Formation of Differential Equations

1 Mark Questions

1. Write the degree of the differential equation

$$\left(\frac{dy}{dx}\right)^4 + 3y\frac{d^2y}{dx^2} = 0.$$ **Delhi 2013C**

> The degree of the differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions sign.

Given differential equation is

$$\left(\frac{dy}{dx}\right)^4 + 3y\frac{d^2y}{dx^2} = 0$$

Here, highest order derivative is d^2y/dx^2, whose degree is one. So, degree of differential equation is 1. **(1)**

2. Write the degree of the differential equation

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0.$$ **Delhi 2013**

Given differential equation is

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$$

Here, all differential coefficients are free from radical sign.

$\therefore$ Degree = 2 **(1)**

3. Write the degree of the differential equation

$$\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0.$$ **Delhi 2013**

Given differential equation is

$$\left(\frac{dy}{dx}\right)^4 + 3x\left(\frac{d^2y}{dx^2}\right) = 0$$

Here, all differential coefficients are free from radical sign.

$\therefore$ Degree = 1 **(1)**

4. Write the differential equation representing the family of curves $y = mx$, where m is an arbitrary constant. **All India 2013**

Given, family of curves is $y = mx$ ...(i)

where, m is an arbitrary constant.

Now, differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = m$$

On putting $m = \frac{dy}{dx}$ in Eq. (i), we get

$$y = x\frac{dy}{dx}$$

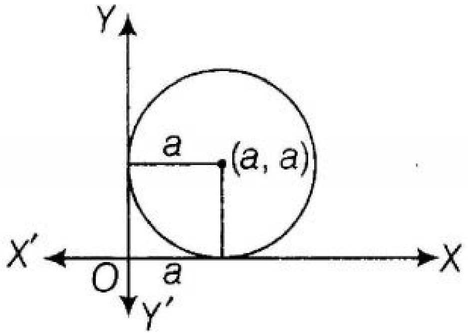
which is the required differential equation. **(1)**

4 Marks Questions

5. Find the differential equation of the family of circles in the first quadrant which touch the coordinate axes. **All India 2010C**

The equation of family of circles in first quadrant, which touch the coordinate axes, is $(x-a)^2+(y-a)^2=a^2$, where a is radius of circle. Differentiate it one time and eliminate the arbitrary constant a.

Let a be the radius of family of circles in the first quadrant, which touch the coordinate axes.



Then, coordinates of centre of circle = (a, a). **(1)**

We know that, equation of circle which has centre (h, k) and radius r is given by

$$(x-h)^2+(y-k)^2=r^2$$

Here, $(h, k) = (a, a)$ and $r = a$

$\therefore$ Equation of family of such circles is

$$(x-a)^2+(y-a)^2=a^2 \quad \text{...(i)} \; \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$2(x-a)+2(y-a)\frac{dy}{dx}=0$$

$$\Rightarrow \quad x-a+(y-a)\cdot y'=0 \quad \left[\because \frac{dy}{dx}=y'\right]$$

$$\Rightarrow \quad x+yy'=a+ay'$$

$$\Rightarrow \quad a=\frac{x+yy'}{1+y'} \quad \textbf{(1)}$$

On putting above value of a in Eq. (i), we get

$$\left[x - \frac{x + yy'}{y' + 1}\right]^2 + \left[y - \frac{x + yy'}{y' + 1}\right]^2 = \left(\frac{x + yy'}{y' + 1}\right)^2$$

$$\Rightarrow \left[\frac{xy' + x - x - yy'}{y' + 1}\right]^2 + \left[\frac{yy' + y - x - yy'}{y' + 1}\right]^2 = \left(\frac{x + yy'}{y' + 1}\right)^2$$

On multiplying both sides by $(y' + 1)^2$, we get

$$(xy' - yy')^2 + (y - x)^2 = (x + yy')^2$$

$$\Rightarrow \quad (x - y)^2 (y')^2 + (x - y)^2 = (x + yy')^2$$

$$[\because (x - y)^2 = (y - x)^2]$$

$$\Rightarrow \quad (x - y)^2 [(y')^2 + 1] = (x + yy')^2$$

which is the required differential equation. **(1)**

6. Find the differential equation of family of circles touching Y-axis at the origin.

HOTS; Delhi 2010; All India 2009, 2008C

The equation of family of circles touching Y-axis at origin is given by $(x-a)^2 + y^2 = a^2$, where a is radius of circle.
Differentiate this equation once because one arbitrary constant is present in the equation and eliminate a.

