Photometry and Geometrical Optics (Part - 1)

Q.1. Making use of the spectral response curve for an eye (see Fig. 5.1), find:
(a) the energy flux corresponding to the luminous flux of 1.0 lm at the wavelengths 0.51 and 0.64 pm;
(b) the luminous flux corresponding to the wavelength interval from 0.58 to 0.63 tim if the respective energy flux, equal to Φ_e = 4.5 mW, is uniformly distributed over all wavelengths of the interval.
The function V (λ) is assumed to be linear in the given spectral interval.

Ans. (a) The relative spectral response V(λ) shown in Fig. (5.11) of the book is so defined that A/V (λ) is the energy flux of light of wave length λ, needed to produce a unit luminous flux at that wavelength. (A is the conversion factor defined in the book.)
At λ = 0·51 μm, we read from the figure

$V(\lambda) = 0{\cdot}50$ so

energy flux corresponding to a luminous flux of 1 lumen $= \frac{1{\cdot}6}{0{\cdot}50} = 3{\cdot}2\text{ mW}$
At

$\lambda = 0{\cdot}64\ \mu\text{m}$, we read

$V(\lambda) = 0{\cdot}17$

And energy flux corresponding to a luminous flux of 1 lumen $= \frac{1{\cdot}6}{{\cdot}17} = 9{\cdot}4\text{ mW}$

(b) Here $d\Phi_e(\lambda) = \frac{\Phi_e}{\lambda_2 - \lambda_1} d\lambda, \quad \lambda_1 \le \lambda \le \lambda_2$

Since energy is distributed uniformly. Then

$$\Phi = \int_{\lambda_1}^{\lambda_2} V(\lambda)\, d\Phi_2(\lambda)/A = \frac{\Phi_e}{A(\lambda_2 - \lambda_1)} \int_{\lambda_1}^{\lambda_2} V(\lambda)\, d\lambda$$

Since V (λ) is assumed to vary linearly in the interval $\lambda_1 \le \lambda \le \lambda_2$, we have

$$\frac{1}{\lambda_1 - \lambda_2} \int_{\lambda_1}^{\lambda_2} V(\lambda)\, d\lambda = \frac{1}{2}(V(\lambda_1) + V(\lambda_2))$$

Thus $\Phi = \frac{\Phi_e}{2A}(V(\lambda_1) + V(\lambda_2))$

Using $V(0{\cdot}58\ \mu\text{m}) = 0{\cdot}85$

$V(0{\cdot}63\ \mu\text{m}) = 0{\cdot}25$

Thus $\Phi = \frac{\Phi_e}{2 \times 1{\cdot}6} \times 1{\cdot}1 = 1{\cdot}55$ lumen.

Q.2. A point isotropic source emits a luminous flux $\Phi = 10$ lm with wavelength λ = 0.59 µm. Find the peak strength values of electric and magnetic fields in the luminous flux at a distance r = 1.0 m from the source. Make use of the curve illustrated in Fig. 5.1.

Ans.

We have $\Phi_e = \frac{\Phi A}{V(\lambda)}$

But $\Phi_e = \underbrace{\frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}}_{\text{mean energy flux vector}} E_m^2 \times \underbrace{4\pi r^2}_{\text{area}}$ or $E_m^2 = \frac{\Phi A}{2\pi r^2 V(\lambda)}\sqrt{\frac{\mu_0}{\varepsilon_0}}$

For $\lambda = 0{\cdot}59\ \mu\text{m}$ $V(\lambda) = 0{\cdot}74$ Thus

$E_m = 1{\cdot}14\ \text{V/m}$

Also $H_m = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_m = 3{\cdot}02\ \text{m A/m}$

Q.3. Find the mean illuminance of the irradiated part of an opaque sphere receiving
(a) a parallel luminous flux resulting in illuminance E_0 at the point of normal incidence;
(b) light from a point isotropic source located at a distance l = = 100 cm from the centre of the sphere; the radius of the sphere is R = 60 cm and the luminous intensity is I = 36 cd.

