Chapter 2

MOTION IN A STRAIGHT LINE

Art 28

Ex. 1. $x = a\cos nt + 2a\cos\left(nt + \frac{\pi}{2}\right) = a\sqrt{5}\cos(nt + \beta)$, where

$$\frac{\cos\beta}{1} = \frac{\sin\beta}{2} = \frac{1}{\sqrt{5}}.$$

This is a simple harmonic motion of amplitude $a\sqrt{5}$, whose phase is in advance of the first by $\frac{\beta}{2\pi}$ of a period.

Ex. 2. $\dot{x} = n\sqrt{a^2 - x^2}$, so that $\dot{x} = \frac{na}{2}$ when $x = \frac{a\sqrt{3}}{2}$. Hence the new velocity at this distance $= \frac{3na}{2}$. The new motion is given by

$$x = A\cos nt + B\sin nt,$$

where $$\frac{a\sqrt{3}}{2} = A \text{ and } \frac{3na}{2} = Bn.$$

Hence $$x = a\sqrt{3}\cos\left(nt - \frac{\pi}{3}\right).$$

Ex. 3. $$\ddot{x} = -\mu\left(x - \tfrac{1}{2}at^2\right),$$

i.e. $$\frac{d^2}{dt^2}\left[x - \left(\tfrac{1}{2}at^2 - \frac{a}{\mu}\right)\right] = -\mu\left[x - \left(\tfrac{1}{2}at^2 - \frac{a}{\mu}\right)\right].$$

Hence as stated.

Ex. 4. $m\ddot{x} = mg - ng\frac{x - l}{l}$, *i.e.* $\ddot{x} = -\frac{ng}{ml}\left[x - \frac{m+n}{n}l\right]$,

so that $$x - \frac{m+n}{n}l = A\cos\left[\sqrt{\frac{ng}{ml}}\,t + B\right].$$

Hence, etc.

Ex. 5. Let $OA = a$ and $AB = b$, where O is the fixed end of the string. When $a < x < a + b$, we have $m\ddot{x} = -\lambda\frac{x - a}{a}$, *i.e.* a S. H. M. about A as centre. Hence, by Art. 22, the time from B to $A = \frac{\pi}{2}\sqrt{\frac{am}{\lambda}}$, and the velocity at $A = \sqrt{\frac{\lambda}{am}}\,b$, so that the time from A to $O = a \div \sqrt{\frac{\lambda}{am}}\,.\,b = \frac{a}{b}\sqrt{\frac{am}{\lambda}}$. The required time is four times the sum of these times.

Ex. 6. When on the one side there are lengths $l - x$ and $l' + x$ of m and m' and similarly $l + x$ and $l' - x$ on the other, then

$$(2ml + 2m'l')\ddot{x} = g\left[m(l - x) + m'(l' + x) - m(l + x) - m'(l' - x)\right]$$
$$= -2(m - m')gx.$$

Hence, etc.

Ex. 7. Let AB be the tunnel, P any point on it, N the centre of AB, and O the centre of the Earth. Then attraction at $P = \mu \, . \, OP = \frac{g}{a} \, . \, OP$. Hence, if $NP = x$, we have $\ddot{x} = -\frac{g}{a} \, . \, NP = -\frac{g}{a} x$. Hence the time

$$= \pi \sqrt{\frac{a}{g}} = \pi \sqrt{\frac{4000 \times 5280}{g}} \text{ secs.} = 100\pi \, . \, \sqrt{66} \text{ secs.} = \text{about } 2550 \text{ secs.}$$
$$= 42\tfrac{1}{2} \text{ mins. approx.}$$

Art 34

Ex. 1. $\ddot{x} = -\frac{\mu}{x^2} = -\frac{ga^2}{x^2}$, so that $\dot{x}^2 = \frac{2ga^2}{x}$, since $\dot{x}$ is zero at infinity. Hence at the surface of the Earth $\dot{x}^2 = 2ga$, etc.

Ex. 2. Here, by Ex. 1, $v = \sqrt{2g \, . \, 4000 \, . \, 5280} = 64 \, . \, 10^2 \sqrt{33}$ ft. per sec. $= \frac{40}{33} \times 5{\cdot}75$ miles per sec. $= 7$ miles per sec. nearly. In the case of the sun $v'^2 = 2g'a'$, where $g' = \lambda \frac{S}{a'^2}$ and $\lambda \frac{S}{d^2}$ = normal acceleration of the Earth $= \omega^2 d$.

$$\therefore \; g' = \frac{\omega^2 d^2}{a'^2}, \text{ and } v' = \omega d \, . \, \sqrt{\frac{2d}{a'}}$$

$$= \frac{2\pi}{365 \times 24 \times 60 \times 60} \times 92500000 \times \sqrt{\frac{2 \times 92500000}{440000}} \text{ miles per sec.}$$

$$= \frac{1850000\pi}{73 \times 864} \sqrt{\frac{185}{11}} \text{ miles per sec.}$$

$$= 377{\cdot}9 \text{ miles per sec.}$$

Ex. 3. $\ddot{x} = -\frac{\lambda}{x^2} = -\frac{ga^2}{x^2}$, so that $\dot{x}^2 = 2ga^2\left[\frac{1}{x} - \frac{1}{a+h}\right]$.

$$\therefore \; \sqrt{\frac{2ga^2}{a+h}} \, t = -\int_{a+h}^{a} dx \sqrt{\frac{x}{a+h-x}} \quad [\text{Put } x = (a+h)\cos^2\theta.]$$

$$= (a+h)\int(1+\cos 2\theta) = (a+h)\left[\theta + \tfrac{1}{2}\sin 2\theta\right]$$

$$= (a+h)\left[\cos^{-1}\sqrt{\frac{x}{a+h}} + \sqrt{\frac{x}{a+h}}\sqrt{1 - \frac{x}{a+h}}\right]_{a+h}^{a}.$$

$$\therefore \; t = \sqrt{\frac{a+h}{2g}} \, \frac{a+h}{a}\left[\cos^{-1}\sqrt{\frac{a}{a+h}} + \sqrt{\frac{a}{a+h}}\sqrt{\frac{h}{a+h}}\right]$$

$$= \sqrt{\frac{a+h}{2g}}\left[\frac{a+h}{a}\sin^{-1}\sqrt{\frac{h}{a+h}} + \sqrt{\frac{h}{a}}\right].$$

Now $$\sin^{-1}\lambda = \lambda + \frac{\lambda^3}{6} + \frac{1 \, . \, 3}{2 \, . \, 4}\frac{\lambda^5}{5} + \ldots .$$

$$\therefore \; t = \sqrt{\frac{a+h}{2g}}\left[\frac{a+h}{a}\left\{\sqrt{\frac{h}{a+h}} + \frac{1}{6}\frac{h}{a+h}\sqrt{\frac{h}{a+h}} + \ldots\right\} + \sqrt{\frac{h}{a}}\right]$$

$$= \sqrt{\frac{h}{2g}}\left[\frac{a+h}{a} + \frac{1}{6}\frac{h}{a} + \ldots + \sqrt{\frac{a+h}{a}}\right]$$

$$= \sqrt{\frac{h}{2g}}\left[1 + \frac{h}{a} + \frac{1}{6}\frac{h}{a} + \ldots + 1 + \frac{h}{2a}\right] = \sqrt{\frac{2h}{g}}\left[1 + \frac{5}{6}\frac{h}{a}\right],$$

squares of $\frac{h}{a}$ being neglected.

End of Art 37

EXAMPLES ON CHAPTER 2

1. $v^2 \propto \frac{a^2}{x^2} - 1$; $\therefore$ $v\frac{dv}{dx} \propto -\frac{a^2}{x^3}$, *i.e.* the acceleration varies inversely as the cube of the distance.

2. $\frac{d^2x}{dt^2} = \frac{\mu}{x^2}$; $\therefore \left(\frac{dx}{dt}\right)^2 = -\frac{2\mu}{x} + \frac{2\mu}{a} = 2\mu\frac{x-a}{ax}$.

$$\therefore t\sqrt{\frac{2\mu}{a}} = \int_a^x \sqrt{\frac{x}{x-a}}\,dx = \int_a^x \left[\frac{2x-a}{2\sqrt{x^2-ax}} + \frac{a}{2\sqrt{x^2-ax}}\right]dx$$

$$= \left[\sqrt{x^2-ax} + a\log\left(\sqrt{x}+\sqrt{x-a}\right)\right]_a^x, \quad \therefore \text{ etc.}$$

3. If possible, let $v = \lambda x$; hence f = acceleration $= \frac{dv}{dt} = \lambda\frac{dx}{dt} = \lambda^2 x$. Hence initially, when $x=0$, we have both v and f zero so that the particle remains at rest.

If $v = \lambda\,.\,x^n$, then $f = \frac{dv}{dt} = n\lambda x^{n-1}\,.\,v = n\lambda\,.\,x^{2n-1}$; hence, if $2n > 1$, both v and f will vanish when $x=0$, and the particle remains at rest.

4. $\ddot{x} = -\frac{\mu}{x^3}$; $\therefore$ $\dot{x}^2 = \frac{\mu}{x^2} - \frac{\mu}{a^2} = \frac{\mu(a^2-x^2)}{a^2x^2}$.

$$\therefore \sqrt{\mu}\,t = -\int_a^x \frac{ax}{\sqrt{a^2-x^2}} = \left[a\sqrt{a^2-x^2}\right]_a^b = a\sqrt{a^2-b^2}.$$

5. $\ddot{x} = -\frac{\mu}{x^{\frac{5}{3}}}$; $\therefore$ $\dot{x}^2 = 3\mu\left[\frac{1}{x^{\frac{2}{3}}} - \frac{1}{a^{\frac{2}{3}}}\right]$.

$$\therefore t\sqrt{3\mu} = -\int_a^0 \frac{a^{\frac{1}{3}}x^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}-x^{\frac{2}{3}}}}\,dx = 3a^{\frac{4}{3}}\int_0^{\frac{\pi}{2}}\cos^3\phi\,d\phi, \text{ if } x = a\cos^3\phi,$$

$$= 3a^{\frac{4}{3}}\,.\,\tfrac{2}{3}. \quad \therefore \text{ etc.}$$

6. $\ddot{x} = -\frac{\mu}{x^{\frac{4}{3}}}$; $\therefore$ $\dot{x}^2 = \frac{6\mu}{x^{\frac{1}{3}}} + C$.

$$\therefore v_1^2 = \frac{6\mu}{a^{\frac{1}{3}}}, \text{ and } v_2^2 = \left(\frac{6\mu}{x^{\frac{1}{3}}}\right)_a^{\frac{a}{8}} = \frac{6\mu}{a^{\frac{1}{3}}}(2-1) ; \quad \therefore v_1 = v_2.$$

7. $\ddot{x} = -\mu\left(x + \frac{a^4}{x^3}\right)$; $\therefore$ $\dot{x}^2 = \mu\left(-x^2 + \frac{a^4}{x^2}\right)$.

$$\therefore t\sqrt{\mu} = -\int_a^0 \frac{x}{\sqrt{a^4-x^4}}\,dx = -\frac{1}{2}\left[\sin^{-1}\frac{x^2}{a^2}\right]_a^0 = \frac{\pi}{4}.$$

8. $\ddot{x} = -\frac{\mu}{x^2} + \frac{\lambda}{x^3}$; $\therefore$ $\dot{x}^2 = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right) - \lambda\left(\frac{1}{x^2} - \frac{1}{a^2}\right)$

$$= \frac{\lambda(a-x)(x-pa)}{pa^2x^2}, \text{ if } \frac{\lambda}{2a\mu - \lambda} = p.$$

The particle is at rest again when $x = pa$.

$$\frac{t}{a}\sqrt{\frac{\lambda}{p}} = -\int_a^{pa} \frac{x\,dx}{\sqrt{(a-x)(x-pa)}} \quad [\text{Put } x = a\sin^2\theta + pa\cos^2\theta.]$$

$$= \int_0^{\frac{\pi}{2}} 2a(\sin^2\theta + p\cos^2\theta)\,d\theta = \frac{\pi a}{2}(1+p).$$

Hence the required time $= 2t = 2\mu\pi a^3 \div (2a\mu - \lambda)^{\frac{3}{2}}$.

9. $\ddot{x} = -\frac{\mu}{x}$; $\therefore$ $\dot{x}^2 = -2\mu \log\frac{x}{a}$.

$$\therefore t\sqrt{2\mu} = -\int_a^0 \frac{dx}{\sqrt{-\log\frac{x}{a}}} = 2a\int_0^\infty e^{-y^2}dy, \text{ if } x = ae^{-y^2},$$

$$= 2a\frac{\sqrt{\pi}}{2}, \text{ so that } t = a\sqrt{\frac{\pi}{2\mu}}.$$

10. $\ddot{x} = -\frac{\mu}{x^n}$, so that $\dot{x}^2 = \frac{\mu}{n-1}\cdot\frac{1}{x^{n-1}} + C$.

Now $C = 0$ in the first case, and $= -\frac{\mu}{n-1}\frac{1}{a^{n-1}}$ in the second case. We are given $\frac{1}{a^{n-1}} = \left(\frac{4}{a}\right)^{n-1} - \frac{1}{a^{n-1}}$, so that $n = \frac{3}{2}$.

11. $\ddot{x} = \mu(a-x) - \mu'(x+a) = -(\mu+\mu')\left(x - a\frac{\mu-\mu'}{\mu+\mu'}\right)$. Hence the result, by Art. 22.

12. When the mass has risen x feet the tension of the rope $= (150-x)g$. Hence $100\ddot{x} = (150-x)g - 100g = -(x-50)g$.

$$\therefore \text{(Art. 22) } x - 50 = A\cos\left(\sqrt{\frac{g}{100}}t + B\right) = -50\cos\left[\frac{2\sqrt{2}}{5}t\right],$$

since initially $x = 0$ and $\dot{x} = 0$.

Hence, when $x = 50$, $\quad t = \frac{5}{2\sqrt{2}}\cdot\frac{\pi}{2} = \frac{5\pi\sqrt{2}}{8}$ secs.,

and then $\quad v = \frac{dx}{dt} = 50\cdot\frac{2\sqrt{2}}{5}\sin\frac{\pi}{2} = 20\sqrt{2}$ ft. per sec.

13. $\ddot{x} = -\frac{\mu}{x^n}$, and hence $\dot{x}^2 = \frac{2\mu}{n-1}\frac{1}{x^{n-1}}$, the constant being zero, since $\dot{x} = 0$ when $x = \infty$.

$$\therefore \sqrt{\frac{2\mu}{n-1}}\,t = -\int_a^0 x^{\frac{n-1}{2}}\,dx = \left[\frac{2}{n+1}x^{\frac{n+1}{2}}\right]_0^a = \frac{2}{n+1}a^{\frac{n+1}{2}}.$$

14. Here $$\dot{x}^2 = \frac{2\mu}{n-1}\left(\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right).$$

(1) $n > 1$. $$\therefore \sqrt{\frac{2\mu}{n-1}}\,t = -\int_a^0 \frac{a^{\frac{n-1}{2}}\,x^{\frac{n-1}{2}}}{\sqrt{a^{n-1} - x^{n-1}}}\,dx. \quad [\text{Put } x = a\xi^{\frac{1}{n-1}}.]$$

$$= \int_0^1 \frac{a^{\frac{n+1}{2}}}{n-1}\,\frac{\xi^{\frac{1}{n-1}-\frac{1}{2}}\,d\xi}{\sqrt{1-\xi}} = \frac{a^{\frac{n+1}{2}}}{n-1}\,B\left[\frac{1}{n-1}+\frac{1}{2},\,\frac{1}{2}\right]$$

$$= \frac{a^{\frac{n+1}{2}}}{n-1}\,\frac{\Gamma\left(\frac{1}{n-1}+\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n-1}+1\right)} = a^{\frac{n+1}{2}}\,\frac{\Gamma\left(\frac{1}{n-1}+\frac{1}{2}\right).\sqrt{\pi}}{\Gamma\left(\frac{1}{n-1}\right)}, \text{ etc.}$$

(2) $n < 1$. $$\therefore \sqrt{\frac{2\mu}{1-n}}\,t = -\int_a^0 \frac{dx}{\sqrt{a^{1-n} - x^{1-n}}} \quad [\text{Put } x = a\xi^{\frac{1}{1-n}}.]$$

$$= \int_0^1 \frac{a^{\frac{n+1}{2}}}{1-n}\,\frac{\xi^{\frac{1}{1-n}-1}}{\sqrt{1-\xi}}\,d\xi = \frac{a^{\frac{n+1}{2}}}{1-n}\,B\left(\frac{1}{1-n},\,\frac{1}{2}\right)$$

$$= \frac{a^{\frac{n+1}{2}}}{1-n}\,\frac{\Gamma\left(\frac{1}{1-n}\right).\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{1-n}+\frac{1}{2}\right)} = \frac{a^{\frac{n+1}{2}}.\sqrt{\pi}}{1-n}\,\frac{\Gamma\left(\frac{1}{1-n}\right)}{\Gamma\left(\frac{1}{1-n}+\frac{1}{2}\right)}, \text{ etc.}$$

15. $50\ddot{x} = \pi . 2^2 . \frac{A}{x}$.

