx$.

Gamma function

$\int_0^\infty x^{n-1}e^{-x}dx$, $n > 0$ is called Gamma function and denoted by Γn.

If m and n are non-negative integers, then

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

where $\Gamma(n)$ is called gamma function which satisfy the following properties $\Gamma(n+1) = n\Gamma(n) = n!$ *i.e.*, $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$

In place of gamma function, we can also use the following formula

$$\int_0^{\pi/2} \sin^m x \cos^n x dx$$

$$= \frac{(m-1)(m-3).....(2 \text{ or } 1)(n-1)(n-3).....(2 \text{ or } 1)}{(m+n)(m+n-2)....(2 \text{ or } 1)}$$

It is important to note that we multiply by (π/2); when both m and n are even.

Reduction formulae for definite integration

(1) $\int_0^\infty e^{-ax} \sin bx dx = \frac{b}{a^2 + b^2}$

(2) $\int_0^\infty e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2}$

(3) $\int_0^\infty e^{-ax} x^n dx = \frac{n!}{a^n + 1}$

Walli's formula

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx$$

$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3}, & \text{when } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots\cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, & \text{when } n \text{ is even} \end{cases}$$

$$\int_0^{\pi/2} \sin^m x \cos^n dx = \frac{(m-1)(m-3).....(n-1)(n-3)....}{(m+n)(m+n-2)...(2 \text{ or } 1)},$$

[If m, n are both odd positive integers or one odd positive integer]

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{(m-1)(m-3)......... ...(n-1)(n-3)}{(m+n)(m+n-2)........ (2 \text{ or } 1)} \cdot \frac{\pi}{2},$$

[If m, n are both positive integers]

Leibnitz's rule

(1) If f(x) is continuous and u(x), v(x) are differentiable functions in the interval [a, b], then,

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f\{v(x)\}\frac{d}{dx}\{v(x)\} - f\{u(x)\}\frac{d}{dx}\{u(x)\}.$$

(2) If the function and are defined on [a, b] and differentiable at a point and is continuous, then,

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)\,dt\right] = \int_{\phi(x)}^{\psi(x)} \frac{d}{dx} f(x,t)dt + \left\{\frac{d\psi(x)}{dx}\right\} f(x,\psi(x)) - \left\{\frac{d\phi(x)}{dx}\right\} f(x,\phi(x)).$$

Some important results of definite integral

(1) If $I_n = \int_0^{\pi/4} \tan^n x dx$ then $I_n + I_{n-2} = \frac{1}{n-1}$

(2) If $I_n = \int_0^{\pi/4} \cot^n x dx$ then $I_n + I_{n-2} = \frac{1}{1-n}$

(3) If $I_n = \int_0^{\pi/4} \sec^n x\, dx$ then $I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$

(4) If $I_n = \int_0^{\pi/4} \text{cosec}^n x\, dx$ then $I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$

(5) If $I_n = \int_0^{\pi/2} \sin^n x\, dx$, then $I_n = \frac{n-1}{n} I_{n-2}$

(6) If $I_n = \int_0^{\pi/2} \cos^n x\, dx$, then $I_n = \frac{n-1}{n} I_{n-2}$.

(7) If $I_n = \int_0^{\pi/2} x^n \sin x\, dx$ then $I_n + n(n-1)I_{n-2} = n(\pi/2)^{n-1}$

(8) If $I_n = \int_0^{\pi/2} x^n \cos x\, dx$ then $I_n + n(n-1)I_{n-2} = (\pi/2)^n$

(9) If $a > b > 0$, then $\int_0^{\pi/2} \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\sqrt{\frac{a+b}{a-b}}$

(10) If $0 < a < b$ then $\int_0^{\pi/2} \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{b^2 - a^2}} \log\left|\frac{\sqrt{b+a} - \sqrt{b-a}}{\sqrt{b+a} + \sqrt{b-a}}\right|$

(11) If $a > b > 0$ then $\int_0^{\pi/2} \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\sqrt{\frac{a-b}{a+b}}$

(12) If $0 < a < b$, then

$$\int_0^{\pi/2} \frac{dx}{a + b\sin x} = \frac{1}{\sqrt{b^2 - a^2}} \log\left|\frac{\sqrt{b+a} + \sqrt{b-a}}{\sqrt{b+a} - \sqrt{b-a}}\right|$$

(13) If $a > b, a^2 > b^2 + c^2$, then $\int_0^{\pi/2} \frac{dx}{a + b\cos x + c\sin x}$

$$= \frac{2}{\sqrt{a^2 - b^2 - c^2}} \tan^{-1} \frac{a - b + c}{\sqrt{a^2 - b^2 - c^2}}$$

(14) If $a > b, a^2 < b^2 + c^2$, then $\int_0^{\pi/2} \frac{dx}{a + b\cos x + c\sin x}$

$$= \frac{1}{\sqrt{b^2 + c^2 - a^2}} \log\left|\frac{a - b + c - \sqrt{b^2 + c^2 - a^2}}{a - b + c + \sqrt{b^2 + c^2 - a^2}}\right|$$

(15) If $a < b,\ a^2 < b^2 + c^2$ then $\int_0^{\pi/2} \frac{dx}{a + b\cos x + c\sin x}$

$$= \frac{-1}{\sqrt{b^2 + c^2 - a^2}} \log\left|\frac{b - a - c - \sqrt{b^2 + c^2 - a^2}}{b - a - c + \sqrt{b^2 + c^2 - a^2}}\right|.$$

Integration of piecewise continuous functions

Any function f(x) which is discontinuous at finite number of points in an interval [*a*, *b*] can be made continuous in sub-intervals by breaking the intervals into these subintervals. If f(x) is discontinuous at points x_1, x_2, x_3, ………. x_n in (*a*, *b*), then we can define subintervals (a, x_1), (x_1, x_2) ……… (x_{n-1}, x_n), (x_n, b) such that f(x) is continuous in each of these subintervals. Such functions are called piecewise continuous functions. For integration of piecewise continuous function, we integrate f(x) in these sub-intervals and finally add all the values.