Chapter 6

TANGENTIAL AND NORMAL ACCELERATIONS. UNIPLANAR CONSTRAINED MOTION

End of Art 90

EXAMPLES

1. $v\frac{dv}{ds}=A$, so that $v^2=2As+B$. $\frac{v^2}{\rho}\div v=C$, so that $C\rho=v$.

$$\therefore\ C\frac{ds}{d\psi}=\sqrt{2As+B},\ i.e.\ A\psi=C(2As+B)^{\frac{1}{2}}+D.$$

If s and ψ vanish together, the equation is $2C^2s=A\left[\psi^2-2\psi\gamma\right]$, where γ is arbitrary.

2. $s=4a\sin\theta$, where $\dot{\theta}=\omega$, a constant.

$$\therefore\ v=\frac{ds}{dt}=4a\omega\cos\theta,\ \frac{d^2s}{dt^2}=-4a\omega^2\sin\theta,\ \text{and}\ \rho=\frac{ds}{d\theta}=4a\cos\theta,$$

so that $\frac{v^2}{\rho}=4a\omega^2\cos\theta$. Hence the resultant acceleration $=4a\omega^2$.

3. $\frac{v\,dv}{ds}=\frac{v^2}{\rho}$, and $\frac{d\psi}{dt}=\omega$, so that $\frac{dv}{v}=d\psi$.

$$\therefore\ v=Ae^{\psi},\ \text{and}\ \frac{ds}{d\psi}=v\div\dot{\psi}=\frac{A}{\omega}e^{\psi}.\quad\therefore\ s=\frac{A}{\omega}e^{\psi}+B,\ \text{giving the path.}$$

4. $b+av^2=(b+ac^2)e^{-2as}$, so that the tangential acceleration

$$=v\frac{dv}{ds}=-(b+ac^2)e^{-2as},$$

$$t=\int\frac{\sqrt{a}e^{as}ds}{\sqrt{b+ac^2-be^{2as}}}=\frac{1}{\sqrt{ab}}\sin^{-1}\left[\sqrt{\frac{b}{b+ac^2}}e^{as}\right]+C.$$

Now $v=V$ when $e^{2as}=\frac{b+ac^2}{b+aV^2}$.

$$\therefore\ t=\frac{1}{\sqrt{ab}}\left[\sin^{-1}\sqrt{\frac{b}{b+aV^2}}-\sin^{-1}\sqrt{\frac{b}{b+ac^2}}\right].$$

5. The curve is $s=4a\sin\theta$, and therefore $\rho=4a\cos\theta$. If P be the constant force, then, since it is inclined at θ to the normal to the path, we have

$$v\frac{dv}{ds}=-P\sin\theta,\ \text{and}\ \frac{v^2}{\rho}=P\cos\theta,\ i.e.\ v^2=4Pa\cos^2\theta.$$

$$\therefore\ v\,dv=-4aP\cos\theta\sin\theta\,d\theta,\ \text{and}\ \therefore\ v^2=4aP\cos^2\theta+C.$$

These agree if $C=0$, i.e. if the velocity of projection at the vertex

$$=\sqrt{4aP}.$$

6. $R = mg\cos\alpha$, and $\mu R = mg\cos\alpha \cdot \tan\alpha = mg\sin\alpha$.

Hence $\quad v\frac{dv}{ds} = g\sin\alpha\sin\psi - g\sin\alpha$, and $\frac{v^2}{\rho} = g\sin\alpha\cos\psi$.

$$\therefore \frac{dv}{v} = d\psi\frac{\sin\psi - 1}{\cos\psi} = -d\psi\frac{\cos\psi}{1+\sin\psi}.$$

$\therefore \log v = -\log(1+\sin\psi) + \log V$, *i.e.* $v = \frac{V}{1+\sin\psi}$, and the maximum value of v is thus $\frac{1}{2}V$.

$$\frac{ds}{d\psi} = v^2 \div g\sin\alpha\cos\psi = \frac{V^2}{g\sin\alpha}\cdot\frac{1}{(1+\sin\psi)^2\cos\psi}.$$

$$\therefore \frac{g\sin\alpha}{V^2}s = \int\frac{1}{8}\frac{(\xi+1)^2}{\xi^2}d\xi, \left[\text{where } \xi = \frac{1+\sin\psi}{1-\sin\psi}\right] = \frac{1}{8}\left[\log\xi - \frac{2}{\xi} - \frac{1}{2\xi^2}\right] + A,$$

where $\quad 0 = \frac{1}{8}\left[-2-\frac{1}{2}\right] + A$, *i.e.* $A = \frac{5}{16}$.

Hence the equation required on substitution.

7. $\dot{x} = v$, and $\ddot{x} = f$; $v = a\dot{\theta}$, and $f = a\ddot{\theta}$.

The accelerations of P, relative to the centre O, are $a\ddot{\theta}\,(=f)$ along the tangent at P, and $a\dot{\theta}^2\left(=\frac{v^2}{a}\right)$ along PO.

Also the acceleration of O is f parallel to the straight line.

Hence the accelerations of P are $\frac{v^2}{a} + f\sin\theta$ along PO, and $f - f\cos\theta$ along the tangent at P.

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EXAMPLES

1. The rod will sink until the total work done is zero, *i.e.* until

$$mga\tan\theta = 2\times\frac{1}{2}\left(\frac{a}{\cos\theta} - a\right)\cdot\lambda\frac{\frac{a}{\cos\theta} - a}{a},$$

i.e. until $\quad \frac{\sin\theta\cos\theta}{(1-\cos\theta)^2} = n$, and then $\cot^3\frac{\theta}{2} - \cot\frac{\theta}{2} = 2n$.

2. When the depth of m below the pulley is x, let the depth of M be y, so that $\sqrt{a^2+x^2} + y = l$.

Motion ceases when the total work done is zero, *i.e.* when

$$mx + My = M(l-a),$$

i.e. when $mx + Ma = M\sqrt{a^2+x^2}$, giving the required value for x.

Also the equation of Energy gives

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M\dot{y}^2 = [mx + My - M(l-a)]g,$$

i.e. $\quad \frac{1}{2}\dot{x}^2\left[m + M\frac{x^2}{a^2+x^2}\right] = [mx - M\sqrt{a^2+x^2} + Ma]g$. Hence $\dot{x}$.

3. $\frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{1}{2}MV^2 + E$.

Also since the momentum is unaltered by the explosion,

$$m_1V_1 + m_2V_2 = MV.$$

Solve for V_1 and V_2.

4. Consider an element PQ, which subtends an angle $\delta\phi$ at the centre and whose mass is thus $\frac{m \cdot \delta\phi}{2\pi}$, and which at time t has expanded into the element PQ of a circle of radius r.

Since the acceleration perpendicular to PQ is zero, we have

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0,$$

so that $r^2\dot{\theta}=\text{const.}=a^2\omega$.

Also, for the motion along OP, we have

$$\frac{m\delta\phi}{2\pi}(\ddot{r}-r\dot{\theta}^2)=-2T\sin\frac{\delta\phi}{2}=-T\delta\phi=-\lambda\frac{r-a}{a}\delta\phi.$$

$$\therefore\ \ddot{r}-\frac{a^4}{r^3}\omega^2=-\frac{2\lambda\pi}{ma}(r-a).\qquad \therefore\ \dot{r}^2=a^2\omega^2\left(1-\frac{a^2}{r^2}\right)-\frac{2\lambda\pi}{ma}(r-a)^2,$$

since $\dot{r}=0$ when $r=a$.

5. Let the particles be A, B, C and D, and let the string BC be cut. When AD is at a distance x from the centre O, let the strings DC and AB be inclined at θ to their original positions. The velocities of C are $\dot{x}+2a\dot{\theta}\sin\theta$ perpendicular to AD, and $2a\dot{\theta}\cos\theta$ parallel to AD. So for B.

Since the centre of gravity of the system is at rest,

$$\therefore\ 2m\dot{x}+2m(\dot{x}+2a\dot{\theta}\sin\theta)=0.$$

Hence $x=a\cos\theta$, since $x=a$ when $\theta=0$.

Also the equation of energy gives

$$2\cdot\tfrac{1}{2}m\dot{x}^2+2\cdot\tfrac{1}{2}m\left[(\dot{x}+2a\dot{\theta}\sin\theta)^2+(2a\dot{\theta}\cos\theta)^2\right]$$

$$=2\int_{OD}^{OD'}4\mu mr\,dr+2\int_{OC}^{OC'}4\mu mr\,dr=4m\mu\left[OD'^2-OD^2+OC'^2-OC^2\right].$$

$$\therefore\ 2m\left[\dot{x}^2+2\dot{x}a\dot{\theta}\sin\theta+2a^2\dot{\theta}^2\right]$$

$$=4\mu m\left[(x^2+a^2)-2a^2+\{(x-2a\cos\theta)^2+(a+2a\sin\theta)^2\}-2a^2\right],$$

i.e. $\quad 2ma^2(2-\sin^2\theta)\dot{\theta}^2=8\mu m\left[x^2-2ax\cos\theta+a^2+2a^2\sin\theta\right]$

$$=8\mu ma^2\left[\sin^2\theta+2\sin\theta\right],\text{ etc.}$$

6. If v is the required velocity, the equation of Energy gives

$$\tfrac{1}{2}mv^2+\tfrac{1}{2}m'v^2+\tfrac{1}{2}Mv^2=mga-m'ga+Mg\left[a-\frac{a}{2}\right],\text{ etc.}$$

7. When the portions on each side are $l+c+x$ and $l-c-x$, the depth of the centre of gravity of the chain

$$=\left\{\frac{m(l+c+x)}{2l}\cdot\frac{l+c+x}{2}+\frac{m(l-c-x)}{2l}\cdot\frac{l-c-x}{2}\right\}\div m=\frac{(x+c)^2+l^2}{2l}.$$

Hence the equation of Energy gives

$$\tfrac{1}{2}m\dot{x}^2=\frac{mg}{2l}\left[(x+c)^2+l^2-c^2-l^2\right].$$

$$\therefore\ t\sqrt{\frac{g}{l}}=\int_0^{l-c}\frac{dx}{\sqrt{(x+c)^2-c^2}}=\left[\log\left(x+c+\sqrt{(x+c)^2-c^2}\right)\right]_0^{l-c}$$

$$=\log\frac{l+\sqrt{l^2-c^2}}{c}.$$

8. When a length x has run over the edge, we have

$$m\ddot{x}=\frac{mx}{l}g+\frac{m(l-x)}{l}g\sin\alpha.$$

Also, if T be the tension at the pulley, we have, for the motion of the straight portion of the chain,

$$m\frac{x}{l}\ddot{x}=\frac{mx}{l}g-T.$$

Substituting for $\ddot{x}$, we obtain $T=\frac{mgx(l-x)}{l^2}(1-\sin\alpha)$.

9. We obtain the relative motion by giving to the whole system an acceleration f downwards, so that the motion is the same as it would be if the pulley were at rest and g changed to $g+f$. When the longer part is $l+a+x$, and the shorter $l-a-x$, the equation of motion is

$$m\,.\,2l\ddot{x}=m(2a+2x)(f+g).$$

$$\therefore\ l\dot{x}^2=\{(a+x)^2-a^2\}(f+g),\text{ since }\dot{x}=0\text{ when }x=0.$$

$$\therefore\ \sqrt{\frac{f+g}{l}}\,.\,t=\int_0^{l-a}\frac{dx}{\sqrt{(a+x)^2-a^2}}=\left[\cosh^{-1}\frac{x+a}{a}\right]_0^{l-a}=\cosh^{-1}\frac{l}{a}.$$

Art 99

Ex. 1. Using the approximation, to squares of the small angle of oscillation, in Art. 97, we have

$$2\pi\sqrt{\frac{l}{g}}\left[1+\frac{1}{16}\frac{\pi^2}{60^2}\right]=1;\quad 2\pi\sqrt{\frac{l}{g}}\left[1+\frac{1}{16}\frac{\pi^2}{45^2}\right]=T_1;$$

and

$$2\pi\sqrt{\frac{l}{g}}\left[1+\frac{1}{16}\frac{\pi^2}{36^2}\right]=T_2.$$

Hence, by division, $T_1=1+\frac{1}{16}\frac{\pi^2}{45^2}-\frac{1}{16}\frac{\pi^2}{60^2}=1+\frac{7\pi^2}{16\,.\,180^2}$.

$$\therefore\ 86400\left(1-\frac{1}{T_1}\right)=86400\left[1-\left(1-\frac{7\pi^2}{16\,.\,180^2}\right)\right]$$

$$=86400\,.\,\frac{7\,.\,10}{16\,.\,32400}=\frac{315}{27}=\frac{35}{3}=\text{nearly }12.$$

Also

$$T_2=1+\frac{1}{16}\frac{\pi^2}{36^2}-\frac{1}{16}\frac{\pi^2}{60^2}=1+\frac{\pi^2}{180^2}.$$

$$\therefore\ 86400\left(1-\frac{1}{T_2}\right)=86400\left[1-\left(1-\frac{\pi^2}{180^2}\right)\right]=\frac{86400\times 10}{32400}=\text{about }27.$$

Ex. 2. We have $a\ddot{\theta}=-g\sin\theta$.

$$\therefore\ a\dot{\theta}^2=2g\cos\theta+C=2g(1+\cos\theta),\text{ since }\dot{\theta}=0\text{ when }\theta=\pi.$$

$$\therefore\ t\sqrt{\frac{g}{a}}=\int\frac{d\theta}{2\cos\frac{\theta}{2}}=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{4}\right),$$

the constant vanishing since t and θ vanish together.

$$\therefore\ e^{t\sqrt{\frac{g}{a}}}-e^{-t\sqrt{\frac{g}{a}}}=\tan\left(\frac{\pi}{4}+\frac{\theta}{4}\right)-\cot\left(\frac{\pi}{4}+\frac{\theta}{4}\right)=\frac{-2\cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)}=2\tan\frac{\theta}{2}.$$

$$\therefore\ \theta=2\tan^{-1}\left(\sinh\sqrt{\frac{g}{a}}\,t\right).$$

End of Art 103

EXAMPLES ON CHAPTER 6

1. $v\dfrac{dv}{ds}=-g\sin\psi$, and $\dfrac{v^2}{\rho}=-\dfrac{R}{m}+g\cos\psi$,

where $$\tan\psi=\frac{dy}{dx}=\cosh\frac{x}{a}=\sqrt{1+\frac{y^2}{a^2}},$$

and $v^2=2g(y-b)$, where $\tan\alpha=\sqrt{1+\dfrac{b^2}{a^2}}$.