If a be the length behind the shot initially, then

$$\frac{A}{a} = 10 \times 2240g, \text{ and } \frac{A}{8} = 2240g, \text{ so that } a = \frac{4}{5}.$$

$$\therefore \ddot{x} = \frac{\pi \times 32 \times 224g}{5x}.$$

$$\therefore \dot{x}^2 = \frac{\pi . 64 . 224g}{5}\log\frac{x}{a}, \text{ since } \dot{x} = 0 \text{ when } x = a.$$

Hence $$V = [\dot{x}]_{x=8} = \left(\frac{\pi . 64 . 224 . 32}{5}\log_e 10\right)^{\frac{1}{2}}$$

$$= 8 \times 32 \times \sqrt{\frac{7\pi}{5} \times 2\cdot3026} = 8 \times 32 \times \sqrt{10\cdot1274}$$

$$= 8 \times 32 \times 3\cdot182 = 814\cdot6 \text{ ft. per sec.}$$

16. Let a and b be the radii of the Earth and Moon respectively, E and M their masses, and d the distance between their centres. Then at distance x from the centre of the Moon,

$$\ddot{x} = \frac{\gamma E}{(d-x)^2} - \frac{\gamma M}{x^2}.$$

$$\therefore \tfrac{1}{2}\dot{x}^2 = \frac{\gamma E}{d-x} + \frac{\gamma M}{x} + \frac{V^2}{2} - \frac{\gamma E}{d-b} - \frac{\gamma M}{b}, \quad \text{............(1)}$$

where V is the required velocity.

Let x_1 be the distance from the centre of the Moon at which a particle would be in equilibrium under the attraction of the Earth and Moon, so that

$$\frac{\gamma M}{x_1^2}=\frac{\gamma E}{(d-x_1)^2}, \text{ and hence } \frac{x_1}{\sqrt{M}}=\frac{d-x_1}{\sqrt{E}}, \text{ i.e. } x_1=\frac{d}{10}.$$

Now the particle will just reach the Earth if the velocity is just sufficient to carry it to the distance x_1, i.e. by (1) if

$$\frac{V^2}{2}=ga^2\left[\frac{1}{d-b}+\frac{1}{81b}-\frac{100}{81d}\right], \text{ since } \frac{\gamma E}{a^2}=g \text{ and } M=\frac{1}{81}E,$$

$$=ga\left[\frac{40}{2389}+\frac{40}{81\times 11}-\frac{10}{81\times 6}\right]$$

$$=32\times 4000\times 1760\times 3\left[\frac{1}{60}+\frac{40}{81\times 11}-\frac{10}{81\times 6}\right] \text{ approximately, in ft.-sec. units.}$$

$$=32\times 4\times 10^4\times 16\times \frac{2200}{81\times 20} \text{ nearly.}$$

$$\therefore\ V=\frac{64\times 10^2}{9}\sqrt{110}=\frac{64\times 10^2}{9}\times 10{\cdot}53=7488 \text{ ft. per sec.}$$

$$=1{\cdot}42 \text{ miles per sec. approx.}$$

17. $\ddot{x}=-\frac{\mu}{b^2+x^2}\cdot\frac{x}{\sqrt{b^2+x^2}}=-\frac{\mu}{b^3}x+\text{higher powers of } x.$

$\therefore$ required time $=2\pi\sqrt{\frac{b^3}{\mu}}$. (Art. 22.)

18. The acceleration at a point outside distant x from the centre $=\frac{\gamma M}{x^2}$.

Hence by Art. 31, the time of falling to the sphere

$$=\sqrt{\frac{b}{2\gamma M}}\left[\sqrt{bx-x^2}+b\cos^{-1}\sqrt{\frac{x}{b}}\right]_{x=b}^{x=a}$$

$$=\sqrt{\frac{b}{2\gamma M}}\left[\sqrt{ab-a^2}+b\cos^{-1}\sqrt{\frac{a}{b}}\right] \quad\text{..................(1)}$$

and the velocity on reaching the sphere $=\sqrt{2\gamma M\frac{b-a}{ab}}=V$. For the motion inside the sphere we have, since the attraction now varies as the distance,

$$\ddot{x}=-\frac{\gamma M}{a^3}x, \text{ so that } \dot{x}^2=-\frac{\gamma M}{a^3}x^2+\left[V^2+\frac{\gamma M}{a}\right]=\frac{\gamma M}{a^3}[c^2-x^2],$$

where $c^2=a^2(3b-2a)/b$.

$$\therefore\ \text{time to the centre}=-\sqrt{\frac{a^3}{\gamma M}}\int_a^0\frac{dx}{\sqrt{c^2-x^2}}=\sqrt{\frac{a^3}{\gamma M}}\sin^{-1}\frac{a}{c}$$

$$=\sqrt{\frac{a^3}{\gamma M}}\sin^{-1}\sqrt{\frac{b}{3b-2a}}. \quad\text{...............(2)}$$

The required time is four times the sum of (1) and (2).

19. $\ddot{x} = -\gamma \cdot \frac{2\pi a\rho}{a^2+x^2} \cdot \frac{x}{\sqrt{a^2+x^2}} = -2\pi a\rho\gamma \cdot \frac{x}{(a^2+x^2)^{\frac{3}{2}}}$ ....(1)

$\therefore\ \dot{x}^2 = 4\pi a\rho\gamma \left[\frac{1}{\sqrt{a^2+x^2}} - \frac{1}{\sqrt{a^2+b^2}}\right].$

Also, when x is small, (1) gives $\ddot{x} = -\frac{2\pi\rho\gamma}{a^2}\, x +$ higher powers.

Hence the time of a small oscillation

$$= 2\pi \sqrt{\frac{a^2}{2\pi\rho\gamma}} = a\sqrt{\frac{2\pi}{\rho\gamma}}, \text{ by Art. 22.}$$

20. $\ddot{x} = -2\int_0^\pi \frac{\gamma\rho a\, d\theta}{a^2+x^2-2ax\cos\theta} \times \frac{a\cos\theta - x}{\sqrt{a^2+x^2-2ax\cos\theta}}$

$= \frac{-2\gamma\rho}{a^2}\int_0^\pi \left(1 - \frac{2x}{a}\cos\theta\right)^{-\frac{3}{2}} (a\cos\theta - x)\, d\theta$, neglecting sqs, etc. of x

$= \frac{-2\gamma\rho}{a^2}\int_0^\pi (a\cos\theta + 3x\cos^2\theta - x)\, d\theta$ „ „ „

$= -\frac{\gamma\rho}{a^2}\int_0^\pi x(1+3\cos 2\theta)\, d\theta = -\frac{\gamma\rho}{a^2}\cdot \pi \cdot x.$

$$\therefore \text{ time of a small oscillation} = 2\pi\sqrt{\frac{a^2}{\pi\rho\gamma}} = 2a\sqrt{\frac{\pi}{\rho\gamma}}.$$

21. $\ddot{x} = -\mu x \pm f = -\mu\left(x \mp \frac{f}{\mu}\right).$

Hence, as in Art. 22, the required time $= \frac{2\pi}{\sqrt{\mu}}.$

22. At time t, $OC = \frac{1}{2}ft^2$, so that $\ddot{x} = -\mu(x - \frac{1}{2}ft^2).$

$$\therefore\ x = A\cos\sqrt{\mu}t + B\sin\sqrt{\mu}t + \frac{f}{2}t^2 - \frac{f}{\mu},$$

where $c = A - \frac{f}{\mu}$, and $V = \left(\frac{dx}{dt}\right)_{t=0} = B\sqrt{\mu}$, etc.

23. $Mg = \lambda\frac{b}{a}$, and $M'g = \lambda\frac{c}{a}$; when both masses are hanging at rest, the length of the string $= a+b+c$.

The equation of motion, when M' has fallen off, is

$$M\ddot{x} = Mg - \lambda\frac{x-a}{a}, \text{ i.e. } \ddot{x} = -\frac{g}{b}(x-a-b).$$

$$\therefore\ x-a-b = A\cos\sqrt{\frac{g}{b}}\,t + B\sin\sqrt{\frac{g}{b}}\,t,$$

where $c = A$, and $0 = \left(\frac{dx}{dt}\right)_{t=0} = B\sqrt{\frac{g}{b}}$. Hence, etc.

24. If a is the amplitude of the original motion, the velocity at the origin is $a\sqrt{\mu}$ (Art. 22). The equation of motion becomes

$$\ddot{x} = -\mu x + \lambda x^3; \quad \therefore \dot{x}^2 = -\mu x^2 + \frac{\lambda}{2} x^4 + \mu a^2.$$

Hence $\dot{x} = 0$ when $x^2 - a^2 = \frac{\lambda}{2\mu} x^4$. The first approximation is $x = a$. Put $x = a + \xi$, where ξ and λ are both small. Substituting, we have $2a\xi = \frac{\lambda}{2\mu} a^4$, on neglecting squares, so that $x = a + \frac{\lambda}{4\mu} a^3$. Hence, etc.

25. Let O be the fixed point, $OACB$ the vertical through O, $OA = a$, $OB = b$ and $OC = a + \frac{a}{n}$.

The vertical acceleration when the particle is at any point P (where $OP = x$)

$$= \lambda \frac{x-a}{a} - g = \frac{ng}{a}\left[x - \frac{1+n}{n} a\right] = \frac{ng}{a} \cdot CP,$$

so that the motion is simple harmonic with C as centre and CB as amplitude, where $CB = b - \frac{1+n}{n} a = \frac{pa}{n}$. By Art. 22,

$$\text{time from } B \text{ to } A = \frac{1}{2} \cdot \frac{\pi}{\sqrt{\frac{ng}{a}}} + \frac{\frac{\pi}{2} - \cos^{-1}\frac{CA}{CB}}{\sqrt{\frac{ng}{a}}}$$

$$= \sqrt{\frac{a}{ng}}\left[\frac{\pi}{2} + \operatorname{cosec}^{-1} p\right]. \quad \ldots\ldots\ldots\ldots(1)$$

At A the string becomes slack, and time to highest point of the path

$$= \frac{\text{vel. at } A}{g} = \frac{\sqrt{\frac{ng}{a}(CB^2 - CA^2)}}{g} = \sqrt{\frac{a}{ng}(p^2 - 1)}. \quad \ldots\ldots(2)$$

This holds provided the velocity at A is not $> \sqrt{2g \cdot 2a}$ (for then the string would again become stretched before the highest point of the path was reached), *i.e.* if $\frac{ng}{a}\left(\frac{p^2a^2}{n^2} - \frac{a^2}{n^2}\right)$ not $> 4ga$, *i.e.* if p^2 not $> 1 + 4n$.

Thus the total time is twice the sum of (1) and (2).

If $p > \sqrt{1+4n}$ the velocity, when the string again becomes stretched above O,

$$= \sqrt{\frac{ng}{a}(p^2 - 1)\frac{a^2}{n^2} - 2g \cdot 2a} = \sqrt{\frac{ag}{n}(p^2 - 1 - 4n)},$$

and we have simple harmonic motion about a point, above O, as centre.

26. Consider the motion of an element of the string subtending a small angle 2θ at the centre. Let T be the tension, and r the radius, then. The equation of motion is

$$\frac{m \cdot 2\theta}{2\pi}\ddot{r} = \mu r \cdot \frac{m \cdot 2\theta}{2\pi} - 2T \cdot \sin\theta = \frac{m\mu r\theta}{\pi} - 2\theta \cdot \lambda\frac{r-c}{c},$$

i.e.
$$\ddot{r} = -\frac{2\pi\lambda - m\mu c}{mc}\left[r - \frac{2\pi\lambda c}{2\pi\lambda - m\mu c}\right].$$

Hence as stated.

If $2\pi\lambda = m\mu c$, the equation becomes $\ddot{r} = \mu c$, *i.e.* the string continually increases in radius until it finally breaks.

27. As in the last example

$$\frac{m \cdot 2\theta}{2\pi} \cdot \ddot{r} = \frac{m \cdot 2\theta}{2\pi} \cdot \frac{\mu}{r^2} - 2\theta \cdot \lambda\frac{r-a}{a}, \quad i.e.\ \ddot{r} = \frac{\mu}{r^2} - \frac{2\pi\lambda}{ma}(r-a).$$

$$\therefore \tfrac{1}{2}\dot{r}^2 = -\frac{\mu}{r} - \frac{\pi\lambda}{ma}(r-a)^2 + \frac{\mu}{a}.$$

The string is at rest again when

$$\mu\left(\frac{1}{a} - \frac{1}{r}\right) = \frac{\pi\lambda}{ma}(r-a)^2, \ i.e.\ \text{when } r^2 - ar = \frac{m\mu}{\pi\lambda}.$$

28. When the string is of length x, let the block have moved through a distance ξ. The equations of motion of the block and particle are

$$M\ddot{\xi} = E\frac{x-a}{a}, \text{ and } m(\ddot{\xi} + \ddot{x}) = -E\frac{x-a}{a}.$$

$$\therefore \ddot{x} = -p^2(x-a), \text{ where } p^2 = \frac{E(M+m)}{Mma}.$$

$$\therefore x - a = A\cos(pt+B),$$

where $na = A\cos B$, and $0 = -A\sin B$, so that $x - a = na\cos pt$.

Now the string is unstretched again when $t_1 = \dfrac{\pi}{2p}$, and then the velocity of the particle along the block $= [\dot{x}]_{t=t_1} = -nap$, so that the time t_2 to the fixed point $= \dfrac{a}{nap} = \dfrac{1}{np}$.

Hence the total time required $= 4(t_1 + t_2) = 2\left(\pi + \dfrac{2}{n}\right) \cdot \dfrac{1}{p} =$ etc.

Also the centre of gravity of the system is clearly fixed, since on the whole system there is no external force.

$$\therefore M\xi_1 + m[\xi_1 + (n+1)a] = M\xi_2 + m[\xi_2 - (n+1)a].$$

$$\therefore \text{twice the required amplitude} = \xi_2 - \xi_1 = \frac{2m(n+1)a}{M+m}.$$

29. When the string is of length x, the equation of motion is

$$m\ddot{x} = mg\sin\alpha - \mu mg\cos\alpha - \lambda\frac{x-a}{a}.$$

$$\therefore \dot{x}^2 = 2g(\sin\alpha - \mu\cos\alpha)x - \frac{\lambda}{ma}(x-a)^2 - 2g(\sin\alpha - \mu\cos\alpha)a.$$

Hence the velocity vanishes again when $2g(\sin\alpha-\mu\cos\alpha)=\frac{\lambda}{ma}(x-a)$, and then the acceleration up the plane

$$=\frac{\lambda}{ma}(x-a)-g(\sin\alpha+\mu\cos\alpha)=g(\sin\alpha-3\mu\cos\alpha).$$

This is positive, *i.e.* particle oscillates, if $\tan\alpha>3\mu$.

30. $m\ddot{x}=\frac{H.550g}{\dot{x}}$, so that $\dot{x}^2=\frac{1100Hg}{m}t+u^2$.

The acceleration is $\frac{1}{n}$th of its original value when the velocity is n times the original value, *i.e.* when

$$u^2+\frac{1100Hg}{m}t=n^2u^2, \text{ etc.}$$

31. $A=$ the section of the bore; $x=$ the distance of the shot from the end at time t; $a=$ the original value of x, so that $V=Aa$.

The pressure of gas behind the shot at time t, by Boyle's Law,

$$=\frac{V.m\Pi}{Ax}=\frac{a}{x}.m\Pi.$$

Then
$$M\ddot{x}=A\left[\frac{a}{x}.m\Pi-\Pi\right]=V\Pi\left(\frac{m}{x}-\frac{1}{a}\right).$$

$$\therefore\ \tfrac{1}{2}M\dot{x}^2=V\Pi\left[m\log x-\frac{x}{a}\right]-V\Pi\left[m\log a-1\right]$$

$$=V\Pi\left[m\log\frac{x}{a}-\frac{x}{a}+1\right].$$

Now the acceleration is zero, and hence the velocity greatest, when $x=am$, and then

$$\dot{x}^2=\frac{2V\Pi}{M}\left[m\log m-m+1\right].$$

32. $m_2\ddot{x}=-\lambda\frac{x-a}{a}$, so that $\frac{1}{n}=2\pi\sqrt{\frac{m_2a}{\lambda}}$, and $\therefore\ \lambda=4m_2\pi^2n^2a$.

In the second case $\frac{1}{n_1}=2\pi\sqrt{\frac{m_1a}{\lambda}}$, and hence $\frac{n_1}{n}=\sqrt{\frac{m_2}{m_1}}$.

When both masses are free, let y and $y+z$ be the distances of m_1 and m_2 from a fixed origin. Then

$$m_1\ddot{y}=T,\text{ and } m_2(\ddot{y}+\ddot{z})=-T.$$

$$\therefore\ \ddot{z}=-T\frac{m_1+m_2}{m_1m_2}=-\frac{m_1+m_2}{m_1m_2}.\lambda\frac{z-a}{a}.$$

$$\therefore\ \frac{1}{n_2}=2\pi\sqrt{\frac{m_1m_2}{m_1+m_2}.\frac{a}{\lambda}}=\frac{1}{n}\sqrt{\frac{m_2}{m_1+m_2}}.$$

33. The acceleration of the end of the string is μx, where $\frac{2\pi}{\sqrt{\mu}}=\frac{1}{n}$. Hence, at the highest point of the motion, the acceleration downwards $=\mu a=4\pi^2n^2a$. Hence the string will become slack if $4\pi^2n^2a>g$.

