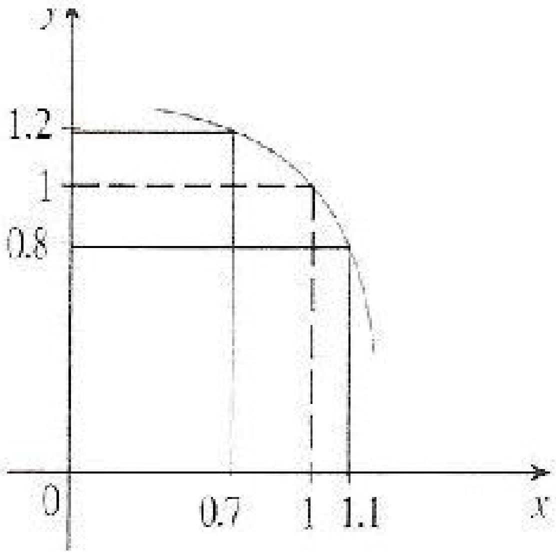
Exercise 1.7

Chapter 1 Functions and Limits Exercise 1.7 1E

Given graph of f



We have to find a number δ such that

If $|x-1|<\delta$ then $|f(x)-1|<0.2$

When

$|f(x)-1|<0.2$

$\Rightarrow -0.2<f(x)-1<0.2$

$\Rightarrow -0.2+1<f(x)<0.2+1$

$\Rightarrow 0.8<f(x)<1.2$

From the graph, we observe that

if $0.7<x<1.1$ then $0.8<f(x)<1.2$

This interval $(0.7, 1.1)$ is not symmetric about $x=1$

The distance from $x=1$ to left end point is $1-0.7=0.3$ and

The distance to the right end point is $1.1-1=0.1$.

So we choose δ to be smaller of these numbers, 0.1.

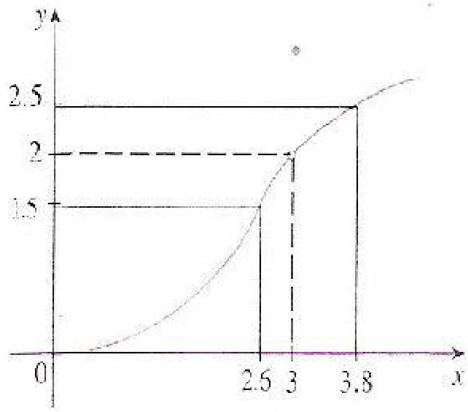
Then we can write

If $|x-1|<0.1$ then $|f(x)-1|<0.2$

Here we have chosen $\delta=0.1$ but any smaller positive value of δ would have also worked.

Chapter 1 Functions and Limits Exercise 1.7 2E

Given graph of f



We have to find a number δ such that

If $0<|x-3|<\delta$ then $|f(x)-2|<0.5$

When

$|f(x)-2|<0.5$

$\Rightarrow -0.5<f(x)-2<0.5$

$\Rightarrow -0.5+2<f(x)<0.5+2$

$\Rightarrow 1.5<f(x)<2.5$

From the graph, we observe that

if $2.6<x<3.8$ then $0.8<f(x)<1.2$

This interval $(2.6, 3.8)$ is not symmetric about $x=3$

The distance from $x=3$ to left end point is $3-2.6=0.4$ and

The distance to the right end point is $3.8-3=0.8$.

So we choose δ to be smaller of these numbers, 0.4.

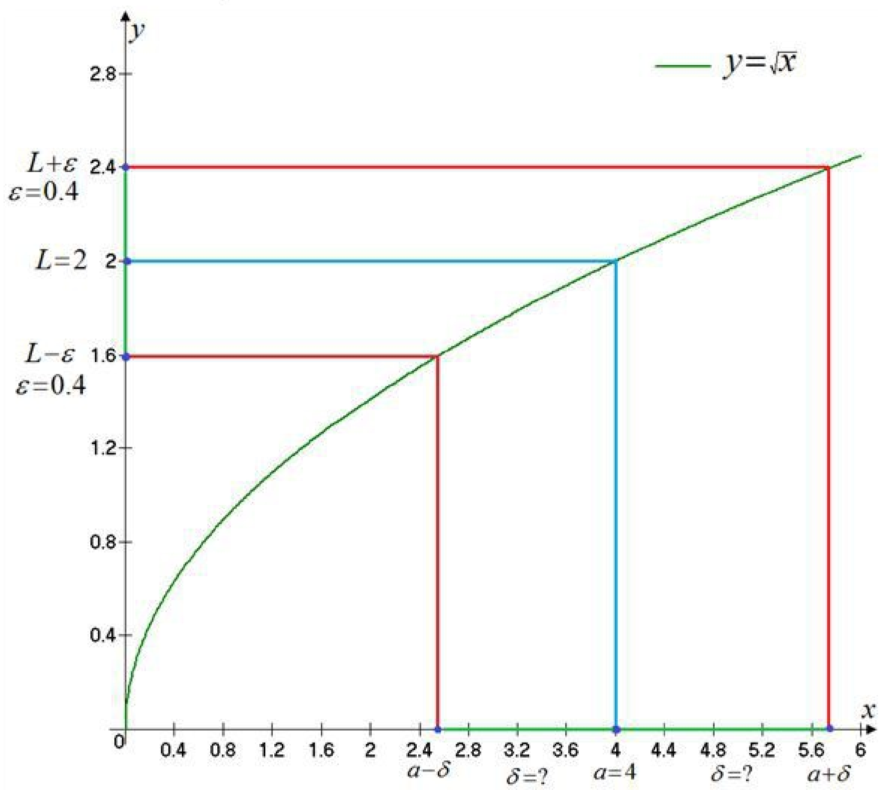
Then we can write

If $0<|x-3|<0.4$ then $|f(x)-2|<0.5$

Here we have chosen $\delta=0.4$ but any smaller positive value of δ would have also worked.

Chapter 1 Functions and Limits Exercise 1.7 3E

Consider the graph of $y=\sqrt{x}$ given below:



Using the above graph of $f(x)=\sqrt{x}$ to find a number δ such that if $|x-4|<\delta$ then $|\sqrt{x}-2|<0.4.$

Here, $a=4, L=2,$ and $\varepsilon=0.4.$

Rewrite the inequality $|\sqrt{x}-2|<0.4$ as shown below:

$$|\sqrt{x}-2|<0.4$$
$$-0.4<\sqrt{x}-2<0.4$$
$$-0.4+2<\sqrt{x}-2+2<0.4+2$$
$$1.6<\sqrt{x}<2.4$$
$$(1.6)^2<\left(\sqrt{x}\right)^2<(2.4)^2$$
$$2.56<x<5.76$$

Therefore, if $2.56<x<5.76$ then $1.6<f(x)<2.4.$

The interval $2.56<x<5.76$ is not symmetric about the line $x=4.$

Find the distance from $x=4$ to the left endpoint $x=2.56$ of the interval $2.56<x<5.76.$

$$\delta=4-2.56$$
$$=1.44$$

Find the distance from $x=4$ to the right endpoint $x=5.76$ of the interval $2.56<x<5.76.$

$$\delta=5.76-4$$
$$=1.76$$

Now choose the value of δ to be the smaller of these two delta values 1.44 and 1.76.

Therefore, $\delta=\boxed{1.44}.$

Note: any smaller positive value of δ can also be worked here.

If for every number $\varepsilon=0.4>0$ there is a number $\delta=1.44>0$ such that if $0<|x-4|<1.44$ then $|\sqrt{x}-2|<0.4.$

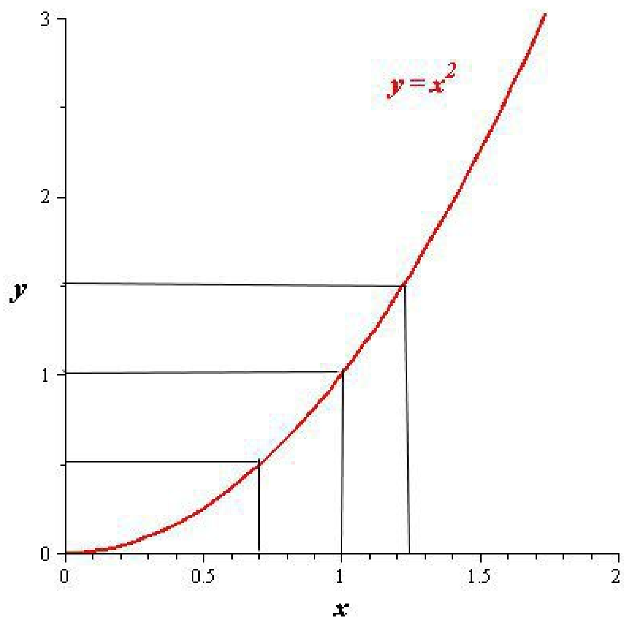
That is, by keeping x within 1.44 of 4, the value of $f(x)$ will be kept within 0.4 of 2.

Hence, the value of δ will be 1.44 or any smaller positive value of δ.

Chapter 1 Functions and Limits Exercise 1.7 4E

Consider the function $f(x)=x^2$

Sketch the figure representing the function and the limit and the ε values so that δ can be determined.



The objective is to find the value of δ

From the graph, to find the limit it is interested in the area near the point $(1,1)$.

Need to determine the values of x for which the curve $y = x^2$ lies between the horizontal lines $y = 0.5$ and $y = 1.5$.

Then find the value of x from the value of y.

When $y = 0.5$, then;

$$\begin{aligned} y &= x^2 \\ 0.5 &= x^2 \\ x &= 0.7071 \end{aligned}$$

Similarly find the value of x when $y = 1.5$;

$$\begin{aligned} y &= x^2 \\ 1.5 &= x^2 \\ x &= 1.2247 \end{aligned}$$

So, x-coordinate corresponding to intersection of $y = 0.5$ is ≈ 0.71 and the x-coordinate corresponding to intersection of $y = 1.5$ is ≈ 1.22.

The minimum of these two values is 0.71.

If $0.71 < x < 1.22$ then $0.5 < x^2 < 1.5$

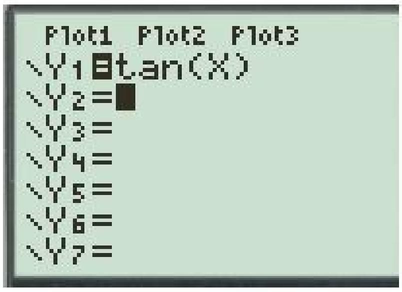
Therefore, $\boxed{\delta = 0.71}$.

Chapter 1 Functions and Limits Exercise 1.7 5E

If $\left|x - \frac{\pi}{4}\right| < \delta$ then $|\tan x - 1| < 0.2$

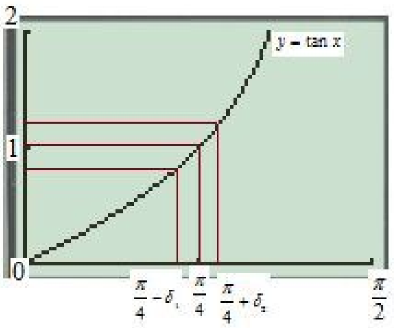
Find a number δ by using graph:

Enter the equations into Y1 in the equation editor $\boxed{\text{Y=}}$.



First set the window as shown in figure.

Now click on the GRAPH button to get the graph.



From the graph observe that $\tan x = 0.8$ when $x \approx 0.675$.

So,

$$\frac{\pi}{4} - \delta_1 = 0.675$$

$$\delta_1 = \frac{\pi}{4} - 0.675$$

$$= 0.1106$$

Again $\tan x = 1.2$ when $x \approx 0.8761$.

So,

$$\frac{\pi}{4} + \delta_2 = 0.8761$$

$$\delta_2 = 0.8761 - \frac{\pi}{4}$$

$$\approx 0.0906$$

Now, choose δ is smaller of δ_1 and δ_2.