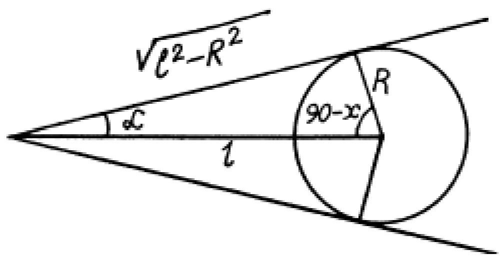
Ans. (a) Mean illuminance

$= \frac{\text{Total luminous flux incident}}{\text{Total area illuminated}}$.

Now, to calculate the total luminous flux incident on the sphere, we note that the illuminance at the point of normal incidence is E_0. Thus the incident flux is $E_o \cdot \pi R^2$.
Thus

Mean illuminance $= \dfrac{\pi R^2 \cdot E_0}{2\pi R^2}$

$$\langle E\rangle = \frac{1}{2}E_0 .$$



(b) The sphere subtends a solid angle

$$2\pi(1-\cos\alpha) = 2\pi\left(1-\frac{\sqrt{l^2-R^2}}{l}\right)$$

at the point source and therefore receives a total flux of

$$2\pi I\left(1-\frac{\sqrt{l^2-R^2}}{l}\right)$$

The area irradiated is: $\displaystyle 2\pi R^2\int_0^{90-\alpha}\sin\theta\, d\theta = 2\pi R^2(1-\sin\alpha) = 2\pi R^2\left(1-\frac{R}{l}\right)$

Thus $\displaystyle \langle E\rangle = \frac{I}{R^2}\,\frac{1-\sqrt{1-(R/l)^2}}{1-\frac{R}{l}}$

Substituting we get $\langle E\rangle = 50$ lux.

Q.4. Determine the luminosity of a surface whose luminance depends on direction as $L = L_0\ \cos\theta$, where θ is the angle between the radiation direction and the normal to the surface.

Ans. Luminance L is the light energy emitted per unit area of the emitting surface in a given direction per unit solid angle divided by cosθ. Luminosity M is simply energy emitted per unit area.

Thus

$$M = \int L\cdot\cos\theta\cdot d\Omega$$

where the integration must be in the forward hemisphere of the emitting surface (assuming light is being emitted in only one direction say outward direction of the surface.) But

$$L = L_0 \cos\theta$$

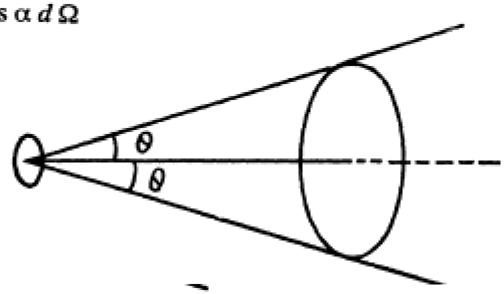
Thus

$$M = \int L_0 \cos^2\theta \cdot d\Omega = 2\pi \int_0^{\pi/2} L_0 \cos^2\theta \sin\theta \, d\theta = \frac{2}{3}\pi L_0$$

Q.5. A certain luminous surface obeys Lambert's law. Its luminance is equal to L. Find:
(a) the luminous flux emitted by an element ΔS of this surface into a cone whose axis is normal to the given element and whose aperture angle is equal to θ;
(b) the luminosity of such a source.

Ans. For a Lambert source L = Const
The flux emitted into the cone is

$$\Phi = L\,\Delta S \cos\alpha \, d\Omega$$



$$= L\,\Delta S \int_0^{\theta} 2\pi \cos\alpha \sin\alpha \, d\alpha$$

$$= L\,\Delta S\, \pi (1 - \cos^2\theta) = \pi L\,\Delta S \sin^2\theta$$

(b) The luminosity is obtained from the previous formula for θ = 90^0

$$M = \frac{\Phi(\theta = 90°)}{\Delta S} = \pi L$$

Q.6. An illuminant shaped as a plane horizontal disc S = 100 cm^2 in area is suspended over the centre of a round table of radius R = 1.0 m. Its luminance does not depend on direction and is equal to L = $1.6.10^4$ cd/m^2. At what height over the table should the illuminant be suspended to provide maximum illuminance at the circumference of the table? How great will that illuminance be? The illuminant is assumed to be a point source.