34. Let P be the force, ξ the length to which it compresses the spring, and a the unstretched length, so that $P=\lambda\frac{a-\xi}{a}$. When P is reversed, let x be the greatest length of the spring during the subsequent motion, so that the total work done during the motion is zero.

$$\therefore\ P(x-\xi)+\int_{\xi}^{a}\frac{\lambda(a-\xi)(-d\xi)}{a}-\int_{a}^{x}\frac{\lambda(x-a)}{a}dx=0,$$

$$\therefore\ (a-\xi)(x-\xi)+\tfrac{1}{2}(a-\xi)^2-\tfrac{1}{2}(x-a)^2=0, \text{ so that } x-a=3(a-\xi), \text{ etc.}$$

35. If l_1 be the unstretched length, then

$$Mg=\lambda\frac{l}{l_1}. \quad\text{..............................(1)}$$

Let u be the velocity when M strikes the table; the spring becomes compressed until the velocity of m is destroyed; it then recovers and when it is again unstretched the velocity of m is u upwards. Let m rise until the length of the spring is x. Then

$$\tfrac{1}{2}mu^2=\text{work done}=\tfrac{1}{2}(x-l_1).\lambda\frac{x-l_1}{l_1}+mg(x-l_1). \quad\text{.........(2)}$$

The mass M will be then *just* on the point of rising, if

$$Mg=\lambda\frac{x-l_1}{l_1}. \quad\text{..............................(3)}$$

Hence (2) gives $\quad \tfrac{1}{2}m.2gh=\frac{l_1}{2\lambda}M^2g^2+\frac{l_1}{\lambda}.Mmg^2.$

$$\therefore\ h=\frac{M+2m}{2m}.\frac{Mgl_1}{\lambda}=\frac{M+2m}{2m}l, \text{ by (1).}$$

If h is greater than this, then, before the velocity u is destroyed, the tension of the spring is $>Mg$, and M rises.

36. $\ddot{x}=-\gamma\frac{m_1+m_2}{x^2}$. Hence, by Art. 31,

$$\sqrt{\frac{2\gamma(m_1+m_2)}{a}}\,t=\sqrt{ax-x^2}+a\cos^{-1}\sqrt{\frac{x}{a}}.$$

Hence, when $x=a_1+a_2$,

$$t=\sqrt{\frac{a}{2\gamma(m_1+m_2)}}\left[\sqrt{(a_1+a_2)(a-a_1-a_2)}+a\cos^{-1}\sqrt{\frac{a_1+a_2}{a}}\right].$$

Also $$g=\gamma\frac{\frac{4}{3}\pi R^3D}{R^2},\ i.e.\ \gamma=\frac{3g}{4\pi RD}.$$

$$\therefore\ t=\sqrt{\frac{2\pi aDR}{3g(m_1+m_2)}}\left[a\cos^{-1}\sqrt{\frac{a_1+a_2}{a}}+\sqrt{(a_1+a_2)(a-a_1-a_2)}\right].$$

If $m_1=m_2=4$; $a_1=a_2=\frac{1}{8}$; and $a=1$, then

$$t=\sqrt{\frac{2\pi.350.4000.1760.3}{3g.8}}\left[\cos^{-1}\frac{1}{2}+\sqrt{\frac{1}{4}.\frac{3}{4}}\right]\text{ secs.}$$

$$=500\sqrt{77\pi}\left[\frac{\pi}{3}+\frac{3}{4}\right]=5500\sqrt{2}\left[\frac{22}{21}+\frac{\sqrt{3}}{4}\right], \text{ taking } \pi \text{ as } \frac{22}{7},$$

$$=\frac{55}{36}\times 1{\cdot}4142\times 1{\cdot}481 \text{ hours}=3{\cdot}2 \text{ hours approx.}$$

37. Here $\frac{\pi DR}{m_1+m_2} = \frac{\pi DR}{\frac{82}{81} \times \frac{4}{3}\pi DR^3} = \frac{3}{328} \times \frac{81}{R^2}$.

$$\therefore\ t = \frac{9}{2R}\sqrt{\frac{a}{41g}}\left[a\cos^{-1}\sqrt{\frac{51}{2400}} + \sqrt{(a_1+a_2)(a-a_1-a_2)}\right]$$

$$= \frac{9}{8000}\sqrt{\frac{240000 \times 5280}{41g}}\left[240000\cos^{-1}\frac{\sqrt{34}}{40} + \sqrt{5100 \times 234900}\right]$$

$$= \frac{3 \times 9000}{4}\sqrt{\frac{110}{41}}\left[24\cos^{-1}(\cdot 1458) + \sqrt{11 \cdot 98}\right] \text{ nearly}$$

$$= \frac{3 \times 9000}{4} \times 1\cdot 638 \times [34\cdot 18 + 3\cdot 46],$$

since $\cos^{-1}(\cdot 1458) = 81°\, 37' = 1\cdot 424$ rad.,

$= 11000 \times 37\cdot 64 = 414000$ secs. approx. $=$ 4 days 19 hrs. approx.

38. $$\ddot{x} = \int_0^\infty \frac{2\pi a\mu}{a^2+(y-x)^2} \cdot \frac{y-x}{\sqrt{a^2+(y-x)^2}}\,dy = \left[\frac{-2\pi a\mu}{\sqrt{a^2+(y-x)^2}}\right]_0^\infty = \frac{2\pi a\mu}{\sqrt{a^2+x^2}}.$$

$$\therefore\ \dot{x}^2 = 4\pi a\mu\,\{\log(x+\sqrt{a^2+x^2}) - \log a\}, \text{ etc.}$$

39. $OB = \xi$; $BP = x$, where P is any point of the string; then

$$M\frac{dx}{2a}.\ddot{\xi} = (T+dT) - T + \frac{M\,dx}{2a}.\mu.(\xi+x). \quad \text{............(1)}$$

Integrating this from 0 to $2a$, we have

$$M\ddot{\xi} = \Big[T\Big]_{x=0}^{x=2a} + M\mu(\xi+a),$$

i.e. since T is zero at each end of the string,

$\ddot{\xi} = \mu(\xi+a) =$ acceleration of a particle placed at the middle point.

Also (1) gives $\frac{dT}{dx} = \frac{M}{2a}[\mu(a-x)]$.

$$\therefore\ T = \frac{M\mu}{4a}[a^2-(a-x)^2] = \frac{M\mu}{4a}.BP.PA.$$

40. If ψ be the inclination to the horizontal at a point whose coordinates are x and y, we have

$$\frac{y}{\sin\psi} = \tfrac{1}{2}g\sin\psi.\lambda^2. \quad \therefore\ \frac{g\lambda^2}{2y} = 1 + \left(\frac{dx}{dy}\right)^2.$$

Putting $y = \frac{1}{2}g\lambda^2\sin^2\theta$, we obtain $x = \frac{g\lambda^2}{4}(2\theta + \sin 2\theta)$, *i.e.* the curve is a cycloid with axis vertical.

41. At time t, let m' have moved through x, and let ξ be the length of the string then. Hence

$$m'\ddot{x} = T, \text{ and } m(\ddot{x}+\ddot{\xi}) = -T. \quad \text{....................(1)}$$

$$\therefore\ \ddot{\xi} = -T\left(\frac{1}{m}+\frac{1}{m'}\right) = -\frac{m+m'}{mm'}\lambda.\frac{\xi-a}{a} = -\frac{\xi-a}{p^2}.$$

$$\therefore\ \xi - a = A\cos\left(\frac{t}{p}+B\right) = pV\sin\frac{t}{p},$$

since $\xi = a$ and $\dot{\xi} = V$, when $t = 0$.

Hence the greatest value of ξ is $a + pV$, and $\xi = a$ again when $t = \pi p$.

Also (1) gives $(m+m')\ddot{x} + m\ddot{\xi} = 0$.

$$\therefore\ (m+m')\dot{x} + m\dot{\xi} = \text{const.} = mV.$$

$$\therefore\ (m+m')x + m\xi = mVt + ma,$$

giving the motion of m'.

42. Let x be the depth of A, and y the length of the elastic string, at time t. Then

$$m\ddot{x} = mg + \lambda\frac{y-a}{a} - T, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$2m(\ddot{x}+\ddot{y}) = 2mg - \lambda\frac{y-a}{a}, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and $$m\frac{d^2}{dt^2}(l-x) = mg - T. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Since $\lambda = 2mg$, these give

$$\ddot{x} = \frac{g}{a}(y-a), \text{ and } \ddot{y} = -\frac{2g}{a}\left(y - \frac{3a}{2}\right).$$

$$\therefore\ y - \frac{3a}{2} = A\cos\left[\sqrt{\frac{2g}{a}}\,t + B\right] = -\frac{a}{2}\cos\left[\sqrt{\frac{2g}{a}}\,t\right],$$

since $y = a$ and $\dot{y} = 0$, when $t = 0$.

$$\therefore\ y = \frac{3a}{2} - \frac{a}{2}\cos\left[\sqrt{\frac{2g}{a}}\,t\right] \text{ and } \ddot{x} = g\sin^2\left[\sqrt{\frac{g}{2a}}\,t\right].$$

43. Let x and y be the lengths of the string hanging vertically, and on the table, at time t. Then

$$m\ddot{x} = mg - T = mg - \lambda\frac{x+y-l}{l}, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and $$m\ddot{y} = -T = -\lambda\frac{x+y-l}{l}. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Hence $\ddot{x} + \ddot{y} = g - \frac{2\lambda}{ml}(x+y-l)$.

$$\therefore\ x + y - l = \frac{lmg}{2\lambda}\left[1 - \cos\sqrt{\frac{2\lambda}{ml}}\,t\right],$$

since initially the string is at rest and unstretched.

(2) gives $$\ddot{y} = -\frac{g}{2} + \frac{g}{2}\cos\sqrt{\frac{2\lambda}{ml}}\,t.$$

$$\therefore\ y = -\frac{g}{4}t^2 - \frac{mlg}{4\lambda}\cos\sqrt{\frac{2\lambda}{ml}}\,t + \left(l + \frac{mlg}{4\lambda}\right),$$

since $y = l$ and $\dot{y} = 0$, initially.

Hence $y = 0$, when $\frac{gt^2}{2} = 2l + \frac{mgl}{\lambda}\sin^2\left[\sqrt{\frac{\lambda}{2ml}}\,t\right]$.

Chapter 2

MOTION IN A STRAIGHT LINE

20. Let the distance of a moving point P from a fixed point be O at any time t. Let its distance similarly at time $t+\triangle t$ be $x+\triangle x$, so that $PQ=\triangle x$.

The velocity of P at time t = Limit, when $\triangle t=0$, of $\dfrac{PQ}{\triangle t}$ = Limit, when $\triangle t=0$, of $\dfrac{\triangle x}{\triangle t}=\dfrac{dx}{dt}$.

Hence the velocity $\quad v=\dfrac{dx}{dt}$.

Let the velocity of the moving point at time $t+\triangle t$ be $v+\triangle v$.

Then the acceleration of P at time t = Limit, when $\triangle t=0$, of $\dfrac{\triangle v}{\triangle t}=\dfrac{dv}{dt}=\dfrac{d^2x}{dt^2}$

21. Motion in a straight line with constant acceleration f

Let x be the distance of the moving point at time t from a fixed point in the straight line.

$$\text{Then} \quad \frac{d^2x}{dt^2}=f \qquad ...(1),$$

$$\text{Hence, on integration} \quad v=\frac{dx}{dt}=ft+A \qquad ...(2),$$

where A is an arbitrary constant.

Integrating again, we have

$$x = \frac{1}{2}ft^2 + At + B \qquad ...(3),$$

where B is an arbitrary constant.

Again, on multiplying (1) by $2\frac{dx}{dt}$, and integrating with respect to t, we have

$$v^2 = \left(\frac{dx}{dt}\right)^2 = 2fx + C \qquad ...(4),$$

where C is an arbitrary constant.

These three equations contain the solution of all questions on motion in a straight line with constant acceleration. The arbitrary constants A, B, C are determined from the initial conditions.

Suppose for example that the particle started at a distance a from a fixed point O on the straight line with velocity u in a direction away from O, and suppose that the time t is reckoned from the instant of projection.

We then have that when $t = 0$, then $v = u$ and $x = a$. Hence the equations (2), (3), and (4) give

$$u = A, \ a = B, \text{ and } u^2 = C + 2fa.$$

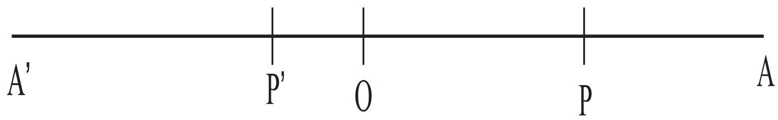
Hence we have $v = u + ft$, $x - a = ut + \frac{1}{2}ft^2$ and $v^2 = u^2 + 2f(x-a)$, the three standard equations of Elementary Dynamics.

22. *A particle moves in a straight line OA starting from rest at A and moving with an acceleration which is always directed towards O and varies as the distance from O; to find the motion.*

Let x be the distance OP of the particle from at any time t; and let the acceleration at this distance be μx.
The equation of motion is then

$$\frac{d^2x}{dt^2} = -\mu x \qquad ...(1).$$

[We have a negative sign on the right-hand side because $\frac{d^2x}{dt^2}$ is the acceleration in the direction of x increasing, *i.e.* in the direction OP; whilst μx is the acceleration towards O, *i.e.* in the direction PO.]



Multiplying by $2\frac{dx}{dt}$ and integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = -\mu x^2 + C.$$

If OA be a, then $\frac{dx}{dt} = 0$ when $x = a$, so that $0 = -\mu a^2 + C$, and

$$\therefore \left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2)$$

$$\therefore \qquad \frac{dx}{dt} = -\sqrt{\mu}\sqrt{a^2 - x^2} \qquad ...(2).$$

[The negative sign is put on the right-hand side because the velocity is clearly negative so long as OP is positive and P is moving towards O.]

Hence, on integration,

$$t\sqrt{\mu} = -\int \frac{dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\frac{x}{a} + C_1,$$

$$\text{where } 0 = \cos^{-1}\frac{a}{a} + C_1, \ i.e.\ C_1 = 0,$$

if the time be measured from the instant when the particle was at A.

$$\therefore \qquad x = \cos\sqrt{\mu}t \qquad \text{...(3).}$$

When the particle arrives at O, x is zero; and then, by (2), the velocity = $-a\sqrt{\mu}$. The particle thus passes through O and immediately the acceleration alters its direction and tends to diminish the velocity; also the velocity is destroyed on the left-hand side of O as rapidly as it was produced on the right-hand side; hence the particle comes to rest at a point A' such that OA and OA' are equal. It then retraces its path, passes through O, and again is instantaneously at rest at A. The whole motion of the particle is thus an oscillation from A to A' and back, continually repeated over and over again.

The time from A to O is obtained by putting x equal to zero in (3). This gives $\cos(\sqrt{\mu}t) = 0$, *i.e.* $t = \dfrac{\pi}{2\sqrt{\mu}}$.

The time from A to A' and back again, *i.e.* the time of a complete oscillation, is four times this, and therefore $= \dfrac{2\pi}{\sqrt{\mu}}$.

This result is independent of the distance a, *i.e. is independent of the distance from the centre at which the particle started*. It depends solely on the quantity μ which is equal to the acceleration at unit distance from the centre.

23. Motion of the kind investigated in the previous article is called **Simple Harmonic Motion**.

The time, $\dfrac{2\pi}{\sqrt{\mu}}$, for a complete oscillation is called the **Periodic Time** of the motion, and the distance, OA or OA', to which the particle vibrates on either side of the centre of the motion is called the **Amplitude** of its motion.

The **Frequency** is the number of complete oscillations that the particle makes in a second, and hence $= \dfrac{1}{\text{Periodic time}} = \dfrac{\sqrt{\mu}}{2\pi}$.

24. The equation of motion when the particle is on the left-hand side of O is

$$\frac{d^2x}{dt^2} = \text{ acceleration in the direction } P'A = \mu . P'O = \mu(-x) = -\mu x.$$

Hence the same equation that holds on the right hand of O holds on the left hand also.

As in Art. 22 it is easily seen that the most general solution of this equation is

$$x = a\ \cos[\sqrt{\mu}t + \varepsilon] \qquad ...(1),$$

which contains two arbitrary constants a and ε.

This gives $$\frac{dx}{dt} = -a\sqrt{\mu}\ \sin\left(\sqrt{\mu}t + \varepsilon\right) \qquad ...(2).$$

(1) and (2) both repeat when t is increased by $\dfrac{2\pi}{\sqrt{\mu}}$, since the sine and cosine of an angle always have the same value when the angle is increased by 2π.

Using the standard expression (1) for the displacement in a simple harmonic motion, the quantity ε is called the **Epoch**, the angle $\sqrt{\mu}t + \varepsilon$ is called the **Argument**, whilst the **Phase** of the motion is the time that has elapsed since the particle was at its maximum distance in the positive direction. Clearly x is a maximum at time t_0 where $\sqrt{\mu}t_0 + \varepsilon = 0$.

Hence the phase at time $t = t - t_0 = t + \dfrac{\varepsilon}{\sqrt{\mu}} = \dfrac{\sqrt{\mu}t + \varepsilon}{\sqrt{\mu}}$.

Motion of the kind considered in this article, in which the time of falling to a given point is the same whatever be the distance through which the particle falls, is called **Tautochronous**.

25. In Art. 22 if the particle, instead of being at rest initially, be projected from A with velocity V in the positive direction, we have

$$V^2 = -\mu a^2 + C.$$

Hence $\left(\frac{dx}{dt}\right)^2 = V^2 + \mu(a^2 - x^2) = \mu(b^2 - x^2)$, where $b^2 = a^2 + \frac{V^2}{\mu}$ ...(1),

$$\therefore \quad \frac{dx}{dt} = \sqrt{\mu}\sqrt{b^2 - x^2} \text{ and } t\sqrt{\mu} = -\cos^{-1}\frac{x}{b} + C_1,$$

$$\text{where } 0 = -\cos^{-1}\frac{a}{b} + C_1.$$

$$\therefore \quad t\sqrt{\mu} = -\cos^{-1}\frac{a}{b} - \cos^{-1}\frac{x}{b} \qquad \text{...(2)}.$$

From (1), the velocity vanishes when $x = b = \sqrt{a^2 + \frac{V^2}{\mu}}$,

and then, from (2),

$$t\sqrt{\mu} = \cos^{-1}\frac{a}{b}, \text{ } i.e., \text{ } t = \frac{1}{\sqrt{\mu}}\cos^{-1}\frac{a}{\sqrt{a^2 + \frac{V^2}{\mu}}}.$$

The particle then retraces its path, and the motion is the same as in Art. 22 with b substituted for a.