Thus, the number $\delta = \boxed{0.0906}$

Chapter 1 Functions and Limits Exercise 1.7 6E

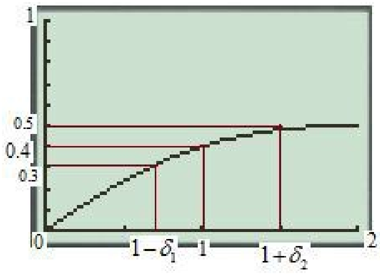
If $|x-1| < \delta$ then $\left|\frac{2x}{x^2+4} - 0.4\right| < 0.1$

Find a number δ by using graph:

Enter the equations into Y1 in the equation editor $\boxed{Y=}$.

First set the window as shown in figure.

Now click on the GRAPH button to get the graph.



From the graph observe that $\frac{2x}{x^2+4}=0.3$ when $x \approx 0.67$.

So,

$$\begin{aligned} 1-\delta_1 &= 0.67 \\ \delta_1 &= 1-0.67 \\ &= 0.33 \end{aligned}$$

Again $\frac{2x}{x^2+4}=0.5$ when $x=2$.

So,

$$\begin{aligned} 1+\delta_2 &= 2 \\ \delta_2 &= 2-1 \\ &= 1 \end{aligned}$$

Now, choose δ is smaller of δ_1 and δ_2.

Thus, the number $\delta = \boxed{0.33}$

Chapter 1 Functions and Limits Exercise 1.7 7E

Given limit

$$\lim_{x\to 2}\left(x^3-3x+4\right)=6$$

Let ε be a given positive number.

We want to find a number δ such that

If $0<|x-2|<\delta$ then $\left|\left(x^3-3x+4\right)-6\right|<\varepsilon$

But

$$\begin{aligned} \left|\left(x^3-3x+4\right)-6\right| &= \left|x^3-3x-2\right| \\ &= \left|x^3-2x^2+2x^2-4x+x-2\right| \\ &= \left|x^2(x-2)+2x(x-2)+(x-2)\right| \\ &= \left|(x-2)\left(x^2+2x+1\right)\right| \\ &= \left|(x-2)(x+1)^2\right| \end{aligned}$$

Then we want that

If $0<|x-2|<\delta$ then $|(x-2)|\left|(x+1)^2\right|<\varepsilon$

If we can find a positive constant C such that $\left|(x+1)^2\right|<C$, then

$$|(x-2)|\left|(x+1)^2\right|<C|(x-2)|$$

And we can make $C|(x-2)|<\varepsilon$ by taking $|(x-2)|<\frac{\varepsilon}{C}=\delta$

We can find such a number C if we restrict x to lie in some interval centered at 2.
In fact, since we are interested only in the values of x that are close to 2, it is reasonable to assume that x is within a distance of 1 from 2, that is, $|(x-2)|<1$. Then $1<x<3$, so

$4<(x+1)^2<16$.

Thus we have $\left|(x+1)^2\right|<16$, and so $C=16$ is suitable choice for the constant.

But now there are two restrictions on $|(x-2)|$, namely

$|(x-2)|<1$ and $|(x-2)|<\frac{\varepsilon}{C}=\frac{\varepsilon}{16}$

To make sure that both of these inequalities are satisfied,

We take δ to be smaller of the two numbers 1 and $\frac{\varepsilon}{16}$.

The notation for this is $\delta=\min\left\{1,\frac{\varepsilon}{16}\right\}$

When $\varepsilon=0.2$

$\delta=\min\left\{1,\frac{0.2}{16}\right\}$

$\Rightarrow\delta=\min\{1,0.0125\}$

$\Rightarrow\delta=0.0125$

$\delta=0.0125$ or any smaller positive value.

Verifying

If $0<|x-2|<0.0125$ then $|(x-2)|\left|(x+1)^2\right|<0.0125\times16=0.2=\varepsilon$

When $\varepsilon=0.1$

$\delta=\min\left\{1,\frac{0.1}{16}\right\}$

$\Rightarrow\delta=\min\{1,0.00625\}$

$\Rightarrow\delta=0.00625$

$\delta=0.00625$ or any smaller positive value.

Verifying

If $0<|x-2|<0.00625$ then $|(x-2)|\left|(x+1)^2\right|<0.00625\times16=0.1=\varepsilon$

Chapter 1 Functions and Limits Exercise 1.7 8E

Part 1:

Here $\epsilon=0.5$

So $\left|\frac{4x+1}{3x-4}-4.5\right|<0.5$ *where* $|x-2|<\delta$

So we have to determine the values of x for which the curve $f(x)=\frac{4x+1}{3x-4}$ lies between the lines y = 4 and y = 5 near the point (2, 4.5)

Therefore $4<\frac{4x+1}{3x-4}<5$

We graph the curves $y=\frac{4x+1}{3x-4}$, y = 4, y = 5 near the point (2, 4.5)

Then we see that the x – coordinate of the point of intersection of the line y = 4 and the curve is about 1.9. Similarly $y=\frac{4x+1}{3x-4}$ intersects the line y = 5 at x = 2.12 (approx.).

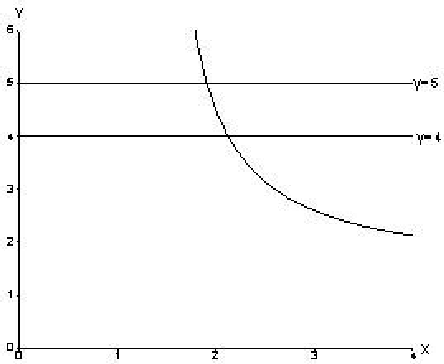


Fig.1

So we say that $4<\frac{4x+1}{3x-4}<5$ where $1.9<x<2.12$

This interval (1.9, 2.12) is not symmetric about x = 2
The distance from left end point is 2-1.9 = 0.1
The distance from right end point is 2.12-2 = 0.12
We can choose δ to be the smaller of these numbers

So $\boxed{\delta = 0.1}$

We write $4 < \frac{4x+1}{3x-4} < 5 \Rightarrow |x-2| < 0.1$

Part 2:
Here $\epsilon = 0.1$

We have to determine the values of x for the curve $f(x) = \frac{4x+1}{3x-4}$ lies between $[4.4, 4.6]$ near the point (2, 4.5)

Therefore $4.4 < \frac{4x+1}{3x-4} < 4.6$

We graph the curves $y = \frac{4x+1}{3x-4}, y = 4.4$ and y = 4.6 near the point (2, 4.5)
Then we see that the x-coordinates of the point of intersection of the line y = 4.4 and curve is about 1.98 and $y = \frac{4x+1}{3x-4}$ intersects the line y = 4.6 at $x \approx 2.03$

So we say that $4.4 < \frac{4x+1}{3x-4} < 4.6$ where 1.98 < x < 2.03

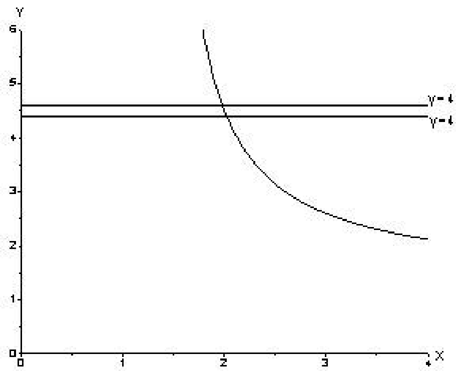


Fig.2

This interval (1.98, 2.03) is not symmetric about x = 2
The distance form left end is 2-1.98 = 0.02
The distance form right end is 2.03-2.0 = 0.03
We can choose δ to be the smaller of these numbers

$\boxed{\delta = 0.02}$ We write $4.4 < \frac{4x+1}{3x-4} < 4.6 \Rightarrow |x-2| < 0.02$

Chapter 1 Functions and Limits Exercise 1.7 9E

Consider $\lim_{x \to \frac{\pi}{2}} \tan^2 x = \infty$

Recall the definition of Infinite Limits:

Let *f* be a function defined on some open interval that contains the number *a*, except possibly at *a* itself. Then $\lim_{x \to a} f(x) = \infty$ means that for every positive number M there is a positive number δ such that if $0 < |x-a| < \delta$ then $f(x) > M$.

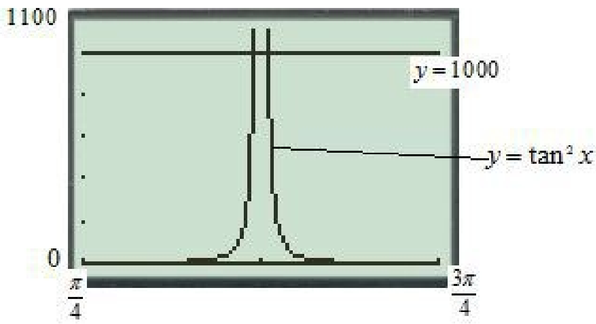
a)

Find the values of δ that correspond to $M = 1000$:

Now enter the two equations in the Y=window.

Here, we're using the window settings

Press GRAPH to graph the equations



From the graph observe that $\tan^2 x = 1000$ when $x \approx\sim 1.539$ and $x \approx\sim 1.602$ for x near $\frac{\pi}{2}$.

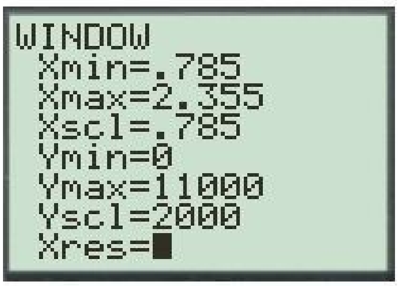
Thus,

$$\delta \approx 1.602 - \frac{\pi}{2}$$
$$\approx \boxed{0.031}$$

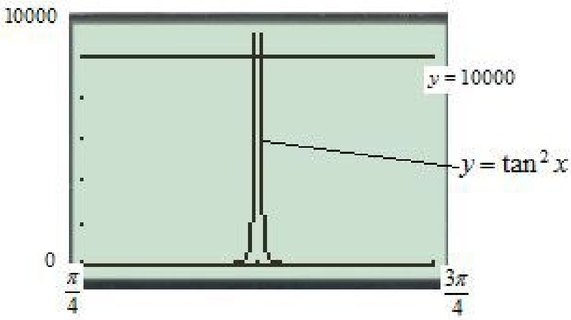
Find the values of δ that correspond to $M = 10{,}000$:

Now enter the two equations in the Y=window.

Here, we're using the window settings,



Press GRAPH to graph the equations



From the graph observe that $\tan^2 x = 10000$

when $x \approx\sim 1.561$ and $x \approx\sim 1.581$ for x near $\frac{\pi}{2}$.

Thus,

$$\delta \approx 1.581 - \frac{\pi}{2}$$
$$\approx \boxed{0.010}$$

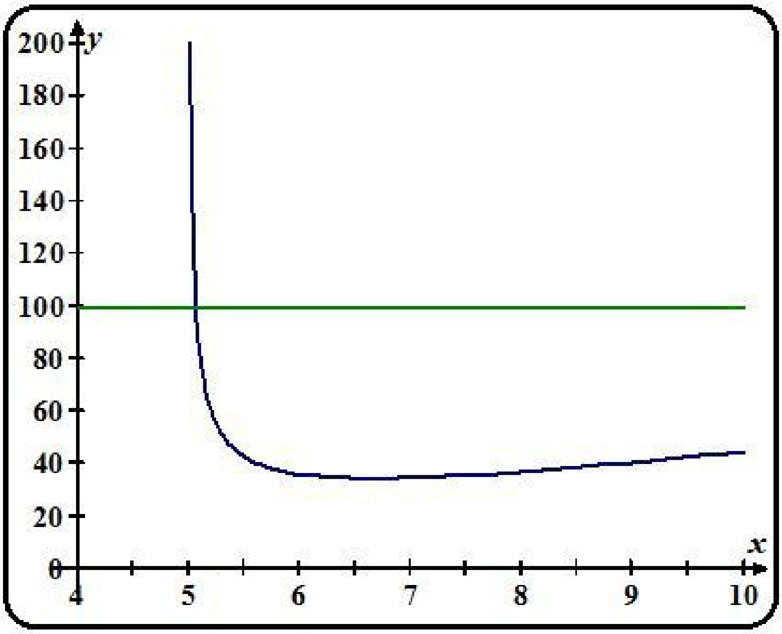
Chapter 1 Functions and Limits Exercise 1.7 10E

Use the graph of to determine the value of δ, for $5 < x < 5+\delta$, then $\frac{x^2}{\sqrt{x-5}} > 100$.