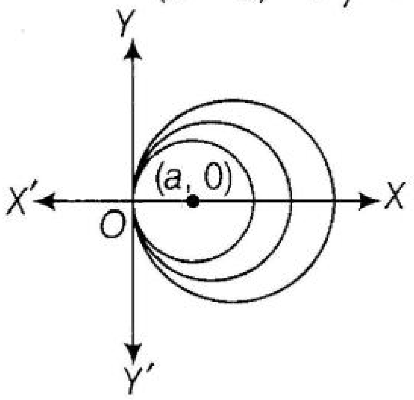
Given, family of circles touch Y-axis at the origin.

Let radius of family of circles be a.

$\therefore$ Centre of circle = (a, 0) **(1)**

Now, equation of family of circles with centre (a, 0) and radius a is

$$(x-a)^2 + y^2 = a^2$$ **(1)**



[putting $(h, k) = (a, 0)$ and $r = a$ in $(x-h)^2 + (y-k)^2 = r^2$]

$$\Rightarrow \quad x^2 + a^2 - 2ax + y^2 = a^2$$

$$\Rightarrow \quad x^2 - 2ax + y^2 = 0 \quad \text{...(i)}$$

On differentiating both sides w.r.t. x, we get

$$2x - 2a + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad x^2 - 2ax + y^2 = 0 \qquad \ldots(i)$$

On differentiating both sides w.r.t. x, we get

$$2x - 2a + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad a = x + y\frac{dy}{dx} \qquad \textbf{(1)}$$

On putting above value of a in Eq. (i), we get

$$x^2 + y^2 - 2\left(x + y\frac{dy}{dx}\right)x = 0$$

$$\Rightarrow \qquad 2xy\frac{dy}{dx} + x^2 - y^2 = 0$$

$$\Rightarrow \qquad 2xyy' + x^2 - y^2 = 0$$

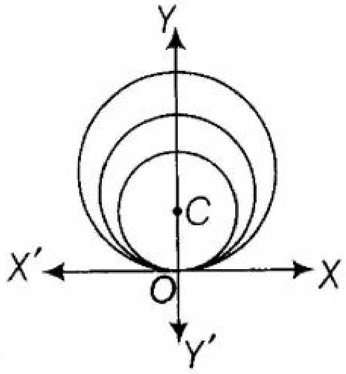
or $$2xy\frac{dy}{dx} + x^2 - y^2 = 0$$

which is the required differential equation. **(1)**

7. Find the differential equation of family of circles touching X-axis at the origin.

HOTS; Delhi 2010C; All India 2009C

Let a be the radius of family of circles which touch X-axis at origin.



$\therefore$ Centre of circle $= (0, a)$

Now, equation of family of such circles is

$$x^2 + (y - a)^2 = a^2 \quad \textbf{(1)}$$

[putting $(h, k) = (0, a)$ and $r = a$ in $(x - h)^2 + (y - k)^2 = r^2$]

$$\Rightarrow \quad x^2 + y^2 - 2ay = 0 \quad \text{...(i)}$$

On differentiating both sides w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + (y - a)\frac{dy}{dx} = 0 \quad \textbf{(1)}$$

$$\Rightarrow \quad x + yy' - ay' = 0, \left[\text{where, } y' = \frac{dy}{dx}\right]$$