26. Compounding of two simple harmonic motions of the same period and in the same straight line

The most general displacements of this kind are given by $a\cos(nt + \varepsilon)$ and $b\cos(nt + \varepsilon')$, so that

$$x = a\cos(nt+\varepsilon) + b\cos(nt+\varepsilon')$$
$$= \cos\ nt(a\cos\varepsilon + b\cos\varepsilon') - \sin nt(a\sin\varepsilon + b\sin\varepsilon')$$

Let $\quad a\cos\varepsilon + b\cos\varepsilon' = A\cos E \quad$ and $\quad a\sin\varepsilon + b\sin\varepsilon' = A\sin\ E \quad$ ...(1),

so that

$$A = \sqrt{a^2+b^2+2ab\cos(\varepsilon-\varepsilon')} \text{ and } \tan E = \frac{a\sin\varepsilon + b\sin\varepsilon'}{a\cos\varepsilon + b\cos\varepsilon'}.$$

Then $x = a\cos(nt+E)$,

so that the composition of the two given motions gives a similar motion of the same period whose amplitude and epoch are known.

If we draw OA $(= a)$ at an angle ε to a fixed line, and OB $(= b)$ at an angle ε' and complete the parallelogram OACB then by equations (1) we see that OC represents A and that it is inclined at an angle E to the fixed line. The line representing the resultant of the two given motions is therefore the geometrical resultant of the lines representing the two component motions.

So with more than two such motions of the same period.

27. We cannot compound two simple harmonic motions of different periods.

The case when the periods are nearly but not quite equal, is of some considerable importance.

In this case we have

$x = a\cos(nt+\varepsilon) + b\cos(n't+\varepsilon')$, where $n'-n$ is small, = λ say.

Then $\quad x = a\cos(nt+\varepsilon) + b\cos[nt+\varepsilon_1]$, $\quad$ where $\varepsilon_1' = \lambda t + \varepsilon'$.

By the last article

$$x = A\cos(nt + E) \qquad ...(1),$$

$$\text{where} \quad A^2 = a^2 + b^2 + 2ab\cos(\varepsilon - \varepsilon_1') = a^2 + b^2 + 2ab\cos(\varepsilon - \varepsilon' - (n' - n)t] \qquad ...(2),$$

$$\text{and} \quad \tan E = \frac{a\sin\varepsilon + b\sin\varepsilon_1'}{a\cos\varepsilon + b\cos\varepsilon_1'} = \frac{a\sin\varepsilon + b\sin[\varepsilon' + (n' - n)t]}{a\cos\varepsilon + b\cos[\varepsilon' + (n' - n)t]} \qquad ...(3).$$

The quantities A and E are now not constant, but they vary *slowly* with the time, since $n' - n$ is very small.

The greatest value of A is when $\varepsilon - \varepsilon' - (n' - n)t =$ any even multiple of π and then its value is $a + b$.

The least value of A is when $\varepsilon - \varepsilon' - (n' - n)t =$ any odd multiple of π and then its value is $a - b$.

At any given time therefore the motion may be taken to be a simple harmonic motion of the same approximate period as either of the given component motions, but with its amplitude A and epoch E gradually changing from definite minimum to definite maximum values, the periodic times of these changes being, $\dfrac{2\pi}{n' - n}$.

[The Student who is acquainted with the theory of Sound may compare the phenomenon of Beats.]

28. Ex. 1. Show that the resultant of two simple harmonic vibrations in the same direction and of equal periodic time, the amplitude of one being twice that of the other and its phase a quarter of a period in advance, is a simple harmonic vibration of amplitude $\sqrt{5}$ times that of the first and whose phase is in advance of the first by $\dfrac{\tan^{-1} 2}{2\pi}$ of a period.

Ex. 2. A particle is oscillating in a straight line about a centre of force O, towards which when at a distance r the force is $m.n^2r$, and a is the amplitude of the oscillation; when at a distance $\frac{a\sqrt{3}}{2}$ from O, the particle receives a blow in the direction of motion which generates a velocity na. If this velocity be away from O, show that the new amplitude is $a\sqrt{3}$.

EX. 3. A particle P, of mass m, moves in a straight line Ox under a force $m\mu(distance)$ directed towards a point A which moves in the straight line Ox with constant acceleration a. Show that the motion of P is simple harmonic, of period $\frac{2\pi}{\sqrt{\mu}}$, about a moving centre which is always at a distance $\frac{a}{\mu}$ behind A.

EX. 4. An elastic string without weight, of which the unstretched length is l and the modulus of elasticity is the weight of n *ozs.*, is suspended by one end, and a mass of m *ozs.* is attached to the other; show that the time of a vertical oscillation is $2\pi\sqrt{\frac{ml}{ng}}$.

EX. 5. One end of an elastic string, whose modulus of elasticity is λ and whose unstretched length is a, is fixed to a point on a smooth horizontal table and the other end is tied to a particle of mass m which is lying on the table. The particle is pulled to a distance where the extension of the string is b and then let go; show that the time of a complete oscillation is $2\left(\pi+\frac{2a}{b}\right)\sqrt{\frac{am}{\lambda}}$.

EX. 6. An endless cord consists of two portions, of lengths $2l$ and $2l'$ respectively, knotted together, their masses per unit of length being m and m'. It is placed in stable equilibrium over a small smooth peg and then slightly displaced. Show that the time of a complete oscillation is $2\pi\sqrt{\frac{ml+m'l'}{(m-m')g}}$.

EX. 7. Assuming that the earth attracts points inside it with a force which varies as the distance from its centre, show that, if a straight frictionless airless tunnel be made from one point of the earth's surface to any other point, a train would traverse the tunnel in slightly less than three-quarters of an hour. (Assume the earth to be a homogeneous sphere of radius 6400 km.)

29. *Motion when the motion is in a straight line and the acceleration is proportional to the distance from a fixed point O in the straight line and is always away from O.*

$$\text{Here the equation of motion is} \quad \frac{d^2x}{dt^2} = \mu x \qquad ...(1).$$

Suppose the velocity of the particle to be zero at a distance a from O at time zero.

The integral of (1) is

$$\left(\frac{dx}{dt}\right)^2 = \mu x^2 + A, \quad \text{where } 0 = \mu a^2 + A$$

$$\therefore \quad \frac{dx}{dt} = \sqrt{\mu(x^2 - a^2)} \qquad ...(2),$$

the positive sign being taken in the right-hand member since the velocity is positive in this case.

$$\therefore \; t\sqrt{\mu} = \int \frac{dx}{\sqrt{x^2 - a^2}} = \log[x + \sqrt{x^2 - a^2}] + B,$$

$$\text{where } \; 0 = \log[a] + B$$

$$\therefore \; t\sqrt{\mu} = \log \frac{x + \sqrt{x^2 - a^2}}{a}. \qquad \therefore \; x + \sqrt{x^2 - a^2} = ae^{t\sqrt{\mu}}.$$

$$\therefore \; x - \sqrt{x^2 - a^2} = \frac{a^2}{x + \sqrt{x^2 - a^2}} = ae^{-t\sqrt{\mu}}$$

Hence, by addition,

$$x = \frac{a}{2}\, e^{t\sqrt{\mu}} + \frac{a}{2} e^{-t\sqrt{\mu}} \qquad ...(3).$$

As t increases, it follows from (3) that x continually increases, and then from (2) that the velocity continually increases also.

Hence the particle would continually move along the positive direction of the axis of x and with continually increasing velocity.

Equation (3) may be written in the form

$$x = a\cosh(\sqrt{\mu}t),$$

and then (2) gives

$$v = a\sqrt{\mu}\sinh(\sqrt{\mu}t).$$

30. In the previous article suppose that the particle were initially projected *towards* the origin O with velocity V; then we should have $\frac{dx}{dt}$ equal to $-V$ when $x = a$; and equation (2) would be more complicated. We may however take the most general solution of (1) in the form

$$x = Ce^{\sqrt{\mu}t} + De^{-\sqrt{\mu}t} \qquad ...(4),$$

where C and D are any constants.

Since, when $t = 0$, we have $x = a$ and $\frac{dx}{dt} = -V$, this gives

$$a = C + D, \text{ and } -V = \sqrt{\mu}C - \sqrt{\mu}D.$$

$$\text{Hence } C = \frac{1}{2}\left(a - \frac{V}{\sqrt{\mu}}\right) \text{ and } D = \frac{1}{2}\left(a + \frac{V}{\sqrt{\mu}}\right).$$

$$\therefore \text{ (4) gives} \quad x = \frac{1}{2}\left(a - \frac{V}{\sqrt{\mu}}\right)e^{\sqrt{\mu}t} + \frac{1}{2}\left(a + \frac{V}{\sqrt{\mu}}\right)e^{-\sqrt{\mu}t} \qquad ...(5)$$

$$= a\cosh(\sqrt{\mu}t) - \frac{V}{\sqrt{\mu}}\sinh(\sqrt{\mu}t) \qquad ...(6).$$

In this case the particle will arrive at the origin O when

$$0 = \frac{1}{2}\left(a - \frac{V}{\sqrt{\mu}}\right)e^{\sqrt{\mu}t} + \frac{1}{2}\left(a + \frac{V}{\sqrt{\mu}}\right)e^{-\sqrt{\mu}t},$$

i.e. when $\quad e^{2\sqrt{\mu}t} = \dfrac{V + a\sqrt{\mu}}{V - a\sqrt{\mu}}$

i.e. when $\quad t = \dfrac{1}{2\sqrt{\mu}}\log\dfrac{V + a\sqrt{\mu}}{V - a\sqrt{\mu}}.$

In the particular case when $V = a\sqrt{\mu}$, this value of t is infinity.

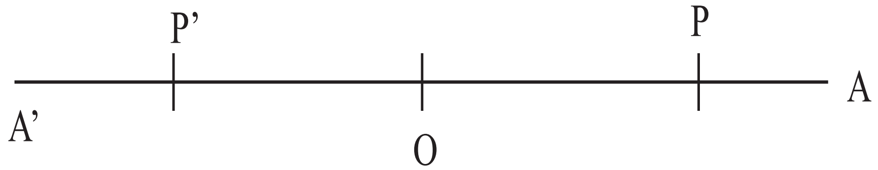
If therefore the particle were projected at distance a towards the origin with the velocity $a\sqrt{\mu}$, it would not arrive at the origin until after an infinite time.

Also, putting $V = a\sqrt{\mu}$ in (5), we have $\quad x = ae^{-\sqrt{\mu}t}$,

and $v = \dfrac{dx}{dt} = -a\sqrt{\mu}e^{-\sqrt{\mu}t}$.

The particle would therefore always be travelling towards O with a continually decreasing velocity, but would take an infinite time to get there.

31. *A particle moves in a straight line OA with an acceleration which is always directed towards O and varies inversely as the square of its distance from O; if initially the particle were at rest at A, find the motion.*



Let OP be x, and let the acceleration of the particle when at P be $\dfrac{\mu}{x^2}$ in the direction PO. The equation of motion is therefore

$$\frac{d^2x}{dt^2} = \text{acceleration along } OP = -\frac{\mu}{x^2} \qquad \ldots(1).$$

Multiplying both sides by $2\dfrac{d^2x}{dt^2}$ and integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} + C,$$

where $0 = \dfrac{2\mu}{a} + C$, from the initial conditions.

$$\text{Subtracting,} \quad \left(\frac{dx}{dt}\right)^2 = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right).$$

$$\therefore \quad \frac{dx}{dt} = -\sqrt{2\mu}\sqrt{\frac{a-x}{ax}} \qquad ...(2),$$

the negative sign being prefixed because the motion of P is towards O, *i.e.* in the direction of x decreasing.

$$\text{Hence} \quad \sqrt{\frac{2\mu}{a}}.t = -\int \sqrt{\frac{x}{a-x}}dx.$$

To integrate the right-hand side, put $x = a\cos^2\theta$, and we have

$$\sqrt{\frac{2\mu}{a}}.t = \int \frac{\cos\theta}{\sin\theta}.2a\cos\theta\sin\theta\, d\theta = a\int(1+\cos 2\theta)d\theta$$

$$= a\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C_1 = a\cos^{-1}\sqrt{\frac{x}{a}} + \sqrt{ax - x^2} + C_1,$$

where $0 = a\cos^{-1}(1) + 0 + C_1,\ i.e.\ C_1 = 0$

$$\therefore \quad t = \sqrt{\frac{a}{2\mu}}\left[\sqrt{ax-x^2} + a\cos^{-1}\sqrt{\frac{x}{a}}\right] \qquad ...(3).$$

Equation (2) gives the velocity at any point P of the path, and (3) gives the time from the commencement of the motion.

The velocity on arriving at the origin O is found, by putting $x = 0$ in (2), to be infinite.

Also the corresponding time, from (3),

$$= \sqrt{\frac{a}{2\mu}}[a\cos^{-1} 0] = \frac{\pi}{2}\,\frac{a^{3/2}}{\sqrt{2\mu}}.$$

The equation of motion (1) will not hold after the particle has passed through O; but it is clear that then the acceleration, being opposite to the direction of the velocity, will destroy the velocity, and the latter will be diminished at the same rate as it was produced on the positive side of O. The particle will therefore, by symmetry, come to rest at a point A' such that AO and OA' are equal. It will then return, pass again through O and come to rest at A.

The total time of the oscillation = four times the time from A to O $= 2\pi \dfrac{a^{3/2}}{\sqrt{2\mu}}$.

32. By the consideration of Dimensions only we can show that the time $\propto \dfrac{a^{3/2}}{\sqrt{\mu}}$. For the only quantities that can appear in the answer are a and μ. Let then the time be $a^p\ \mu^q$.

Since $\dfrac{\mu}{(\text{ distance })^2}$ is an acceleration, whose dimensions are $[L][T]^{-2}$, the dimensions of μ are $[L]^3[T]^{-2}$; hence the dimensions of $a^p\mu^q$ are $[L]^{p+3q}\ [T]^{-2q}$. Since this is a time, we have $p+3q=0$ and $-2q=1$.

$$\therefore\ \ q = -\frac{1}{2} \text{ and } p = \frac{3}{2}. \qquad \text{Hence the required time} \propto \frac{a^{3/2}}{\sqrt{\mu}}.$$

33. As an illustration of Art. 31 let us consider the motion of a particle let fall towards the earth (assumed at rest) from a point outside it. It is shown in treatises on Attractions that the attraction on a particle outside the earth (assumed to be a homogeneous sphere), varies inversely as the square of its distance from the centre. The acceleration of a particle outside the earth at distance x may therefore be taken to be $\frac{\mu}{x^2}$.

If a be the radius of the earth this quantity at the earth's surface is equal to g, and hence $\frac{\mu}{a^2} = g$. *i.e.* $\mu = ga^2$.

For a point P outside the earth the equation of motion is therefore

$$\frac{d^2x}{dt^2} = -\frac{ga^2}{x^2} \qquad ...(1),$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{2ga^2}{x} + C.$$

If the particle started from rest at a distance b from the centre of the earth, this gives

$$\left(\frac{dx}{dt}\right)^2 = 2ga^2\left(\frac{1}{x} - \frac{1}{b}\right) \qquad ...(2),$$

and hence the square of the velocity on reaching the surface of the earth

$$= 2ga\left(1 - \frac{a}{b}\right) \qquad ...(3).$$

Let us now assume that there is a hole going down to the earth's centre just sufficient to admit of the passing of the particle.

On a particle inside the earth the attraction can be shown to vary directly as the distance from the centre, so that the acceleration at distance x from its centre is $\mu_1 x$,, where $\mu_1 a$ = its value at the earth's surface = g.

The equation of motion of the particle when inside the earth therefore is $\frac{d^2x}{dt^2} = -\frac{g}{a}x$,

and therefore $\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + C_1$.

Now when $x = a$, the square of the velocity is given by (3), since there was no instantaneous change of velocity at the earth's surface.

$$\therefore \qquad 2ga\left(1-\frac{a}{b}\right) = -\frac{g}{a}.a^2 + C_1,$$

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + ga\left[3-\frac{2a}{b}\right].$$

On reaching the centre of the earth the square of the velocity is therefore $ga\left(3-\dfrac{2a}{b}\right)$.

34. EX. 1. A particle falls towards the earth from infinity; show that its velocity on reaching the earth is the same as it would have acquired in falling with constant acceleration g through a distance equal to the earth's radius.

EX. 2. Show that the velocity with which a body falling from infinity reaches the surface of the earth (assumed to be a homogeneous sphere of radius 6400 km) is about 11.25 km per second.

In the case of the sun show that it is about 610 km per second, the radius of the sun being 708,000 km and the distance of the earth from it 149,000,000 km.

EX. 3. If the earth's attraction vary inversely as the square of the distance from its centre, and g be its magnitude at the surface, the time of falling from a height h above the surface to the surface is

$$\sqrt{\frac{a+h}{2g}}\left[\frac{a+h}{a}\sin^{-1}\sqrt{\frac{h}{a+h}}+\sqrt{\frac{h}{a}}\right],$$

where a is the radius of the earth and the resistance of the air is neglected.

If h be small compared with a, show that this result is approximately $\sqrt{\frac{2h}{g}}\left[1+\frac{5}{6}\frac{h}{a}\right]$.