Consider $f(x) = \frac{x^2}{\sqrt{x-5}}$.

The graph of f is drawn close to $x = 5$ below.

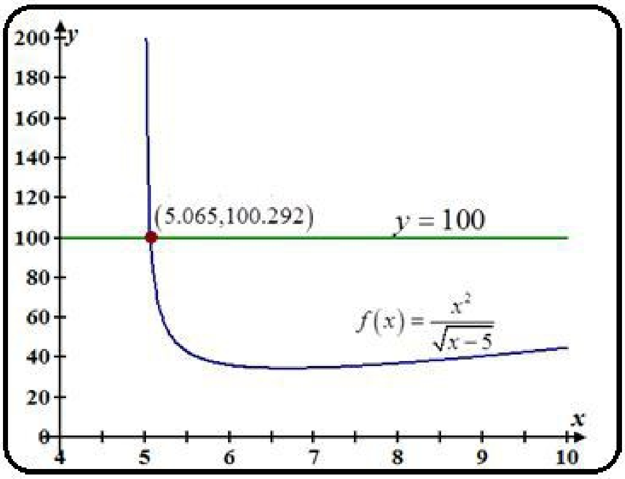
Sketch the figure representing the function and the line $y = 100$ on the graph.



Find the intersection of the curve $f(x)$ and line $y=100$ to estimate the value of x.

Then use the cursor to estimate the x-coordinate of the point of intersection of the line $y=100$ and the curve $y=\frac{x^2}{\sqrt{x-5}}$.

The graph with the point of intersection is as follows.



From the figure, the graph of the function $f(x)$ and $y=100$ intersects close to the point $(5.065,100.292)$.

This suggests that if $5<x<5.065$, the graph of f is below the line $y=100$.

But this cannot be sure that taking $\delta=.065$ by using graphing.

Evaluate *f* at this value of $x=5.065$.

The value of $f(5.065)$ is

$$\begin{aligned} f(5.065) &= \frac{(5.065)^2}{\sqrt{(5.065)-5}} \\ &= 100.62 \end{aligned}$$

Observe $f(5.065)>100$.

It also appears that the graph of f is decreasing between 5 and 5.065.

This implies that $f(x)>100$ for $5<x<5.065$.

Therefore, the value is $\boxed{\delta=0.065}$.

Chapter 1 Functions and Limits Exercise 1.7 11E

Consider a circular metal disk with area 1000cm^2

a)

What radius produces such a disk:

Recall the area of the circle is $A=\pi r^2$

The area of metal disk is $A=1000\text{cm}^2$

Thus

$$\begin{aligned} \pi r^2 &= 1000\text{cm}^2 \\ r^2 &= \frac{1000\text{cm}^2}{\pi} \\ r &= \sqrt{\frac{1000\text{cm}^2}{\pi}} \\ &\approx 17.8412\text{ cm} \end{aligned}$$

Therefore, the radius produces such disk is $\boxed{17.8412\text{ cm}}$

b)

Consider an error tolerance of $\pm 5\text{ cm}^2$ in the area of disk.

That is

$$|A-1000| \le 5$$
$$|\pi r^2 - 1000| \le 5$$
$$-5 \le \pi r^2 - 1000 \le 5$$
$$1000 - 5 \le \pi r^2 \le 5 + 1000$$
$$995 \le \pi r^2 \le 1005$$
$$\frac{995}{\pi} \le r^2 \le \frac{1005}{\pi}$$
$$\sqrt{\frac{995}{\pi}} \le r \le \sqrt{\frac{1005}{\pi}}$$
$$17.7966 \le r \le 17.8858$$

Thus, the difference of 17.7966 cm and 17.8412 cm is 0.0446 and the difference of 17.8858 and 17.8412 cm is 0.0445.

So,

if the machinist gets the radius within 0.0445cm of 17.8412, the area will be within 5 cm^2 of 1000.

c)

In terms of the ε, δ definition of $\lim_{x \to a} f(x) = L$

Here *x* is the radius of the circular metal disk and $f(x)$ is the area of the circular metal disk.

And *a* is target radius 17.8412 cm, *L* is target area 1000cm^2, ε is tolerance in the area (5), δ is the tolerance in the radius (0.0445cm).

Chapter 1 Functions and Limits Exercise 1.7 12E

Consider the relation between temperature and input power is

$$T(w) = 0.1w^2 + 2.155w + 20$$

Where

The temperature in degree Celsius is *T*

The power input in watts is *w*

a)

Consider that the temperature *T* is $200°C$

It is required to find the power needed to maintain the temperature given.

Plug in 200 for $T(w)$ in (1) to find the power in watts.

$T(w) = 0.1w^2 + 2.155w + 20$ From (1)

$200 = 0.1w^2 + 2.155w + 20$

$0.1w^2 + 2.155w + 20 - 200 = 0$ Subtract 200 on each side

$0.1w^2 + 2.155w - 180 = 0$

This represents a quadratic equation. Solve this quadratic equation using the quadratic formula.

$$w = \frac{-(2.155) \pm \sqrt{(2.155)^2 - 4 \cdot 0.1 \cdot (-180)}}{2(0.1)}$$

$$= \frac{-(2.155) \pm \sqrt{(2.155)^2 - 72}}{0.2}$$

$$= \frac{-2.155 \pm 8.755}{0.2}$$

$$w = \frac{-2.155 + 8.755}{0.2} \quad \text{or} \quad w = \frac{-2.155 - 8.755}{0.2}$$ Use calculator

$$w = 33 \quad \text{or} \quad w = -54.55$$

Power needed to maintain the temperature cannot be negative.

Therefore, the power needed to maintain the temperature at $200°\text{C}$ is $\boxed{33 \text{ watts}}$.

b)

Consider the temperature varies $\pm 1°\text{C}$ up of $200°\text{C}$.

So, the temperature varies from $199°\text{C}$ to $201°\text{C}$ (that is $199 \le T \le 201$).

Find the value of *w* for these values of *T*, so that the range of *w* is obtained.

Given

$T(w) = 0.1w^2 + 2.155w + 20$

To determine the range of input power, use graph.

First enter the function in $\boxed{\text{Y=}}$ screen.

Hit the sequence of keys to enter the function $T(w), 199, 201$.

The key strokes are

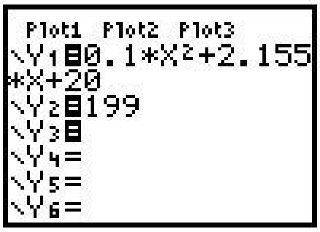
In Y1,

$\boxed{0}\boxed{.}\boxed{1}\boxed{\text{X,T,}\theta\text{,n}}\boxed{x^2}\boxed{+}\boxed{2}\boxed{.}\boxed{1}\boxed{5}\boxed{5}\boxed{\times}\boxed{\text{X,T,}\theta\text{,n}}\boxed{+}\boxed{2}\boxed{0}$

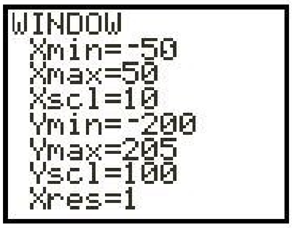
In Y2,

$\boxed{1}\boxed{9}\boxed{9}$

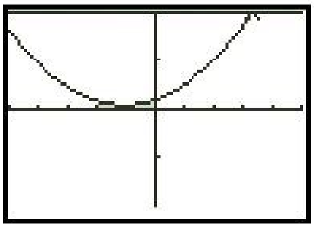
The function is entered as follows.



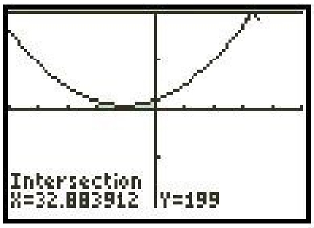
Adjust the scaling in WINDOW.



Hit the graph button to view.

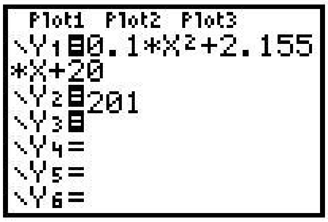


Click 2nd + TRACE and then select 5: intersect to find the point of intersection of Y1 and Y2.

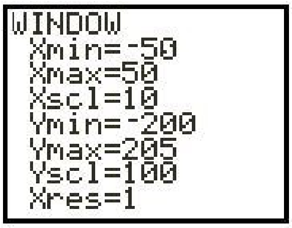


Do the same with Y2=201.

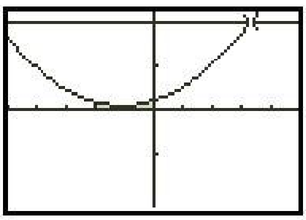
The function is entered as follows.



Adjust the scaling in WINDOW.

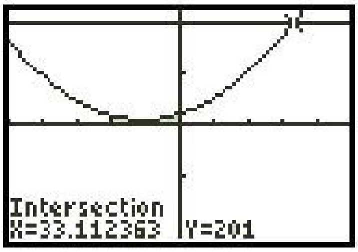


Hit the graph button to view.



Click $\boxed{\text{2nd}}+\boxed{\text{TRACE}}$ and then select 5: intersect to find the point of intersection of

Y1 and Y2.



Therefore, the points of intersection are $\boxed{(32.89,199) \text{ and } (33.11,200)}$.

Hence, conclude that the temperature will be between $199°\text{C}$ and $201°\text{C}$ if the wattage is between approximately 32.89 watts and 33.11 watts. As an interval, the range of wattage is $\boxed{32.89 < w < 33.11}$.

c)

Recall that, **the limit of** $f(x)$ **as *x* approaches *a* is *L*** is

$$\lim_{x \to a} f(x) = L$$

Or

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad 0 < |x-a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Basically, this definition says that as a values for *x* is chosen extremely close to *a*, then the corresponding $f(x)$ values that are extremely close to *L*.

The input to the function needed for this problem is power input in watts. So, $\boxed{x \text{ is the power input}}$.

The function evaluated is $T(w) = 0.1w^2 + 2.155w + 20$ where *T* is temperature in degrees Celsius. Hence, $\boxed{f(x) \text{ is the temperature}}$.

The target input power from part (a) is 33 watts. Hence, $\boxed{a = 33 \text{ watts}}$.

The target temperature is $200°\text{C}$. Hence, $\boxed{L = 200°\text{C}}$.

The error tolerance for the temperature is $1°\text{C}$. Hence, $\boxed{\varepsilon = 1°\text{C}}$.

Finally, the error tolerance for the power input is δ.

Find δ by examining the interval $32.89 < w < 33.11$ and comparing distances from each endpoint to the target power input 33 watts.

The distance from 32.89 to 33 is:

$$33 - 32.89 = 0.11$$

And the distance from 33.11 to 33 is:

$$33.11 - 33 = 0.11$$

Both distances are 0.11. Hence, $\boxed{\delta = 0.11 \text{ watts}}$.

Chapter 1 Functions and Limits Exercise 1.7 13E

(a)

Consider,

$|4x-8|<\varepsilon$

$|4x-8|<0.1$ since $\varepsilon=0.1$

Now simplify

$|4x-8|<0.1$

$4|x-2|<0.1$

$|x-2|<\frac{0.1}{4}=0.025$

Recollect the definition

If for every number $\varepsilon>0$ there is a number $\delta>0$ such that

If $0<|x-a|<\delta$ Then $|f(x)-L|<\varepsilon$

So,

$|x-2|<\delta$

Comparing the above two expression,

$\boxed{\delta=0.025}$.