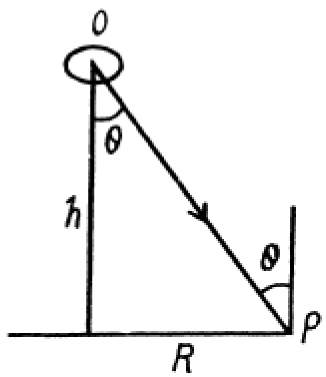
Ans. The equivalent luminous intensity in the direction OP is

$$L\,S\cos\theta$$

and the illuminance at P is

$$\frac{L\,S\cos\theta}{(R^2+h^2)}\cos\theta = \frac{L\,S\,h^2}{(R^2+h^2)^2}$$

$$= \frac{L\,S}{\left(\frac{R^2}{h}+h\right)^2} = \frac{L\,S}{\left[\left(\frac{R}{\sqrt{h}}-\sqrt{h}\right)^2+2R\right]^2}$$



This is maximum when and the maximum illuminance is R = h

$$\frac{L\,S}{4R^2} = \frac{1{\cdot}6\times 10^2}{4} = 40\text{ lux}$$

Q.7. A point source is suspended at a height h = 1.0 m over the centre of a round table of radius R = 1.0 m. The luminous intensity I of the source depends on direction so that illuminance at all points of the table is the same. Find the function I (θ), where θ is the angle between the radiation direction and the vertical, as well as the luminous flux reaching the table if I (0) = I_0 = 100 cd.

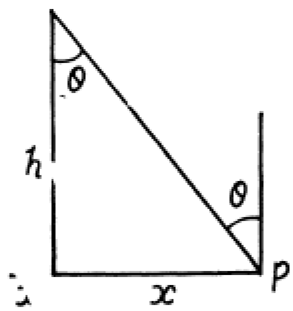
Ans. The illuminance at P is

$$E_p = \frac{I(\theta)}{(x^2+h^2)}\cos\theta = \frac{I(\theta)\cos^3\theta}{h^2}$$

Since this is constant at all x, we must have

$$I(\theta)\cos^3\theta = \text{const} = I_0$$

$$\text{or}\, I(\theta) = I_0/\cos^3\theta$$



The luminous flux reaching the table is

$$\Phi = \pi R^2 \times \frac{I_0}{h^2} = 314 \text{ lumen}$$

Q.8. A vertical shaft of light from a projector forms a light spot S = 100 cm^2 in area on the ceiling of a round room of radius R = = 2.0 m. The illuminance of the spot is equal to E = 1000 l_x. The reflection coefficient of the ceiling is equal to p = 0.80. Find the maximum illuminance of the wall produced by the light reflected from the ceiling. The reflection is assumed to obey Lambert's law.

Ans. The illuminated area acts as a Lambert source of luminosity M = πL where

$MS = \rho ES$ = total reflected light

Thus, the luminance

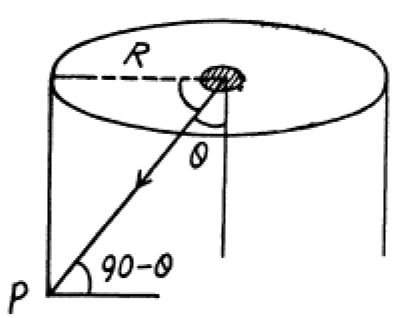
$$L = \frac{\rho E}{\pi}$$

The equivalent luminous intensity in the direction making an angle θ from the vertical is

$$LS\cos\theta = \frac{\rho ES}{\pi}\cos\theta$$

And the illuminance at the point P is

$$\frac{\rho ES}{\pi}\cos\theta\sin\theta / R^2\operatorname{cosec}^2\theta = \frac{\rho ES}{\pi R^2}\cos\theta\sin^3\theta$$



This is maximum when

$$\frac{d}{d\theta}(\cos\theta \sin^3\theta) = -\sin^4\theta + 3\sin^2\theta\cos^2\theta = 0$$

Or $\tan^2\theta = 3 \Rightarrow \tan\theta = \sqrt{3}$

Then the maximum illuminance is

$$\frac{3\sqrt{3}}{16\pi}\frac{\rho E S}{R^2}$$

This illuminance is obtained at a distance $R\cot\theta = R/\sqrt{3}$ from the ceiling. Substitution gives the value
0·21 lux

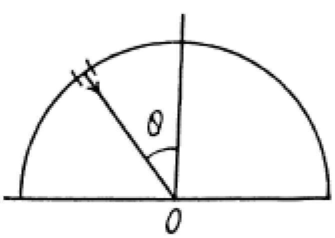
Q.9. A luminous dome shaped as a hemisphere rests on a horizontal plane. Its luminosity is uniform. Determine the illuminance at the centre of that plane if its luminance equals L and is independent of direction.