35. It is clear that equations (2) and (3) of Art. 31 cannot be true after the particle has passed O; for on giving x negative values these equations give impossible values for v and t.

When the particle is at P', to the left of O, the acceleration is $\frac{\mu}{OP'^2}$, *i.e.* $\frac{\mu}{x^2}$, towards the right. Now $\frac{d^2x}{dt^2}$ means the acceleration towards the positive direction of x. Hence, when P' is on the left of O, the equation of motion is

$$\frac{d^2x}{dt^2}=\frac{\mu}{x^2},$$

giving a different solution from (2) and (3).

The general case can be easily considered. Let the acceleration be $\mu(\text{distance})^n$ towards O. The equation of motion when the particle is on the right hand of O is clearly

$$\frac{d^2x}{dt^2}=-\mu.x^n.$$

When P' is on the left of O, the equation is

$$\frac{d^2x}{dt^2}=\text{ acceleration in direction } OA=\mu(P'O)^n=\mu(-x)^n.$$

These two equations are the same if

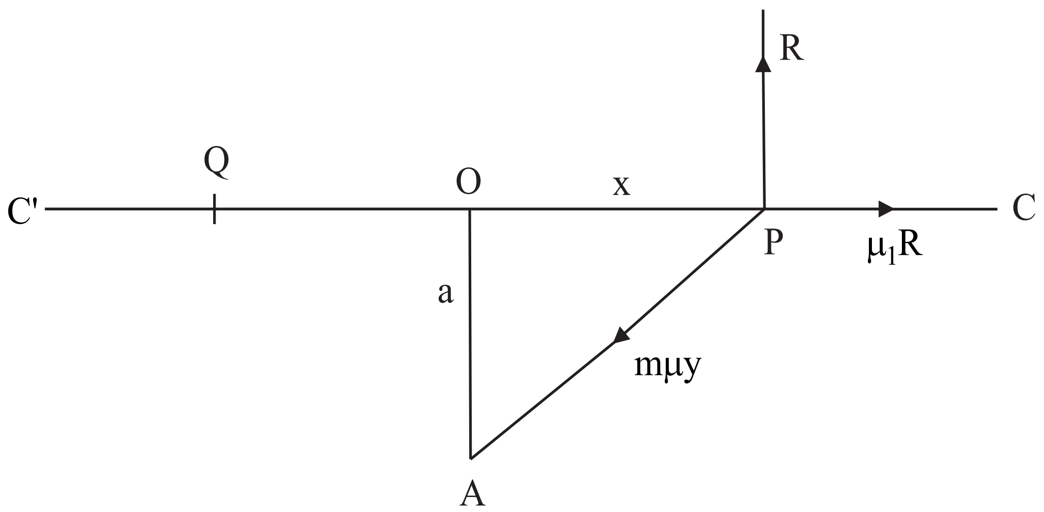
$$-\mu.x^n=\mu(-x)^n,\ i.e.,\ \text{if}\quad(-1)^n=-1,$$

i.e. if n be an odd integer, or if it be of the form $\frac{2p+1}{2q+1}$, where p and q are integers; in these cases the same equation holds on both sides of the origin; otherwise it does not.

36. EX. *A small bead, of mass m, moves on a straight rough wire under the action of a force equal to mμ times the distance of the bead from a fixed point A outside the wire at a perpendicular distance a from it. Find the motion if the bead start from rest at a distance c from the foot, O, of the perpendicular from A upon the wire.*

Let P be the position of the bead at any time t, where $OP = x$ and $AP = y$.

Let R be the normal reaction of the wire and μ_1 the coefficient of friction.



Resolving forces perpendicular to the wire, we have

$R = m\mu y \sin OPA = m\mu a.$

Hence the friction $\mu_1 R = m\mu\mu_1 a.$

The resolved part of the force $m\mu y$ along the wire

$= m\mu y \cos OPA = m\mu x.$

Hence the total acceleration = $\mu\mu_1 a - \mu x.$

The equation of motion is thus

$$\frac{d^2x}{dt^2} = \mu\mu_1 a - \mu x = -\mu(x - \mu_1 a) \qquad ...(1),$$

so long as P is to the right of O.

[If P be to the left of O and moving towards the left, the equation of motion is

$$\frac{d^2x}{dt^2} = \text{ acceleration in the direction } OC$$
$$= \mu\mu_1 a + \mu(-x), \text{ as in the last article,}$$

and this is the same as (1) which therefore holds on both sides of O.]

Integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = -\mu(x-\mu_1 a)^2 + C, \quad \text{where } 0 = -\mu(c-\mu_1 a)^2 + C.$$

$$\therefore\ v^2 = \left(\frac{dx}{dt}\right)^2 = \mu[(c-\mu_1 a)^2 - (x-\mu_1 a)^2] \qquad ...(2),$$

and therefore, as in Art. 22,

$$\sqrt{\mu}t = \cos^{-1}\frac{x-\mu_1 a}{c-\mu_1 a} + C_1,$$

where $0 = \cos^{-1}\frac{c-\mu_1 a}{c-\mu_1 a} + C_1$, *i.e.* $C_1 = 0$.

$$\therefore\ \sqrt{\mu}t = \cos^{-1}\frac{x-\mu_1 a}{c-\mu_1 a} \qquad ...(3),$$

(2) and (3) give the velocity and time for any position.

From (2) the velocity vanishes when $x-\mu_1 a = \pm(c-\mu_1 a)$,
i.e. when $x = c = OC$, and when $x = -(c-2a\mu_1)$,
i.e. at the point C', where $OC' = c-2a\mu_1$,
and then from (3) the corresponding time

$$= \frac{1}{\sqrt{\mu}}\cos^{-1}\frac{-c+\mu_1 a}{c-\mu_1 a} = \frac{1}{\sqrt{\mu}}\cos^{-1}(-1) = \frac{\mu}{\sqrt{\mu}}.$$

The motion now reverses and the particle comes to rest at a point C'' on the right of O where $OC'' = OC' - 2\mu_1 a = OC - 4\mu_1 a$.

Finally, when one of the positions of instantaneous rest is at a distance which is equal to or less than $\mu_1 a$ from O, the particle remains at rest. For at this point the force towards the centre is less than the limiting friction and therefore only just sufficient friction will be exerted to keep the particle at rest.

It will be noted that the periodic time $\frac{2\pi}{\sqrt{\mu}}$ is not affected by the friction, but the amplitude of the motion is altered by it.

37. EX. *A particle, of mass m, rests in equilibrium at a point N, being attracted by two forces equal to* $m\mu^n$ *(distance)*^{n} *and* $m\mu'^n$ *(distance)*^{n} *towards two fixed centres O and* O'. *If the particle be slightly displaced from N, and if n be positive, show that it oscillates, and find the time of a small oscillation.*

Let $OO' = a$, ON = d, and $NO' = d'$, so that

$$\mu^n.d^n = \mu'^n.d'^n \qquad ...(1),$$

since there is equilibrium at N.

$$\therefore \frac{d}{\mu'} = \frac{d'}{\mu} = \frac{a}{\mu + \mu'} \qquad ...(2).$$

Let the particle be at a distance x from N towards O'.

The equation of motion is then

$$\frac{d^2x}{dt^2} = -\mu^n.OP^n + \mu'^n.PO'^n = \mu^n(d+x)^n + \mu'^n(d'-x)^n \qquad ...(3).$$

If x is positive, the right-hand side is negative; if x is negative, it is positive; the acceleration is towards N in either case.

Expanding by the Binomial Theorem, (3) gives

$$\frac{d^2x}{dt^2} = -\mu^n(d^n + nd^{n-1}x + \cdots) + \mu'^n(d'^n - nd'^{n-1}x + \cdots)$$

$$= -nx[\mu^n d^{n-1} + \mu'^n d'^{n-1}] + \text{ terms involving higher powers of } x$$

$$= -nxa^{n-1}\frac{(\mu\mu')^{n-1}}{(\mu+\mu')^{n-2}} + \cdots \qquad \text{by (2).}$$

If x be so small that its squares and higher powers may be neglected, this gives

$$\frac{d^2x}{dt^2} = -n\frac{(\mu\mu' a)^{n-1}}{(\mu+\mu')^{n-2}}x \qquad \text{...(4).}$$

Hence, as in Art. 22, the time of a small oscillation

$$2\pi \div \sqrt{n\frac{(\mu\mu' a)^{n-1}}{(\mu+\mu')^{n-2}}} = 2\pi\sqrt{\frac{(\mu+\mu')^{n-2}}{n(\mu\ \mu' a)^{n-1}}}.$$

If n be negative, the right-hand member of (4) is positive and the motion is not one of oscillation.

EXAMPLES ON CHAPTER 2

1. A particle moves towards a centre of attraction starting from rest at a distance a from the centre; if its velocity when at any distance x from the centre vary as $\sqrt{\frac{a^2-x^2}{x^2}}$, find the law of force.
2. A particle starts from rest at a distance a from a centre of force where the repulsion at distance x is μx^{-2}; show that is velocity at distance x is $\sqrt{\frac{2\mu(x-a)}{ax}}$ and that the time it has taken is

$$\sqrt{\frac{a}{2\mu}}\left[\sqrt{x^2-ax}+a\log_e\left(\sqrt{\frac{x}{a}}+\sqrt{\frac{x}{a}-1}\right)\right].$$

3. Prove that it is impossible for a particle to move from rest so that its velocity varies as the distance described from the commencement of the motion.

   If the velocity vary as (distance)^n, show that n cannot be greater than $\frac{1}{2}$.

4. A point moves in a straight line towards a centre of force $\left\{\frac{\mu}{(\text{distance})^3}\right\}$, starting from rest at a distance a from the centre of force; show that the time of reaching a point distant b from the centre of force is $\frac{a\sqrt{a^2-b^2}}{\sqrt{\mu}}$, and that its velocity then is $\frac{\sqrt{\mu}}{ab}\sqrt{a^2-b^2}$.

5. A particle falls from rest at a distance a from a centre of force, where the acceleration at distance x is $\mu n^{-5/3}$; when it reaches the centre show that its velocity is infinite and that the time it has taken is $\frac{2a^{1/3}}{\sqrt{3\mu}}$.

6. A particle moves in a straight line under a force to a point in it varying as (distance)$^{-4/3}$; show that the velocity in falling from rest at infinity to a distance a is equal to that acquired in falling from rest at a distance a to a distance $\frac{a}{8}$.

7. A particle, whose mass is m, is acted upon by a force $\mu\left(x+\frac{a^4}{x^3}\right)$ towards the origin; if it start from rest at a distance a, show that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$.

8. A particle moves in a straight line with an acceleration towards a fixed point in the straight line, which is equal to $\frac{\mu}{x^2} - \frac{\lambda}{x^3}$ when the particle is at a distance x from the given point; it starts from rest at a distance a; show that it oscillates between this distance and the distance $\frac{\lambda a}{2\mu a - \lambda}$, and that its periodic time is $\frac{2\pi\mu a^3}{(2a\mu - \lambda)^{3/2}}$.
9. A particle moves with an acceleration which is always towards, and equal to μ divided by the distance from, a fixed point O. If it start from rest at a distance a from O, show that it will arrive at O in time $a\sqrt{\frac{\pi}{2\mu}}$.

   $\left[\text{Assume that } \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.\right]$
10. A particle is attracted by a force to a fixed point varying inversely as the nth power of the distance; if the velocity acquired by it in falling from an infinite distance to a distance a from the centre is equal to the velocity that would be acquired by it in falling from rest at a distance a to a distance $\frac{a}{4}$, show that $n = \frac{3}{2}$.
11. A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their attractions per unit of mass at unit distance being μ and μ'; the particle is slightly displaced towards one of them; show that the time of a small oscillation is $\frac{2\pi}{\sqrt{\mu + \mu'}}$.
12. A mass of 100 kg. hangs freely from the end of a rope. The mass is hauled up vertically from rest by winding up the rope, the pull of which starts at 150 kg. weight and diminishes uniformly at the rate of 1 kg weight for each metre wound up. Neglecting the weight of

the rope, show that the mass has described 50 metres at the end of time $\dfrac{5\pi\sqrt{2}}{4.428}$ secs. and that its velocity then is $11.07\sqrt{2}$ metres/sec.

13. A particle moves in a straight line with an acceleration equal to μ $\div$ the nth power of the distance from a fixed point O in the straight line. If it be projected towards O, from a point at a distance a, with the velocity it would have acquired in falling from infinity, show that it will reach O in time $\dfrac{2}{n+1}\sqrt{\dfrac{n-1}{2\mu}}.a^{\frac{n+1}{2}}$.

14. In the previous question if the particle started from rest at distance a, show that it would reach O in time

$$\sqrt{\frac{n-1}{2\mu}}\pi a^{\frac{n+1}{2}}\frac{\Gamma\left(\frac{1}{n-1}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n-1}\right)}, \text{ or } \sqrt{\frac{\pi}{2\mu(1-n)}}a^{\frac{n+1}{2}}\frac{\Gamma\left(\frac{1}{1-n}\right)}{\Gamma\left(\frac{1}{1-n}+\frac{1}{2}\right)}.$$

according as n is $>$ or $<$ unity.

15. A shot, whose mass is 25 kg is fired from a gun, 2.5 metres in length. The pressure of the powder gas is inversely proportional to the volume behind the shot and changes from an initial value of 1600 kg weight per square centimetre to 160 kg weight per square centimetre as the shot leaves the gun. Show that the muzzle velocity of the shot is approximately 240 metres per second, having given $\log_e 10 = 2.3026$.

16. If the Moon and Earth were at rest, show that the least velocity with which a particle could be projected from the Moon, in order to reach the Earth, is about $2\frac{1}{4}$ km per second, assuming their radii to be 1760 and 6400 km respectively, the distance between their centres 385,000 km, and the mass of the Moon to be 1/81 that of the Earth.

17. A small bead can slide on a smooth wire AB, being acted upon by a force per unit of mass equal to $\mu \div$ the square of its distance from a point O which is outside AB. Show that the time of a small oscillation about its position of equilibrium is $\frac{2\pi}{\sqrt{\mu}}b^{3/2}$, where b is the perpendicular distance of O from AB.

18. A solid attracting sphere, of radius a and mass M, has a fine hole bored straight through its centre; a particle starts from rest at a distance b from the centre of the sphere in the direction of the hole produced, and moves under the attraction of the sphere entering the hole and going through the sphere; show that the time of a complete oscillation is

$$\frac{4}{\sqrt{2\gamma M}}\left[\sqrt{2}a^{3/2}\sin^{-1}\sqrt{\frac{b}{3b-2a}}+b^{3/2}\cos^{-1}\sqrt{\frac{a}{b}}+\sqrt{ab(b-a)}\right],$$

where γ is the constant of gravitation.

19. A circular wire of radius a and density ρ attracts a particle according to the Newtonian law $\gamma\frac{m_1m_2}{(\text{distance})^2}$; if the particle be placed on the axis of the wire at a distance b from the centre, find its velocity when it is at any distance x.

If it be placed on the axis at a small distance from the centre, show that the time of a complete oscillation is $a\sqrt{\frac{2\pi}{\gamma\rho}}$.

20. In the preceding question if the wire repels instead of attracting, and the particle be placed in the plane of the wire at a small distance from its centre, show that the time of an oscillation $2a\sqrt{\frac{\pi}{\gamma\rho}}$.

21. A particle moves in a straight line with an acceleration directed towards, and equal to μ times the distance from, a point in the straight line, and with a constant acceleration f in a direction op-

posite to that of its initial motion; show that its time of oscillation is the same as it is when f does not exist.

22. A particle P moves in a straight line OCP being attracted by a force $m\mu.\,PC$ always directed towards C, whilst C moves along OC with constant acceleration f. If initially C was at rest at the origin O, and P was at a distance c from O and moving with velocity V, prove that the distance of P from O at any time t is

$$\left(\frac{f}{\mu}+c\right)\cos\sqrt{\mu}t+\frac{V}{\sqrt{\mu}}\sin\sqrt{\mu}t-\frac{f}{\mu}+\frac{f}{2}t^2.$$

23. Two bodies, of masses M and M', are attached to the lower end of an elastic string whose upper end is fixed and hang at rest; M' falls off; show that the distance of M from the upper end of the string at time t is $a+b+c\cos\left(\sqrt{\frac{g}{b}}t\right)$, where a is the unstretched length of the string, and b and c the distances by which it would be stretched when supporting M and M' respectively.

24. A point is performing a simple harmonic motion. An additional acceleration is given to the point which is very small and varies as the cube of the distance from the origin. Show that the increase in the amplitude of the vibration is proportional to the cube of the original amplitude if the velocity at the origin is the same in the two motions.

25. One end of a light extensible string is fastened to a fixed point and the other end carries a heavy particle; the string is of unstretched length a and its modulus of elasticity is n times the weight of the particle. The particle is pulled down till it is at a depth b below the fixed point and then released.

    Show that it will return to this position at the end of time

$$2\sqrt{\frac{a}{ng}}\left[\frac{\pi}{2}+\operatorname{cosec}^{-1}p+\sqrt{p^2-1}\right],$$

where $p=\dfrac{nb}{a}-(n+1)$, provided that p is not $>\sqrt{1+4n}$.
If $p>\sqrt{1+4n}$, show how to find the corresponding time.

26. An endless elastic string, whose modulus of elasticity is λ and natural length is $2\pi c$, is placed in the form of a circle on a smooth horizontal plane and is acted upon by a force from the centre equal to μ times the distance per unit mass of the string. Show that its radius will vary harmonically about a mean length $\dfrac{2\pi\lambda c}{2\pi\lambda-m\mu c}$, where m is the mass of the string, assuming that $2\pi\lambda>m\mu c$. Examine the case when $2\pi\lambda=m\mu c$.