(b)

Consider,

$|4x-8|<\varepsilon$

$|4x-8|<0.01$ since $\varepsilon=0.01$

Now simplify

$|4x-8|<0.01$

$4|x-2|<0.01$

$|x-2|<\frac{0.01}{4}=0.0025$

Recollect the definition

If for every number $\varepsilon>0$ there is a number $\delta>0$ such that

If $0<|x-a|<\delta$ Then $|f(x)-L|<\varepsilon$

Then, according to definition

$|x-2|<\delta$

Comparing the above two expression,

$\boxed{\delta=0.0025}$.

Chapter 1 Functions and Limits Exercise 1.7 14E

Consider the limit $\lim_{x\to 2}(5x-7)=3$

Recall the definition limit of $f(x)$ as *x* approaches *a* as *L*:

$\lim_{x\to a} f(x)=L$

If for every number $\varepsilon>0$ there is a number $\delta>0$ such that if $0<|x-a|<\delta$ then

$|f(x)-L|<\varepsilon$.

Find the value of δ that corresponds to $\varepsilon = 0.1$:

If $0 < |x-2| < \delta$ then $|(5x-7)-3| < \varepsilon$

But $|(5x-7)-3| = |5x-10|$

$= |5(x-2)|$
$= 5|x-2|$

Therefore, we want δ such that

if $0 < |x-2| < \delta$ then $5|x-2| < 0.1$ Substitute $\varepsilon = 0.1$

$|x-2| < \frac{0.1}{5}$ Divide each side by 5

That is, if $0 < |x-2| < \delta$ then $|x-2| < 0.02$

Therefore, the number $\delta = \boxed{0.02}$

Find the value of δ that corresponds to $\varepsilon = 0.05$:

If $0 < |x-2| < \delta$ then $|(5x-7)-3| < \varepsilon$

But $|(5x-7)-3| = |5x-10|$

$= |5(x-2)|$
$= 5|x-2|$

Therefore, we want δ such that

if $0 < |x-2| < \delta$ then $5|x-2| < 0.05$ Substitute $\varepsilon = 0.05$

$|x-2| < \frac{0.05}{5}$ Divide each side by 5

That is, if $0 < |x-2| < \delta$ then $|x-2| < 0.01$

Therefore, the number $\delta = \boxed{0.01}$

Find the value of δ that corresponds to $\varepsilon = 0.01$:

If $0 < |x-2| < \delta$ then $|(5x-7)-3| < \varepsilon$

But $|(5x-7)-3| = |5x-10|$

$= |5(x-2)|$
$= 5|x-2|$

Therefore, we want δ such that

if $0 < |x-2| < \delta$ then $5|x-2| < 0.01$ Substitute $\varepsilon = 0.01$

$|x-2| < \frac{0.01}{5}$ Divide each side by 5

That is, if $0 < |x-2| < \delta$ then $|x-2| < 0.002$

Therefore, the number $\delta = \boxed{0.002}$

Chapter 1 Functions and Limits Exercise 1.7 15E

Recall the definition of limit,

$$\lim_{x\to a} f(x) = L$$

If for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a-\delta < x < a+\delta$ then $|f(x)-L| < \varepsilon$.

Use this definition to prove that $\lim_{x\to 3}\left(1+\frac{1}{3}x\right) = 2$

Guessing a value of δ:

Let ε be a given positive number.

Here, $a = 3, L = 2$

So, it is required to find a number δ such that

If $0 < |x-3| < \delta$ then $\left|\left(1+\frac{1}{3}x\right)-2\right| < \varepsilon$

Now,

$$\left|\left(1+\tfrac{1}{3}x\right)-2\right| < \varepsilon$$
$$\left|\tfrac{1}{3}x-1\right| < \varepsilon$$
$$\tfrac{1}{3}|x-3| < \varepsilon$$
$$|x-3| < 3\varepsilon$$

This suggests choosing $\delta = 3\varepsilon$.

Showing that this δ works:

For given $\varepsilon > 0$, choose $\delta = 3\varepsilon$.

If $0 < |x-3| < \delta$, then

$$\left|\left(1+\tfrac{1}{3}x\right)-2\right| = \left|\tfrac{1}{3}x-1\right|$$
$$= \tfrac{1}{3}|x-3|$$
$$< \tfrac{1}{3}\delta$$
$$= \tfrac{1}{3}(3\varepsilon)$$
$$= \varepsilon$$

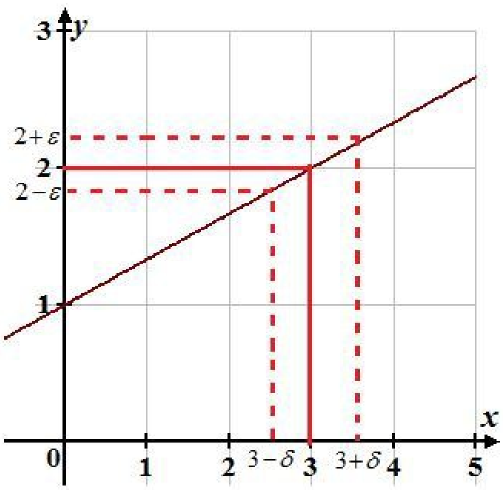
Thus

if $0 < |x-3| < \delta$ then $\left|\left(1+\frac{1}{3}x\right)-2\right| < \varepsilon$

Therefore, by definition of limit,

$$\lim_{x\to 3}\left(1+\tfrac{1}{3}x\right) = 2$$

The graphical illustration of the limit $\lim_{x\to 3}\left(1+\frac{1}{3}x\right) = 2$ is as follows.



Recall the definition of limit,

$$\lim_{x \to a} f(x) = L$$

If for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a - \delta < x < a + \delta$ then $|f(x) - L| < \varepsilon$.

Use this definition to prove that $\lim_{x \to 4}(2x - 5) = 3$

Guessing a value of δ:

Let ε be a given positive number.

Here, $a = 4, L = 3$

So, it is required to find a number δ such that

If $0 < |x - 4| < \delta$ then $|(2x - 5) - 3| < \varepsilon$

Now,

$$|(2x-5)-3| < \varepsilon$$
$$|2x-8| < \varepsilon$$
$$|2||x-4| < \varepsilon$$
$$2|x-4| < \varepsilon$$

$$|x-4| < \frac{\varepsilon}{2}$$

This suggests choosing $\delta = \frac{\varepsilon}{2}$.

Showing that this δ works:

For given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2}$.

If $0 < |x - 4| < \delta$, then

$$|(2x-5)-3| = |2x-5-3|$$

$$= |2x-8|$$
$$= 2|x-4|$$

$$< 2\frac{\varepsilon}{2}$$

$$= \varepsilon$$

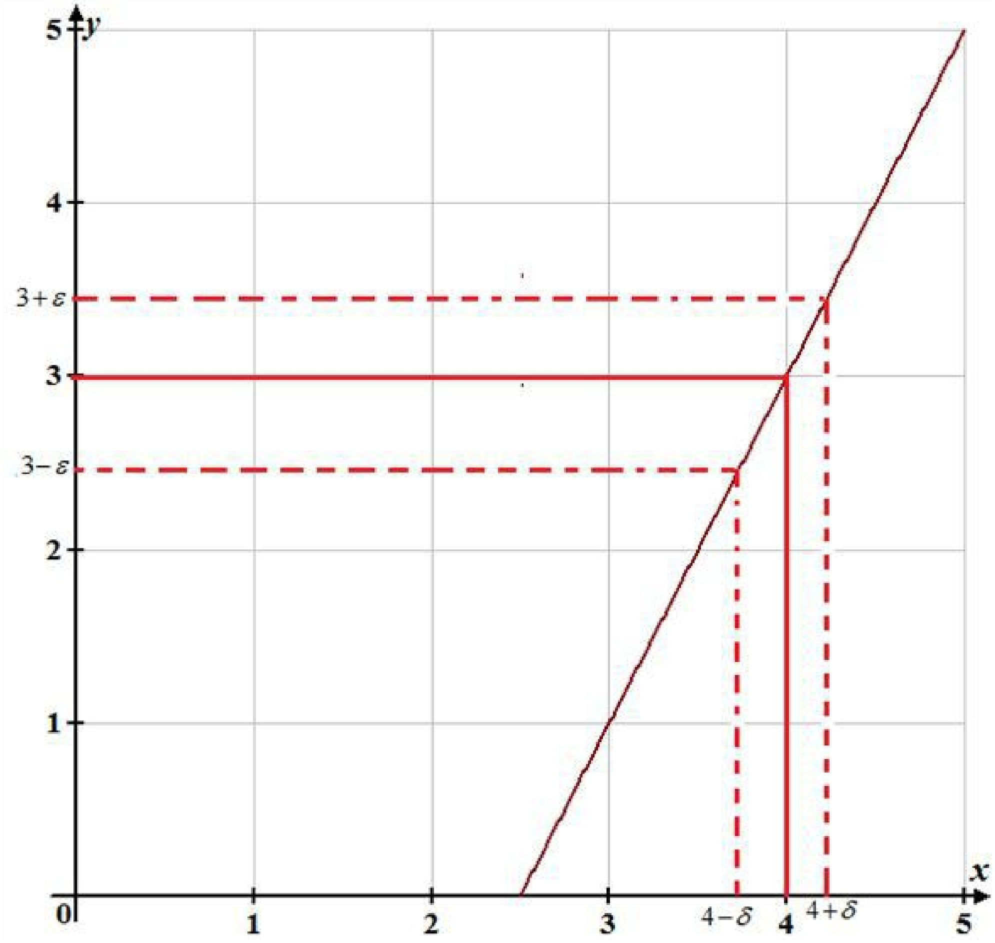
Thus

if $0 < |x - 4| < \delta$ then $|(2x - 5) - 3| < \varepsilon$

Therefore, by definition of limit,

$$\lim_{x \to 4}(2x - 5) = 3$$

The graphical illustration of the limit $\lim_{x \to 4}(2x-5)=3$ is as follows.



Chapter 1 Functions and Limits Exercise 1.7 17E

Consider the limit $\lim_{x \to -3}(1-4x)=13$

Prove the statement using the ε, δ definition of a limit:

Recall the definition of limit.

Let *f* be a function defined on some open interval that contains the number *a*, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is *L*, and we write

$\lim_{x \to a} f(x) = L$

If, for every number $\varepsilon > 0$, there is a number $\delta > 0$, such that if

$0 < |x-a| < \delta$ then $|f(x)-L| < \varepsilon$

Given $\varepsilon > 0$, we need $\delta > 0$, such that $0 < |x-(-3)| < \delta$, then $|(1-4x)-13| < \varepsilon$

