* Differential Equations *

- An equation consisting of differential co-efficient $\frac{dv}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^ny}{dx^n}$ is called differential equation.

$\frac{dv}{dx}$ → 1^{st} order, $\frac{d^2y}{dx^2}$ → 2^{nd} order, $\frac{d^ny}{dx^n}$ → n^{th} order

Order of eq^n^: The order of differential equation is the order of highest derivative present in it.

Degree of eq^n^:- The degree of differential equation is the degree of highest order (after expressed in form free from radicals and fractions).

<u>NOTE</u>: Observe the order first & then degree.

Q:- $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 2y = 0$

Order = 2
Degree = 1

Q:- $\left(\frac{d^2y}{dx^2}\right)^3 + 3\frac{dy}{dx} + 2y = K\left(\frac{d^3y}{dx^3}\right)^2$

Order = 3
Degree = 2

Q:- $\left(\frac{d^2y}{dx^2}\right)^2 + \sqrt{1+\left(\frac{dy}{dx}\right)^2} = 5y$

Order = 2

Degree = 4

Q:- $\left(\frac{d^3y}{dx^2}\right)^2 + 5\left(\frac{d^2y}{dx^2}\right) + 3\cos\left(\frac{dy}{dx}\right) + 2y = 0$

Order = 3

Degree = not possible

Q:- Blasius eq^n^ $\frac{d^3f}{d\eta^3} + \frac{f}{2}\frac{d^2f}{d\eta^2} = 0$

(a) 2^{nd} order non linear O.D.E.

(b) 3^{rd} order " " "

(c) 3^{rd} order linear O.D.E.

(d) mixed order non linear O.D.E.

Order = 3

But $\frac{f}{2} \cdot \frac{d^2 f}{dn^2}$

Dependent value (product) so you can't get lineal O.D.E.

$\therefore$ Non-linear.

Q:- The partial differential eq^n i.e. $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$

(a) Linear eq^n of order 2

(b) Non-linear " " " 1

(c) Linear " " " 1

(d) Non-linear " " " 2

* Formation of differential equation:-

- A differential eq^n is formed by eliminating the arbitrary constant in the solution. [arbritary constant = order of equation].

$$y^2 = 4ax$$

$$2y\frac{dy}{dx} = 4a$$

$$y^2 = 2y\frac{dy}{dx}x$$

$$\boxed{y - 2x\frac{dy}{dx} = 0}$$

Q:- $x^2 + y^2 = a^2$

$$2x + 2y\frac{dy}{dx} = 0$$

$$x + y\frac{dx}{dx} = 0$$

Q:- $y = ae^{x} + be^{-x}$

$y' = ae^{x} - be^{-x}$

$y'' = ae^{x} + be^{-x}$

$y' + y'' = 2ae^{x}$

$a = \frac{y' + y''}{2e^{x}}$ $\quad -2be^{-x} = y' - y''$

$b = \frac{y'' - y'}{2e^{-x}}$

$y = \frac{y' + y''}{2} + \frac{y'' - y'}{2}$

$y = \frac{2y''}{\cancel{2}}$. $\boxed{y'' - y = 0.}$

Q:- $y = e^{x}(a\cos x + b\sin x)$

$y' = e^{x}[-a\sin x + b\cos x] + e^{x}[a\cos x + b\sin x]$

$= -e^{x}a\sin x + e^{x}b\cos x + \underline{e^{x}a\cos x} + \underline{e^{x}b\sin x}$

$y' = -e^{x}a\sin x + e^{x}b\cos x + y$

$y'' = -\underline{e^{x}a\cos x} + be^{x}\cos x + -e^{x}a\sin x - \underline{be^{x}\sin x} + y'$

$y'' = -y + y' - y + y'$

$y'' = -2y + 2y'$

$\boxed{y'' - 2y' + 2y = 0}$

Solⁿ of differential equation

-A solⁿ (integral) of differential equation is the relation between which satisfies given differential eqⁿ.

$y = c e^{x^3/3}$ —(i) is the solⁿ of $\frac{dy}{dx} = x^2 y$ —(ii)

① The general (complete) solution of differential eqⁿ is that in which the no. of arbritary constant is equal to the order of D.E.

② Thus the first eqⁿ is the general solⁿ of second eqⁿ as the no. of arbritary constant (c) is the same as order of eqⁿ (2). [First order]. Similarly in the general solution of the second order D.E. there will be 2 arbritary constant.

③ A particular solⁿ is that which can be obtained from general solⁿ by giving particular value to arbritary constant.