27. An elastic string of mass m and modulus of elasticity λ rests unstretched in the form of a circle whose radius is a. It is now acted on by a repulsive force situated in its centre whose magnitude per unit mass of the string is $\dfrac{\mu}{(\text{distance})^2}$. Show that when the circle next comes to rest its radius is a root of the quadratic equation

$$r^2-ar=\frac{m\mu}{\pi\lambda}.$$

28. A smooth block, of mass M, with its upper and lower faces horizontal planes, is free to move in a groove in a parallel plane, and a particle of mass m is attached to a fixed point in the upper face by an elastic string whose natural length is a and modulus E. If the system starts from rest with the particle on the upper face and the string stretched parallel to the groove to $(n+1)$ times its natural length, show that the block will perform oscillations of amplitude $\dfrac{(n+1)am}{M+m}$ in the periodic time $2\left(\pi+\dfrac{2}{n}\right)\sqrt{\dfrac{aMm}{E(M+m)}}$.

29. A particle is attached to a point in a rough plane inclined at an angle α to the horizon; originally the string was unstretched and lay along a line of greatest slope; show that the particle will oscillate only if the coefficient of friction is $< \frac{1}{3}\tan\alpha$.
30. A mass of m kg moves initially with a velocity of u metres per sec. A constant power equal to H horse-power is applied so as to increase its velocity; show that the time that elapses before the acceleration is reduced to $\frac{1}{n}$th of its original value is
$$\frac{m(n^2-1)u^2}{150gH}.$$
31. Show that the greatest velocity which can be given to a bullet of mass M fired from a smooth-bore gun is
$$\sqrt{\frac{2\Pi V}{M}(m\log m + 1 - m)},$$
where changes of temperature are neglected, and the pressure Π in front of the bullet is supposed constant, the volume V of the powder in the cartridge being assumed to turn at once, when fired, into gas of pressure $m\Pi$ will and of volume V.
32. Two masses, m_1 and m_2, are connected by a spring of such a strength that when m_1 is held fixed m_2 performs n complete vibrations per second. Show that if m_2 be held fixed, m_1 will make $n\sqrt{\frac{m_2}{m_1}}$, and, if both be free, they will make $n\sqrt{\frac{m_1+m_2}{m_1}}$, vibrations per second, the vibrations in each case being in the line of the spring.
33. A body is attached to one end of an inextensible string, and the other end moves in a vertical line with simple harmonic motion of

amplitude a and makes n complete oscillations per second. Show that the string will not remain tight during the motion unless $n^2 < \frac{g}{4\pi^2 a}$.

34. A light spring is kept compressed by the action of a given force; the force is suddenly reversed; prove that the greatest subsequent extension of the spring is three times its initial contraction.

35. Two masses, M and m, connected by a light spring, fall in a vertical line with the spring unstretched until M strikes an inelastic table. Show that if the height through which M falls is greater than $\frac{M+2m}{2m}l$, the mass M will after an interval be lifted from the table, l being the length by which the spring would be extended by the weight of M.

36. Two uniform spheres, of masses m_1 and m_2 and of radii a_1 and a_2, are placed with their centres at a distance a apart and are left to their mutual attractions; show that they will have come together at the end of time

$$\sqrt{\frac{2\pi aDR}{3g(m_1+m_2)}}\left[a\cos^{-1}\sqrt{\frac{a_1+a_2}{a}}+\sqrt{(a_1+a_2)(a-a_1-a_2)}\right],$$

where R is the radius, and D the mean density of the Earth.
If $m_1 = m_2 = 4$ kg, $a_1 = a_2 = 6.25$ cm, and $a = 0.5$ metre, show that the time is about $4\frac{1}{2}$ hours, assuming R = 6400 km. and D = 5600 kg per cubic metre.
[When the spheres have their centres at a distance x, the acceleration of m_1 due to the attraction of m_2 is $\gamma\frac{m_2}{x^2}$ and that of m_2, due to m_1 is $\gamma\frac{m_1}{x^2}$. Hence the acceleration of m_2 relative to m_1 is $\gamma\frac{m_1+m_2}{x^2}$ and the equation of relative motion is $\ddot{x} = -\gamma\frac{m_1+m_2}{x^2}$.]

37. Assuming the mass of the Moon to be 1/81 that of the Earth, that their radii are respectively 1760 and 6400 km, and the distance between their centres 385,000 km, show that, if they were instantaneously reduced to rest and allowed to fall towards one another under their mutual attraction only, they would meet in about $4\frac{1}{2}$ days.
38. A particle is placed at the end of the axis of a thin attracting cylinder of radius a and of infinite length; show that its kinetic energy when it has described a distance x varies as $\log \dfrac{x+\sqrt{x^2+a^2}}{a}$.
39. AB is a uniform string of mass M and length $2a$; every element of it is repelled with a force, $= \mu.$ distance, acting from a point O in the direction of AB produced; show that the acceleration of the string is the same as that of a particle placed at its middle point, and that the tension at any point P of the string varies as $AP.PB$.
40. Show that the curve which is such that a particle will slide down each of its tangents to the horizontal axis in a given time is a cycloid whose axis is vertical.
41. Two particles, of masses m and m' are connected by an elastic string whose coefficient of elasticity is λ; they are placed on a smooth table, the distance between them being a, the natural length of the string. The particle m is projected with velocity V along the direction of the string produced; find the motion of each particle, and show that in the subsequent motion the greatest length of the string is $a+Vp$, and that the string is next at its natural length after time πp, where $p^2 = \dfrac{mm'}{m+m'}\dfrac{a}{\lambda}$.
42. Two particles, each of mass m, are attached to the ends of an inextensible string which hangs over a smooth pulley; to one of them.

A, another particle of mass $2m$ is attached by means of an elastic string of natural length a, and modulus of elasticity $2mg$. If the system be supported with the elastic string just unstretched and be then released, show that A will descend with acceleration

$$g\sin^2\left[\sqrt{\frac{g}{2a}}.t\right].$$

43. A weightless elastic string, of natural length l and modulus λ, has two equal particles of mass m at its ends and lies on a smooth horizontal table perpendicular to an edge with one particle just hanging over. Show that the other particle will pass over at the end of time t given by the equation

$$2l+\frac{mgl}{\lambda}\sin^2\sqrt{\frac{\lambda}{2ml}}t=\frac{1}{2}gt^2.$$

ANSWERS WITH HINTS

Art. 28

Ex. 1 $x=a\cos nt+2a\cos\left(nt+\frac{\pi}{2}\right)=a\sqrt{5}\cos(nt+\beta)$
where $\frac{\cos\beta}{1}=\frac{\sin\beta}{2}=\frac{1}{\sqrt{5}}$.

Ex. 2 $x=a\sqrt{3}\cos\left(nt-\frac{\pi}{3}\right)$.

Ex. 3 $\ddot{x}=-\mu\left(x-\frac{1}{2}at^2\right)$.

Ex. 4 $m\ddot{x}=mg-ng\frac{x-l}{l}$.

Ex. 5 $m\ddot{x}=-\lambda\frac{x-a}{a}$, See Art. 22.

Ex. 6 $(2ml+2m'l')\ddot{x}=g[m(l-x)+m'(l'+x)-m(l+x)-m'(l'-$

$x)]$.

Ex. 7 $\ddot{x} = -\frac{g}{a}x$, $42\frac{1}{2}$ mins. (approx.)

Art. 34

Ex. 1 $\ddot{x} = -\frac{ga^2}{x^3}$.

Ex. 2 608.17 km/sec.

Ex. 3 $\ddot{x} = -\frac{ga^2}{x^2}$, i.e., $\dot{x}^2 = 2ga^2\left[\frac{1}{x} - \frac{1}{a+h}\right]$

Examples on Chapter 2 (End of Art. 37).

1. The acceleration varies inversely as the cubic of the distance.

3. $v = \lambda x^n$, $f = n\lambda x^{2n-1}$, if $2n-1 > 0$ both $v = 0, f = 0$ when $x = 0$

Chapter 2

MOTION IN A STRAIGHT LINE

Art 28

Ex. 1. $x = a\cos nt + 2a\cos\left(nt + \frac{\pi}{2}\right) - a\sqrt{5}\cos(nt + \beta)$, where

$$\frac{\cos\beta}{1} = \frac{\sin\beta}{2} = \frac{1}{\sqrt{5}}.$$

This is a simple harmonic motion of amplitude $a\sqrt{5}$, whose phase is in advance of the first by $\frac{\beta}{2\pi}$ of a period.

Ex. 2. $\dot{x} = n\sqrt{a^2 - x^2}$, so that $\dot{x} = \frac{na}{2}$ when $x = \frac{a\sqrt{3}}{2}$. Hence the new velocity at this distance $= \frac{3na}{2}$. The new motion is given by

$$x = A\cos nt + B\sin nt,$$

where $$\frac{a\sqrt{3}}{2} = A \text{ and } \frac{3na}{2} = Bn.$$

Hence $$x = a\sqrt{3}\cos\left(nt - \frac{\pi}{3}\right).$$

Ex. 3. $$\ddot{x} = -\mu\left(x - \tfrac{1}{2}at^2\right),$$

i.e. $$\frac{d^2}{dt^2}\left[x - \left(\tfrac{1}{2}at^2 - \frac{a}{\mu}\right)\right] = -\mu\left[x - \left(\tfrac{1}{2}at^2 - \frac{a}{\mu}\right)\right].$$

Hence as stated.

Ex. 4. $m\ddot{x} = mg - ng\frac{x - l}{l}$, *i.e.* $\ddot{x} = -\frac{ng}{ml}\left[x - \frac{m+n}{n}l\right]$,

so that $$x - \frac{m+n}{n}l = A\cos\left[\sqrt{\frac{ng}{ml}}\,t + B\right].$$

Hence, etc.

Ex. 5. Let $OA = a$ and $AB = b$, where O is the fixed end of the string. When $a < x < a + b$, we have $m\ddot{x} = -\lambda\frac{x - a}{a}$, *i.e.* a S. H. M. about A as centre. Hence, by Art. 22, the time from B to $A = \frac{\pi}{2}\sqrt{\frac{am}{\lambda}}$, and the velocity at $A = \sqrt{\frac{\lambda}{am}}\,b$, so that the time from A to $O = a \div \sqrt{\frac{\lambda}{am}}\,.\,b = \frac{a}{b}\sqrt{\frac{am}{\lambda}}$. The required time is four times the sum of these times.

Ex. 6. When on the one side there are lengths $l - x$ and $l' + x$ of m and m' and similarly $l + x$ and $l' - x$ on the other, then

$$\begin{aligned}(2ml + 2m'l')\ddot{x} &= g\left[m(l - x) + m'(l' + x) - m(l + x) - m'(l' - x)\right]\\ &= -2(m - m')gx.\end{aligned}$$

Hence, etc.

Ex. 7. Let AB be the tunnel, P any point on it, N the centre of AB, and O the centre of the Earth. Then attraction at $P=\mu \,.\, OP=\frac{g}{a}\,.\,OP$. Hence, if $NP=x$, we have $\ddot{x}=-\frac{g}{a}\,.\,NP=-\frac{g}{a}x$. Hence the time

$$=\pi\sqrt{\frac{a}{g}}=\pi\sqrt{\frac{4000\times 5280}{g}}\text{ secs.}=100\pi\,.\,\sqrt{66}\text{ secs.}=\text{about }2550\text{ secs.}$$
$$=42\tfrac{1}{2}\text{ mins. approx.}$$

Art 34

Ex. 1. $\ddot{x}=-\frac{\mu}{x^2}=-\frac{ga^2}{x^2}$, so that $\dot{x}^2=\frac{2ga^2}{x}$, since $\dot{x}$ is zero at infinity. Hence at the surface of the Earth $\dot{x}^2=2ga$, etc.

Ex. 2. Here, by Ex. 1, $v=\sqrt{2g\,.\,4000\,.\,5280}=64\,.\,10^2\sqrt{33}$ ft. per sec. $=\frac{40}{33}\times 5{\cdot}75$ miles per sec. $=7$ miles per sec. nearly. In the case of the sun $v'^2=2g'a'$, where $g'=\lambda\frac{S}{a'^2}$ and $\lambda\frac{S}{d^2}=$ normal acceleration of the Earth $=\omega^2 d$.

$\therefore\ g'=\frac{\omega^2 d^3}{a'^2}$, and $v'=\omega d\,.\,\sqrt{\frac{2d}{a'}}$

$$=\frac{2\pi}{365\times 24\times 60\times 60}\times 92500000\times\sqrt{\frac{2\times 92500000}{440000}}\text{ miles per sec.}$$
$$=\frac{1850000\pi}{73\times 864}\sqrt{\frac{185}{11}}\text{ miles per sec.}$$
$$=377{\cdot}9\text{ miles per sec.}$$

Ex. 3. $\ddot{x}=-\frac{\lambda}{x^2}=-\frac{ga^2}{x^2}$, so that $\dot{x}^2=2ga^2\left[\frac{1}{x}-\frac{1}{a+h}\right]$.

$$\therefore\ \sqrt{\frac{2ga^2}{a+h}}\,t=-\int_{a+h}^{a}dx\sqrt{\frac{x}{a+h-x}}\quad[\text{Put } x=(a+h)\cos^2\theta.]$$
$$=(a+h)\int(1+\cos 2\theta)=(a+h)\left[\theta+\tfrac{1}{2}\sin 2\theta\right]$$
$$=(a+h)\left[\cos^{-1}\sqrt{\frac{x}{a+h}}+\sqrt{\frac{x}{a+h}}\sqrt{1-\frac{x}{a+h}}\right]_{a+h}^{a}.$$
$$\therefore\ t=\sqrt{\frac{a+h}{2g}}\,\frac{a+h}{a}\left[\cos^{-1}\sqrt{\frac{a}{a+h}}+\sqrt{\frac{a}{a+h}}\sqrt{\frac{h}{a+h}}\right]$$
$$=\sqrt{\frac{a+h}{2g}}\left[\frac{a+h}{a}\sin^{-1}\sqrt{\frac{h}{a+h}}+\sqrt{\frac{h}{a}}\right].$$

Now $$\sin^{-1}\lambda=\lambda+\frac{\lambda^3}{6}+\frac{1\,.\,3}{2\,.\,4}\frac{\lambda^5}{5}+\ldots.$$

$$\therefore\ t=\sqrt{\frac{a+h}{2g}}\left[\frac{a+h}{a}\left\{\sqrt{\frac{h}{a+h}}+\frac{1}{6}\frac{h}{a+h}\sqrt{\frac{h}{a+h}}+\ldots\right\}+\sqrt{\frac{h}{a}}\right]$$
$$=\sqrt{\frac{h}{2g}}\left[\frac{a+h}{a}+\frac{1}{6}\frac{h}{a}+\ldots+\sqrt{\frac{a+h}{a}}\right]$$
$$=\sqrt{\frac{h}{2g}}\left[1+\frac{h}{a}+\frac{1}{6}\frac{h}{a}+\ldots+1+\frac{h}{2a}\right]=\sqrt{\frac{2h}{g}}\left[1+\frac{5}{6}\frac{h}{a}\right],$$

squares of $\frac{h}{a}$ being neglected.

End of Art 37

EXAMPLES ON CHAPTER 2

1. $v^2 \propto \frac{a^2}{x^2} - 1$; $\therefore$ $v\frac{dv}{dx} \propto -\frac{a^2}{x^3}$, *i.e.* the acceleration varies inversely as the cube of the distance.

2. $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$; $\therefore$ $\left(\frac{dx}{dt}\right)^2 = -\frac{2\mu}{x} + \frac{2\mu}{a} = 2\mu\,\frac{x-a}{ax}$.

$$\therefore\ t\sqrt{\frac{2\mu}{a}} = \int_a^x \sqrt{\frac{x}{x-a}}\,dx = \int_a^x \left[\frac{2x-a}{2\sqrt{x^2-ax}} + \frac{a}{2\sqrt{x^2-ax}}\right]dx$$

$$= \left[\sqrt{x^2-ax} + a\log\left(\sqrt{x}+\sqrt{x-a}\right)\right]_a^x, \quad \therefore \text{ etc.}$$

3. If possible, let $v = \lambda x$; hence f = acceleration $= \frac{dv}{dt} = \lambda\frac{dx}{dt} = \lambda^2 x$. Hence initially, when $x=0$, we have both v and f zero so that the particle remains at rest.

If $v = \lambda . x^n$, then $f = \frac{dv}{dt} = n\lambda x^{n-1} . v = n\lambda^2 . x^{2n-1}$; hence, if $2n > 1$, both v and f will vanish when $x=0$, and the particle remains at rest.