$$\Rightarrow \quad x + yy' = ay'$$

$$\Rightarrow \quad a = \frac{x + yy'}{y'} \quad \textbf{(1)}$$

On putting above value of a in Eq. (i), we get

$$x^2 + y^2 = 2y\left(\frac{x + yy'}{y'}\right)$$

$$\Rightarrow \quad (x^2 + y^2) \cdot y' = 2xy + 2y^2 \cdot y'$$

$$\Rightarrow \quad x^2y' + y^2y' - 2xy - 2y^2y' = 0$$

$$\Rightarrow \quad x^2y' - 2xy - y^2y' = 0$$

$$\Rightarrow \quad y'(x^2 - y^2) = 2xy$$

$$\Rightarrow \quad y' = \frac{2xy}{x^2 - y^2} \text{ or } \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

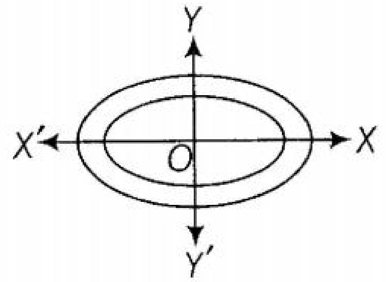
which is the required differential equation. **(1)**

8. Form the differential equation representing family of ellipses having foci on X-axis and centre at the origin. **HOTS; Delhi 2009C**

The equation of family of ellipses having foci on X-axis and centre at origin is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$.

Differentiate this equation two times and eliminate two arbitrary constants a and b to get the required result.

We know that, the equation of family of ellipse having foci on X-axis and centre at origin is given by



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots(i)$$

where, $a>b$ **(1)**

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0 \qquad \left[\text{put } \frac{dy}{dx}=y'\right]$$

$$\Rightarrow \quad \frac{x}{a^2}=\frac{-yy'}{b^2}$$

$$\Rightarrow \quad \frac{yy'}{x}=\frac{-b^2}{a^2} \qquad \ldots(ii)\ \textbf{(1)}$$

Again, differentiating both sides of Eq. (ii) w.r.t. x, we get

$$\frac{\left[x\cdot\frac{d}{dx}(yy')-yy'\cdot\frac{d}{dx}(x)\right]}{x^2}=0$$

$$\left[\begin{array}{l}\text{using quotient rule of differentiation}\\ \text{in LHS and } \frac{d}{dx}\left(\frac{-b^2}{a^2}\right)=0\end{array}\right]$$

$$\Rightarrow x\left[y\cdot\frac{d}{dx}(y')+y'\cdot\frac{d}{dx}(y)\right]-yy'\cdot 1=0 \qquad \textbf{(1)}$$

$$\Rightarrow \quad xyy'' + x(y')^2 - yy' = 0$$

or $$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

which is the required differential equation. **(1)**

9. Form the differential equation representing family of curves given by $(x-a)^2 + 2y^2 = a^2$ where, a is an arbitrary constant. **All India 2009**

Given equation of family of curves is

$$(x-a)^2 + 2y^2 = a^2 \quad \text{...(i)}$$

On differentiating both sides w.r.t. x in Eq. (i), we get

$$2(x-a) + 4yy' = 0 \quad \left[\because \frac{d}{dx}(y^2) = 2yy'\right] \textbf{(1)}$$

$$\Rightarrow \quad x - a + 2yy' = 0$$

$$\Rightarrow \quad a = x + 2yy' \quad \textbf{(1)}$$

On putting above value of a in Eq. (i), we get

$$(x - x - 2yy')^2 + 2y^2 = (x + 2yy')^2$$

$$\Rightarrow \quad 4y^2(y')^2 + 2y^2 = x^2 + 4y^2(y')^2 + 4xyy'$$

$$\Rightarrow \quad 2y^2 = x^2 + 4xyy' \quad \textbf{(1)}$$

Hence, the required differential equation is

$$x^2 + 4xyy' = 2y^2$$

or $$x^2 + 4xy\frac{dy}{dx} = 2y^2 \quad \textbf{(1)}$$