Ans. From the definition of luminance, the energy emitted in the radial direction by an element ds of the surface of the dome is

$$d\Phi = L\, dS\, d\Omega$$

Here L = constant. The solid angle $d\Omega$ is given by

$$d\Omega = \frac{dA\cos\theta}{R^2}$$



where dA is the area of an element on the plane illuminated by the radial light. Then

$$d\Phi = \frac{L\, dS\, dA}{R^2}\cos\theta$$

The illuminance at 0 is then

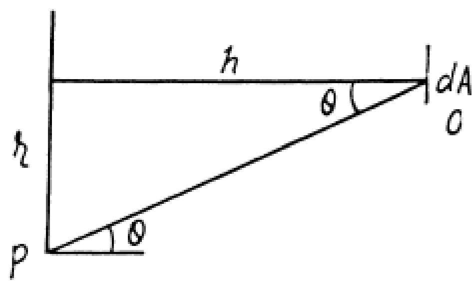
$$E = \int \frac{d\Phi}{dA} = \int_0^{\pi/2} \frac{L}{R^2} 2\pi R^2 \sin\theta \, d\theta \cos\theta = 2\pi L \int_0^1 x \, dx = \pi L$$

Q.10. A Lambert source has the form of an infinite plane. Its luminance is equal to L. Find the illuminance of an area element oriented parallel to the given source.

Ans. Consider an element of area dS at point P.

It emits light of flux

$$d\Phi = L \, dS \, d\Omega \cos\theta$$
$$= L \, dS \frac{dA}{h^2 \sec^2\theta} \cdot \cos^2\theta$$
$$= \frac{L \, dS \, dA}{h^2} \cos^4\theta$$



In the direction of the surface element dA at O.

The total illuminance at O is then

$$E = \int \frac{L \, dS}{h^2} \cos^4\theta$$

But

$$dS = 2\pi r \, dr = 2\pi h \tan\theta \, d(h\tan\theta)$$
$$= 2\pi h^2 \sec^2\theta \tan\theta \, d\theta$$

Substitution gives $$E = 2\pi L \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta = \pi L$$

Q.11. An illuminant shaped as a plane horizontal disc of radius R = 25 cm is suspended over a table at a height h = 75 cm. The illuminance of the table below

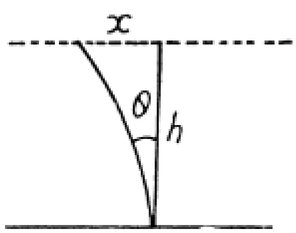
the centre of the illuminant is equal to E_0 = 70 l_x. Assuming the source to obey Lambert's law, find its luminosity.

Ans. Consider an angular element of area

$$2\pi x\,dx = 2\pi h^2 \tan\theta \sec^2\theta\, d\theta$$

Light emitted from this ring is

$$d\Phi = L\,d\Omega\,(2\pi h^2 \tan\theta \sec^2\theta\, d\theta)\cdot\cos\theta$$



Now $$d\Omega = \frac{dA\cos\theta}{h^2\sec^2\theta}$$

Where dA = an element of area of the table just below the untre of the illuminant Then the illuminance at the element dA will be

$$E_0 = \int_{\theta=0}^{\theta=\alpha} 2\pi L \sin\theta\cos\theta\, d\theta$$

where $$\sin\alpha = \frac{R}{\sqrt{h^2+R^2}}$$. Finally using luminosity M = πL

$$E_0 = M\sin^2\alpha = M\frac{R^2}{h^2+R^2}$$

Or $$M = E_0\left(1+\frac{h^2}{R^2}\right) = 700\ \text{lm/m}^2 \quad *\left(1\ \text{lx} = 1\frac{\text{lm}}{\text{m}^2}\ \text{dimensionally}\right).$$

Q.12. A small lamp having the form of a uniformly luminous sphere of radius R = 6.0 cm is suspended at a height h = 3.0 m above the floor. The luminance of the lamp is equal to L = 2.0.10^4 cd/m^2 and is independent of direction. Find the illuminance of the floor directly below the lamp.