4. $\ddot{x} = -\frac{\mu}{x^3}$; $\therefore$ $\dot{x}^2 = \frac{\mu}{x^2} - \frac{\mu}{a^2} = \frac{\mu(a^2-x^2)}{a^2x^2}$.

$$\therefore\ \sqrt{\mu}\,t = -\int_a^x \frac{ax}{\sqrt{a^2-x^2}} = \left[a\sqrt{a^2-x^2}\right]_a^b = a\sqrt{a^2-b^2}.$$

5. $\ddot{x} = -\frac{\mu}{x^{\frac{5}{3}}}$; $\therefore$ $\dot{x}^2 = 3\mu\left[\frac{1}{x^{\frac{2}{3}}} - \frac{1}{a^{\frac{2}{3}}}\right]$.

$$\therefore\ t\sqrt{3\mu} = -\int_a^0 \frac{a^{\frac{1}{3}}x^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}-x^{\frac{2}{3}}}}\,dx = 3a^{\frac{4}{3}}\int_0^{\frac{\pi}{2}} \cos^3\phi\,d\phi, \text{ if } x = a\cos^3\phi,$$

$$= 3a^{\frac{4}{3}} . \tfrac{2}{3}. \quad \therefore \text{ etc.}$$

6. $\ddot{x} = -\frac{\mu}{x^{\frac{4}{3}}}$; $\therefore$ $\dot{x}^2 = \frac{6\mu}{x^{\frac{1}{3}}} + C$.

$$\therefore\ v_1^2 = \frac{6\mu}{a^{\frac{1}{3}}}, \text{ and } v_2^2 = \left(\frac{6\mu}{x^{\frac{1}{3}}}\right)_a^{\frac{a}{8}} = \frac{6\mu}{a^{\frac{1}{3}}}(2-1); \quad \therefore\ v_1 = v_2.$$

7. $\ddot{x} = -\mu\left(x + \frac{a^4}{x^3}\right)$; $\therefore$ $\dot{x}^2 = \mu\left(-x^2 + \frac{a^4}{x^2}\right)$.

$$\therefore\ t\sqrt{\mu} = -\int_a^0 \frac{x}{\sqrt{a^4-x^4}}\,dx = -\frac{1}{2}\left[\sin^{-1}\frac{x^2}{a^2}\right]_a^0 = \frac{\pi}{4}.$$

8. $\ddot{x} = -\frac{\mu}{x^2} + \frac{\lambda}{x^3}$; $\therefore$ $\dot{x}^2 = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right) - \lambda\left(\frac{1}{x^2} - \frac{1}{a^2}\right)$

$$= \frac{\lambda(a-x)(x-pa)}{pa^2x^2}, \text{ if } \frac{\lambda}{2a\mu - \lambda} = p.$$

The particle is at rest again when $x = pa$.

$$\frac{t}{a}\sqrt{\frac{\lambda}{p}} = -\int_a^{pa} \frac{x\,dx}{\sqrt{(a-x)(x-pa)}} \quad [\text{Put } x = a\sin^2\theta + pa\cos^2\theta.]$$

$$= \int_0^{\frac{\pi}{2}} 2a(\sin^2\theta + p\cos^2\theta)\,d\theta = \frac{\pi a}{2}(1+p).$$

Hence the required time $= 2t = 2\mu\pi a^3 \div (2a\mu - \lambda)^{\frac{3}{2}}$.

9. $\ddot{x} = -\frac{\mu}{x}$; $\therefore$ $\dot{x}^2 = -2\mu \log\frac{x}{a}$.

$$\therefore t\sqrt{2\mu} = -\int_a^0 \frac{dx}{\sqrt{-\log\frac{x}{a}}} = 2a\int_0^\infty e^{-y^2}dy, \text{ if } x = ae^{-y^2},$$

$$= 2a\frac{\sqrt{\pi}}{2}, \text{ so that } t = a\sqrt{\frac{\pi}{2\mu}}.$$

10. $\ddot{x} = -\frac{\mu}{x^n}$, so that $\dot{x}^2 = \frac{\mu}{n-1} \cdot \frac{1}{x^{n-1}} + C$.

Now $C = 0$ in the first case, and $= -\frac{\mu}{n-1}\frac{1}{a^{n-1}}$ in the second case. We are given $\frac{1}{a^{n-1}} = \left(\frac{4}{a}\right)^{n-1} - \frac{1}{a^{n-1}}$, so that $n = \frac{3}{2}$.

11. $\ddot{x} = \mu(a-x) - \mu'(x+a) = -(\mu+\mu')\left(x - a\frac{\mu-\mu'}{\mu+\mu'}\right)$. Hence the result, by Art. 22.

12. When the mass has risen x feet the tension of the rope $= (150 - x)g$. Hence $100\ddot{x} = (150 - x)g - 100g = -(x-50)g$.

$$\therefore \text{(Art. 22) } x - 50 = A\cos\left(\sqrt{\frac{g}{100}}t + B\right) = -50\cos\left[\frac{2\sqrt{2}}{5}t\right],$$

since initially $x = 0$ and $\dot{x} = 0$.

Hence, when $x = 50$, $\quad t = \frac{5}{2\sqrt{2}} \cdot \frac{\pi}{2} = \frac{5\pi\sqrt{2}}{8}$ secs.,

and then $\quad v = \frac{dx}{dt} = 50 \cdot \frac{2\sqrt{2}}{5}\sin\frac{\pi}{2} = 20\sqrt{2}$ ft. per sec.

13. $\ddot{x} = -\frac{\mu}{x^n}$, and hence $\dot{x}^2 = \frac{2\mu}{n-1}\frac{1}{x^{n-1}}$, the constant being zero, since $\dot{x} = 0$ when $x = \infty$.

$$\therefore \sqrt{\frac{2\mu}{n-1}}\,t = -\int_a^0 x^{\frac{n-1}{2}}\,dx = \left[\frac{2}{n+1}x^{\frac{n+1}{2}}\right]_0^a = \frac{2}{n+1}a^{\frac{n+1}{2}}.$$

14. Here
$$\dot{x}^2 = \frac{2\mu}{n-1}\left(\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right).$$

(1) $n > 1$. $\quad \therefore \sqrt{\frac{2\mu}{n-1}}\,t = -\int_a^0 \frac{a^{\frac{n-1}{2}}\, x^{\frac{n-1}{2}}}{\sqrt{a^{n-1} - x^{n-1}}}\,dx.$ $\quad$ [Put $x = a\xi^{\frac{1}{n-1}}$.]

$$= \int_0^1 \frac{a^{\frac{n+1}{2}}}{n-1}\,\frac{\xi^{\frac{1}{n-1}-\frac{1}{2}}\,d\xi}{\sqrt{1-\xi}} = \frac{a^{\frac{n+1}{2}}}{n-1}\,B\left[\frac{1}{n-1} + \frac{1}{2},\ \frac{1}{2}\right]$$

$$= \frac{a^{\frac{n+1}{2}}}{n-1}\,\frac{\Gamma\left(\frac{1}{n-1} + \frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n-1} + 1\right)} = a^{\frac{n+1}{2}}\,\frac{\Gamma\left(\frac{1}{n-1} + \frac{1}{2}\right).\sqrt{\pi}}{\Gamma\left(\frac{1}{n-1}\right)}, \text{ etc.}$$

(2) $n < 1$. $\quad \therefore \sqrt{\frac{2\mu}{1-n}}\,t = -\int_a^0 \frac{dx}{\sqrt{a^{1-n} - x^{1-n}}}$ $\quad$ [Put $x = a\xi^{\frac{1}{1-n}}$.]

$$= \int_0^1 \frac{a^{\frac{n+1}{2}}}{1-n}\,\frac{\xi^{\frac{1}{1-n}-1}}{\sqrt{1-\xi}}\,d\xi = \frac{a^{\frac{n+1}{2}}}{1-n}\,B\left(\frac{1}{1-n},\ \frac{1}{2}\right)$$

$$= \frac{a^{\frac{n+1}{2}}}{1-n}\,\frac{\Gamma\left(\frac{1}{1-n}\right).\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{1-n} + \frac{1}{2}\right)} = \frac{a^{\frac{n+1}{2}}.\sqrt{\pi}}{1-n}\,\frac{\Gamma\left(\frac{1}{1-n}\right)}{\Gamma\left(\frac{1}{1-n} + \frac{1}{2}\right)}, \text{ etc.}$$

15. $50\ddot{x} = \pi . 2^2 . \frac{A}{x}$.

If a be the length behind the shot initially, then
$$\frac{A}{a} = 10 \times 2240g, \text{ and } \frac{A}{8} = 2240g, \text{ so that } a = \frac{4}{5}.$$
$$\therefore \ddot{x} = \frac{\pi \times 32 \times 224g}{5x}.$$
$$\therefore \dot{x}^2 = \frac{\pi . 64 . 224g}{5}\log\frac{x}{a}, \text{ since } \dot{x} = 0 \text{ when } x = a.$$

Hence $V = [\dot{x}]_{x=8} = \left(\frac{\pi . 64 . 224 . 32}{5}\log_e 10\right)^{\frac{1}{2}}$
$$= 8 \times 32 \times \sqrt{\frac{7\pi}{5} \times 2\cdot3026} = 8 \times 32 \times \sqrt{10\cdot1274}$$
$$= 8 \times 32 \times 3\cdot182 = 814\cdot6 \text{ ft. per sec.}$$

16. Let a and b be the radii of the Earth and Moon respectively, E and M their masses, and d the distance between their centres. Then at distance x from the centre of the Moon,
$$\ddot{x} = \frac{\gamma E}{(d-x)^2} - \frac{\gamma M}{x^2}.$$
$$\therefore \tfrac{1}{2}\dot{x}^2 = \frac{\gamma E}{d-x} + \frac{\gamma M}{x} + \frac{V^2}{2} - \frac{\gamma E}{d-b} - \frac{\gamma M}{b}, \quad \ldots\ldots\ldots\ldots(1)$$

where V is the required velocity.

Let x_1 be the distance from the centre of the Moon at which a particle would be in equilibrium under the attraction of the Earth and Moon, so that

$$\frac{\gamma M}{x_1^2}=\frac{\gamma E}{(d-x_1)^2}, \text{ and hence } \frac{x_1}{\sqrt{M}}=\frac{d-x_1}{\sqrt{E}}, \text{ i.e. } x_1=\frac{d}{10}.$$

Now the particle will just reach the Earth if the velocity is just sufficient to carry it to the distance x_1, i.e. by (1) if

$$\frac{V^2}{2}=ga^2\left[\frac{1}{d-b}+\frac{1}{81b}-\frac{100}{81d}\right], \text{ since } \frac{\gamma E}{a^2}=g \text{ and } M=\frac{1}{81}E,$$

$$=ga\left[\frac{40}{2389}+\frac{40}{81\times 11}-\frac{10}{81\times 6}\right]$$

$$=32\times 4000\times 1760\times 3\left[\frac{1}{60}+\frac{40}{81\times 11}-\frac{10}{81\times 6}\right] \text{ approximately, in ft.-sec. units.}$$

$$=32\times 4\times 10^4\times 16\times\frac{2200}{81\times 20} \text{ nearly.}$$

$$\therefore V=\frac{64\times 10^2}{9}\sqrt{110}=\frac{64\times 10^2}{9}\times 10\cdot 53=7488 \text{ ft. per sec.}$$

$$=1\cdot 42 \text{ miles per sec. approx.}$$

17. $\ddot{x}=-\frac{\mu}{b^2+x^2}\cdot\frac{x}{\sqrt{b^2+x^2}}=-\frac{\mu}{b^3}x+$ higher powers of x.

$\therefore$ required time $=2\pi\sqrt{\frac{b^3}{\mu}}$. (Art. 22.)

18. The acceleration at a point outside distant x from the centre $=\frac{\gamma M}{x^2}$.

Hence by Art. 31, the time of falling to the sphere

$$=\sqrt{\frac{b}{2\gamma M}}\left[\sqrt{bx-x^2}+b\cos^{-1}\sqrt{\frac{x}{b}}\right]_{x=b}^{x=a}$$

$$=\sqrt{\frac{b}{2\gamma M}}\left[\sqrt{ab-a^2}+b\cos^{-1}\sqrt{\frac{a}{b}}\right] \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and the velocity on reaching the sphere $=\sqrt{2\gamma M\frac{b-a}{ab}}=V$. For the motion inside the sphere we have, since the attraction now varies as the distance,

$$\ddot{x}=-\frac{\gamma M}{a^3}x, \text{ so that } \dot{x}^2=-\frac{\gamma M}{a^3}x^2+\left[V^2+\frac{\gamma M}{a}\right]=\frac{\gamma M}{a^3}[c^2-x^2],$$

where $c^2=a^2(3b-2a)/b$.

$$\therefore \text{ time to the centre}=-\sqrt{\frac{a^3}{\gamma M}}\int_a^0\frac{dx}{\sqrt{c^2-x^2}}=\sqrt{\frac{a^3}{\gamma M}}\sin^{-1}\frac{a}{c}$$

$$=\sqrt{\frac{a^3}{\gamma M}}\sin^{-1}\sqrt{\frac{b}{3b-2a}}. \quad\ldots\ldots\ldots\ldots\ldots(2)$$

The required time is four times the sum of (1) and (2).

19. $\ddot{x} = -\gamma \cdot \frac{2\pi a\rho}{a^2+x^2} \cdot \frac{x}{\sqrt{a^2+x^2}} = -2\pi a\rho\gamma \cdot \frac{x}{(a^2+x^2)^{\frac{3}{2}}}$ ....(1)

$\therefore\ \dot{x}^2 = 4\pi a\rho\gamma\left[\frac{1}{\sqrt{a^2+x^2}} - \frac{1}{\sqrt{a^2+b^2}}\right].$

Also, when x is small, (1) gives $\ddot{x} = -\frac{2\pi\rho\gamma}{a^2}x +$ higher powers.

Hence the time of a small oscillation

$$= 2\pi\sqrt{\frac{a^2}{2\pi\rho\gamma}} = a\sqrt{\frac{2\pi}{\rho\gamma}}, \text{ by Art. 22.}$$

20. $\ddot{x} = -2\int_0^\pi \frac{\gamma\rho a\,d\theta}{a^2+x^2-2ax\cos\theta} \times \frac{a\cos\theta - x}{\sqrt{a^2+x^2-2ax\cos\theta}}$

$= \frac{-2\gamma\rho}{a^2}\int_0^\pi \left(1-\frac{2x}{a}\cos\theta\right)^{-\frac{3}{2}}(a\cos\theta - x)\,d\theta$, neglecting sqs, etc. of x

$= \frac{-2\gamma\rho}{a^2}\int_0^\pi (a\cos\theta + 3x\cos^2\theta - x)\,d\theta$ " " "

$= -\frac{\gamma\rho}{a^2}\int_0^\pi x(1+3\cos 2\theta)\,d\theta = -\frac{\gamma\rho}{a^2}\cdot\pi\cdot x.$

$\therefore$ time of a small oscillation $= 2\pi\sqrt{\frac{a^2}{\pi\rho\gamma}} = 2a\sqrt{\frac{\pi}{\rho\gamma}}.$

21. $\ddot{x} = -\mu x \pm f = -\mu\left(x \mp \frac{f}{\mu}\right).$

Hence, as in Art. 22, the required time $= \frac{2\pi}{\sqrt{\mu}}.$

22. At time t, $OC = \frac{1}{2}ft^2$, so that $\ddot{x} = -\mu(x - \frac{1}{2}ft^2)$.

$$\therefore\ x = A\cos\sqrt{\mu}t + B\sin\sqrt{\mu}t + \frac{f}{2}t^2 - \frac{f}{\mu},$$

where $c = A - \frac{f}{\mu}$, and $V = \left(\frac{dx}{dt}\right)_{t=0} = B\sqrt{\mu}$, etc.

23. $Mg = \lambda\frac{b}{a}$, and $M'g = \lambda\frac{c}{a}$; when both masses are hanging at rest, the length of the string $= a+b+c$.

The equation of motion, when M' has fallen off, is

$$M\ddot{x} = Mg - \lambda\frac{x-a}{a}, \text{ i.e. } \ddot{x} = -\frac{g}{b}(x-a-b).$$

$$\therefore\ x-a-b = A\cos\sqrt{\frac{g}{b}}t + B\sin\sqrt{\frac{g}{b}}t,$$

where $c = A$, and $0 = \left(\frac{dx}{dt}\right)_{t=0} = B\sqrt{\frac{g}{b}}$. Hence, etc.

24. If a is the amplitude of the original motion, the velocity at the origin is $a\sqrt{\mu}$ (Art. 22). The equation of motion becomes

$$\ddot{x} = -\mu x + \lambda x^3 ; \quad \therefore \; \dot{x}^2 = -\mu x^2 + \frac{\lambda}{2} x^4 + \mu a^2 .$$

Hence $\dot{x} = 0$ when $x^2 - a^2 = \frac{\lambda}{2\mu} x^4$. The first approximation is $x = a$. Put $x = a + \xi$, where ξ and λ are both small. Substituting, we have $2a\xi = \frac{\lambda}{2\mu} a^4$, on neglecting squares, so that $x = a + \frac{\lambda}{4\mu} a^3$. Hence, etc.

25. Let O be the fixed point, $OACB$ the vertical through O, $OA = a$, $OB = b$ and $OC = a + \frac{a}{n}$.

The vertical acceleration when the particle is at any point P (where $OP = x$)

$$= \lambda \frac{x - a}{a} - g = \frac{ng}{a} \left[x - \frac{1+n}{n} a \right] = \frac{ng}{a} \, . \, CP,$$

so that the motion is simple harmonic with C as centre and CB as amplitude, where $CB = b - \frac{1+n}{n} a = \frac{pa}{n}$. By Art. 22,

$$\text{time from } B \text{ to } A = \frac{1}{2} \cdot \frac{\pi}{\sqrt{\frac{ng}{a}}} + \frac{\frac{\pi}{2} - \cos^{-1} \frac{CA}{CB}}{\sqrt{\frac{ng}{a}}}$$

$$= \sqrt{\frac{a}{ng}} \left[\frac{\pi}{2} + \operatorname{cosec}^{-1} p \right]. \quad \ldots\ldots\ldots\ldots(1)$$

At A the string becomes slack, and time to highest point of the path

$$= \frac{\text{vel. at } A}{g} = \frac{\sqrt{\frac{ng}{a} (CB^2 - CA^2)}}{g} = \sqrt{\frac{a}{ng} (p^2 - 1)}. \quad \ldots\ldots(2)$$

This holds provided the velocity at A is not $> \sqrt{2g \, . \, 2a}$ (for then the string would again become stretched before the highest point of the path was reached), *i.e.* if $\frac{ng}{a} \left(\frac{p^2 a^2}{n^2} - \frac{a^2}{n^2} \right)$ not $> 4ga$, *i.e.* if p^2 not $> 1 + 4n$.