Now $$\frac{d^2y}{dx^2}=\frac{1}{a}\sinh\frac{x}{a}=\frac{y}{a^2}\text{, so that }\rho=\left(1+\cosh^2\frac{x}{a}\right)^{\frac{3}{2}}\cdot\frac{a^2}{y}.$$

The particle leaves the curve when $R=0$, *i.e.* when

$$2g(y-b)=v^2=g\rho\cos\psi=g\left(1+\cosh^2\frac{x}{a}\right)\frac{a^2}{y}=g\left(2+\frac{y^2}{a^2}\right)\frac{a^2}{y},$$

i.e. when $(y-b)^2=b^2+2a^2=a^2+a^2\tan^2\alpha=a^2\sec^2\alpha$, and then $y-b=a\sec\alpha$.

2. Let the axis of x be horizontal and the axis of y vertically downwards.

Then $\dot{x}^2+\dot{y}^2=2gy+V_1^2$, and $y=Vt$, so that $\dfrac{dy}{dt}=V$.

Also initially $\dot{x}=0$, so that $V=V_1$.

Hence $$\left(\frac{dx}{dy}V\right)^2+V^2=2gy+V^2,\ i.e.\ Vdx=\sqrt{2gy}\,dy.$$

$$\therefore\ Vx=\sqrt{2g}\cdot\frac{2}{3}\cdot y^{\frac{3}{2}}\text{, since } x \text{ and } y \text{ vanish together.}$$

$$\therefore\ y^3=\frac{9V^2}{8g}x^2\text{, a semi-cubical parabola.}$$

3. As in Art. 100, $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s$. Hence $\left(\dfrac{ds}{dt}\right)^2-V^2=-\dfrac{g}{4a}(s^2-16a^2)$.

$$\therefore\ \sqrt{\frac{g}{4a}}\cdot t=-\int_{4a}^{0}\frac{ds}{\sqrt{16a^2+\frac{4aV^2}{g}-s^2}}=\left[\sin^{-1}\frac{s}{\sqrt{16a^2+\frac{4aV^2}{g}}}\right]^{4a},$$

$$=\sin^{-1}\frac{2\sqrt{ga}}{\sqrt{4ag+V^2}}=\tan^{-1}\frac{\sqrt{4ag}}{V}\text{, etc.}$$

4. As in Art. 100, $\ddot{s}=-\dfrac{g}{4a}s$.

$\therefore\ s=A\cos\sqrt{\dfrac{g}{4a}}t+B\sin\sqrt{\dfrac{g}{4a}}t$, where $4a=A$ and $0=\left(\dfrac{ds}{dt}\right)_0=B$.

$$\therefore\ s=4a\cos\sqrt{\frac{g}{4a}}t.$$

Now when $AN=a$, $AQ=\sqrt{2}a$, and arc $AP=2\sqrt{2}a$.

Hence the time from C to $P=\sqrt{\dfrac{4a}{g}}\cos^{-1}\dfrac{2\sqrt{2}a}{4a}=\sqrt{\dfrac{4a}{g}}\cdot\dfrac{\pi}{4}=\dfrac{1}{2}\pi\sqrt{\dfrac{a}{g}}$

$=$ one half the time from C to the lowest point (Art. 100).

5. $\ddot{s} = g \sin\theta = \frac{g}{4a} s$, so that $v^2 = \dot{s}^2 = \frac{g}{4a} s^2$.

$$g \cos\theta - \frac{R}{m} = \frac{v^2}{\rho} = \frac{g}{4a} \cdot \frac{s^2}{4a\cos\theta} = g \cdot \frac{\sin^2\theta}{\cos\theta},$$

since $\qquad s = 4a \sin\theta$ and $\rho = \frac{ds}{d\theta} = 4a\cos\theta$.

Hence R is zero when $\theta = 45^\circ$.

6. $\ddot{s} = \frac{gs}{4a}$, so that $s = A \cosh \sqrt{\frac{g}{4a}}\, t + B \sinh \sqrt{\frac{g}{4a}}\, t$,

where $0 = A$ and $v = [\dot{s}]_{t=0} = \sqrt{\frac{g}{4a}}\, B$.

Hence, when $s = 4a$, $\quad t_1 = \sqrt{\frac{4a}{g}} \sinh^{-1}\left(\frac{\sqrt{4ag}}{v}\right)$.

Also, at the end of the first cycloid,

$$V = \frac{ds}{dt} = B\sqrt{\frac{g}{4a}} \cosh \sqrt{\frac{g}{4a}}\, t_1 = v\sqrt{1 + \frac{4ag}{v^2}} = \sqrt{v^2 + 4ag}.$$

Hence, as in Ex. 3,

$$t_2 = 2\sqrt{\frac{a}{g}} \tan^{-1}\left(\frac{\sqrt{4ag}}{\sqrt{v^2+4ag}}\right) = 2\sqrt{\frac{a}{g}} \sin^{-1}\sqrt{\frac{4ag}{v^2+8ag}}.$$

7. A the lowest point and P any point of the catenary; PN the perpendicular to the directrix and NY perpendicular to the tangent PT. Then $PY = \text{arc } AP = y$, and the equation of motion is $\frac{d^2s}{dt^2} = -\mu y \cdot \cos YPN = -\mu s$.
Hence, etc.

8. Let O be the centre, C the fixed point, $A'COA$ a diameter and

$$\angle POA = \theta.$$

Then $\qquad a\ddot{\theta} = \mu . CP \sin OPC = \mu c \sin\theta.$

$$\therefore \dot{\theta}^2 = \frac{2c\mu}{a}(1 - \cos\theta), \text{ since } \theta \text{ and } \dot{\theta} \text{ vanish together,}$$

$$\therefore t\sqrt{\frac{c\mu}{a}} = \int_{\frac{\pi}{2}}^{\pi} \frac{d\theta}{2\sin\frac{\theta}{2}} = \left[\log\tan\frac{\theta}{4}\right]_{\frac{\pi}{2}}^{\pi} = -\log(\sqrt{2}-1) = \log(\sqrt{2}+1).$$

9. $v\frac{dv}{ds} = -\frac{\mu}{r^2}\frac{dr}{ds}$, and $\frac{v^2}{\rho} = \frac{R}{m} + \frac{\mu}{r^2}\sin\alpha$.

$$\therefore \dot{s}^2 = v^2 = 2\mu\left(\frac{1}{r} - \frac{1}{b}\right), \text{ since } v = 0 \text{ when } r = b.$$

$$\therefore \sqrt{2\mu} . t = -\int_b^0 \sqrt{\frac{rb}{b-r}}\, ds = \frac{\sqrt{b}}{\cos\alpha}\int_0^{\frac{\pi}{2}} 2b\sin^2\theta\, d\theta, \text{ if } r = b\sin^2\theta,$$

and hence $ds = dr \sec\alpha = 2b \sin\theta\cos\theta\sec\alpha\, d\theta$. $\quad \therefore t = \frac{\pi}{2}\sqrt{\frac{b^3}{2\mu}} \sec\alpha$.

Also $\qquad \frac{R}{m} = \frac{v^2 . \sin\alpha}{r} - \frac{\mu\sin\alpha}{r^2} = \frac{\mu\sin\alpha\,(b-2r)}{br^2}.$

10. $\dot{s}^2 = 2g(h-y)$, and $R - mg\cos\theta = \frac{mv^2}{\rho}$.

Now $\quad y = \frac{x^2}{4a}$; $\tan\theta = \frac{dy}{dx} = \frac{x}{2a}$; $\frac{d^2y}{dx^2} = \frac{1}{2a}$ and $\therefore\ \rho = \frac{2(y+a)}{\cos\theta}$.

$$\therefore\ \frac{R}{m} = \frac{2g(h-y)}{\rho} + g\cdot\frac{2(y+a)}{\rho} = 2g\cdot\frac{h+a}{\rho}.$$

11. If ψ is the inclination of the normal to the major axis, and θ the eccentric angle, then

$$\frac{R}{m} + g\sin\psi = \frac{v^2}{\rho} = \frac{ab[V^2 - 2gb(1+\sin\theta)]}{(a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{3}{2}}},\quad \text{since } \rho = \frac{CD^3}{ab}.$$

The particle will *just* leave the curve at the highest point, where $\theta = \psi = \frac{\pi}{2}$, if $R = 0$ there, *i.e.* if $V = \sqrt{g\frac{a^2+4b^2}{b}}$, and the particle will reach the highest point since this velocity is $> \sqrt{2g\cdot 2b}$.

Also the particle must reach the end of the major axis, *i.e.* $V > \sqrt{2gb}$; for it clearly will not leave the curve otherwise. Also, if ψ is positive, *i.e.* if the particle gets beyond the end of the major axis, R is negative when v vanishes, *i.e.* R vanishes before v does, and then the particle leaves the curve.

12. Let C be the centre of the circle, O the centre of attraction, and $AOCB$ a diameter.

Then $\quad a\ddot{\theta} = -\frac{\mu}{OP^2}\sin CPO = -\frac{\mu b\sin\theta}{(a^2+b^2-2ab\cos\theta)^{\frac{3}{2}}}$.

$$\therefore\ a^2\dot{\theta}^2 = \frac{2\mu}{\sqrt{a^2+b^2-2ab\cos\theta}} + V^2 - \frac{2\mu}{a-b},$$

where V is the required velocity at A.

Then $\quad (\text{Velocity at } B)^2 = \frac{2\mu}{a+b} + V^2 - \frac{2\mu}{a-b} = V^2 - \frac{4\mu b}{a^2-b^2}$.

Hence V must be not less than $\sqrt{\frac{4\mu b}{a^2-b^2}}$.

13. $\frac{v\,dv}{ds} = -\left(\frac{\mu}{r^2}+\frac{\lambda}{r^3}\right)\frac{dr}{ds}$, so that $v^2 = \frac{2\mu}{r} + \frac{\lambda}{r^2} + A$,

where $\quad \mu\left(\frac{2}{R}-\frac{1}{a}\right) = \frac{2\mu}{R} + \frac{\lambda}{R^2} + A$. Also $\frac{v^2}{\rho} = \left(\frac{\mu}{r^2}+\frac{\lambda}{r^3}\right)\sin\phi - S$.

Now $\frac{b^2}{p^2} = \frac{2a}{r} - 1$, so that $\rho = r\frac{dr}{dp} = \frac{r^3}{p^3}\frac{b^2}{a}$, and $\rho\sin\phi = \rho\cdot\frac{p}{r} = \frac{r^2}{p^2}\frac{b^2}{a}$.

$$\therefore\ S\cdot\rho = \frac{r^2b^2}{ap^2}\left(\frac{\mu}{r^2}+\frac{\lambda}{r^3}\right) - \frac{2\mu}{r} - \frac{\lambda}{r^2} + \frac{\mu}{a} + \frac{\lambda}{R^2}, \text{ etc.}$$

14. Let $x = a\cos^3\theta$, and $y = a\sin^3\theta$.

Then $\ddot{s} = -\mu y^{\frac{1}{3}}\sin\phi$, where $\tan\phi = -\frac{dy}{dx} = \tan\theta$,

and $\left(\frac{ds}{d\theta}\right)^2 = 9a^2\sin^4\theta\cos^2\theta + 9a^2\sin^2\theta\cos^4\theta = 9a^2\sin^2\theta\cos^2\theta$.

$$\therefore \frac{ds}{d\theta} = 3a\sin\theta\cos\theta, \text{ and } s = \frac{3a}{2}\sin^2\theta.$$

$$\therefore \ddot{s} = -\mu a^{\frac{1}{3}}\sin^2\theta = -\frac{2\mu}{3a^{\frac{2}{3}}}s. \quad \text{Hence, etc.}$$

15. With the usual notation $r^2 = a^2 + \frac{4b(a+b)}{(a+2b)^2}p^2$, and $\psi = \theta \cdot \frac{a+2b}{2b}$.

$$\therefore \frac{ds}{d\theta} = \frac{a+2b}{2b}\cdot\frac{ds}{d\psi} = \frac{a+2b}{2b}\cdot r\frac{dr}{dp} = \frac{2(a+b)}{a+2b}p = 2(a+b)\sin\frac{a\theta}{2b}.$$

Hence, if A is the vertex of the curve, where $\theta = \frac{\pi b}{a}$, then

$$\text{arc } PA = \int_\theta^{\frac{\pi b}{a}} 2(a+b)\sin\frac{a\theta}{2b} = \frac{4b(a+b)}{a}\cos\frac{a\theta}{2b}.$$

Hence the acceleration at P along the arc towards A

$$= \mu r\cos OPT = \mu\sqrt{r^2 - p^2} = \mu\sqrt{a^2 - \frac{a^2p^2}{(a+2b)^2}} = \mu a\cos\frac{a\theta}{2b}$$

$$= \frac{\mu a^2}{4b(a+b)} \times \text{arc } PA. \quad \text{Hence, etc.}$$

If the curve is a hypocycloid, and the force attractive, we change the signs of μ and b, and the acceleration similarly $= \frac{\mu a^2}{4b(a-b)} \times \text{arc } PA$. Hence, etc.