Ans. See the figure below. The light emitted by an element of the illuminant towards the point 0 under consideration is

$$d\Phi = L\, dS\, d\Omega \cos(\alpha + \beta)$$

The element dS has the area

$$dS = 2\pi R^2 \sin\alpha\, d\alpha$$

The distance

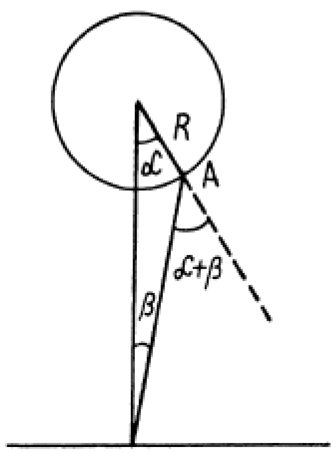
$$OA = \left[h^2 + R^2 - 2hR\cos\alpha\right]^{1/2}$$

we also have

$$\frac{OA}{\sin\alpha} = \frac{h}{\sin(\alpha+\beta)} = \frac{R}{\sin\beta}$$

From the diagram

$$\cos(\alpha+\beta) = \frac{h\cos\alpha - R}{OA}$$



$$\cos\beta = \frac{h - R\cos\alpha}{OA}$$

If we imagine a small area $d\Sigma$ at O then

$$\frac{d\Sigma\cos\beta}{OA^2} = d\Omega$$

Hence, the illuminance at O is

$$\int \frac{d\Phi}{d\Sigma} = \int L\, 2\pi R^2 \sin\alpha\, d\alpha\, \frac{(h\cos\alpha - R)(h - R\cos\alpha)}{(OA)^4}$$

The limit of α is $\alpha = 0$ to that value for which $\alpha + \beta = 90°$, for then light is emitted tangentially. Thus

$$\alpha_{max} = \cos^{-1}\frac{R}{h}.$$

$$\cos^{-1}\frac{R}{h}$$

Thus $$E = \int_0 L \cdot 2\pi R^2 \sin\alpha\, d\alpha \,\frac{(h - R\cos\alpha)(h\cos\alpha - R)}{(h^2 + R^2 - 2hR\cos\alpha)^2}$$

we put $$y = h^2 + R^2 - 2hR\cos\alpha$$

So, $$dy = 2hR\sin\alpha\, d\alpha$$

$$E = \int_{(h-R)^2}^{h^2 - R^2} L \cdot 2\pi R^2 \frac{dy}{2hR} \frac{\left(h - \frac{h^2 + R^2 - y}{2h}\right)\left(\frac{h^2 + R^2 - y}{2R} - R\right)}{y^2}$$

$$= \frac{L \cdot 2\pi R^2}{8h^2R^2} \int_{(h-R)^2}^{h^2 - R^2} \frac{(h^2 - R^2 + y)(h^2 - R^2 - y)}{y^2}\, dy$$

$$= \frac{\pi L}{4h^2} \int_{(h-R)^2}^{h^2 - R^2} \left[\frac{(h^2 - R^2)^2}{y^2} - 1\right] dy = \frac{\pi L}{4h^2}\left[-\frac{(h^2 - R^2)^2}{y} - y\right]_{(h-R)^2}^{h^2 - R^2}$$

$$= \frac{\pi L}{4h^2}\left[(h + R)^2 - (h^2 - R^2) - (h^2 - R^2) + (h - R)^2\right]$$

$$= \frac{\pi L}{4h^2}\left[2h^2 + 2R^2 - 2h^2 + 2R^2\right] = \frac{\pi L R^2}{h^2}$$

Substitution gives: $E = 25 \cdot 1$ lux

Q.13. Write the law of reflection of a light beam from a mirror in vector form, using the directing unit vectors e and e' of the inci- dent and reflected beams and the unit vector n of the outside normal to the mirror surface.

Ans. We see from the diagram that because of the law of reflection, the component of the incident unit vector $\vec{e}$ along $\cdot\vec{n}$ changes sign on reflection while the component || to the mirror remains unchanged.