For eg:- $y = 4e^{x^3/3}$

④ A differential eqⁿ may sometimes have an additional solⁿ which cannot be obtained from general solⁿ by assigning particular value to arbritary constant. Such a solⁿ is called singular solⁿ.

*Equations of 1st order and 1st degree

Methods of solⁿ:-

(1) Variable separable

(2) Homogeneous eqⁿ & non-homogeneous eqⁿ

(3) Linear eq^n

(4) Exact eq^n

[1] Variable separable

<u>Steps:</u> (1) Collect all function of x and dx one side and y and dy on other side

(2) Integrating on both the sides

(3) Find the value of arbitrary constant with help of initial conditions

For eg:- $y(0) = 1$

At $x=0$ $y=1$

(4) Substitute the value of arbitrary constant in eq

Q:- $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

$$\frac{dy}{dx} = e^y (e^x + x^2)$$

$$\frac{1}{e^y}\frac{dy}{dx} = e^x + x^2$$

$$\int \frac{1}{e^y} dy = \int (e^x + x^2)\, dx$$

$$-e^{-y} = e^x + \frac{x^3}{3} + c$$

$$-e^{-y} = e^x + \frac{x^3}{3} + c$$

$$\boxed{-3e^{-y} = 3e^x + x^3 + c'}$$

Q:- Consider the following differential eqⁿ

$\frac{dy}{dt} = -5y$ Initial condition $y = 2$ at $t = 0$. The value of y at $t = 3$ is ___.

(a) $-5e^{-10}$
(b) $2e^{-10}$
(c) $2e^{-15}$ ✓
(d) $-15e^{2}$

$$\int \frac{dy}{y} = \int -5\,dt$$

$$\log y = -5t + \log c$$

$$\log y - \log c = -5t$$

$$y = e^{-5t} c.$$

$$\boxed{2 = c}$$

$$y = 2e^{-5t}\Big|_{t=3}$$

$$y = 2e^{-15}$$ ✓

Q:- Consider the following differential eqⁿ $x(y\,dx + x\,dy)\cos \frac{y}{x} = y(x\,dy - y\,dx)\sin y/x$.

$$x(y\,dx + x\,dy)\cos y/x = y(x\,dy - y\,dx)\sin y/x$$

$$\tan y/x = \frac{x(y\,dx + x\,dy)}{y(x\,dy - y\,dx)}$$

$$\tan y/x = \frac{dx + \frac{1}{y}dy}{dy - \frac{1}{x}dx}$$

$$\left(\sin y/x\right)\left(dy - \frac{1}{x}dx\right) = \left(dx + \frac{1}{y}dy\right)\cos y/x$$

$y/x = t$

$\cancel{\sin t \left[\quad \right] = \left[\quad \right] \cos t}$

$y = tx$

$dy = t\,dx + x\,dt$

$\left[(t\,dx + x\,dt) - \frac{1}{x} \cdot \left[\frac{dy - x\,dt}{t}\right]\right] = \left[\frac{dy - x\,dt}{t}\right] \cot t$

$xt^2 dx + x^2 t\,dt - dy + x\,dt = (x\,dy - x^2 dt) \cos t$

$\cancel{x} \cdot \frac{y^2}{x^{\cancel{2}}} dx + x^2 \cdot \frac{y}{x} dt - dy + x\,dt = (x\,dy - x^2 dt) \cos t$

$\frac{y}{x} \tan y/x = \frac{y\,dx + x\,dy}{x\,dy - y\,dx}$

Take $y/x = v$

$y = xv$

$dy = v\,dx + x\,dv$

$\therefore \frac{y\,dx + x[v\,dx + x\,dv]}{x[v\,dx + x\,dv] - y\,dx} = v \tan v$

$\frac{vx\,dx + vx\,dx + x^2 dv}{vx\,dx + x^2 dv - vx\,dx} = v \tan v$

$\frac{2v\,dx \cdot x + x^2 dv}{x^2 dv} = v \tan v$

$\frac{2v\cancel{x}}{x^{\cancel{2}}} \cdot \frac{dx}{dv} + 1 = v \tan v$

$$\therefore\ 2\frac{v}{x}\frac{dx}{dv} = v\tan v - 1$$

$$\frac{2}{x}\frac{dx}{dv} = \tan v - 1/v$$

$$\int \frac{2}{x}\,dx = \int\left(\tan v - 1/v\right)dv$$

$$2\ln x = \log\sec v - \ln v + \ln c$$

$$2\ln x = \ln\left(\frac{\sec v}{v}\cdot c\right)$$

$$x^2 = \frac{\sec v}{v}\cdot c$$

$$x^2\cos v \cdot v = c$$

$$x^2\cos(y/x)\cdot y/x = c$$

$$\boxed{xy\cos(y/x) = c}$$

Q:- The sol^n of differential eq^n $3y\frac{dy}{dx} + 2x = 0$ represent a family of (a) ellipse (c) parabola
(b) circle (d) hyperbola

$$3y\frac{dy}{dx} = -2x$$

$$\int 3y\,dy = -\int 2x\,dx$$

$$\frac{3y^2}{2} = \frac{-2x^2}{2} + \frac{c}{2}$$

$$3y^2 = -2x^2 + c'$$

$$x^2 + \frac{3}{2}y^2 = c^2$$

$$2x^2 + 3y^2 = 2c$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$ ← ellipse