16. $\dot{r}^2 + r^2\dot{\theta}^2 = \dot{s}^2 = 2gy = 2gr\cos\theta$. Hence, if $r = f(\theta)$, we have given that

$$\int_0^\alpha \sqrt{\frac{f^2 + f'^2}{2gf\cos\theta}} = t = \sqrt{\frac{2r}{g\cos\alpha}} = \sqrt{\frac{2f(\alpha)}{g\cos\alpha}}.$$

Differentiate with respect to α, and we have

$$\sqrt{\frac{f^2 + f'^2}{f\cos\alpha}} = \frac{1}{2}\sqrt{\frac{\cos\alpha}{f}}\left[\frac{2f'\cos\alpha + 2f\sin\alpha}{\cos^2\alpha}\right],$$

and hence $\frac{2f'(\alpha)}{f(\alpha)} = (\cot\alpha - \tan\alpha)$, i.e. $2\log f(\alpha) = \log\sin\alpha + \log\cos\alpha + \text{const.}$

$$\therefore f(\theta) = A\sqrt{\sin\theta\cos\theta},$$

so that the curve is $r^2 = \frac{A^2}{2}\sin 2\theta = \frac{A^2}{2}\cos\left[2\left(\frac{\pi}{4} - \theta\right)\right]$, etc.

17. Since there are no forces the particle will continue to move in the plane section, *i.e.* a circle, in which it starts.

The equations of motion are $a\ddot{\theta} = -\mu R$, and $a\dot{\theta}^2 = R$.

Hence $\qquad \ddot{\theta} + \mu\dot{\theta}^2 = 0. \quad \therefore \log \dot{\theta} + \mu\theta - \text{const.} = \log \frac{V}{a}.$

$$\therefore \dot{\theta} = \frac{V}{a} e^{-\mu\theta}, \text{ and } t = \int_0^{2\pi} \frac{a}{V} e^{\mu\theta} \, d\theta = \frac{a}{\mu V}(e^{2\mu\pi} - 1).$$

18. $a\ddot{\theta} = g\cos\theta - \mu\frac{R}{m}$, and $a\dot{\theta}^2 = \frac{R}{m} - g\sin\theta$.

$$\therefore \ddot{\theta} + \mu\dot{\theta}^2 = \frac{g}{a}(\cos\theta - \mu\sin\theta). \quad \therefore \dot{\theta}^2 e^{2\mu\theta} = \frac{2g}{a}\int e^{2\mu\theta}(\cos\theta - \mu\sin\theta)$$

$$= \frac{2g}{a(1+4\mu^2)} \cdot e^{2\mu\theta}[3\mu\cos\theta + (1-2\mu^2)\sin\theta] + C,$$

where $\qquad 0 = \frac{2g}{a(1+4\mu^2)} \times 3\mu + C$, etc.

19. $a\ddot{\theta} = -\frac{\mu R}{m} + g\sin\theta$, and $a\dot{\theta}^2 = -\frac{R}{m} + g\cos\theta$.

$$\therefore \ddot{\theta} - \mu\dot{\theta}^2 = \frac{g}{a}(\sin\theta - \mu\cos\theta).$$

$$\therefore \dot{\theta}^2 e^{-2\mu\theta} = -\frac{2g}{a(1+4\mu^2)}\{[(1-2\mu^2)\cos\theta + 3\mu\sin\theta]e^{-2\mu\theta} - (1-2\mu^2)\}.$$

The particle leaves the sphere when $R=0$, *i.e.* when $\dot{\theta}^2 = \frac{g}{a}\cos\theta$, *i.e.* when, on expanding and neglecting squares of μ,

$$\cos\theta(1-2\mu\theta) = -2[(\cos\theta + 3\mu\sin\theta)(1-2\mu\theta) - 1],$$

i.e. when $\qquad \cos\theta(1-2\mu\theta) + 2\mu\sin\theta = \frac{2}{3}.$

Here put $\theta = \alpha + \beta$, where β is small, and we have

$$(\cos\alpha - \beta\sin\alpha)(1-2\mu\alpha) + 2\mu\sin\alpha = \tfrac{2}{3} = \cos\alpha.$$

$$\therefore \beta = 2\mu\left[\frac{\sin\alpha - \alpha\cos\alpha}{\sin\alpha}\right] = \mu\left[2 - \frac{4}{3}\frac{\alpha}{\sin\alpha}\right]. \text{ Hence, etc.}$$

20. On the way up, the equations of motion are

$$a\ddot{\theta} = -g\sin\theta - \frac{\mu R}{m}, \text{ and } a\dot{\theta}^2 = \frac{R}{m} - g\cos\theta.$$

$$\therefore \ddot{\theta} + \mu\dot{\theta}^2 = -\frac{g}{a}(\sin\theta + \mu\cos\theta).$$

$$\therefore \dot{\theta}^2 e^{2\mu\theta} = -\frac{2g}{a(1+4\mu^2)} e^{2\mu\theta}[3\mu\sin\theta - (1-2\mu^2)\cos\theta] + C,$$

where $\qquad \frac{V^2}{a^2} - \frac{2g}{a}\frac{1-2\mu^2}{1+4\mu^2} = C = \frac{2g}{a}\frac{3\mu\sin\alpha - (1-2\mu^2)\cos\alpha}{1+4\mu^2}e^{2\mu\alpha}. \quad \ldots\ldots(1)$

On the way down, the equations are

$$a\ddot{\theta} = \frac{\mu R}{m} - g\sin\theta, \text{ and } a\dot{\theta}^2 = \frac{R}{m} - g\cos\theta.$$

$$\therefore \ddot{\theta} - \mu\dot{\theta}^2 = -\frac{g}{a}(\sin\theta - \mu\cos\theta),$$

and hence $\dot{\theta}^2 e^{-2\mu\theta} = \frac{2g}{a(1+4\mu^2)} e^{-2\mu\theta}[3\mu\sin\theta + (1-2\mu^2)\cos\theta] + B.$

Now $\dot{\theta}$ is zero both when $\theta = \alpha$ and when $\theta = 0$. Hence

$$e^{2\mu\alpha} = \frac{3\mu\sin\alpha + (1-2\mu^2)\cos\alpha}{1-2\mu^2}.$$

Hence V^2, on substitution in (1).

21. Taking the result of Art. 103, we have

$$v^2 = \frac{4ag}{1+\mu^2}[A^2 e^{2\mu\theta} - (\sin\theta - \mu\cos\theta)^2].$$

We are given that $v = 0$ when $\theta = \frac{\pi}{2}$ and when $\theta = 0$.

Hence $\quad 0 = A^2 e^{\mu\pi} - 1$, and $0 = A^2 - \mu^2$, so that $\mu^2 e^{\mu\pi} = 1$.

The solution of this is approximately $\mu = \cdot 475$.

22. Here $v = 0$ when $\theta = 0$.

$$\therefore v^2 = \frac{4ag}{1+\mu^2}[\mu^2 e^{2\mu\theta} - (\sin\theta - \mu\cos\theta)^2].$$

v was therefore zero initially when $\mu e^{\mu\theta} = \sin\theta - \mu\cos\theta$.

23. As in the two previous questions, if V is the velocity at the lowest point, then

$$A^2 = e^{-\mu\pi}, \text{ and } V^2 = \frac{4ag}{1+\mu^2}[e^{-\mu\pi} - \mu^2].$$

For the motion upwards on the other side of the vertex, we have to change the sign of μ, and thus

$$v^2 = \frac{4ag}{1+\mu^2}[Ce^{-2\mu\theta} - (\sin\theta + \mu\cos\theta)^2],$$

where $\quad \frac{4ag}{1+\mu^2}(e^{-\mu\pi} - \mu^2) = V^2 = \frac{4ag}{1+\mu^2}[C - \mu^2]$, i.e. $C = e^{-\mu\pi}$.

Also $v = 0$ when $\theta = \frac{\pi}{4}$.

$$\therefore 0 = e^{-\mu\pi} \cdot e^{-\frac{\mu\pi}{2}} - \frac{(1+\mu)^2}{2}. \text{ Hence, etc.}$$

24. Given $\quad v\frac{dv}{ds} = -\frac{\mu R}{m}, \ \frac{v^2}{\rho} = \frac{R}{m}$, and $\frac{d\psi}{dt} = \omega$, i.e. $\rho = \frac{v}{\omega}$.

$$\therefore \frac{dv}{ds} = -\mu\omega, \text{ so that } v = -\mu\omega s + A.$$

$$\therefore \frac{ds}{d\psi}\omega = A - \mu\omega s, \text{ giving } A - \mu\omega s = Be^{-\mu\psi}, \text{ etc.}$$

25. Equation (5) gives

$$\sqrt{\frac{2g}{c}}\,t=\sqrt{2}\left[\frac{\cos\frac{\phi}{2}}{1-2\sin^2\frac{\phi}{2}}-\log\frac{1+\sqrt{2}\sin\frac{\phi}{2}}{1-\sqrt{2}\sin\frac{\phi}{2}}\right].$$

Also $\qquad v^2=s^2\dot{\phi}^2=c^2\tan^2\phi\dot{\phi}^2=2cg\,(1-\cos\phi)=4cg\sin^2\frac{\phi}{2},$

and equation (6) gives

$$T=mg\,\frac{1+2\cos\phi-3\cos^2\phi}{\sin\phi}=mg\,(1+3\cos\phi)\sqrt{\frac{1-\cos\phi}{1+\cos\phi}}.$$

26. As in Ex. 25, $\qquad \frac{T}{m}-g\sin\theta=s\dot{\theta}^2=a\theta\dot{\theta}^2,$

and $\qquad g\cos\theta=s\ddot{\theta}+\dot{s}\dot{\theta}=a\,(\theta\ddot{\theta}+\dot{\theta}^2).$

Hence $\qquad \theta^2\dot{\theta}^2=\int\frac{2g}{a}\,\theta\cos\theta\,d\theta=\frac{2g}{a}\,(\theta\sin\theta+\cos\theta-1).$

$$\therefore\ v^2=s^2\dot{\theta}^2=a^2\theta^2\dot{\theta}^2=2ga\,(\theta\sin\theta+\cos\theta-1),$$

and $\qquad \frac{T}{m}=g\sin\theta+\frac{2g}{\theta}\,(\theta\sin\theta+\cos\theta-1)$, etc.

27. As in Ex. 25, $a\theta\dot{\theta}^2=\frac{T}{m}-\mu\,.\,OP\cos OPQ=\frac{T}{m}-\mu s=\frac{T}{m}-\mu a\theta,$

and $\qquad a\theta\ddot{\theta}+a\dot{\theta}^2=\mu\,.\,OP\sin OPQ=\mu a.$

$\therefore\ \theta\dot{\theta}=\mu t$, since $\dot{\theta}=0$ when $t=0$.

$\therefore\ \theta^2=\mu t^2$, and the time of unwinding $=\frac{2\pi}{\sqrt{\mu}}$.

Also $\qquad \frac{T}{m}=\mu a\theta+\frac{a}{\theta}\cdot\mu^2t^2=\mu a\sqrt{\mu}\,t+\frac{a\mu^2t^2}{\sqrt{\mu}\,t}=2\mu^{\frac{3}{2}}at.$

28. Let θ be measured from the horizontal line through the original point of contact, so that at any instant $r=PQ=a(\beta-\theta)$.

Then, O being the centre, the acceleration of Q along $QO=a\dot{\theta}^2$, and the acc. of P relative to Q in this direction $-\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=-2a\dot{\theta}^2+a(\beta-\theta)\ddot{\theta}$.

$$\therefore\ a(\beta-\theta)\ddot{\theta}-a\dot{\theta}^2=-g\sin\theta.$$

$$\therefore\ (\beta-\theta)^2\dot{\theta}^2=-\frac{2g}{a}\int\sin\theta\,(\beta-\theta)\,d\theta=\frac{2g}{a}\left[(\beta-\theta)\cos\theta+\sin\theta\right]+A.$$

When $\theta=0$, then $a\beta\dot{\theta}=V$, so that $\frac{V^2}{a^2}=\frac{2g}{a}\beta+A$.