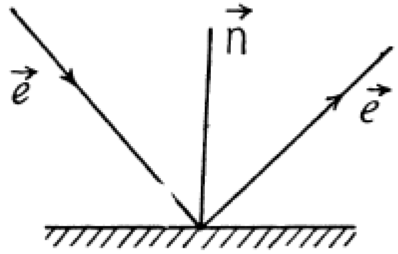
Writing $\vec{e} = \vec{e}_{\parallel} + \vec{e}_{\perp}$

where $\vec{e}_{\perp} = \vec{n}(\vec{e}\cdot\vec{n})$

$$\vec{e}_{\parallel} = \vec{e} - \vec{n}(\vec{e}\cdot\vec{n})$$

we see that the reflected unit vector is

$$\vec{e}\,' = \vec{e}_{\parallel} - \vec{e}_{\perp} = \vec{e} - 2\,\vec{n}(\vec{e}\cdot\vec{n})$$



Q.14. Demonstrate that a light beam reflected from three mutually perpendicular plane mirrors in succession reverses its direction.

Ans. We choose the unit vectors perpendicular to the mirror as the x, y,-z axes in space. Then after reflection from the mirror with normal along x axis

$$\vec{e}\,' = \vec{e} - 2\,\hat{i}\,(\hat{i}\cdot\vec{e}) = -\vec{e}_x\,\hat{i} + e_y\,\hat{j} + e_z\,\hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ are the basic unit vectors. After a second reflection from the 2nd mirror say along y axis.

$$\vec{e}\,'' = \vec{e}\,' - 2\,\hat{j}\,(\hat{j}\cdot\vec{e}\,') = -e_x\,\hat{i} - e_y\,\hat{h} + e_z\,\hat{k}$$

Finally after the third reflection

$$\vec{e}\,''' = -e_x\,\hat{i} - e_y\,\hat{j} - e_z\,\hat{k} = -\,\vec{e}.$$

Q.15. At what value of the angle of incident θ_1 is a shaft of light reflected from the surface of water perpendicular to the refracted shaft?

Ans. Let PQ be the surface of water and n be the R.I. of water. Let AO is the shaft of light with incident angle θ_1 and OB and OC are the reflected and refracted light rays at angles

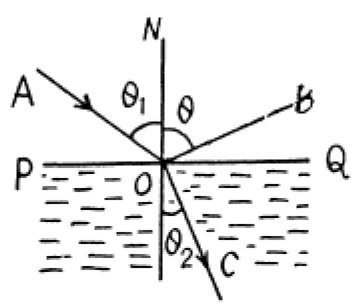
θ_1 and θ_2 respectively (Fig.). From the figure $\theta_2 = \frac{\pi}{2} - \theta_1$

From the law of refraction at the interface *PQ*

$$n = \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin\theta_1}{\sin\left(\frac{\pi}{2} - \theta_1\right)}$$

$$\text{or, } n = \frac{\sin\theta_1}{\cos\theta_1} = \tan\theta_1$$

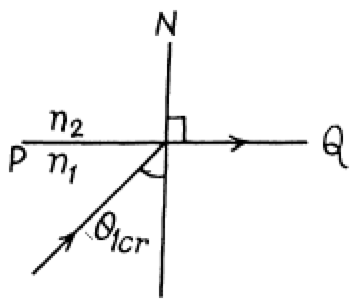
Hence $\theta_1 = \tan^{-1} n$



Q.16. Two optical media have a plane boundary between them. Suppose θ_{icr} is the critical angle of incidence of a beam and θ_1 is the angle of incidence at which the refracted beam is perpendicular to the reflected one (the beam is assumed to come from an optically denser medium). Find the relative refractive index of these media if $\sin\theta_{1cr}/\sin\theta_1 = \eta = 1.28$.