But

$|(1-4x)-13| < \varepsilon$

$|-4x-12| < \varepsilon$

$|(-4)(x+3)| < \varepsilon$ Factor -4

$|-4||x+3| < \varepsilon$

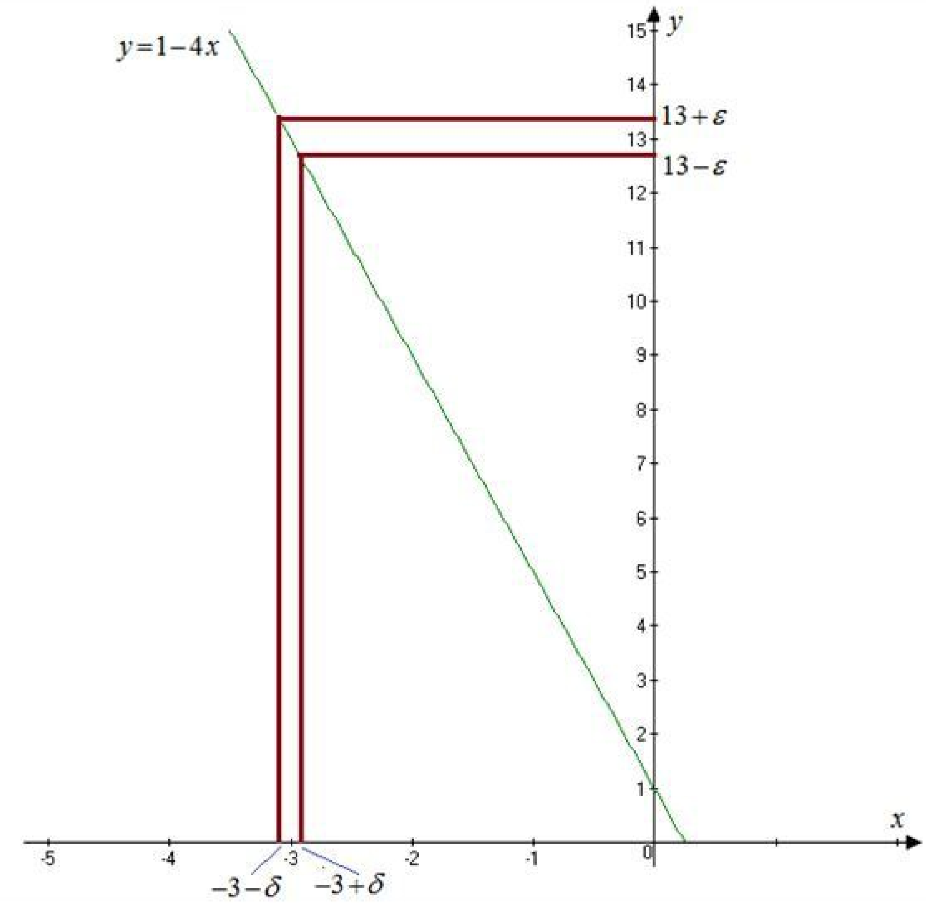
$4|x-(-3)| < \varepsilon$

$|x-(-3)| < \frac{\varepsilon}{4}$ Divide each side by 4

Now, choose $\delta = \frac{\varepsilon}{4}$, then $0 < |x-(-3)| < \delta$ implies $|(1-4x)-13| < \varepsilon$

Therefore, $\lim_{x \to -3}(1-4x)=13$ by the definition of limit

Sketch the graph of $y = 1 - 4x$ as follows:



Chapter 1 Functions and Limits Exercise 1.7 18E

To prove

$\lim_{x \to -2} (3x+5) = -1$

Let ε be a given positive number.

We want to find a number δ such that

If $0 < |x+2| < \delta$ then $|(3x+5)+1| < \varepsilon$

But

$$|(3x+5)+1| = |3x+6|$$
$$= 3|x+2|$$

Then we want δ such that

if $0 < |x+2| < \delta$ then $3|x+2| < \varepsilon$

that is, if $0 < |x+2| < \delta$ then $|x+2| < \frac{\varepsilon}{3}$

This suggests that we should choose $\delta = \frac{\varepsilon}{3}$

Given $\varepsilon > 0$ choose $\delta = \frac{\varepsilon}{3}$. If $0 < |x+2| < \delta$, then

$$|(3x+5)+1| = |3x+6|$$
$$= 3|x+2|$$
$$= 3\delta$$
$$= 3\left(\frac{\varepsilon}{3}\right)$$
$$= \varepsilon$$

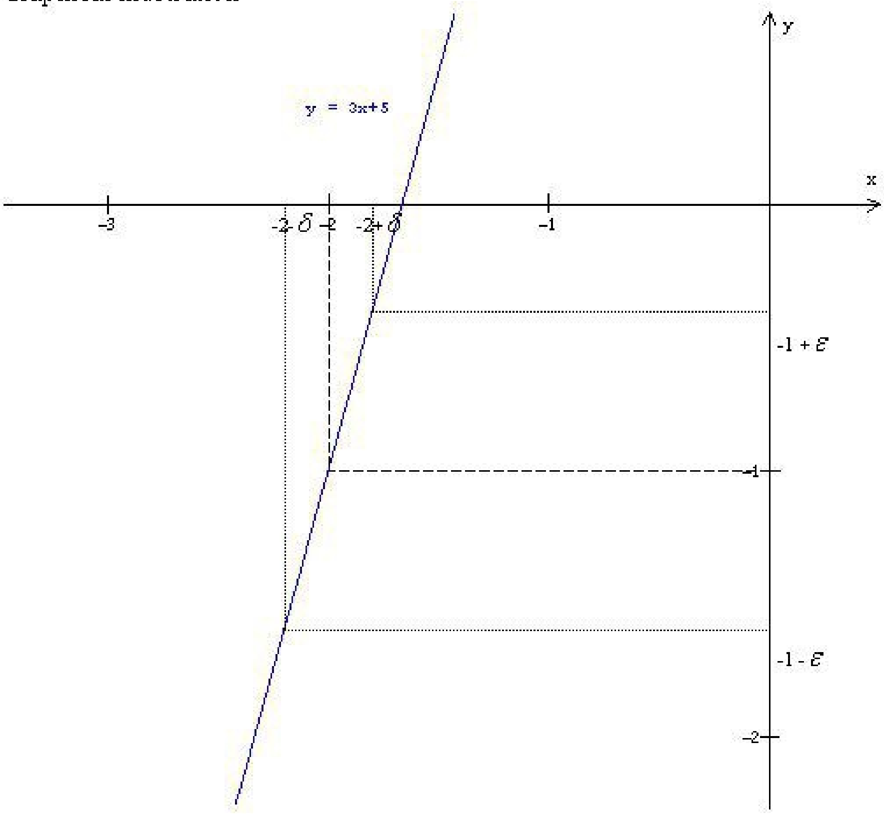
Thus

if $0 < |x+2| < \delta$ then $|(3x+5)+1| < \varepsilon$

Therefore, by definition of limit,

$$\lim_{x \to -2} (3x+5) = -1$$

Graphical illustration



Chapter 1 Functions and Limits Exercise 1.7 19E

To prove

$$\lim_{x\to 1}\left(\frac{2+4x}{3}\right)=2$$

Let ε be a given positive number.

We want to find a number δ such that

If $\quad 0<|x-1|<\delta \quad$ then $\quad \left|\frac{2+4x}{3}-2\right|<\varepsilon$

But

$$\left|\frac{2+4x}{3}-2\right|=\left|\frac{2+4x-6}{3}\right|$$

$$=\left|\frac{4x-4}{3}\right|$$

$$=\frac{4}{3}|x-1|$$

Then we want δ such that

if $\quad 0<|x-1|<\delta \quad$ then $\quad \frac{4}{3}|x-1|<\varepsilon$

that is, if $\quad 0<|x-1|<\delta \quad$ then $\quad |x-1|<\frac{3\varepsilon}{4}$

This suggests that we should choose $\delta=\frac{3\varepsilon}{4}$

Then we want δ such that

if $\quad 0<|x-1|<\delta \quad$ then $\quad \frac{4}{3}|x-1|<\varepsilon$

that is, if $\quad 0<|x-1|<\delta \quad$ then $\quad |x-1|<\frac{3\varepsilon}{4}$

This suggests that we should choose $\delta=\frac{3\varepsilon}{4}$

Given $\varepsilon>0$ choose $\delta=\frac{3\varepsilon}{4}$. If $0<|x-1|<\delta$, then

$$\begin{aligned}\left|\frac{2+4x}{3}-2\right| &= \left|\frac{2+4x-6}{3}\right| \\ &= \left|\frac{4x-4}{3}\right| \\ &= \frac{4}{3}|x-1| \\ &= \frac{4}{3}\delta\end{aligned}$$

Thus

if $\quad 0<|x-1|<\delta \quad$ then $\quad \left|\frac{2+4x}{3}-2\right|<\varepsilon$

Therefore, by definition of limit,

$$\lim_{x\to 1}\left(\frac{2+4x}{3}\right)=2$$

Chapter 1 Functions and Limits Exercise 1.7 20E

To prove

$$\lim_{x\to 10}\left(3-\frac{4}{5}x\right)=-5$$

Let ε be a given positive number. We want to find a number δ such that

If $\quad 0<|x-10|<\delta \quad$ then $\quad \left|\left(3-\frac{4}{5}x\right)+5\right|<\varepsilon$

But

$$\begin{aligned}\left|\left(3-\frac{4}{5}x\right)+5\right| &= \left|8-\frac{4x}{5}\right| \\ &= \left|\frac{40-4x}{5}\right| \\ &= \frac{4}{5}|10-x| \\ &= \frac{4}{5}|x-10|\end{aligned}$$

Then we want δ such that

if $\quad 0<|x-10|<\delta \quad$ then $\quad \frac{4}{5}|x-10|<\varepsilon$

that is, $\quad$ if $\quad 0<|x-10|<\delta \quad$ then $\quad |x-10|<\frac{5\varepsilon}{4}$

This suggests that we should choose $\delta=\frac{5\varepsilon}{4}$

Given $\varepsilon>0$ choose $\delta=\frac{5\varepsilon}{4}$. If $0<|x-10|<\delta$, then

$$\begin{aligned}\left|\left(3-\frac{4}{5}x\right)+5\right| &= \left|8-\frac{4x}{5}\right| \\ &= \left|\frac{40-4x}{5}\right| \\ &= \frac{4}{5}|10-x| \\ &= \frac{4}{5}\delta \\ &= \frac{4}{5}\left(\frac{5\varepsilon}{4}\right) \\ &= \varepsilon\end{aligned}$$

Thus

if $\quad 0<|x-10|<\delta \quad$ then $\quad \left|\left(3-\frac{4}{5}x\right)+5\right|<\varepsilon$

Therefore, by definition of limit,

$$\lim_{x\to 10}\left(3-\frac{4}{5}x\right)=-5$$

Chapter 1 Functions and Limits Exercise 1.7 21E

Consider the limit,

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}=5.$$

Use $\varepsilon-\delta$ definition to prove the statement $\lim_{x\to 2}\frac{x^2+x-6}{x-2}=5$.

According to the definition of a limit,

$\lim_{x\to c} f(x)=L$ means that for every $\varepsilon>0$, there exists a $\delta>0$, such that for every *x*, the expression $0<|x-a|<\delta$ implies $0<|f(x)-L|<\varepsilon$.

From the given statement $\lim_{x\to 2}\left(\frac{x^2+x-6}{x-2}\right)=5$, let

$$f(x)=\frac{x^2+x-6}{x-2},\ L=5,\ \text{and}\ a=2.$$

So, consider the absolute value inequality : $\left|\left(\frac{x^2+x-6}{x-2}\right)-5\right|<\varepsilon$, to get the expression $|x-2|$ on the left hand side:

$$|f(x)-L|<\varepsilon$$
$$\left|\left(\frac{x^2+x-6}{x-2}\right)-5\right|<\varepsilon$$
$$\left|\frac{(x+3)(x-2)}{x-2}-5\right|<\varepsilon$$
$$|x+3-5|<\varepsilon$$
$$|x-2|<\varepsilon$$

Choose $\delta=\varepsilon$, then the first inequality becomes:

$$0<|x-2|<\varepsilon$$

$$0<|x+3-5|<\varepsilon$$

$$0<\left|\frac{(x+3)(x-2)}{x-2}-5\right|<\varepsilon$$

$$0<\left|\frac{x^2+x-6}{x-2}-5\right|<\varepsilon$$

Therefore, by the definition of a limit, it can be shown that $\boxed{\lim_{x\to 2}\left(\frac{x^2+x-6}{x-2}\right)=5}$.

Chapter 1 Functions and Limits Exercise 1.7 22E

Take a function *f* defined on the open interval which includes the number *a*, except possibly at the point *a*.

$$\lim_{x\to a} f(x)=L$$

The limit is defined such that for a number $\varepsilon>0$, there is a number $\delta>0$ which satisfies:

$$\text{if } 0<|x-a|<\delta, \text{then} \quad |f(x)-L|<\varepsilon$$

Consider the limit to be evaluated:

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}=5$$

Evaluate the expression shown below:

$$\begin{aligned}\left|\frac{x^2+x-6}{x-2}-5\right| &= \left|\frac{(x+3)(x-2)}{x-2}-5\right| \\ &= |(x+3)-5| \\ &= |x-2| \\ &< \varepsilon\end{aligned}$$

So, the expression obtained is $|x-2|<\varepsilon$.