$3y^2 + 2x^2 = c'$

$\frac{y^2}{c'/3} + \frac{x^2}{c'/2} = 1$

Q:- The solⁿ of $\frac{dy}{dx} = y^2$ with initial value $y(0) = 1$ bounded in the interval

(a) $-\infty \le x \le \infty$

(b) $-\infty \le x \le 1$

✓(c) $x < 1, x > 1$

(d) $-2 \le x \le 2$

Solⁿ:- $\int \frac{dy}{y^2} = \int dx$

$\frac{y^{-2+1}}{-2+1} = x + c$

$\frac{y^{-1}}{-1} = x + c$

$-y^{-1} = x + c$

$\frac{-1}{1} = 0 + c$

$\boxed{c = -1}$

$\frac{-1}{y} = x - 1$

$y = \frac{-1}{x-1} = \frac{1}{-x+1} = \frac{-1}{x-1}$

$x - 1 \neq 0$

$x \neq 1$

✓(c) $x < 1, x > 1$

☆ Q:- Consider the following 2nd order linear differential eqⁿ $\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$. The boundary condition are $x = 2, y = 21$ & $x = 0, y = 5$

The value of y at $x = 1$ is ______.

$$\frac{dy^2}{dx^2} = -12x^2+24x-20$$

$$\int y'' = \int -12x^2 x'' + 24x \cdot x'' - 20x''$$

$$y' = -12\frac{x^1 \cdot x^2}{3} + 24\frac{x \cdot x}{2} - 20x^1 + c_1$$

$$y = -\frac{\cancel{12}x^4}{\cancel{3}\cdot 4} + \frac{\cancel{24}x^3}{\cancel{2}\cdot\cancel{3}} - \frac{\cancel{20}x^2}{\cancel{2}} + c_1 x + c_2$$

$$\cancel{21 = -12(4) + 48 - 20 + c_1 + c_2}$$

$$\cancel{21 = -20 + c_1 + c_2} \qquad \boxed{5 = c_2}$$

$$\cancel{c_1 + c_2 = 41}$$

5

$$21 = -16 + 32 - 40 + 2c_1 + 5$$

$$\boxed{c_1 = 20}$$

$$y = -x^4 + 4x^3 - 10x^2 + 20x + 5$$

$$y = -1 + 4 - 10 + 20 + 5$$

$$\boxed{y = 18}$$

*Homogeneous Equations:-

-Homogeneous eq^n are of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ where $f(x,y)$ and $\phi(x,y)$ homogeneous function of same degree in x and y.

To solve homogeneous eq^n steps are:-

(1) Put $y = vx$

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

(2) Separate the variable v and x and integrate

Q:- Solve $(y^2 - x^2)\,dx - 2xy\,dy = 0$ represents

(a) family of ellipse
(b) family of parabola
(c) " " circle
(d) " " hyperbola

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Homogeneous eq^n

Sol^n:- Substitute $y = vx$

$$(v^2x^2 - x^2)\,dx - 2x(vx)\,dy = 0$$

$$\frac{dy}{dx} = \frac{\cancel{x^2}(v^2-1)}{\cancel{x^2}(+2v)} = \frac{v^2-1}{2v}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{v^2-1}{2v} = v + x\frac{dv}{dx}$$

$$\frac{-1+v^2-2v^2}{2v} = x\frac{dv}{dx}$$

$$\frac{-v^2-1}{2v} = x\frac{dv}{dx}$$

$$\int \frac{1}{x}\,dx = \int \frac{2v}{-(v^2+1)}\,dv$$

$$\log x = -\log|v^2+1| + \log c$$

$$\cancel{x = \frac{c}{v^2+1}}$$

$$\cancel{x\left(\frac{y}{x}\right)^2 + x = c}$$

$$\cancel{x\,\frac{y^2}{x^2} + x = c}$$

$$\cancel{\frac{y^2+x^2}{x} = c}$$

$$x(v^2+1) = c$$

$$x\left(\frac{y^2}{x^2}+1\right) = c$$

$$y^2 + x^2 = c \cdot x$$

$$y^2 + x^2 = c'$$

$$\boxed{y^2+x^2 = c} \leftarrow \text{circle}$$

Exact differential equation

- A differential eq^n of the form $Mdx + Ndy = 0$ is said to be exact if $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