Also the string will *just* wind itself up if $\dot{\theta}=0$, when $\theta=\beta$, *i.e.* if

$$0=\frac{2g}{a}\sin\beta+A.\ \text{ Hence, etc.}$$

29. Let PG and PT be the tangent and normal at P, so that $\tan OPT=\tan\phi=\frac{rd\theta}{dr}=-\cot 2\theta$, and $\therefore\ \phi=\frac{\pi}{2}+2\theta$, so that $OPG=2\theta$.

$$\therefore \frac{v^2}{\rho} - \frac{R}{m} - g\cos(\pi - 3\theta) = \frac{R}{m} + g\cos 3\theta.$$

Now $v^2 = V^2 - 2gr\cos\theta$. Also $p = r\sin\phi = r\cos 2\theta = \frac{r^3}{a^2}$.

$$\therefore \rho = r\frac{dr}{dp} = \frac{a^2}{3r}. \qquad \therefore \frac{R}{m} = (V^2 - 2gr\cos\theta)\cdot\frac{3r}{a^2} - g\cos 3\theta.$$

Hence R is just zero at the highest point if

$$0 = (V^2 - 2ga)\frac{3}{a} - g, \text{ i.e. if } 3V^2 = 7ag, \text{ etc.}$$

31. Draw a circle through the two centres of force, O and O', and the starting point A. As in Page 58, Ex. 3, this will be described with an acceleration $\frac{\mu}{r^5}$ towards O if the velocity at any point is $\sqrt{\frac{\mu}{2r^4}}$.

So for the other centre.

Hence, by the last Example, the circle will be described with both accelerations if the velocity V at the point of projection A is given by

$$V^2 = \frac{\mu}{2\,.\,OA^4} + \frac{\mu'}{2\,.\,O'A^4}.$$

32. A particle will describe the ellipse with an acceleration $2\lambda\,.\,CP$ towards the centre, and its velocity at $P = \sqrt{2\lambda}\,.\,CD = \sqrt{2\lambda rr'}$.

So it will describe the ellipse with acceleration $\frac{\mu}{r^2}$ towards S, and the velocity $= \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$. So for the third acceleration.

Hence it will describe the ellipse with all three accelerations if the velocity, V, at any point P is, by Ex. 30, given by

$$V^2 = 2\lambda rr' + \mu\left(\frac{2}{r} - \frac{1}{a}\right) + \mu'\left(\frac{2}{r'} - \frac{1}{a}\right).$$

If we take P at B then $V_B^2 = 2\lambda a^2 + \frac{\mu + \mu'}{a}$.

Also the acceleration $2\lambda\,.\,CP$ is equivalent to the two accelerations λr and $\lambda r'$ towards the foci. Hence, etc.

33. $$v\frac{dv}{ds} = -\frac{\mu}{r^5}\frac{dr}{ds}, \text{ and hence } v^2 = \frac{\mu}{2r^4} + K = \frac{\mu}{2}\left(\frac{1}{r^4} - \frac{1}{r_0^4}\right),$$

and $$\frac{v^2}{a} = -\frac{\mu'}{a^5} + \frac{\mu}{r^5}\cos OPC = -\frac{\mu'}{a^5} + \frac{\mu}{2ar^4}.$$

These agree if $r_0 = a\,.\sqrt[4]{\frac{\mu}{2\mu'}}$.

34. Let O be the centre, C and C' the inverse points, $OC = f$ and $OC' = f'$, so that, by similar triangles, $\frac{r_1}{r_2} = \frac{f}{a} = \frac{a}{f'}$, and hence $\frac{f}{f'} = \frac{r_1^2}{r_2^2}$.

The equations of motion are

$$m\frac{v\,dv}{ds} = -\frac{\mu f^2}{r_1^5}\frac{dr_1}{ds} - \frac{\mu f'^2}{r_2^5}\frac{dr_2}{ds}, \text{ so that } mv^2 = \frac{\mu f^2}{2r_1^4} + \frac{\mu f'^2}{2r_2^4} + A = \frac{\mu f^2}{r_1^4} + A \;\ldots(1)$$

and $$\frac{mv^2}{a} = \frac{\mu f^2}{r_1^5}\cos OPC + \frac{\mu f'^2}{r_2^5}\cos OPC' = \frac{\mu f^2}{r_1^4}\left[\frac{\cos OPC}{r_1} + \frac{f}{ar_1}\cos OCP\right].$$

$$\therefore\; mv^2 = \frac{\mu f^2}{r_1^4}\cdot\frac{a\cos OPC + f\cos OCP}{r_1} = \frac{\mu f^2}{r_1^4}. \quad \ldots\ldots\ldots\ldots(2)$$

(1) and (2) agree if $A = 0$ and then $mv^2 = \frac{\mu f^2}{r_1^4} = \frac{\mu f'^2}{r_2^4}$.

35. To obtain the motion of the bead relative to the wire we must give to the whole system an acceleration which will reduce the wire to rest. If R is the normal reaction the acceleration of the wire due to it is $\frac{R}{M}$ normally outwards; hence upon the ring we must impress an extra force $m\,.\,\frac{R}{M}$ normally inwards. The velocity v relative to the wire remains constant, so that $m\frac{v^2}{a} = R + m\,.\,\frac{R}{M}$, etc.

36. When OA makes an angle θ with its original direction OX, let AB make an angle ϕ with OX and an angle ψ with OA produced. Then

$$\dot\theta = \frac{v}{a} = \omega,\; \theta = \omega t, \text{ and } \ddot\theta = 0.$$

The acceleration of A is $a\omega^2$ along AO, and that of B relative to A is $b\dot\phi^2$ along BA and $b\ddot\phi$ perpendicular to BA. Hence

$$b\ddot\phi + a\omega^2\sin\psi = 0 \text{ and } \frac{T}{m} = b\dot\phi^2 + a\omega^2\cos\psi.$$

Then $$\ddot\psi = \ddot\phi + \ddot\theta = \ddot\phi = -\frac{a\omega^2}{b}\sin\psi = -\frac{v^2}{ab}\sin\psi,$$

the equation of motion of a simple pendulum of length $\frac{abg}{v^2}$.

Again $$\frac{d^2}{dt^2}(\phi + \omega t) = \ddot\phi = -\frac{v^2}{ab}\sin\psi = -\frac{v^2}{ab}\sin(\phi + \omega t).$$

$$\therefore\; \left\{\frac{d}{dt}(\phi + \omega t)\right\}^2 = \frac{2v^2}{ab}\cos(\phi + \omega t) + v^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

since initially $b\dot\phi = a\dot\theta = v$.

$$\therefore\; \dot\phi = -\omega + \frac{v}{ab}r = \frac{\omega}{b}(r - b), \text{ where } r = OB.$$

$$\therefore\; \frac{T}{m} = a\omega^2\cos\psi + \frac{\omega^2}{b}(r - b)^2 = \omega^2\left[\frac{r^2 - a^2 - b^2}{2b} + \frac{(r - b)^2}{b}\right]$$

$$= \frac{\omega^2}{2b}\left[3\left(r - \frac{2b}{3}\right)^2 - a^2 - \frac{b^2}{3}\right],$$

and is therefore least when r is least, *i.e.* when $r = a - b$, and then $\frac{T}{m} = \frac{\omega^2}{b}(a - b)(a - 4b)$. Hence, etc.

37. Taking the result of Ex. 21, we see that the particle comes to rest at the lowest point if the coefficient of friction, μ_1, is given by $\mu_1 e^{\mu_1 \frac{\pi}{2}} = 1$. If the coefficient of friction is greater than this, the particle will not reach the lowest point.

If $\mu e^{\mu \frac{\pi}{2}} > 1$, we have, on putting $\mu = \mu_1 + x$,

$$(\mu_1 + x) e^{\mu_1 \frac{\pi}{2}} . e^{\frac{\pi}{2} x} > 1, \quad i.e. \ (1 + x e^{\mu_1 \frac{\pi}{2}}) e^{\frac{\pi}{2} x} > 1.$$

Hence x must be positive; for otherwise both factors would be less than unity. Hence μ is greater than μ_1 and the given result follows.

If $\mu = \frac{1}{2}$, then $\mu e^{\mu \frac{\pi}{2}} = \frac{1}{2} e^{\frac{\pi}{4}} = \frac{1}{2} . e^{\cdot 7854} = \frac{1}{2}(2 \cdot 1933)$, by the Tables, $= 1 \cdot 0966$, and the inequality is satisfied.

38. $\left(\frac{ds}{dt}\right)^2 = v^2 = 2g(a - x)$, and $y^2 = 4ax$, so that $\left(\frac{ds}{dx}\right)^2 = 1 + \frac{4a^2}{y^2} = \frac{x + a}{x}$.

$$\therefore \sqrt{2g} . t = -\int_a^0 dx \sqrt{\frac{x + a}{x(a - x)}} \quad [\text{Put } x = a \sin^2 \theta.]$$

$$= 2\sqrt{a} \int_0^{\frac{\pi}{2}} \sqrt{1 + \sin^2 \theta}\, d\theta$$

$$= 2\sqrt{a} \int_0^{\frac{\pi}{2}} \left(1 + \frac{1}{2}\sin^2 \theta - \frac{1}{8}\sin^4 \theta + \frac{1}{16}\sin^6 \theta - \dots\right) d\theta$$

$$= 2\sqrt{a}\left[1 + \frac{1}{2} . \frac{1}{2} - \frac{1}{8} . \frac{3.1}{4.2} + \frac{1}{16} . \frac{5.3.1}{6.4.2} - \dots\right] \frac{\pi}{2}.$$

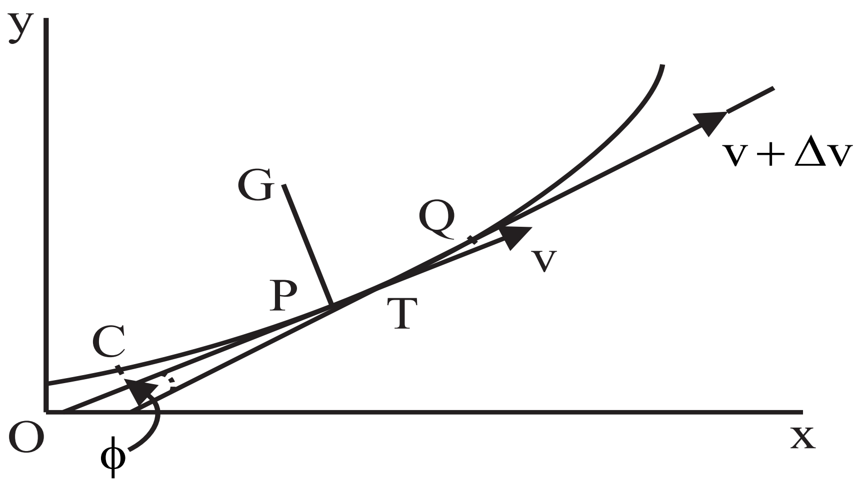
$$\therefore t = \sqrt{\frac{2a}{g}} \left[\frac{313\pi}{512}\right] = 2 \cdot 716 \times \sqrt{\frac{a}{g}} \text{ secs.}$$

Chapter 6

TANGENTIAL AND NORMAL ACCELERATIONS: UNIPLANAR CONSTRAINED MOTION

87. In the present chapter will be considered questions which chiefly involve motions where the particle is constrained to move in definite curves. In these cases the accelerations are often best measured along the tangent and normal to the curve. We must therefore first determine the tangential and normal accelerations in the case of any plane curve.

88. *To show that the accelerations along the tangent and normal to the path of a particle are* $\dfrac{d^2s}{dt^2}\left(=v\dfrac{dv}{ds}\right)$ *and* $\dfrac{v^2}{\rho}$, *where* ρ *is the radius of curvature of the curve at the point considered.*



Let v be the velocity at time t along the tangent at any point P, whose arcual distance from a fixed point C on the path is s, and let $v+\triangle v$ be the velocity at time $t+\triangle t$ along the tangent at Q, where $PQ=\triangle s$.

Let ϕ and $\phi + \triangle\phi$ be the angles that the tangents at P and Q make with a fixed line Ox, so that $\triangle\phi$ is the angle between the tangents at P and Q.

Then, by definition, the acceleration along the tangent at P

$$=\lim_{\triangle t=0}\frac{\begin{bmatrix}\text{Velocity along the tangent at time } (t+\triangle t)\\ -\text{ the same at time } t\end{bmatrix}}{\triangle t}$$

$$=\lim_{\triangle t=0}\frac{(v+\triangle v)\cos\triangle\phi - v}{\triangle t}$$

$$=\lim_{\triangle t=0}\frac{v+\triangle v - v}{\triangle t}, \text{on neglecting small quantities of the second order,}$$

$$=\frac{dv}{dt}=\frac{d^2s}{dt^2}$$

Also $$\frac{dv}{dt}=\frac{dv}{ds}\,\frac{ds}{dt}=v\frac{dv}{ds}.$$

Again the acceleration along the normal at P

$$=\lim_{\triangle t=0}\frac{\begin{bmatrix}\text{Velocity along the normal at time } (t+\triangle t)\\ -\text{ the same at time } t\end{bmatrix}}{\triangle t}$$

$$= \lim_{\triangle t=0} \frac{(v+\triangle v)\sin\triangle\phi}{\triangle t} = \lim_{\triangle t=0} (v+\triangle v).\frac{\sin\triangle\phi}{\triangle\phi}.\frac{\triangle\phi}{\triangle s}.\frac{\triangle s}{\triangle t}$$

$$= v.1.\frac{1}{\rho}.v = \frac{v^2}{\rho}.$$

COR. In the case of a circle we have $\rho = a, s = a\theta, v = a\dot{\theta}$ and the accelerations are $a\ddot{\theta}$ and $\dot{\theta}^2$.