The above inequality obtained is of the form $|x-a|<\delta$.

$$\begin{aligned}&|x-a|<\delta \\ &\quad\Rightarrow |x-2|<\varepsilon \\ &\quad\Rightarrow |f(x)-L|<\varepsilon\end{aligned}$$

Hence, the statement is proved.

Chapter 1 Functions and Limits Exercise 1.7 23E

Consider the following limit:

$$\lim_{x\to a} x = a$$

Guess a value for δ as follows:

Let ε be a given positive number.

Compute a number δ as follows:

If $0<x-a<\delta$ then $|x-a|<\varepsilon$

This suggests that we should choose $\delta=\varepsilon$.

Show that this δ works as follows:

Given $\varepsilon>0$, assume that $\delta=\varepsilon$.

If $0<x-a<\delta$ then $|x-a|<\delta=\varepsilon$

Therefore, by the definition of a limit,

$$\boxed{\lim_{x\to a} x = a}$$

Chapter 1 Functions and Limits Exercise 1.7 24E

Let $\varepsilon>0$ and $\delta>0$ be any positive numbers

If $|x-a|<\delta$ then we have $|c-c|<\varepsilon$

Hence $|c-c|=0<\varepsilon$ whenever $|x-a|<\delta$

This is true for any value of δ.

So by the definition of limits we have

$$\lim_{x\to a} c = c$$

Chapter 1 Functions and Limits Exercise 1.7 25E

Consider the limit,

$\lim_{x\to 0} x^2 = 0$

Guess a value for δ as follows,

Assume that ε be a given positive number. Compute a number δ such that

If $0 < x < \delta$ then $|x^2 - 0| < \varepsilon$

Or if $0 < x < \delta$ then $x^2 < \varepsilon$

Take square root of the inequality $x^2 < \varepsilon$,

If $0 < x < \delta$ then $x < \sqrt{\varepsilon}$

Thus, choose $\delta = \sqrt{\varepsilon}$

Show that this δ works as follows:

Given $\varepsilon > 0$, assume that $\delta = \sqrt{\varepsilon}$. If $0 < x < \delta$, then

$x^2 < \delta^2$

$x^2 < \left(\sqrt{\varepsilon}\right)^2$

$x^2 < \varepsilon$

Or, $|x^2 - 0| < \varepsilon$

Therefore, by definition of a limit,

$\boxed{\lim_{x\to 0} x^2 = 0}$

Chapter 1 Functions and Limits Exercise 1.7 26E

Let $\varepsilon > 0$ by any number. Choose $\delta = \sqrt[3]{\varepsilon}$

If $|x-0| = |x| < \delta$ then $|x^3 - 0| = |x^3| < \delta^3$

$$= \left(\sqrt[3]{\varepsilon}\right)^3$$

$$= \varepsilon$$

Hence $|x^3 - 0| < \varepsilon$ when ever $|x-0| < \delta$

By definition of limit, we have $\lim\limits_{x\to 0} x^3 = 0$

Chapter 1 Functions and Limits Exercise 1.7 27E

Guessing the value of δ : - here $L = 0$, $a = 0$

So by the definition of limit

$||x| - 0| < \epsilon \Rightarrow 0 < |x - 0| < \delta$

$\Rightarrow ||x|| < \epsilon \Rightarrow 0 < |x| < \delta$

$\Rightarrow |x| < \epsilon \Rightarrow 0 < |x| < \delta$

So we would choose $\delta = \epsilon$

(2) Proof (Showing that δ works)

We have $|x| < \delta$

So $||x|| < |\delta|$

But we know that $\delta > 0$ therefore $|\delta| = \delta$

So we have $||x|| < \delta$

$\Rightarrow ||x| - 0| < \epsilon$

By the definition of limit we can write

$\lim\limits_{x\to 0} |x| = 0$

Chapter 1 Functions and Limits Exercise 1.7 28E

Consider the limit,

$\lim_{x\to -6^+} \sqrt[8]{6+x} = 0$.

Use $\varepsilon - \delta$ definition to prove the statement $\lim_{x\to -6^+} \sqrt[8]{6+x} = 0$.

According to the definition of a right-hand limit,

$\lim_{x\to a^+} f(x) = L$ means that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that for every *x*, the expression $a < x < a+\delta$ implies $|f(x) - L| < \varepsilon$.

From the given statement $\lim_{x\to -6^+} \sqrt[8]{6+x} = 0$, let

$f(x) = \sqrt[8]{6+x},\ L = 0,$ and $a = -6.$

So, consider the absolute value inequality : $\left|\sqrt[8]{6+x} - 0\right| < \varepsilon$, to get the inequality

$-6 < x < -6+\delta$:

$$|f(x) - L| < \varepsilon$$
$$\left|\sqrt[8]{6+x} - 0\right| < \varepsilon$$
$$\left|\sqrt[8]{6+x}\right| < \varepsilon$$
$$\sqrt[8]{6+x} < \varepsilon$$
$$6 + x < \varepsilon^8$$
$$x < \varepsilon^8 - 6$$

Let $\delta = \varepsilon^8 - 6$. If $0 < x < \delta$ then,

$$x < \delta$$
$$x < \varepsilon^8 - 6$$
$$6 + x < \varepsilon^8$$
$$\sqrt[8]{6+x} < \varepsilon$$
$$\left|\sqrt[8]{6+x} - 0\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon$$

Therefore, by the definition of a right-hand limit, it can be shown that

$\boxed{\lim_{x\to -6^+} \sqrt[8]{6+x} = 0}$.

Chapter 1 Functions and Limits Exercise 1.7 29E

Guessing the value of δ : -

Here $L = 1 \quad a = 2$

By the definition of limit

$$\left|\left(x^2 - 4x + 5\right) - 1\right| < \epsilon \quad \textit{whenever} \quad |x-2| < \delta$$
$$\Rightarrow \left|x^2 - 4x + 4\right| < \epsilon \quad \textit{whenever} \quad |x-2| < \delta$$
$$\Rightarrow \left|(x-2)^2\right| < \epsilon \quad \textit{whenever} \quad |x-2| < \delta$$

Taking square root of both side

$$|(x-2)| < \left|\sqrt{\epsilon}\right| \quad \textit{whenever} \quad |x-2| < \delta$$

We would choose $\delta = \left|\sqrt{\epsilon}\right|$

Proof (showing that δ works): -

If $|x-2|<\delta \Rightarrow 2-\delta<x<2+\delta$

$\Rightarrow -\delta<x-2<\delta$

By squaring both sides

$$\left|(x-2)^2\right|<\delta^2$$

$$\Rightarrow \left|x^2-4x+4\right|<\left(\left|\sqrt{\epsilon}\right|\right)^2$$

$$\Rightarrow \left|x^2-4x+5-1\right|<\epsilon$$

$$\Rightarrow \left|\left(x^2-4x+5\right)-1\right|<\epsilon$$

Then by the definition of limit

$$\lim_{x\to 2}\left(x^2-4x+5\right)=1$$

Chapter 1 Functions and Limits Exercise 1.7 30E

Using the ε,δ definition of a limit, prove the statement

$$\lim_{x\to 2}\left(x^2+2x-7\right)=1$$

Recall the definition of limit using the ε,δ that,

"Let f be function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L,

and we write

$$\lim_{x\to a} f(x)=L$$

if for every number $\varepsilon>0$ there is a number $\delta>0$ such that

if $0<|x-a|<\delta$ then $|f(x)-L|<\varepsilon$"

1. Guessing a value for δ.

Let ε be a given positive number.

It is need to find a number $\delta>0$ such that

if $0<|x-2|<\delta$ then $\left|\left(x^2+2x-7\right)-1\right|<\varepsilon$

To connect $\left|\left(x^2+2x-7\right)-1\right|$ with $|x-2|$, write

$$\left|\left(x^2+2x-7\right)-1\right|=\left|x^2+2x-8\right|$$

$$=\left|(x+4)(x-2)\right|$$

Then it is need to be that

if $0<|x-2|<\delta$ then $|x+4||x-2|<\varepsilon$

Notice that, If we can find a positive constant C such that $|x+4|<C$, then

$$|x+4||x-2|<C|x-2|$$

And make $C|x-2|<\varepsilon$ by taking $|x-2|<\frac{\varepsilon}{C}=\delta$

Find such a number C if restrict x to lie in some interval centered at 2.

In fact, since we are interested only in values of x that are close to 2, it is reasonable to assume that x is within a distance of 1 from 2, that is,

$$|x-2|<1$$

$$-1<x-2<1 \qquad \text{since } |x|<a \text{ means } -a<x<a$$

$$2-1<x<2+1$$

$$1<x<3$$

So, $5<x+4<7$

Thus, $|x+4|<7$, and so $C=7$ is suitable choice for the constant.

But now there are two restrictions on $|x-2|$, namely

$|x-2|<1$ and $|x-2|<\frac{\varepsilon}{C}=\frac{\varepsilon}{7}$

To make sure that both of these inequalities are satisfied, we take δ to be smaller of the two numbers 1 and $\frac{\varepsilon}{7}$. The notation for this is $\delta=\min\left\{1,\frac{\varepsilon}{7}\right\}$

2. Showing that this δ works.

Given $\varepsilon>0$, let $\delta=\min\left\{1,\frac{\varepsilon}{7}\right\}$

If $0<|x-2|<\delta$, then

$$\begin{aligned} |x-2|&<1 \\ -1<x-2&<1 \qquad \text{since } |x|<a \text{ means } -a<x<a \\ 2-1<x&<2+1 \\ 1<x&<3 \end{aligned}$$

So, $5<x+4<7$

Thus, $|x+4|<7$

Also, we have $|x-2|<\frac{\varepsilon}{7}$, so

$$\begin{aligned} \left|\left(x^2+2x-7\right)-1\right| &= |(x+4)(x-2)| \\ &< 7\cdot\frac{\varepsilon}{7} \\ &= \varepsilon \end{aligned}$$

This shows that

$$\lim_{x\to 2}\left(x^2+2x-7\right)=1$$

Chapter 1 Functions and Limits Exercise 1.7 31E

Consider the following statement.

$\lim_{x\to -2}\left(x^2-1\right)=3$.

Recollect the definition of the limit of the function as follows,

Let *f* be a function defined on some open interval that contains the number *a*, except possibly at *a* itself. Then say that the limit of $f(x)$ as *x* approaches *a* is *L*, and write

$$\lim_{x\to a} f(x)=L$$

If for every number $\varepsilon>0$ there is a number $\delta>0$ such that if

$0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$.

To prove the statement using the ε, δ definition of limit:

Guessing a value for δ:

Let $\varepsilon > 0$ be given.

Find a number $\delta > 0$, such that if $0 < |x+2| < \delta$ then $|(x^2-1)-3| < \varepsilon$.

To connect $|(x^2-1)-3|$ with $|x+2|$ we write

$$\begin{aligned} |(x^2-1)-3| &= |x^2-1-3| \\ &= |x^2-4| \\ &= |x^2-2^2| \\ &= |(x+2)(x-2)| \qquad a^2-b^2=(a-b)(a+b). \end{aligned}$$

$= |x+2||x-2|$.

Then we want if $0 < |x+2| < \delta$ then $|(x^2-1)-3| < \varepsilon$. So

$$\begin{aligned} |(x^2-1)-3| &< \varepsilon \\ |x+2||x-2| &< \varepsilon \\ |x+2| &< \frac{\varepsilon}{|x-2|} \\ &= \delta \end{aligned}$$

Showing that this δ works:

Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{|x-2|}$.

If $0 < |x+2| < \delta$, then $|x+2| < \delta$. So

$$\begin{aligned} |x+2| &< \delta \\ |x+2||x-2| &< |x-2|\delta \\ &= |x-2| \cdot \frac{\varepsilon}{|x-2|} \\ &= \varepsilon \end{aligned}$$

This shows that $\boxed{\lim_{x \to -2}(x^2-1) = 3}$.

Chapter 1 Functions and Limits Exercise 1.7 32E

Consider the following limit:

$$\lim_{x \to 2} x^3 = 8$$

The objective is to prove the statement by using the definition of ε, δ limit.