The general sol^n is,

$$G.S. = \int M\,dx + \int \left(\text{terms in } N \text{ not containing } x\right) dy = \text{constant}$$

Q:- Find the general solution $(x^2 - ay)dx + (y^2 - ax)dy = 0$

$$\frac{\partial M}{\partial y} = -a \qquad \frac{\partial N}{\partial x} = -a$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\downarrow$

exact diff^n eq^n

$$G.S. = \frac{x^3}{3} - axy + \frac{y^3}{3} = c$$

$$= x^3 - 3axy + y^3 = 3c$$

Q:- $(ax + hy + g)\,dx + (hx + by + f)\,dy = 0$

$$\frac{\partial m}{\partial y} = h$$

$$\frac{\partial N}{\partial x} = h$$

$$\frac{\partial m}{\partial y} = \frac{\partial N}{\partial x}, \leftarrow \text{exact diff. eq}^n$$

$$\frac{ax^2}{2} + hyx + gx + \frac{by^2}{2} + fy = c$$

$$ax^2 + by^2 + 2hxy + 2gx + 2fy = 2c$$

Q:- $(1+e^{x/y})\,dx + e^{x/y}(1 - x/y)\,dy = 0$

$$\frac{\partial M}{\partial y} = e^{x/y} \cdot \frac{-x}{y^2}$$

$$N = \left(e^{x/y} - e^{x/y} \cdot x/y\right)$$

$$\frac{\partial N}{\partial x} = e^{x/y}\frac{1}{y} - e^{x/y} \cdot \frac{1}{y}\frac{x}{y} - e^{x/y}\frac{1}{y} = -e^{x/y} \cdot \frac{x}{y^2}$$

$$x + \frac{e^{x/y}}{1/y} = \text{constant}$$

$$x + ye^{x/y} = c$$

Non-exact differential equations

-A diff. eq^n of form $Mdx + Ndy$ is said to be non-exact if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

-Sometimes in differential eq^n which is not exact can be made so on multiplication of suitable factor called as integrable factor.

Rules for finding integrating factor for $Mdx + Ndy = 0$ are as follows:-

(1) Integrating factor found by inspection

- In no. of cases, the integrating factor can be found after regrouping the term of the eq^n and recognizing each group as being a part of exact differential equation.

- In this connection the following integrable combination prove quite useful

$$\cancel{x\,dy + y\,dx = d(xy)}$$

$$\Rightarrow x\,dy + y\,dx = d(xy)$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = d(y/x)$$

$$\Rightarrow \frac{x\,dy - y\,dx}{xy} = d\left(\log(y/x)\right)$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}(y/x)\right)$$

$$\Rightarrow \frac{x\,dy - y\,dx}{y^2} = -d(x/y)$$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2-y^2} = d\left(\frac{1}{2}\log\left(\frac{x+y}{x-y}\right)\right)$$

Q:- $y\,dx - x\,dy + (1+x^2)\,dx - x^2\sin y\,dy = 0$

$$\cancel{y - x^2 \sin} \; (y+1+x^2)\,dx + dy(-x - x^2\sin y) = 0$$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -1 - 2x\sin y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Non-exact diff}^n \text{ eq}^n$$

$$\left(\frac{y\,dx - x\,dy}{x^2}\right) + \left(\frac{1+x^2}{x^2}\right)dx - \frac{\cancel{x^2}\sin y\,dy}{\cancel{x^2}} = 0$$

$$-\left(\frac{x\,dy - y\,dx}{x^2}\right)$$

$$\int -d\left(y/x\right) + \int\left(\frac{1}{x^2} + 1\right)dx - \int \sin y\,dy = 0$$

$$-y/x + \frac{-1}{x} + x + \cos y = c$$

Q:- $(y^2e^x + 2xy)\,dx - x^2\,dy = 0$

$$\frac{\partial M}{\partial y} = 2ye^x + 2x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{Non-exact}$$

$$\frac{\partial N}{\partial x} = -2x$$

$$\frac{y^2e^x\,dx}{y^2} + \frac{2xy\,dx - x^2\,dy}{y^2} = 0$$

$$\int e^x\,dx + \int d\left(\frac{x^2}{y}\right) = 0$$

$$\boxed{e^x + \frac{x^2}{y} = c}$$