89. The tangential and normal accelerations may also be directly obtained from the accelerations parallel to the axes.

For $\quad \dfrac{dx}{dt} = \dfrac{dx}{ds}.\dfrac{ds}{dt}.$

$$\therefore \quad \frac{d^2x}{dt^2} = \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dx}{ds}\frac{d^2s}{dt^2}.$$

So $\quad \dfrac{d^2y}{dt^2} = \dfrac{d^2y}{ds^2}\left(\dfrac{ds}{dt}\right)^2 + \dfrac{dy}{ds}\dfrac{d^2s}{dt^2}.$

But, by Differential Calculus,

$$\frac{1}{\rho} = \frac{-\dfrac{d^2x}{ds^2}}{\dfrac{dy}{ds}} = \frac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}}$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{dy}{ds}.\frac{1}{\rho}.\left(\frac{ds}{dt}\right)^2 + \frac{dx}{ds}.\frac{d^2s}{dt^2} = -\frac{\sin\phi}{\rho}v^2 + \frac{d^2s}{dt}.\cos\phi,$$

$$\text{and} \quad \frac{d^2y}{dt^2} = \frac{dx}{ds}.\frac{1}{\rho}.\left(\frac{ds}{dt}\right)^2 + \frac{dy}{ds}.\frac{d^2s}{dt^2} = \frac{\cos\phi}{\rho}.v^2 + \frac{d^2s}{dt^2}\sin\phi.$$

Therefore the acceleration along the tangent

$$= \frac{d^2x}{dt^2}\cos\phi + \frac{d^2y}{dt^2}\sin\phi = \frac{d^2s}{dt^2} = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds},$$

and the acceleration along the normal
$= -\frac{d^2x}{dt^2}\sin\phi + \frac{d^2y}{dt^2}\cos\phi = \frac{v^2}{\rho}.$

90. Ex. *A curve is described by a particle having a constant acceleration in a direction inclined at a constant angle to the tangent; show that the curve is an equiangular spiral.*

Here $\frac{vdv}{ds} = f\cos\alpha$ and $\frac{v^2}{\rho} = f\sin\alpha$, where f and α are constants,

$$\therefore \;\; 2f\cos\alpha s + \text{ const.} = v^2 = f\sin\alpha.\rho = f\sin\alpha\frac{ds}{d\psi}.$$

$$\therefore \;\; \frac{1}{2}\frac{ds}{d\psi} = s\cot\;\alpha + A, \text{ where } A \text{ is constant.}$$

$$\therefore \;\; \log(s\cot\alpha + A) = 2\psi\cot\alpha + \text{ const.}$$

$$\therefore \;\; s = -A\tan\alpha + Be^{2\psi\cot\alpha},$$

which is the intrinsic equation of an equiangular spiral.

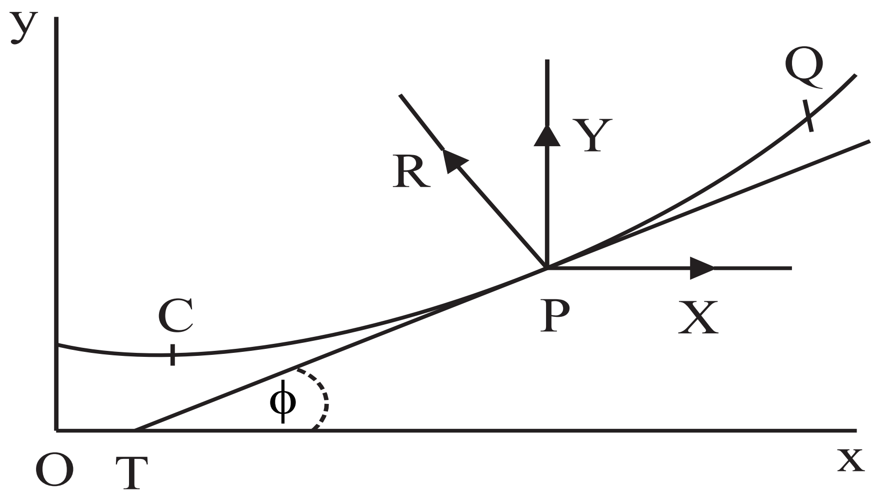
EXAMPLES

1. Find the intrinsic equation to a curve such that, when a point moves on it with constant tangential acceleration, the magnitudes of the tangential velocity and the normal acceleration are in a constant ratio.
2. A point moves along the arc of a cycloid in such a manner that the tangent at it rotates with constant angular velocity; show that the acceleration of the moving point is constant in magnitude.
3. A point moves in a curve so that its tangential and normal accelerations are equal and the tangent rotates with constant angular velocity; find the path.

4. If the relation between the velocity of a particle and the arc it has described be $2as = \log\frac{b+ac^2}{b+av^2}$, find the tangential force acting on the particle and the time that must elapse from the beginning of the motion till the velocity has the value V.
5. Show that a cycloid can be a free path for a particle acted on at each point by a constant force parallel to the corresponding radius of the generating circle, this circle being placed at the vertex.
6. A heavy particle lying in limiting equilibrium on a rough plane, inclined at an angle α to the horizontal, is projected with velocity V horizontally along the plane; show that the limiting velocity is $\frac{1}{2}V$ and find the intrinsic equation to the path.
7. A circle rolls on a straight line, the velocity of its centre at any instant being v and its acceleration f; find the tangential and normal accelerations of a point on the edge of the circle whose angular distance from the point of contact is θ.

91. *A particle is compelled to move on a given smooth plane curve under the action of given forces in the plane; to find the motion.*

Let P be a point of the curve whose actual distance from a fixed point C is s, and let v be the velocity at P.



Let X, Y be the components parallel to two rectangular axes Ox, Oy of the forces acting on the particle when at P; since the curve is smooth the only reaction will be a force R along the normal at P.

Resolving along the tangent and normal, we have

$$m\frac{vdv}{ds} = \text{ force along } TP = X\cos\phi + Y\sin\phi = X\frac{dx}{ds} + Y\frac{dy}{ds} \quad ...(1),$$

and $$m.\frac{v^2}{\rho} = -X\sin\phi + Y\cos\phi + R = -X\frac{dy}{ds} + Y\frac{dx}{ds} + R \quad ...(2),$$

When v is known, equation (2) gives R at any point.

Equation (1) gives

$$\frac{1}{2}mv^2 = \int (Xdx + Ydy) \quad ...(3).$$

Suppose that $Xdx + Ydy$ is the complete differential of some function $\phi(x,y)$, so that

$$X = \frac{d\phi}{dx} \text{ and } Y = \frac{d\phi}{dy}.$$

Then $$\frac{1}{2}mv^2 = \int \left(\frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy\right) = \phi(x,y) + C \quad ...(4).$$

Suppose that the particle started with a velocity V from a point whose coordinates are x_0, y_0. Then $\frac{1}{2}mV^2 = \phi(x_0, y_0) + C$.

Hence, by subtraction,

$$\frac{1}{2}mv^2 - \frac{1}{2}mV^2 = \phi(x,y) - \phi(x_0,y_0) \quad ...(5).$$

This result is quite independent of the path pursued between the initial point and P, and would therefore be the same whatever be the form of the restraining curve.

From the definition of Work it is clear that $Xdx + Ydy$ represents the work done by the forces X, Y during a small displacement ds along the curve. Hence the right-hand side of (3) or of (4) represents the total work done on the particle by the external forces, during its motion from the point of projection to P, added to an arbitrary constant.

Hence, when the components of the forces are equal to the differentials with respect to x and y of some function $\phi(x, y)$, it follows from (5) that

The change in the Kinetic Energy of the particle = the Work done by the External Forces.

Forces of this kind are called Conservative Forces.

The quantity $\phi(x, y)$ is known as the Work-Function of the system of forces. From the ordinary definition of a Potential Function, it is clear that $\phi(x, y)$ is equal to the Potential of the given system of forces added to some constant.

If the motion be in three dimensions we have, similarly, that the forces are Conservative when $\int (Xdx + Ydy + Zdz)$ is a perfect differential, and an equation similar to (5) will also be true. [See Art. 131.]

92. The Potential Energy of the particle, due to the given system of forces, when it is in the position P = the work done by the forces as the particle moves to some standard position.

Let the latter position be the point (x_1, y_1). Then the potential energy of the particle at P

$$= \int_{(x,y)}^{(x_1,y_1)} (Xdx + Ydy) = \int_{(x,y)}^{(x_1,y_1)} \left(\frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy\right)$$

$$= [\phi(x,y)]_{(x,y)}^{(x_1,y_1)} = \phi(x_1,y_1) - \phi(x,y).$$

Hence, from equation (4) of the last article,

(Kinetic Energy + Potential Energy) of the particle when at P

$$= \phi(x,y) + C + \phi(x_1,y_1) - \phi(x,y) = C + \phi(x,y_1) = a \text{ constant.}$$

Hence, when a particle moves under the action of a Conservative System of Forces, the sum of its Kinetic and Potential Energies is constant throughout the motion.

93. In the particular case when gravity is the only force acting we have, if the axis of y be vertical, $X = 0$ and $Y = -mg$.

Equation (3) then gives $\frac{1}{2}mv^2 = -mgy + C$

Hence, if Q be a point of the path, this gives kinetic energy at P - kinetic energy at Q

$= mgx \times$ difference of the ordinates at P and Q

$=$ the work done by gravity as the particle passes from Q to P.

This result is important; from it, given the kinetic energy at any known point of the curve, we have the kinetic energy at any other point of the path, if the curve be smooth.

94. If the only forces acting on a particle be perpendicular to its direction of motion (as in the case of a particle tethered by an inextensible string, or moving on a smooth surface) its velocity is constant; for the work done by the string or reaction is zero.

95. *All forces which are one-valued functions of distances from fixed points are Conservative Forces.*

Let a force acting on a particle at the point (x,y) be a function $\psi(r)$ of the distance r from a fixed point (a,b) so that

$r^2 = (x-a)^2 + (y-b)^2$.

Also let the force act towards the point (a,b).

Then $r\dfrac{dr}{dx} = (x-a)$, and $r\dfrac{dr}{dy} = y-b$.

The component X of this force parallel to the axis of x

$= -\psi(r) \times \dfrac{x-a}{r}$,

if the force be an attraction, and the component Y parallel to y

$= -\psi(r) \times \dfrac{y-b}{r}$.

Hence

$$Xdx + Ydy = -\psi(r) \times \frac{(x-a)dx + (y-b)dy}{r}$$
$$= -\psi(r)\frac{rdr}{r} = -\psi(r)dr.$$

Hence, if $F(r)$ be such that $\quad \dfrac{d}{dr}F(r) = -\psi(r) \quad$ ...(1),

we have $\quad \displaystyle\int (Xdx + Ydy) = \int \frac{d}{dr}F(r)dr = F(r) +$ const.

Such a force therefore satisfies the condition of being a Conservative Force.

If the force be a central one and follow the law of the inverse square, so that $\psi(r) = \dfrac{\mu}{r^2}$, then $F(r) = -\displaystyle\int \psi(r)dr = \dfrac{\mu}{r}$ and hence $\displaystyle\int (Xdx + Ydy) = \frac{\mu}{r} +$ constant.

96. *The work done in stretching an elastic string is equal to the extension produced multiplied by the mean of the initial and final tensions.*

Let a be the unstretched length of the string, and λ its modulus of elasticity, so that, when its length is x, its tension

$$= \lambda . \frac{x-a}{a}, \text{ by Hooke's law.}$$

The work done in stretching it from a length b to a length c

$$= \int_b^c T.dx = \int_b^c \lambda \frac{x-a}{a} dx = \frac{\lambda}{2a}\left[(x-a)^2\right]_b^c = \frac{\lambda}{2a}[(c-a)^2-(b-a)^2]$$

$$= (c-b)\left[\lambda \frac{b-a}{a} + \lambda \frac{c-a}{a}\right] \times \frac{1}{2}$$

$$= (c-b) \times \text{ mean of the initial and final tensions.}$$

EX. *A and B are two points in the same horizontal plane at a distance* 2*a apart; AB is an elastic string whose unstretched length is* 2*a. To O, the middle point of AB, is attached a particle of mass m which is allowed to fall under gravity; find its velocity when it has fallen a distance x and the greatest vertical distance through which it moves.*

When the particle is at P, where $OP = x$, let its velocity be v, so that its kinetic energy then is $\frac{1}{2}mv^2$.

The work done by gravity $= mg.x$.