Thus the total time is twice the sum of (1) and (2).

If $p > \sqrt{1 + 4n}$ the velocity, when the string again becomes stretched above O,

$$= \sqrt{\frac{ng}{a} (p^2 - 1) \frac{a^2}{n^2} - 2g \, . \, 2a} = \sqrt{\frac{ag}{n} (p^2 - 1 - 4n)},$$

and we have simple harmonic motion about a point, above O, as centre.

26. Consider the motion of an element of the string subtending a small angle 2θ at the centre. Let T be the tension, and r the radius, then. The equation of motion is

$$\frac{m\,.\,2\theta}{2\pi}\,\ddot{r}=\mu r\,.\,\frac{m\,.\,2\theta}{2\pi}-2T\,.\sin\theta=\frac{m\mu r\theta}{\pi}-2\theta\,.\,\lambda\,\frac{r-c}{c},$$

i.e.

$$\ddot{r}=-\frac{2\pi\lambda-m\mu c}{mc}\left[r-\frac{2\pi\lambda c}{2\pi\lambda-m\mu c}\right].$$

Hence as stated.

If $2\pi\lambda=m\mu c$, the equation becomes $\ddot{r}=\mu c$, *i.e.* the string continually increases in radius until it finally breaks.

27. As in the last example

$$\frac{m\,.\,2\theta}{2\pi}\,.\,\ddot{r}=\frac{m\,.\,2\theta}{2\pi}\,.\,\frac{\mu}{r^2}-2\theta\,.\,\lambda\,\frac{r-a}{a},\ \ i.e.\ \ddot{r}=\frac{\mu}{r^2}-\frac{2\pi\lambda}{ma}\,(r-a).$$

$$\therefore\ \tfrac{1}{2}\dot{r}^2=-\frac{\mu}{r}-\frac{\pi\lambda}{ma}\,(r-a)^2+\frac{\mu}{a}.$$

The string is at rest again when

$$\mu\left(\frac{1}{a}-\frac{1}{r}\right)=\frac{\pi\lambda}{ma}\,(r-a)^2,\ i.e.\ \text{when}\ r^2-ar=\frac{m\mu}{\pi\lambda}.$$

28. When the string is of length x, let the block have moved through a distance ξ. The equations of motion of the block and particle are

$$M\ddot{\xi}=E\,\frac{x-a}{a},\ \text{and}\ m\,(\ddot{\xi}+\ddot{x})=-E\,\frac{x-a}{a}.$$

$$\therefore\ \ddot{x}=-p^2\,(x-a),\ \text{where}\ p^2=\frac{E\,(M+m)}{Mma}.$$

$$\therefore\ x-a=A\cos(pt+B),$$

where $na=A\cos B$, and $0=-A\sin B$, so that $x-a=na\cos pt$.

Now the string is unstretched again when $t_1=\frac{\pi}{2p}$, and then the velocity of the particle along the block $=[\dot{x}]_{t=t_1}=-nap$, so that the time t_2 to the fixed point $=\frac{a}{nap}=\frac{1}{np}$.

Hence the total time required $=4\,(t_1+t_2)=2\left(\pi+\frac{2}{n}\right).\frac{1}{p}=$ etc.

Also the centre of gravity of the system is clearly fixed, since on the whole system there is no external force.

$$\therefore\ M\xi_1+m\,[\xi_1+(n+1)\,a]=M\xi_2+m\,[\xi_2-(n+1)\,a].$$

$$\therefore\ \text{twice the required amplitude}=\xi_2-\xi_1=\frac{2m\,(n+1)\,a}{M+m}.$$

29. When the string is of length x, the equation of motion is

$$m\ddot{x}=mg\sin\alpha-\mu mg\cos\alpha-\lambda\,\frac{x-a}{a}.$$

$$\therefore\ \dot{x}^2=2g\,(\sin\alpha-\mu\cos\alpha)\,x-\frac{\lambda}{ma}\,(x-a)^2-2g\,(\sin\alpha-\mu\cos\alpha)\,a.$$

Hence the velocity vanishes again when $2g(\sin\alpha-\mu\cos\alpha)=\frac{\lambda}{ma}(x-a)$, and then the acceleration up the plane

$$=\frac{\lambda}{ma}(x-a)-g(\sin\alpha+\mu\cos\alpha)=g(\sin\alpha-3\mu\cos\alpha).$$

This is positive, *i.e.* particle oscillates, if $\tan\alpha>3\mu$.

30. $m\ddot{x}=\frac{H\,.\,550g}{\dot{x}}$, so that $\dot{x}^2=\frac{1100Hg}{m}t+u^2$.

The acceleration is $\frac{1}{n}$th of its original value when the velocity is n times the original value, *i.e.* when

$$u^2+\frac{1100Hg}{m}t=n^2u^2, \text{ etc.}$$

31. A = the section of the bore; x = the distance of the shot from the end at time t; a = the original value of x, so that $V=Aa$.

The pressure of gas behind the shot at time t, by Boyle's Law,

$$=\frac{V\,.\,m\Pi}{Ax}=\frac{a}{x}\,.\,m\Pi.$$

Then $$M\ddot{x}=A\left[\frac{a}{x}\,.\,m\Pi-\Pi\right]=V\Pi\left(\frac{m}{x}-\frac{1}{a}\right).$$

$$\therefore\ \tfrac{1}{2}M\dot{x}^2=V\Pi\left[m\log x-\frac{x}{a}\right]-V\Pi\,[m\log a-1]$$

$$=V\Pi\left[m\log\frac{x}{a}-\frac{x}{a}+1\right].$$

Now the acceleration is zero, and hence the velocity greatest, when $x=am$, and then

$$\dot{x}^2=\frac{2V\Pi}{M}[m\log m-m+1].$$

32. $m_2\ddot{x}=-\lambda\frac{x-a}{a}$, so that $\frac{1}{n}=2\pi\sqrt{\frac{m_2a}{\lambda}}$, and $\therefore\ \lambda=4m_2\pi^2n^2a$.

In the second case $\frac{1}{n_1}=2\pi\sqrt{\frac{m_1a}{\lambda}}$, and hence $\frac{n_1}{n}=\sqrt{\frac{m_2}{m_1}}$.

When both masses are free, let y and $y+z$ be the distances of m_1 and m_2 from a fixed origin. Then

$$m_1\ddot{y}=T, \text{ and } m_2(\ddot{y}+\ddot{z})=-T.$$

$$\therefore\ \ddot{z}=-T\frac{m_1+m_2}{m_1m_2}=-\frac{m_1+m_2}{m_1m_2}\,.\,\lambda\frac{z-a}{a}.$$

$$\therefore\ \frac{1}{n_2}=2\pi\sqrt{\frac{m_1m_2}{m_1+m_2}\,.\,\frac{a}{\lambda}}=\frac{1}{n}\sqrt{\frac{m_1}{m_1+m_2}}.$$

33. The acceleration of the end of the string is μx, where $\frac{2\pi}{\sqrt{\mu}}=\frac{1}{n}$. Hence, at the highest point of the motion, the acceleration downwards $=\mu a=4\pi^2n^2a$. Hence the string will become slack if $4\pi^2n^2a>g$.

34. Let P be the force, ξ the length to which it compresses the spring, and a the unstretched length, so that $P=\lambda\frac{a-\xi}{a}$. When P is reversed, let x be the greatest length of the spring during the subsequent motion, so that the total work done during the motion is zero.

$$\therefore\ P(x-\xi)+\int_\xi^a \frac{\lambda(a-\xi)(-d\xi)}{a}-\int_a^x \frac{\lambda(x-a)}{a}\,dx=0,$$

$\therefore\ (a-\xi)(x-\xi)+\frac{1}{2}(a-\xi)^2-\frac{1}{2}(x-a)^2=0$, so that $x-a=3(a-\xi)$, etc.

35. If l_1 be the unstretched length, then

$$Mg=\lambda\frac{l}{l_1}. \quad \text{.......................}(1)$$

Let u be the velocity when M strikes the table; the spring becomes compressed until the velocity of m is destroyed; it then recovers and when it is again unstretched the velocity of m is u upwards. Let m rise until the length of the spring is x. Then

$$\tfrac{1}{2}mu^2=\text{work done}=\tfrac{1}{2}(x-l_1)\,.\,\lambda\frac{x-l_1}{l_1}+mg(x-l_1). \quad \text{.........}(2)$$

The mass M will be then *just* on the point of rising, if

$$Mg=\lambda\frac{x-l_1}{l_1}. \quad \text{.......................}(3)$$

Hence (2) gives $\quad \frac{1}{2}m\,.\,2gh=\frac{l_1}{2\lambda}M^2g^2+\frac{l_1}{\lambda}\,.\,Mmg^2.$

$$\therefore\ h=\frac{M+2m}{2m}\,.\,\frac{Mgl_1}{\lambda}=\frac{M+2m}{2m}\,l, \text{ by (1).}$$

If h is greater than this, then, before the velocity u is destroyed, the tension of the spring is $>Mg$, and M rises.

36. $\ddot{x}=-\gamma\frac{m_1+m_2}{x^2}$. Hence, by Art. 31,

$$\sqrt{\frac{2\gamma(m_1+m_2)}{a}}\,t=\sqrt{ax-x^2}+a\cos^{-1}\sqrt{\frac{x}{a}}.$$

Hence, when $x=a_1+a_2$,

$$t=\sqrt{\frac{a}{2\gamma(m_1+m_2)}}\left[\sqrt{(a_1+a_2)(a-a_1-a_2)}+a\cos^{-1}\sqrt{\frac{a_1+a_2}{a}}\right].$$

Also $\quad g=\gamma\frac{\frac{4}{3}\pi R^3D}{R^2}$, *i.e.* $\gamma=\frac{3g}{4\pi RD}$.

$$\therefore\ t=\sqrt{\frac{2\pi aDR}{3g(m_1+m_2)}}\left[a\cos^{-1}\sqrt{\frac{a_1+a_2}{a}}+\sqrt{(a_1+a_2)(a-a_1-a_2)}\right].$$

If $m_1=m_2=4$; $a_1=a_2=\frac{1}{8}$; and $a=1$, then

$$t=\sqrt{\frac{2\pi\,.\,350\,.\,4000\,.\,1760\,.\,3}{3g\,.\,8}}\left[\cos^{-1}\frac{1}{2}+\sqrt{\frac{1}{4}\cdot\frac{3}{4}}\right]\text{ secs.}$$

$$=500\sqrt{77\pi}\left[\frac{\pi}{3}+\frac{3}{4}\right]=5500\sqrt{2}\left[\frac{22}{21}+\frac{\sqrt{3}}{4}\right], \text{ taking } \pi \text{ as } \frac{22}{7},$$

$$=\frac{55}{36}\times 1{\cdot}4142\times 1{\cdot}481 \text{ hours}=3{\cdot}2 \text{ hours approx.}$$

37. Here $\frac{\pi DR}{m_1+m_2}=\frac{\pi DR}{\frac{82}{81}\times\frac{4}{3}\pi DR^3}=\frac{3}{328}\times\frac{81}{R^2}$.

$$\therefore\ t=\frac{9}{2R}\sqrt{\frac{a}{41g}}\left[a\cos^{-1}\sqrt{\frac{51}{2400}}+\sqrt{(a_1+a_2)(a-a_1-a_2)}\right]$$

$$=\frac{9}{8000}\sqrt{\frac{240000\times5280}{41g}}\left[240000\cos^{-1}\frac{\sqrt{34}}{40}+\sqrt{5100\times234900}\right]$$

$$=\frac{3\times9000}{4}\sqrt{\frac{110}{41}}\,[24\cos^{-1}(\cdot1458)+\sqrt{11\cdot98}]\text{ nearly}$$

$$=\frac{3\times9000}{4}\times1\cdot638\times[34\cdot18+3\cdot46],$$

since $\cos^{-1}(\cdot1458)=81^\circ\,37'=1\cdot424$ rad.,

$=11000\times37\cdot64=414000$ secs. approx. $=4$ days 19 hrs. approx.

38. $$\ddot{x}=\int_0^\infty\frac{2\pi a\mu}{a^2+(y-x)^2}\cdot\frac{y-x}{\sqrt{a^2+(y-x)^2}}\,dy=\left[\frac{-2\pi a\mu}{\sqrt{a^2+(y-x)^2}}\right]_0^\infty=\frac{2\pi a\mu}{\sqrt{a^2+x^2}}.$$

$$\therefore\ \dot{x}^2=4\pi a\mu\,\{\log(x+\sqrt{a^2+x^2})-\log a\},\text{ etc.}$$

39. $OB=\xi$; $BP=x$, where P is any point of the string; then

$$M\frac{dx}{2a}\cdot\ddot{\xi}=(T+dT)-T+\frac{M\,dx}{2a}\cdot\mu\cdot(\xi+x).\quad\ldots\ldots\ldots\ldots(1)$$

Integrating this from 0 to $2a$, we have

$$M\ddot{\xi}=\Big[T\Big]_{x=0}^{x=2a}+M\mu(\xi+a),$$

i.e. since T is zero at each end of the string,

$\ddot{\xi}=\mu(\xi+a)=$ acceleration of a particle placed at the middle point.

Also (1) gives $\frac{dT}{dx}=\frac{M}{2a}[\mu(a-x)]$.

$$\therefore\ T=\frac{M\mu}{4a}[a^2-(a-x)^2]=\frac{M\mu}{4a}\cdot BP\cdot PA.$$

40. If ψ be the inclination to the horizontal at a point whose coordinates are x and y, we have

$$\frac{y}{\sin\psi}=\tfrac{1}{2}g\sin\psi\cdot\lambda^2.\quad\therefore\ \frac{g\lambda^2}{2y}=1+\left(\frac{dx}{dy}\right)^2.$$

Putting $y=\frac{1}{2}g\lambda^2\sin^2\theta$, we obtain $x=\frac{g\lambda^2}{4}(2\theta+\sin2\theta)$, *i.e.* the curve is a cycloid with axis vertical.

41. At time t, let m' have moved through x, and let ξ be the length of the string then. Hence

$$m'\ddot{x}=T,\text{ and }m(\ddot{x}+\ddot{\xi})=-T.\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore\ \ddot{\xi}=-T\left(\frac{1}{m}+\frac{1}{m'}\right)=-\frac{m+m'}{mm'}\lambda\cdot\frac{\xi-a}{a}=-\frac{\xi-a}{p^2}.$$

$$\therefore\ \xi-a=A\cos\left(\frac{t}{p}+B\right)=pV\sin\frac{t}{p},$$

since $\xi=a$ and $\dot{\xi}=V$, when $t=0$.

Hence the greatest value of ξ is $a+pV$, and $\xi=a$ again when $t=\pi p$.

Also (1) gives $(m+m')\ddot{x}+m\ddot{\xi}=0$.

$$\therefore\ (m+m')\dot{x}+m\dot{\xi}=\text{const.}=mV.$$

$$\therefore\ (m+m')x+m\xi=mVt+ma,$$

giving the motion of m'.

42. Let x be the depth of A, and y the length of the elastic string, at time t. Then

$$m\ddot{x}=mg+\lambda\frac{y-a}{a}-T, \quad \text{......................(1)}$$

$$2m(\ddot{x}+\ddot{y})=2mg-\lambda\frac{y-a}{a}, \quad \text{......................(2)}$$

and
$$m\frac{d^2}{dt^2}(l-x)=mg-T. \quad \text{..................................(3)}$$

Since $\lambda=2mg$, these give

$$\ddot{x}=\frac{g}{a}(y-a), \text{ and } \ddot{y}=-\frac{2g}{a}\left(y-\frac{3a}{2}\right).$$

$$\therefore\ y-\frac{3a}{2}=A\cos\left[\sqrt{\frac{2g}{a}}\,t+B\right]=-\frac{a}{2}\cos\left[\sqrt{\frac{2g}{a}}\,t\right],$$

since $y=a$ and $\dot{y}=0$, when $t=0$.

$$\therefore\ y=\frac{3a}{2}-\frac{a}{2}\cos\left[\sqrt{\frac{2g}{a}}\,t\right] \text{ and } \ddot{x}=g\sin^2\left[\sqrt{\frac{g}{2a}}\,t\right].$$

43. Let x and y be the lengths of the string hanging vertically, and on the table, at time t. Then

$$m\ddot{x}=mg-T=mg-\lambda\frac{x+y-l}{l}, \quad \text{......................(1)}$$

and
$$m\ddot{y}=-T=-\lambda\frac{x+y-l}{l}. \quad \text{..............................(2)}$$

Hence $\ddot{x}+\ddot{y}=g-\dfrac{2\lambda}{ml}(x+y-l)$.

$$\therefore\ x+y-l=\frac{lmg}{2\lambda}\left[1-\cos\sqrt{\frac{2\lambda}{ml}}\,t\right],$$

since initially the string is at rest and unstretched.

(2) gives
$$\ddot{y}=-\frac{g}{2}+\frac{g}{2}\cos\sqrt{\frac{2\lambda}{ml}}\,t.$$

$$\therefore\ y=-\frac{g}{4}t^2-\frac{mlg}{4\lambda}\cos\sqrt{\frac{2\lambda}{ml}}\,t+\left(l+\frac{mlg}{4\lambda}\right),$$

since $y=l$ and $\dot{y}=0$, initially.

Hence $y=0$, when $\dfrac{gt^2}{2}=2l+\dfrac{mgl}{\lambda}\sin^2\left[\sqrt{\dfrac{\lambda}{2ml}}\,t\right]$.