The definition of ε, δ limit, the function f be defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L.

Write the form is, $\lim_{x \to a} f(x) = L$

for every number $\varepsilon > 0$ then there is a number $\delta > 0$ such that $0 < |x-a| < \delta$ then $|f(x)-L| < \varepsilon$

Step1: Guessing the value of δ.

Let $f(x)=x^3$

Let $\varepsilon>0$ and to find a number $\delta>0$ such that $0<|x-2|<\delta$ then

$|x^3-8|<\varepsilon \left(\lim_{x\to a} f(x)=L\right)$

Now write $|x^3-8|=|(x-2)(x^2+2x+4)|$ $\left(a^3-b^3=(a-b)(a^2+ab+b^2)\right)$

If $0<|x-2|<\delta$ then $|(x-2)||(x^2+2x+4)|<\varepsilon$

Here, observing that if can find a positive constant C such that $|(x^2+2x+4)|<C$

Then $|(x-2)(x^2+2x+4)|<C|x-2|$

Now it is $C|x-2|<\varepsilon$ $\left(\text{Since}|x-2|<\frac{\varepsilon}{C}=\delta\right)$

Now find a number C if restrict x lie in some interval centered at 2.

Since the value of x approaches to 2 and it is reasonable to assume that is within a distance of 1 from 2.

So, it is $|x-2|<1$

$$\begin{aligned}
&|x-2|<1 \\
&-1<x-2<1 \qquad \text{since } |x|<a \text{ means } -a<x<a \\
&2-1<x<2+1 \\
&1<x<3
\end{aligned}$$

Then $x^2+2x+4<3^2+2(3)+4$

$=19$

$x^2+2x+4<19$

Since, $x^2+2x+4<C$, then $C=19$ is a suitable choice for the constant.

Now $|x-2|<1$ and $|x-2|<\frac{\varepsilon}{C}=\frac{\varepsilon}{19}$

And so, $0<|x-2|<\delta$ then by choosing the smaller of two numbers 1 and $\frac{\varepsilon}{19}$.

By notation $\delta=\min\left\{1,\frac{\varepsilon}{19}\right\}$

Step 2: showing the work of δ.

Given $\varepsilon>0$ and let $\delta=\min\left\{1,\frac{\varepsilon}{19}\right\}$

If $0<|x-2|<\delta$ then $|x^2+2x+4|<19$

Now, $|x-2|<\frac{\varepsilon}{19}$

$$\begin{aligned}
|x^3-8|&<|x-2||x^2+2x+4| \\
&<\frac{\varepsilon}{19}\cdot 19 \\
&<\varepsilon
\end{aligned}$$

Therefore, by the definition of ε,δ limit is, $\lim_{x\to 2} x^3=8$

Chapter 1 Functions and Limits Exercise 1.7 33E

To prove that $\lim_{x\to 3} x^2 = 9$

1

Guessing a value of δ :- Let $\in > 0$ be given. We have to find a number $\delta > 0$ such that

$|x^2 - 9| < \in$ when ever $0 < |x-3| < \delta$

We have $|(x^2-9)| = |(x+3)(x-3)|$

Then $|x+3|.|x-3| < \in$ when ever $0 < |x-3| < \delta$

Let $|x+3| < C$ where C is any positive constant

Then $|x+3|.|x-3| < C|x-3|$

And we can make $C|x-3| < \in$ by taking $|x-3| < \frac{\in}{C} = \delta$

Assume that x is within a distance 2 from 3 that is $|x-3| < 2$

Then $1 < x < 5$

$\Rightarrow 4 < x+3 < 8$

We have $|x+3| < 8$ and so C = 8 is a suitable choice

But there are two restrictions on $|x-3|$ namely,

$|x-3| < 2$ and $|x-3| < \frac{\in}{C} = \frac{\in}{8}$

So $\delta = \min\left\{2, \frac{\in}{8}\right\}$

2

Showing that this δ works: - Given $\in > 0$

let $\delta = \min\left\{2, \frac{\in}{8}\right\}$

If $0 < |x-3| < \delta$

Then $|x-3| < 2 \Rightarrow 1 < x < 5 \Rightarrow |x+3| < 8$ we have also $|x-3| < \frac{\in}{8}$

So $|x^2-9| = |x+3||x-3| < 8.\frac{\in}{8} = \in$

This shows that $\lim_{x\to 13} x^2 = 9$, and $\delta = \min\left\{2, \frac{\in}{8}\right\}$ works.

Chapter 1 Functions and Limits Exercise 1.7 34E

Consider the limit $\lim_{x\to 3} x^2 = 9$

Show that the largest possible choice of δ is $\delta = \sqrt{9+\varepsilon} - 3$:

Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-3| < \delta$, then $|x^2 - 9| < \varepsilon$

$$|x^2 - 9| < \varepsilon$$

$$-\varepsilon < x^2 - 9 < \varepsilon$$

$$9 - \varepsilon < x^2 < 9 + \varepsilon$$

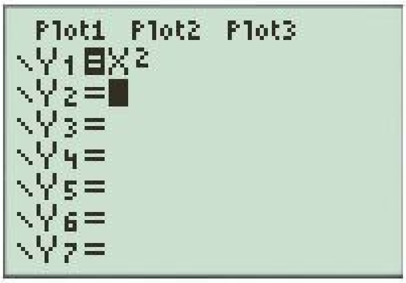
$$\sqrt{9-\varepsilon} < x < \sqrt{9+\varepsilon}$$

$$\sqrt{9-\varepsilon} - 3 < x - 3 < \sqrt{9+\varepsilon} - 3$$

Therefore, the largest possible choice of $\delta = \sqrt{9+\varepsilon} - 3$

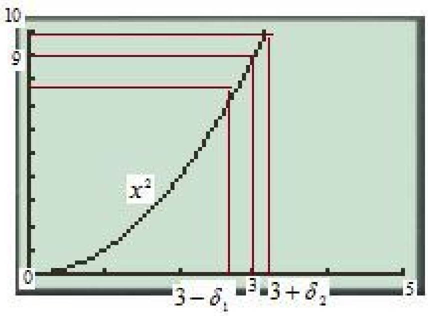
To find the large possible choice of δ , use TI-83Plus calculator

Enter the equations into Y1 in the equation editor $\boxed{\text{Y=}}$.



First set the window as shown in figure.

Now click on the GRAPH button to get the graph.



From the graph observe that $x^2 = 9 - \varepsilon$ when $x = \sqrt{9-\varepsilon}$.

So,

$$3 - \delta_1 = \sqrt{9-\varepsilon}$$
$$\delta_1 = 3 - \sqrt{9-\varepsilon}$$

Again $x^2 = 9 + \varepsilon$ when $x = \sqrt{9+\varepsilon}$.

So,

$$3 + \delta_2 = \sqrt{9+\varepsilon}$$
$$\delta_2 = \sqrt{9+\varepsilon} - 3$$

Now, choose δ is larger of δ_1 and δ_2.

Thus, the number $\delta = \boxed{\sqrt{9+\varepsilon} - 3}$ by graphically

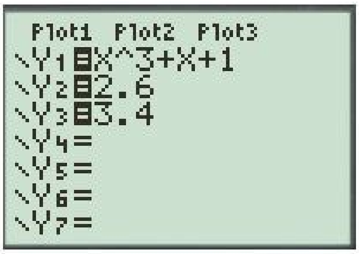
Chapter 1 Functions and Limits Exercise 1.7 35E

Consider the limit $\lim_{x\to 1}(x^3+x+1)=3$

a)

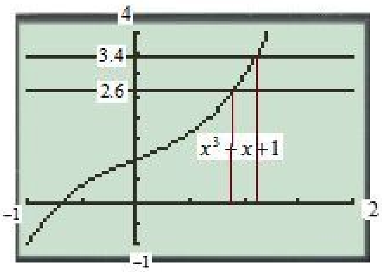
Find a value of δ that corresponds to $\varepsilon=0.4$ by using graph:

Enter the equations into Y1, Y2, and Y3 in the equation editor $\boxed{Y=}$.



First set the window as shown in figure.

Now click on the GRAPH button to get the graph.



From the graph observe that the points of intersection in graph are $(x_1,2.6)$ and $(x_2,3.4)$ with $x_1\approx 0.891$ and $x_2\approx 1.093$.

Now choose δ to be the smaller of $1-x_1$ and x_2-1.

So,

$$\begin{aligned}\delta &= x_2-1\\ &=1.093-1\\ &=0.093\end{aligned}$$

Therefore, the number $\delta=\boxed{0.093}$

b)

Consider the cubic equation $x^3+x+1=3+\varepsilon$

Find the largest possible value of δ that works for any given $\varepsilon>0$:

Thus, the equation gives us two complex roots and one real root.

That one real root is $x(\varepsilon)=\dfrac{\left(216+108\varepsilon+12\sqrt{336+324\varepsilon+81\varepsilon^2}\right)^{\frac{2}{3}}-12}{6\left(216+108\varepsilon+12\sqrt{336+324\varepsilon+81\varepsilon^2}\right)^{\frac{1}{3}}}$ ...... (1)

Thus, $\delta=\boxed{x(\varepsilon)-1}$

c)

Put $\varepsilon = 0.4$ in (1), it gives $x(\varepsilon) = 1.093$

And the value

$$\delta = x(\varepsilon) - 1$$
$$= 1.093 - 1$$
$$= \boxed{0.093}$$

This is the same answer in part (a).

Chapter 1 Functions and Limits Exercise 1.7 36E

To prove $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$

1 Guessing a, value of δ : Let $\in > 0$ be given. We have to find a number $\delta > 0$ such that

$\left|\frac{1}{x} - \frac{1}{2}\right| - \in$ when ever $0 < |x-2| < \delta$

We write $\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right|$

$= \frac{|x-2|}{|2x|}$

Then we have $\frac{|x-2|}{|2x|} < \in$ whenever $0 < |x-2| < \delta$

If we can find a positive constant C such that $\frac{1}{|2x|} < C$

Then $|x-2| . \frac{1}{|2x|} < C|x-2|$

$\Rightarrow C|x-2| < \in$

$\Rightarrow |x-2| < \frac{\in}{C} = \delta$

Assume that $|x-2| < 1$ then $1 < x < 3$

$\Rightarrow 2 < 2x < 6$

$\Rightarrow \frac{1}{2} > \left|\frac{1}{2x}\right| > \frac{1}{6}$

This means $\frac{1}{|2x|} < \frac{1}{2}$ so $C = \frac{1}{2}$

But $|x-2| < 1$ and $|x-2| < \frac{\in}{C} = 2\in$ then $\delta = \min\{1, 2\varepsilon\}$

2 Showing that this δ works: - Given $\in > 0$ if $0 < |x-2| < \delta$

Then $|x-2| < 1 \Rightarrow 1 < x < 3 \Rightarrow \left|\frac{1}{2x}\right| < \frac{1}{2}$ and also $|x-2| < 2\in$

So $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{|2x|} < \frac{1}{2}.2\in = \in$

This shows that $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$

Chapter 1 Functions and Limits Exercise 1.7 37E

1

Guessing a value for δ : - Let $\in > 0$ we have to find $\delta > 0$
Such that

$\left|\sqrt{x} - \sqrt{a}\right| < \in$ *whenever* $0 < |x-a| < \delta$

We use

$\left|\sqrt{x} - \sqrt{a}\right| = \frac{|x-a|}{\sqrt{x} + \sqrt{a}}$

So

$$\frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \epsilon \qquad \textit{whenever} \quad 0<|x-a|<\delta$$

If we find a positive constant C such that $\frac{1}{\sqrt{x}+\sqrt{a}} < C$

Then

$$\frac{|x-a|}{\sqrt{x}+\sqrt{a}} < C.|x-a|$$
$$\Rightarrow C.|x-a| < \epsilon$$
$$\Rightarrow |x-a| < \frac{\epsilon}{C} = \delta$$