The work done against the tension of the string

$$= 2 \times (BP - BO) \times \frac{1}{2}\lambda \frac{BP - BO}{a} = \frac{\lambda}{a}(BP-a)^2 = \frac{\lambda}{a}\left[\sqrt{x^2+a^2}-a\right]^2.$$

Hence, by the Principle of Energy,

$$\frac{1}{2}mv^2 = mgx - \frac{\lambda}{a}\left[\sqrt{x^2+a^2} - a\right]^2.$$

The particle comes to rest when $v = 0$, and then x is given by the equation

$$mgxa = \lambda\left[\sqrt{x^2+a^2} - a\right]^2.$$

EXAMPLES

1. If an elastic string, whose natural length is that of a uniform rod, be attached to the rod at both ends and suspended by the middle point, show by means of the Principle of Energy, that the rod will sink until the strings are inclined to the horizon at an angle θ given by the equation
$$\cot^3\frac{\theta}{2} - \cot\frac{\theta}{2} = 2n,$$
given that the modulus of elasticity of the string is n times the weight of the rod.
2. A heavy ring, of mass m, slides on a smooth vertical rod and is attached to a light string which passes over a small pulley distant a from the rod and has a mass $M(> m)$ fastened to its other end. Show that, if the ring be dropped from a point in the rod in the same horizontal plane as the pulley, it will descend a distance $\frac{2Mma}{M^2-m^2}$ before coming to rest. Find the velocity of m when it has fallen through any distance x.
3. A shell of mass M is moving with velocity V. An internal explosion generates an amount of energy E and breaks the shell into two portions whose masses are in the ratio $m_1 : m_2$. The fragments continue to move in the original line of motion of the shell. Show that their velocities are

$$V+\sqrt{\frac{2m_2E}{m_1M}} \text{ and } V-\sqrt{\frac{2m_1E}{m_2M}}.$$

4. An endless elastic string, of natural length $2\pi a$, lies on a smooth horizontal table in a circle of radius a. The string is suddenly set in motion about its centre with angular velocity ω. Show that if left to itself the string will expand and that, when its radius is r, its angular velocity is $\frac{a^2}{r^2}\omega$, and the square of its radial velocity from the centre is $\frac{a^2\omega^2}{r^2}(r^2-a^2)-\frac{2\pi\lambda(r-a)^2}{ma}$, where m is the mass and λ the modulus of elasticity of the string.
5. Four equal particles are connected by strings, which form the sides of a square, and repel one another with a force equal to $\mu\times$ distance; if one string be cut, show that, when either string makes an angle θ with its original position, its angular velocity is

$$\sqrt{\frac{4\mu\sin\theta(2+\sin\theta)}{2-\sin^2\theta}}.$$

[As in Art 47 the centre of mass of the whole system remains at rest; also the repulsion, by the well-known property, on each particle is the same as if the whole of the four particles were collected at the centre and $=4\mu\times$ distance from the fixed centre of mass. Equate the total kinetic energy to the total work done by the repulsion.]
6. A uniform string, of mass M and length $2a$, is placed symmetrically over a smooth peg and has particles of masses m and m' attached to its extremities; show that when the string runs off the peg its velocity is

$$\sqrt{\frac{M+2(m-m')}{M+m+m'}ag}.$$

7. A heavy uniform chain, of length $2l$, hangs over a small smooth fixed pulley, the length $l+c$ being at one side and $l-c$ at the other; if the end of the shorter portion be held, and then let go, show that the chain will slip off the pulley in time

$$\left(\frac{l}{g}\right)^{1/2}\log\,\frac{l+\sqrt{l^2-c^2}}{c}.$$

8. A uniform chain, of length l and weight W, is placed on a line of greatest slope of a smooth plane, whose inclination to the horizontal is α, and just reaches the bottom of the plane where there is a small smooth pulley over which it can run. Show that, when a length x has run off the tension at the bottom of the plane is

$$W(1-\sin\alpha)\frac{x(l-x)}{l^2}.$$

9. Over a small smooth pulley is placed a uniform flexible cord; the latter is initially at rest and lengths $l-a$ and $l+a$ hang down on the two sides. The pulley is now made to move with constant vertical acceleration f. Show that the string will leave the pulley after a time

$$\sqrt{\frac{l}{f+g}}\cosh^{-1}\frac{l}{a}.$$

97. **Oscillations of a Simple Pendulum.**

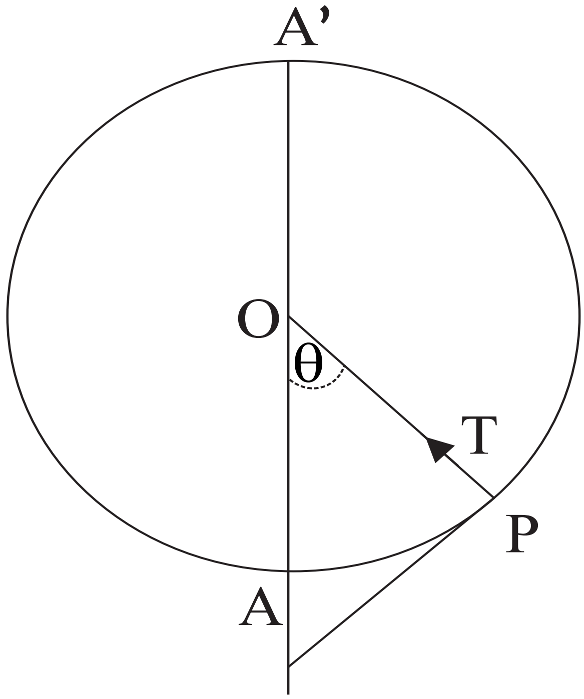
A particle m is attached by a light string, of length l, to a fixed point and oscillates under gravity through a small angle; to find the period of its motion.

When the string makes an angle θ with the vertical, the equation of motion is

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad ...(1).$$

But $s = l\theta$.

$\therefore \quad \ddot{\theta} = -\frac{g}{l}\sin\theta = -\frac{g}{l}\theta$, to a, first approximation.



If the pendulum swings through a small angle α on each side of the vertical, so that $\theta = \alpha$ and $\dot{\theta} = 0$ when $t = 0$, this equation gives $\theta = \alpha\cos\left[\sqrt{\frac{g}{l}}t\right]$, so that the motion is simple harmonic and the time, T_1, of a very small oscillation $= 2\pi\sqrt{\frac{l}{g}}$, as in Art. 22.

For a higher approximation we have, from equation (1),

$$l\dot{\theta}^2 = 2g(\cos\theta - \cos\alpha) \qquad ...(2),$$

since $\dot{\theta}$ is zero when $\theta = \alpha$.

[This equation follows at once from the Principle of Energy.]

$\therefore\ \sqrt{\frac{2g}{l}}.t = \int_0^a \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}$, where t is the time of a quarter-swing.

$$\therefore\ 2\sqrt{\frac{g}{l}}.t = \int_0^a \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}.$$

Put $\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}.\sin\phi$. $\quad\therefore\ 2\sqrt{\frac{g}{l}}.t = \int_0^{\pi/2} \frac{2\sin\frac{\alpha}{2}\cos\phi d\phi}{\cos\frac{\theta}{2}.\sin\frac{\alpha}{2}\cos\phi}$

$$\therefore\ t = \sqrt{\frac{l}{g}}.\int_0^{\pi/2} \frac{d\phi}{\left(1 - \sin^2\frac{\alpha}{2}\sin^2\phi\right)^{1/2}} \quad ...(3)$$

$$= \sqrt{\frac{l}{g}}\int_0^{\pi/2}\left[1 + \frac{1}{2}\sin^2\frac{\alpha}{2}.\sin^2\phi + \frac{1.3}{2.4}\sin^4\frac{\alpha}{2}\sin^4\phi + \cdots\right]d\phi$$

$$= \sqrt{\frac{l}{g}}.\frac{\pi}{2}\left[1 + \frac{1}{2^2}\sin^2\frac{\alpha}{2} + \left(\frac{1.3}{2.4}\right)^2\sin^4\frac{\alpha}{2} + \left(\tfrac{1.3.5}{2.4.6}\right)^2\sin^6\tfrac{\alpha}{2} + \cdots\right] \quad ...(4)$$

Hence a second approximation to the required period, T_2,

$$= T_1\left[1 + \frac{1}{4}.\sin^2\frac{\alpha}{2}\right] = T_1\left[1 + \frac{a^2}{16}\right],$$

if powers of a higher than the second are neglected.

Even if α be not very small, the second term in the bracket of (4) is usually a sufficient approximation. For example, suppose $\alpha = 30^o$, so that the pendulum swings through an angle of 60^o; then $\sin^2\frac{\alpha}{2} = \sin^2 15^o = .067$, and (4) gives

$$t = \frac{\pi}{2}\sqrt{\frac{l}{g}}[1 + .017 + .00063 + ...].$$

[The student who is acquainted with Elliptic Functions will see that (3) gives

$$\sin\phi = \operatorname{sin}\left(t\sqrt{\frac{g}{l}}\right), \left(mod.\ \sin\frac{\alpha}{2}\right),$$

$$\text{so that}\ \ \sin\frac{\theta}{2} = \sin\frac{a}{2}\operatorname{sin}\left(t\sqrt{\frac{g}{l}}\right), \left(mod.\ \sin\frac{\alpha}{2}\right).$$

The time of a complete oscillation is also, by (3), equal to $\sqrt{\frac{l}{g}}$ multiplied by the real period of the elliptic function with modulus $\sin\frac{\alpha}{2}$.]

98. The equations (1) and (2) of the previous article give the motion in a circle in any case, when α is not necessarily small. If ω be the angular velocity of the particle when passing through the lowest point A, we have

$$l\,\dot{\theta}^2 = 2g\cos\theta + \text{ const. } = l\omega^2 - 2g(1-\cos\theta) \qquad ...(5).$$

This equation cannot in general be integrated without the use of Elliptic Functions, which are beyond the scope of this book.

If T be the tension of the string, we have

$$T - mg\cos\theta = \text{ force along the normal } PO$$
$$= ml\,\dot{\theta}^2 = ml\omega^2 - 2mg(1-\cos\theta),$$
$$\therefore\quad T = m\{l\omega^2 - g(2-3\cos\theta)\} \qquad ...(6).$$

Hence T vanishes and becomes negative, and hence circular motion ceases, when

$$\cos\theta = \frac{2g - l\omega^2}{3g}.$$

Particular Case. Let the angular velocity at A be that due to a fall from the highest point A', so that

$$l^2\omega^2 = 2g.2l, \quad i.e. \quad \omega^2 = \frac{4g}{l}.$$

Then (5) gives $\quad \dot{\theta}^2 = \frac{2g}{l}(1+\cos\theta)$

$$\therefore \quad t\sqrt{\frac{2g}{l}} = \int \frac{d\theta}{\sqrt{1+\cos\theta}} = \frac{1}{\sqrt{2}}\int \frac{d\theta}{\cos\frac{\theta}{2}}.$$

$$\therefore \quad t = \frac{1}{2}\sqrt{\frac{l}{g}}\left[2\log\tan\left(\frac{\pi}{4}+\frac{\theta}{4}\right)\right]_0^\theta = \sqrt{\frac{l}{g}}\log\frac{\cos\frac{\theta}{4}+\sin\frac{\theta}{4}}{\cos\frac{\theta}{4}-\sin\frac{\theta}{4}}$$

$$= \sqrt{\frac{l}{g}}\log\frac{1+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \sqrt{\frac{l}{g}}\log\left[\sec\frac{\theta}{2}+\tan\frac{\theta}{2}\right],$$

giving the time t of describing an angle θ from the lowest point.

Also in this case $\quad T = m\{4g-2g+3g\cos\theta\} = mg[2+3\cos\theta]$.

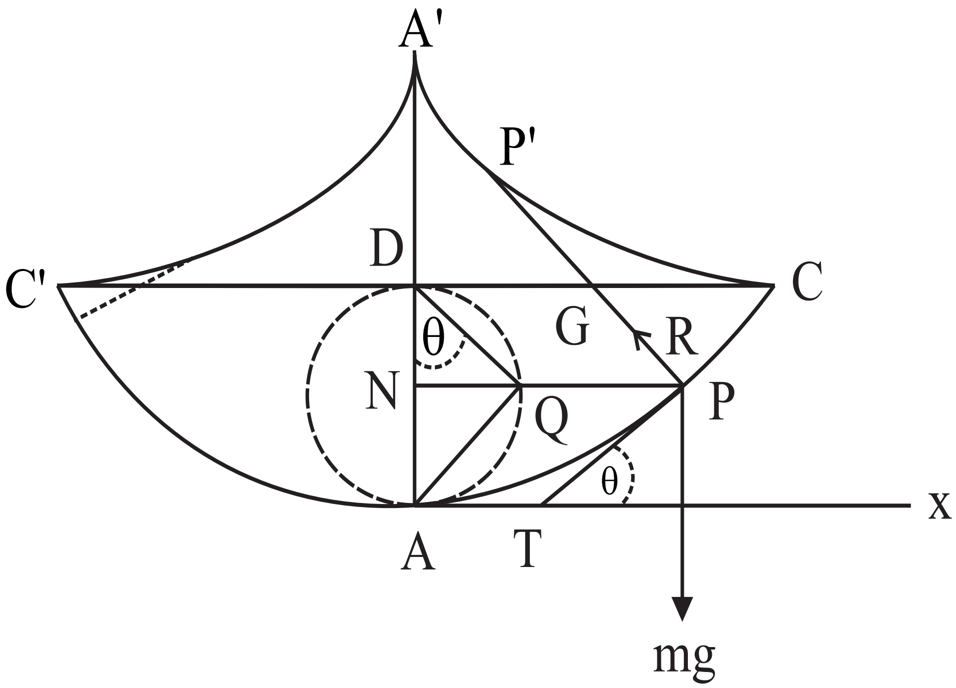
Circular motion therefore ceases when $\cos\theta = -\frac{2}{3}$, and then $\sec\frac{\theta}{2} = \sqrt{6}$ and $\tan\frac{\theta}{2} = \sqrt{5}$. Therefore the time during which is circular motion lasts $= \sqrt{\frac{l}{g}}\log_e(\sqrt{5}+\sqrt{6})$.

99. Ex. 1. Show that a pendulum, which beats seconds when it swings through 3^o on each side of the vertical, will lose about 12 secs. per day if the angle be 4^o and about 27 secs. per day if the angle be 5^o.

Ex. 2. A heavy bead slides on a smooth fixed vertical circular wire of radius a; if it be projected from the lowest point with velocity just sufficient to carry it to the highest point, show that the radius to the bead is at time t inclined to the vertical at an angle $2\tan^{-1}\left[\sinh\sqrt{\frac{g}{a}}t\right]$, and that the bead will be an infinite time in arriving at the highest point.

100. *Motion on a smooth cycloid whose axis is vertical and vertex lowest.*

Let AQD be the generating circle of the cycloid $CPAC'$, P being any point on it; let PT be the tangent at P and PQN perpendicular to the axis meeting the generating circle in Q. The two principal properties of the cycloid are that the tangent TP is parallel to AQ, and that the arc AP is equal to twice the line AQ.



Hence, if PTx be θ, we have $\theta = \angle QAx = ADQ$, and

$s =$ arc $AP = 2.AQ = 4a\sin\theta$ ...(1),

if a be the radius of the generating circle.

If R be the reaction of the curve along the normal, and m the particle at P, the equations of motion are then

$$m\frac{d^2s}{dt^2} = \text{ force along } PT = -mg\sin\theta \qquad ...(2),$$

$$\text{and} \quad m.\frac{v^2}{\rho} = \text{ force along the normal } = R - mg\cos\theta. \qquad ...(3).$$

From (1) and (2), we then have

$$\frac{d^2s}{dt^2} = -\frac{g}{4a}s \qquad ...(4),$$

so that the motion is simple harmonic, and hence, as in Art. 22, the time to the lowest point

$$= \frac{\frac{\pi}{2}}{\sqrt{\frac{g}{4a}}} = \pi\sqrt{\frac{a}{g}},$$

and is therefore always the same whatever be the point of the curve at which the particle started from rest.

Integrating equation (4), we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + C = -g.4a\sin^2\theta + C = 4ag(\sin^2\theta_0 - \sin^2\theta),$$

if the particle started from rest at the point where $\theta = \theta_0$.

[This equation can be written down at once by the Principle of Energy.]

Also $$\rho = \frac{ds}{d\theta} = 4a\cos\theta.$$
Therefore (3) gives

$$R = mg\cos\theta + mg\frac{\sin^2\theta_0 - \sin^2\theta}{\cos\theta} = mg\frac{\cos 2\theta + \sin^2\theta_0}{\cos\theta},$$

giving the reaction of the curve at any point of the path.

On passing the lowest point the particle ascends the other side until it is at the height from which it started, and thus it oscillates backwards and forwards.

101. The property proved in the previous article will be still true if, instead of the material curve, we substitute a string tied to the particle in such a way that the particle describes a cycloid and the string is always normal to the curve. This will be the case if the string unwraps and wraps itself on the evolute of the cycloid. It can be easily shown that the evolute of a cycloid is two halves of an equal cycloid.

For, since $\rho = 4a\cos\theta$, the points on the evolute corresponding to A and C are A', where $AD = DA'$, and C itself. Let the normal PG meet this evolute in P', and let the arc CP' be σ. By the property of the evolute

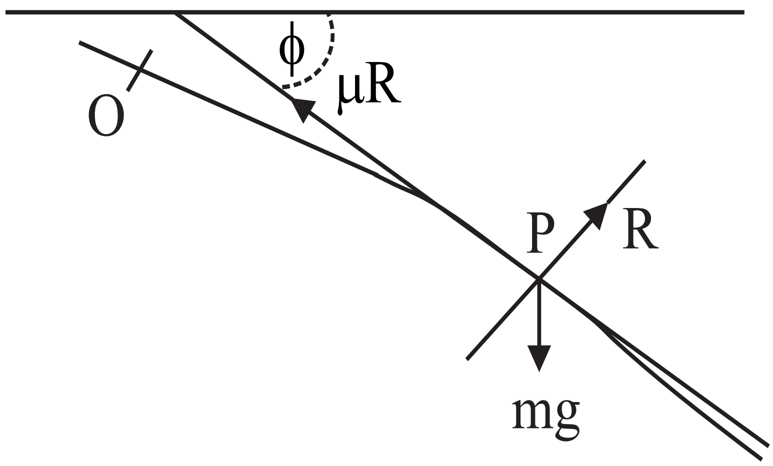
$$\begin{aligned}\sigma = {} & \text{ arc } P'C = P'P, \text{ the radius of curvature at } P \\ & = 4a\cos\theta = 4a\sin P'GD.\end{aligned}$$

Hence, by (1) of the last article, the curve is a similar cycloid whose vertex is at C and whose axis is vertical. This holds for the arc CA. The evolute for the arc $C'A$ is the similar semi-cycloid $C'A'$.

Hence if a string, or flexible wire, of length equal to the arc CA', *i.e.* $4a$, be attached at A' and allowed to wind and unwind itself upon fixed metal cheeks in the form of the curve $CA'C'$, a particle P attached to its other end will describe the cycloid CAC', and the string will always be normal to the curve CAC'; the times of oscillation will therefore be always isochronous, whatever be the angle through which the string oscillates. In actual practice, a pendulum is only required to swing through a small angle, so that only small portions of the two arcs near A' are required. This arrangement is often adopted in the case of the pendulum of a small clock, the upper end of the supporting wire consisting of a thin flat spring which coils and uncoils itself from the two metal cheeks at A'.

102. Motion on a rough curve under gravity.

Whatever be the curve described under gravity with friction, we have, if ϕ be the angle measured from the horizontal made by the tangent, and if s increases with ϕ.



$$v\frac{dv}{ds} = g\sin - \frac{\mu R}{m} \qquad ...(1),$$

and $$\frac{v^2}{\rho} = g\cos\phi - \frac{R}{m} \qquad ...(2).$$

$$\therefore \quad \frac{1}{2}\frac{d.v^2}{ds} - \mu\frac{v^2}{\rho} = g(\sin\phi - \mu\cos\phi).$$

$$\therefore \quad \frac{dv^2}{d\phi} - 2\mu v^2 = 2g\rho(\sin\phi - \mu\cos\phi).$$

Multiplying by $e^{-2\mu\phi}$ and integrating, we have

$$v^2 e^{-2\mu\phi} = 2g\int \rho e^{-2\mu\phi}(\sin\phi - \mu\cos\phi)d\phi + \text{ Constant.}$$

When the curve is given, so that ρ is known in terms of ϕ, this gives v^2 and hence $\left(\frac{ds}{d\phi}\right)^2 \left(\frac{d\phi}{dt}\right)^2$. Hence $\frac{d\phi}{dt}$ is known, and therefore theoretically t in terms of ϕ.

103. *If the cycloid of Art 100 be rough with a coefficient of friction μ, to find the motion, the particle sliding downwards.*

In this case the friction, μR, acts in the direction TP produced. Since $s = 4a\sin\theta$, we have $\rho = 4a\cos\theta$, and $v = 4a\cos\theta$, so that the equations of motion are

$$m\frac{d}{dt}(4a\cos\theta.\dot{\theta}) = \mu R - mg\sin\theta \qquad ...(1),$$

and $$mv^2/\rho = m.4a\cos\theta.\dot{\theta}^2 = R - mg\cos\theta \qquad ...(2).$$

$$\therefore \quad \frac{d}{dt}(\dot{\theta}\cos\theta) - \mu\cos\theta.\dot{\theta}^2 = -\frac{g}{4a}(\sin\theta - \mu\cos\theta),$$

i.e. $$\frac{d}{dt}[\dot{\theta}\cos\theta e^{-\mu\theta}] = -\frac{g}{4a}(\sin\theta - \mu\cos\theta)e^{-\mu\theta} \qquad ...(3).$$

Now $$\frac{d}{dt}[e^{-\mu\theta}(\sin\theta - \mu\cos\theta)] = (1+\mu^2)e^{-\mu\theta}\cos\theta.\dot{\theta}.$$

Hence (3) gives

$$\frac{d^2}{dt^2}[e^{-\mu\theta}(\sin\theta - \mu\cos\theta)] = -(1+\mu^2)\frac{g}{4a}[e^{-\mu\theta}\sin\theta - \mu\cos\theta)].$$

$$\therefore \quad e^{-\mu\theta}(\sin\theta - \mu\cos\theta) = A\cos\left[\sqrt{\frac{g(1+\mu^2)}{4a}}t + B\right] \qquad ...(4),$$

where A and B are constant depending on the initial conditions.

Differentiating (4), we obtain

$$v^2 = 16a^2\cos^2\theta.\dot{\theta}^2 = \frac{4ag}{1+\mu^2}[A^2e^{2\mu\theta} - (\sin\theta - \mu\cos\theta)^2].$$

EXAMPLES

1. A particle slides down the smooth curve $y = a\sinh\frac{x}{a}$, the axis of x being horizontal, starting from rest at the point where the tangent is inclined at α to the horizon; show that it will leave the curve when it has fallen through a vertical distance $a\sec\alpha$.
2. A particle descends a smooth curve under the action of gravity, describing equal vertical distances in equal times, and starting in a vertical direction. Show that the curve is a semi-cubical parabola, the tangent at the cusp of which is vertical.
3. A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc; show that the time of reaching the vertex is
$$2\sqrt{\frac{a}{g}}\tan^{-1}\left[\frac{\sqrt{4ag}}{V}\right].$$
4. A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest; prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.
5. A particle is placed very close to the vertex of a smooth cycloid whose axis is vertical and vertex upwards, and is allowed to run

down the curve. Show that it leaves the curve when it is moving in a direction making with the horizontal an angle of 45^o.

6. A ring is strung on a smooth closed wire which is in the shape of two equal cycloids joined cusp to cusp, in the same plane and symmetrically situated with respect to the line of cusps. The plane of the wire is vertical, the line of cusps horizontal, and the radius of the generating circle is a. The ring starts from the highest point with velocity v. Prove that the times from the upper vertex to the cusp, and from the cusp to the lower vertex are respectively $2\sqrt{\frac{a}{g}}\sinh^{-1}\left(\frac{\sqrt{4ag}}{V}\right)$ and $2\sqrt{\frac{a}{g}}\sin^{-1}\sqrt{\frac{4ag}{v^2+8ag}}$.
7. A particle moves in a smooth tube in the form of a catenary, being attracted to the directrix by a force proportional to the distance from it. Show that the motion is simple harmonic.
8. A particle, of mass m, moves in a smooth circular tube, of radius a, under the action of a force, equal to $m\mu \times$ distance, to a point inside the tube at a distance c from its centre; if the particle be placed very nearly at its greatest distance from the centre of force, show that it will describe the quadrant ending at its least distance in time $\sqrt{\frac{a}{\mu c}}\log(\sqrt{2}+1)$.
9. A bead is constrained to move on a smooth wire in the form of an equiangular spiral. It is attracted to the pole of the spiral by a force, $= m\mu \times (\text{distance})^{-2}$, and starts from rest at a distance b from the pole. Show that, if the equation to the spiral be $r = ae^{\theta\cot\alpha}$, the time of arriving at the pole is $\frac{\pi}{2}\sqrt{\frac{b^3}{2\mu}}.\sec\alpha$. Find also the reaction of the curve at any instant.

10. A smooth parabolic tube is placed, vertex downwards, in a vertical plane; a particle slides down the tube from rest under the influence of gravity; prove that in any position the reaction of the tube is $2w\frac{h+a}{\rho}$, where w is the weight of the particle, ρ the radius of curvature, $4a$ the latus rectum, and h the original vertical height of the particle above the vertex.
11. From the lowest point of a smooth hollow cylinder whose cross-section is an ellipse, of major axis $2a$ and minor axis $2b$, and whose minor axis is vertical, a particle is projected from the lowest point in a vertical plane perpendicular to the axis of the cylinder; show that it will leave the cylinder if the velocity of projection lie between $\sqrt{2gb}$ and $\sqrt{g\frac{a^2+4b^2}{b}}$.
12. A small bead, of mass m, moves on a smooth circular wire, being acted upon by a central attraction $\frac{m\mu}{(\text{distance})^2}$ to a point within the circle situated at a distance b from its centre. Show that, in order that the bead may move completely round the circle, its velocity at the point of the wire nearest the centre of force must not be less than $\sqrt{\frac{4\mu b}{a^2-b^2}}$.
13. A small bead moves on a thin elliptic wire under a force to the focus equal to $\frac{\mu}{r^2}+\frac{\lambda}{r^3}$. It is projected from a point on the wire distant R from the focus with the velocity which would cause it to describe the ellipse freely under a force $\frac{\mu}{r^2}$. Show that the reaction of the wire is $\frac{\lambda}{\rho}\left[\frac{1}{r^2}-\frac{1}{ar}+\frac{1}{R^2}\right]$, where ρ is the radius of curvature.
14. If a particle is made to describe a curve in the form of the four-cusped hypocycloid $x^{2/3}+y^{2/3}=a^{2/3}$ under the action of an at-

traction perpendicular to the axis and varying as the cube root of the distance from it, show that the time of descent from any point to the axis of x is the same, *i.e.,* that the curve is a Tautochrone for this law of force.