If we assume a positive number k such that k < a, k > 0 and then

$$|x-a|<k$$
$$\Rightarrow a-k<x<a+k$$
$$\Rightarrow \sqrt{a-k}<\sqrt{x}<\sqrt{a+k}$$

Then

$$\frac{1}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{a-k}+\sqrt{a}}$$

Then

$$C = \frac{1}{\sqrt{a-k}+\sqrt{a}} \qquad \text{Where k < a and k > 0}$$

But $\quad |x-a|<k \textit{ and } |x-a| < \frac{\epsilon}{C} = \left(\sqrt{a-k}+\sqrt{a}\right)\epsilon$

Then $\quad \delta = \min\left\{k, \left(\sqrt{a-k}+\sqrt{a}\right)\right\}\epsilon$

Showing that this δ works: - given $\epsilon > 0$ let $\delta = \min\left\{k, \left(\sqrt{a-k}+\sqrt{a}\right)\right\}\epsilon$

If $\quad 0<|x-a|<\delta$

Then

$$|x-a|<k \Rightarrow a-k<x<a+k$$
$$\Rightarrow \frac{1}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{a-k}+\sqrt{a}}$$

And also $\quad |x-a| < \left(\sqrt{a-k}+\sqrt{a}\right)\epsilon$

Then

$$\frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{a-k}+\sqrt{a}}.\left(\sqrt{a-k}+\sqrt{a}\right)\epsilon = \epsilon \qquad \text{Where } 0 < k < a$$

This shows

$$\lim_{x\to a}\sqrt{x}=\sqrt{a} \qquad \text{When } a>0$$

Chapter 1 Functions and Limits Exercise 1.7 38E

The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \textit{if } t<0 \\ 1 & \textit{if } t\ge 0 \end{cases} \qquad \text{--- (1)}$$

Suppose that the limit $H(t)$ as $t \to 0$ exist and is L, then for any positive number $\epsilon > 0$, there is a positive number δ such that

$$|H(t)-L| < \epsilon \quad \textit{whenever} \quad 0<|t-0|<\delta$$

Now suppose $\epsilon = \frac{1}{2}$. Three cases arise

Let t < 0 then $H(t)=0$

So we have

$$|H(t)-L|<\frac{1}{2} \quad whenever \quad 0<|t-0|<\delta$$

$$\Rightarrow |0-L|<\frac{1}{2} \quad whenever \quad 0<|t|<\delta$$

$$\Rightarrow |-L|<\frac{1}{2} \quad whenever \quad 0<|t|<\delta$$

$$\Rightarrow \boxed{L<\frac{1}{2}} \qquad \text{--- (2)}$$

Let t > 0, then $H(t)=1$

So we have

$$|H(t)-L|<\frac{1}{2} \quad Whenever \quad 0<|t-0|<\delta$$

$$\Rightarrow |1-L|<\frac{1}{2} \quad Whenever \quad 0<|t|<\delta$$

$$\Rightarrow 1-L<\frac{1}{2} \quad Whenever \quad 0<|t|<\delta$$

$$\Rightarrow -L<-\frac{1}{2} \quad Whenever \quad 0<|t|<\delta$$

$$\Rightarrow \boxed{L>\frac{1}{2}} \quad Whenever \quad 0<|t|<\delta$$

Let t = 0, then $H(t)=1$

$\lim_{t\to 0} H(t)$ Will be found when $t\to 0$ from the left and from the right i.e. we will find $\lim_{t\to 0^-} H(t)=\lim_{t\to 0^+} H(t)$

Now

$$\lim_{t\to 0^-} H(t)=0 \qquad given$$

$$\lim_{t\to 0^+} H(t)=1 \qquad given$$

Hence

$$\lim_{t\to 0^+} H(t)\neq \lim_{t\to 0^+} H(t)$$

Hence from all the three cases it is concluded that the limit L does not exist i.e.

$$\boxed{\lim_{t\to 0} H(t)\; does\, not\; exist}$$

Chapter 1 Functions and Limits Exercise 1.7 39E

Given $f(x)=\begin{cases} 0 & if\ x\ is\ rational \\ 1 & if\ x\ is\ irrational \end{cases}$

Suppose that $\lim_{x\to 0} f(x)$ exist and equal to L

That is $\lim_{x\to 0} f(x)=L$

Then by the definition of limit, for any $\in >0$ these is a number $\delta>0$ such that

$$|f(x)-L|<\in \quad whenever \ 0<|x-0|<\delta$$

Let $\in =\frac{1}{2}$

Then we have $|f(x)-L|<\frac{1}{2} \quad whenever \ 0<|x-0|<\delta$

Now we have two cases

When x is a rational number then $f(x)=0$

So we have

$$|f(x)-L|<\frac{1}{2} \quad whenever \ 0<|x-0|<\delta$$

$$\Rightarrow |0-L|<\frac{1}{2}$$

$$\Rightarrow |-L|<\frac{1}{2}$$

$$\Rightarrow L<\frac{1}{2} \qquad \text{--- (1)} \qquad \left[|x|=-x \ if \ x<0\right]$$

When x is an irrational number then $f(x)=1$

So we have

$$|f(x)-L|<\frac{1}{2} \quad whenever \ 0<|x-0|<\delta$$

$$\Rightarrow |1-L|<\frac{1}{2}$$

$$\Rightarrow 1-L<\frac{1}{2}$$

$$\Rightarrow -L<-\frac{1}{2}$$

$$\Rightarrow \boxed{L>\frac{1}{2}} \qquad \text{--- (2)}$$

There is a contradiction in (1) and (2) so

$\lim_{x\to 0} f(x)=$ does not exist

Chapter 1 Functions and Limits Exercise 1.7 40E

By the definition of left hand limit

$\lim_{x\to a^-} f(x)=L$ --- (A)

If for every number $\in>0$ there is a number $\delta>0$ such that

$|f(x)-L|<\in$ when ever $a-\delta<x<a$ --- (1)

And by the definition of right hand limit

$\lim_{x\to a^+} f(x)=L$ --- (B)

If for every number $\in>0$ there is a number $\delta>0$ such that

$|f(x)-L|<\in$ when ever $a<x<a+\delta$ --- (2)

By adding the inequalities (1) and (2)

$2|f(x)-L|<2\in$ when ever $a-\delta<x<a$ and $a<x<a+\delta$

So $|f(x)-L|<\in$ when ever $a-\delta<x<a+\delta$

This means $|f(x)-L|<\in$ whenever $0<|x-a|<\delta$

Then by the definition of the limit

We have $\lim_{x\to a} f(x)=L$ --- (C)

From (A), (B) and (C) we can have

If $\lim_{x\to a^-} f(x)=L=\lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)=L$

Or we can say $\lim_{x\to a} f(x)=L$ if and only if $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=L$

Chapter 1 Functions and Limits Exercise 1.7 41E

We have $\lim_{x\to -3}\frac{1}{(x+3)^4}=\infty$

So by the definition of infinite limit, for every positive number M, there is a positive number δ such that

$$\frac{1}{(x+3)^4}>M \quad whenever \ 0<|x+3|<\delta \qquad [0<|x-(-3)|<\delta]$$

Guessing the value of δ for M = 10,000

Here M = 10,000 (Given)

So we have $\frac{1}{(x+3)^4}>10,000$ *whenever* $0<|x+3|<\delta$

$$\Rightarrow (x+3)^4<\frac{1}{10,000} \quad whenever \ 0<|x+3|<\delta$$

$$\Rightarrow |x+3|<\frac{1}{\sqrt[4]{10,000}} \quad whenever \ 0<|x+3|<\delta$$

$$\Rightarrow |x+3|<\frac{1}{10} \quad whenever \ 0<|x+3|<\delta$$

$$\Rightarrow |x+3|<0.1 \quad whenever \ 0<|x+3|<\delta$$

We should take $\delta=0.1$

So we have to take x within $\boxed{0.1}$

Chapter 1 Functions and Limits Exercise 1.7 42E

To prove $\lim_{x \to -3} \frac{1}{(x+3)^4} = \infty$

Assuming the value of δ : - by definition for every positive number M, there is a positive number δ

Such that

$$\frac{1}{(x+3)^4} > M \text{ when ever } 0 < |x+3| < \delta$$

$$\Rightarrow (x+3)^4 < \frac{1}{M} \text{ when ever } 0 < |x+3| < \delta$$

$$\Rightarrow |x+3| < \frac{1}{\sqrt[4]{M}} \text{ when ever } 0 < |x+3| < \delta$$

We should take $\delta = \frac{1}{\sqrt[4]{M}}$

Showing, this δ works:-

If $M > 0$, let $\delta = \frac{1}{\sqrt[4]{M}}$ of $0 < |x+3| < \delta$

Then $\Rightarrow |x+3| < \delta \Rightarrow (x+3)^2 < \delta^2$

$$\Rightarrow (x+3)^4 < \delta^4$$

$$\Rightarrow \frac{1}{(x+3)^4} > \frac{1}{\delta^4} = M$$

Thus $\frac{1}{(x+3)^4} > M$ Whenever $0 < |x+3| < \delta$

Therefore by the definition

$$\lim_{x \to -3} \frac{1}{(x+3)^4} = \infty$$

Chapter 1 Functions and Limits Exercise 1.7 43E

To prove $\lim_{x \to -1^-} \frac{5}{(x+1)^3} = -\infty$

1

Guessing the value of δ : - By the definition for every negative number N there is a positive number δ Such that

$$\frac{5}{(x+1)^3} < N \quad \begin{cases} \text{when ever } (-1) - \delta < x < (-1) \\ \Rightarrow -\delta < x+1 < 0 \Rightarrow |x+1| > \delta \end{cases}$$

$$\Rightarrow \frac{5}{(x+1)^3} < N \quad \text{Whenever } |x+1| > \delta$$

$$\Rightarrow \frac{1}{(x+1)^3} < \frac{N}{5} \text{ when ever } 0 > |x+1| > \delta$$

$$\Rightarrow (x+1)^3 > \frac{5}{N} \qquad 0 > |x+1| > \delta$$

$$\Rightarrow |x+1| > \sqrt[3]{\frac{5}{N}} \qquad 0 > |x+1| > \delta$$

So we should take $\sqrt[3]{\frac{5}{N}} = \delta$

2 Showing, this δ works: - of $N<0$ let $\delta=\sqrt[3]{\frac{5}{N}}$

Then $|x+1|>\sqrt[3]{\frac{5}{N}}$

$$\Rightarrow (x+1)^3>\frac{5}{N}$$

$$\Rightarrow \frac{1}{(x+1)^3}<\frac{N}{5}$$

$$\Rightarrow \frac{5}{(x+1)^3}<N \text{ when, } -1-\delta<x<-1$$

This show $\lim_{x\to -1^-}\frac{5}{(x+1)^3}=-\infty$

Chapter 1 Functions and Limits Exercise 1.7 44E

(A)

By this limit law, we know that if $\lim_{x\to a} f(x)=L$ $\lim_{x\to a} g(x)=M$

Then $\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)=L+M$

Suppose $\lim_{x\to a} f(x)=\infty$ *and* $\lim_{x\to a} g(x)=C$, C is a real number

Then $\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a} f(x)\lim_{x\to a} g(x)$

$$=\infty+C$$

$$=\infty$$

So $\boxed{\lim_{x\to a}[f(x)+g(x)]=\infty}$ Proved

(B)

Here $\lim_{x\to a} f(x)=\infty$ *and* $\lim_{x\to a} g(x)=C$ *is* $C>0$

By limit law we have

$$\lim_{x\to a}[f(x)g(x)]=\lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

$$=\infty.C$$

$$=\infty$$

So $\boxed{\lim_{x\to a}[f(x)g(x)]=\infty}$ proved

(C) $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=C$ where $C<0$

Here C is a negative constant let C = -K

Then $\lim_{x\to a} g(x)=-K$

Now by limit law we have

$$\lim_{x\to a}[f(x).g(x)]=\lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

$$=\infty\times(-K)$$

By simple arithmetic we know $(x)(-y)=-xy$

So here $\boxed{\lim_{x\to a}[f(x).g(x)]=-\infty}$ proved