15. A small bead moves on a smooth wire in the form of an epicycloid, being acted upon by a force, varying as the distance, toward the centre of the epicycloid; show that its oscillations are always isochronous. Show that the same is true if the curve be a hypocycloid and the force always from, instead of towards, the centre.
16. A curve in a vertical plane is such that the time of describing any arc, measured from a fixed point O, is equal to the time of sliding down the chord of the arc; show that the curve is a lemniscate of Bernouilli, whose node is at O and whose axis is inclined at 45^o to the vertical.
17. A particle is projected along the inner surface of a rough sphere and is acted on by no forces; show that it will return to the point of projection at the end of time $\frac{a}{\mu V}(e^{2\mu\pi}-1)$, where a is the radius of the sphere, V is the velocity of projection and μ is the coefficient of friction.
18. A bead slides down a rough circular wire, which is in a vertical plane, starting from rest at the end of a horizontal diameter. When it has described an angle θ about the centre, show that the square of its angular velocity is
$$\frac{2g}{a(1+4\mu^2)}[(1-2\mu^2)\sin\theta+3\mu(\cos\theta-e^{-2\mu\theta})],$$
where μ is the coefficient of friction and a the radius of the rod.
19. A particle falls from a position of limiting equilibrium near the top of a nearly smooth glass sphere. Show that it will leave the sphere

at the point whose radius is inclined to the vertical at an angle

$$\alpha+\mu\left(2-\frac{4}{3}\,\frac{\alpha}{\sin\alpha}\right),$$

where $\cos\alpha=\frac{2}{3}$, and μ is the small coefficient of friction.

20. A particle is projected horizontally from the lowest point of a rough sphere of radius a. After describing an arc less than a quadrant it returns and comes to rest at the lowest point. Show that the initial velocity must be

$$\sin\alpha\sqrt{2ga\frac{1+\mu^2}{1-2\mu^2}},$$

where μ is the coefficient of friction and $a\alpha$ is the arc through which the particle moves.

21. The base of a rough cycloidal arc is horizontal and its vertex downwards; a bead slides along it starting from rest at the cusp and coming to rest at the vertex. Show that

$$\mu^2 e^{\mu\pi}=1.$$

22. A particle slides in a vertical plane down a rough cycloidal arc whose axis is vertical and vertex downwards, starting from a point where the tangent makes an angle θ with the horizon and coming to rest at the vertex. Show that

$$\mu e^{\mu\theta}=\sin\theta-\mu\cos\theta.$$

23. A rough cycloid has its plane vertical and the line joining its cusps horizontal. A heavy particle slides down the curve from rest at a cusp and comes to rest again at the point on the other side of the vertex where the tangent is inclined at 45^o to the vertical. Show

that the coefficient of friction satisfies the equation

$$3\mu\pi + 4\log_e(1+\mu) = 2\log_e 2.$$

24. A bead moves along a rough curved wire which is such that it changes its direction of motion with constant angular velocity. Show that the wire is in the form of an equiangular spiral.

25. *A particle is held at the lowest point of a catenary, whose axis is vertical, and is attached to a string which lies along the catenary but is free to unwind from it. If the particle be released, show that the time that elapses before it is moving at an angle ϕ to the vertical is*

$$\sqrt{\frac{c}{2g}}\log\left(\frac{1+\sqrt{2}\sin\dfrac{\phi}{2}}{1-\sqrt{2}\sin\dfrac{\phi}{2}}\right),$$

and that its velocity then is $2\sqrt{gc}\sin\dfrac{\phi}{2}$, where c is the parameter of the catenary. Find also the tension of the string in terms of ϕ.

At time t, let the string PQ be inclined at an angle ϕ to the horizontal, where P is the particle and Q the point where the string touches the catenary. A being the lowest point, let

$$s = \text{ arc } AQ = \text{line } PQ.$$

The velocity of P along QP = vel. of Q along the tangent + the vel. of P relative to Q

$$= (-\dot{s}) + \dot{s} = 0 \qquad ...(1)$$

The velocity of P perpendicular to QP similarly

$$= s.\dot{\phi} \qquad ...(2).$$

The acceleration of P along QP (by Arts. 4 and 49)

$=$ acc. of Q along the tangent QP + the acc. of P relative to Q

$$= -\ddot{s} + (\ddot{s} - s\dot{\phi}^2) = -s\dot{\phi}^2 \qquad ...(3)$$

The acceleration of P perpendicular to QP

$=$ acceleration of Q in this direction + acceleration of P relative to Q

$$= -\frac{\dot{s}^2}{\rho} + \frac{1}{s}\frac{d}{dt}(s^2\dot{\phi}) = -\dot{s}\dot{\phi} + [s\ddot{\phi} + 2\dot{s}\dot{\phi}] = s\ddot{\phi} + \dot{s}\dot{\phi} \qquad ...(4).$$

These are the component velocities and accelerations for any curve, whether a catenary or not.

The equation of energy gives for the catenary

$$\frac{1}{2}m.(c\tan\phi\,\dot{\phi})^2 = mg(c - c\cos\phi) \qquad ...(5).$$

Resolving along the line PQ, we have

$$mc\tan\phi\,\dot{\phi}^2 = T - mg\sin\phi \qquad ...(6).$$

(5) and (6) give the results required.

26. A particle is attached to the end of a light string wrapped round a vertical circular hoop and is initially at rest on the outside of the hoop at its lowest point. When a length $a\theta$ of the string has become unwound, show that the velocity v of the particle then is $\sqrt{2ag(\theta\sin\theta + \cos\theta - 1)}$, and that the tension of the string is $\left(3\sin\theta + \frac{2\cos\theta}{\theta} - \frac{2}{\theta}\right)$ times the weight of the particle.

27. A particle is attached to the end of a fine thread which just winds round the circumference of a circle from the centre of which acts a repulsive force $m\mu$(distance); show that the time of unwinding is $\frac{2\pi}{\sqrt{\mu}}$, and that the tension of the thread at any time t is $2\mu^{3/2}.a.t$, where a is the radius of the circle.

28. A particle is suspended by a light string from the circumference of a cylinder, of radius a, whose axis is horizontal, the string being tangential to the cylinder and its unwound length being $a\beta$. The particle is projected horizontally in a plane perpendicular to the axis of the cylinder so as to pass cylinder it; show that the least velocity it can have so that the string may wind itself completely up is $\sqrt{2ga(\beta - \sin\beta)}$.

29. From the lowest point of a smooth hollow cylinder whose cross-section is one-half of the lemniscate $r^2 = a^2\cos 2\theta$, with axis vertical and node downwards, a particle is projected with velocity V along the inner surface in the plane of a cross-section; show that it will make a complete revolution if $3V^2 > 7ag$.

30. *If a particle can describe a certain plane curve freely under one set of forces and can also describe it freely under a second set, then it can describe it freely when both sets act, provided that the initial kinetic energy in the last case is equal to the sum of the initial kinetic energies in the first two cases.*

Let the arc s be measured from the point of projection, and let the initial velocities of projection in the first two cases be U_1 and U_2. Let the tangential and normal forces in the first case be T_1 and N_1 when an arc s has been described, and T_2 and N^2 similarly in the second case; let the velocities at this point be v_1 and v_2. Then

$mv_1\dfrac{dv_1}{ds} = T_1$; $m\dfrac{v_1^2}{\rho} = N_1$; $mv_2\dfrac{dv_2}{ds} = T_2$; and $m\dfrac{v_2^2}{\rho} = N_2$.

$\therefore\ \dfrac{1}{2}mv_1^2 = \int_0^s T_1 ds + \dfrac{1}{2}mU_1^2$, and $\dfrac{1}{2}mv_2^2 = \int_0^s T_2 ds + \dfrac{1}{2}mU_2^2$

$$\therefore\ \frac{1}{2}m(v_1^2 + v_2^2) = \int_0^s T_1 ds + \int_0^s T_2 ds + \frac{1}{2}mU_1^2 + \frac{1}{2}mU_2^2 \quad ...(1),$$

and $$m\frac{v_1^2+v_2^2}{\rho} = N_1 + N_2 \qquad ...(2).$$

If the same curve be described freely when both sets of forces are acting, and the velocity be v at arcual distance s, and U be the initial velocity, we must have similarly

$$\frac{1}{2}mv^2 = \int_0^s (T_1+T_2)ds + \frac{1}{2}mU^2 \qquad ...(3),$$

and $$m\frac{v^2}{\rho} = N_1 + N_2 \qquad ...(4)$$

Provided that $\frac{1}{2}mU^2 = \frac{1}{2}mU_1^2 + \frac{1}{2}mU_2^2$ equations (1) and (3) give $v^2 = v_1^2 + v_2^2$ and then (4) is the same as (2), which is true.

Hence the conditions of motion are satisfied for the last case, if the initial kinetic energy for it is equal to the sum of the kinetic energies in the first two cases.

The same proof would clearly hold for more than two sets of forces.

COR. The theorem may be extended as follows.

If particles of masses $m_1, m_2, m_3, \dots$ all describe one path under forces $F_1, F_2, F_3, \dots$; then the same path can be described by a particle of mass M under all the forces acting simultaneously, provided its kinetic energy at the point of projection is equal to the sum of the kinetic energies of the particles $m_1, m_2, m_3, \dots$ at the same point of projection.

31. A particle moves under the influence of two forces $\frac{\mu}{r^5}$ to one point and $\frac{\mu}{r'^5}$ to another point; show that it is possible for the particle to describe a circle, and find the circle.

32. Show that a particle can be made to describe an ellipse freely under the action of forces $\lambda r + \frac{\mu}{r^2}, \lambda r' + \frac{\mu'}{r'^2}$ directed towards its foci.
33. A circle, of radius a, is described by a particle under a force $\frac{\mu}{(\text{distance})^5}$ to a point on its circumference. If, in addition, there be a constant normal repulsive force $\frac{\mu'}{a^5}$ show that the circle will still be described freely if the particle start from rest at a point where $r = a\sqrt[4]{\frac{\mu}{2\mu'}}$.
34. Show that a particle can describe a circle under two forces $\frac{uf^2}{r_1^5}$ and $\frac{uf'^2}{r_2^5}$ directed to two centres of force, which are inverse points for the circle at distances f and f' from the centre, and that the velocity at any point is
$$\frac{\sqrt{u}f}{r_1^2}\left(\text{or } \frac{\sqrt{u}f'}{r_2^2}\right)$$
35. A ring, of mass m, is strung on a smooth circular wire, of mass M and radius a; if the system rests on a smooth table, and the ring be started with velocity v in the direction of the tangent to the wire, show that the reaction of the wire is always
$$\frac{Mm}{M+m}\,\frac{v^2}{a}.$$
36. O, A and B are three collinear points on a smooth table, such that $OA = a$ and $AB = b$. A string is laid along AB and to B is attached a particle. If the end A be made to describe a circle, whose centre is O, with uniform velocity v, show that the motion of the string relative to the revolving radius OA is the same as that of a pendu-

lum of length $\frac{gab}{v^2}$, and further that the string will not remain taut unless $a > 4b$.

37. A particle slides under gravity down a rough cycloid, whose axis is vertical and vertex downwards, starting from rest at the cusp. Show that it will come to rest before reaching the lowest point if $\mu e^{\mu\pi/2} > 1$, where μ is the coefficient of friction.
Prove that this inequality is satisfied if $\mu = \frac{1}{2}$.

38. A smooth parabolic tube is fixed in a vertical plane with its vertex downwards. A particle starts from rest at the extremity of the latus rectum and slides down the tube; express as a definite integral the time taken to reach the vertex, and show that this time is approximately $2.7 \times \sqrt{\frac{a}{g}}$ seconds, where $4a$ is length of the latus rectum.

ANSWERS WITH HINTS

Art. 90 EXAMPLES

1. $2C^2 s = A(\psi^2 - 2\psi\gamma)$ where γ is arbitrary.

3. $s = \frac{A}{\omega} e^{\psi} + B$

4. $-(b + ac^2)e^{2as}$, $\frac{1}{\sqrt{ab}}\left(\sin^{-1}\sqrt{\frac{b}{b+aV^2}} - \sin^{-1}\sqrt{\frac{b}{b+ac^2}}\right)$

7. $\frac{v^2}{a} + f\sin\theta$, $f - f\cos\theta$

Art. 103 EXAMPLES

31. Draw a circle through the two centres of force, O and O', and the starting point A. This will be described with an acceleration μ/r^5 towards O if the velocity at any point is $\sqrt{\dfrac{\mu}{2r^4}}$. So for the other centre. Hence the circle will be described with both accelerations if the velocity V at the point of projection A is given by $V^2 = \dfrac{\mu}{2.OA^4} + \dfrac{\mu'}{2.O'A^4}$.