Ans. Let two optical mediums of R.I. n_1 and n_2 respectively be such that $n_1 > n_2$. In the case when angle of incidence is θ_{1cr} (Fig.), from the law of refraction

$$n_1 \sin\theta_{1cr} = n_2 \qquad (1)$$



In the case, when the angle of incidence is θ_1, from the law of refraction at the interface of mediums 1 and 2.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

But in accordance with the problem $\theta_2 = (\pi/2 - \theta_1)$

so, $$n_1 \sin\theta_1 = n_2 \cos\theta_1 \qquad (2)$$

Dividing Eqn (1) by (2)

$$\frac{\sin\theta_{1cr}}{\sin\theta_1} = \frac{1}{\cos\theta_1}$$

or, $$\eta = \frac{1}{\cos\theta_1}, \quad \text{so} \quad \cos\theta_1 = \frac{1}{\eta} \text{ and } \sin\theta_1 = \frac{\sqrt{\eta^2-1}}{\eta} \qquad (3)$$

But $$\frac{n_1}{n_2} = \frac{\cos\theta_1}{\sin\theta_1}$$

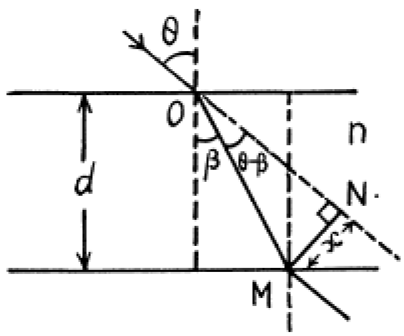
So, $$\frac{n_1}{n_2} = \frac{1}{\eta}\frac{\eta}{\sqrt{\eta^2-1}} \qquad \text{(Using 3)}$$

Thus $$\frac{n_1}{n_2} = \frac{1}{\sqrt{\eta^2-1}}$$

Q.17. A light beam falls upon a plane-parallel glass plate d=6.0 cm in thickness. The angle of incidence is θ = 60°. Find the value of deflection of the beam which passed through that plate.

Ans. From the Fig. the sought lateral shift

$$x = OM \sin(\theta - \beta)$$
$$= d \sec\beta \sin(\theta - \beta)$$
$$= d \sec\beta (\sin\theta \cos\beta - \cos\theta \sin\beta)$$
$$= d(\sin\theta - \cos\theta \tan\beta) \qquad (1)$$



But from the law of refraction

$$\sin\theta = n \sin\beta \quad \text{or,} \quad \sin\beta = \frac{\sin\theta}{n}$$

So, $$\cos\beta = \frac{\sqrt{n^2 - \sin^2\theta}}{n}$$

and $$\tan\beta = \frac{\sin\theta}{\sqrt{n^2 - \sin^2\theta}}$$

Thus $$x = d(\sin\theta - \cos\theta \tan\beta) = d\left(\sin\theta - \cos\theta \frac{\sin\theta}{\sqrt{n^2 - \sin^2\theta}}\right)$$

$$= d \sin\theta \left[1 - \sqrt{\frac{1 - \sin^2\theta}{n^2 - \sin^2\theta}}\right]$$

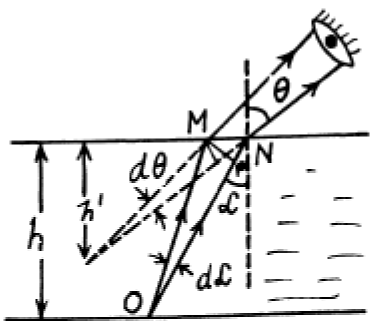
Q.18. A man standing on the edge of a swimming pool looks at a stone lying on the bottom. The depth of the swimming pool is equal to h. At what distance from the surface of water is the image of the stone formed if the line of vision makes an angle θ with the normal to the surface?

Ans. From the Fig.

$$\sin d\alpha = \frac{MP}{OM} = \frac{MN\cos\alpha}{h\sec(\alpha + d\alpha)}$$

As $d\alpha$ is very small, so

$$d\alpha = \frac{MN\cos\alpha}{h\ \sec\alpha} = \frac{MN\cos^2\alpha}{h} \quad (1)$$



Similarly

$$d\theta = \frac{MN\cos^2\theta}{h'} \qquad (2)$$

From Eqns (1) and (2)

$$\frac{d\alpha}{d\theta} = \frac{h'\cos^2\alpha}{h\cos^2\theta} \quad \text{or,} \quad h' = \frac{h\cos^2\theta}{\cos^2\alpha}\frac{d\alpha}{d\theta} \qquad (3)$$

From the law of refraction

$$n\sin\alpha = \sin\theta \qquad (A)$$

$$\sin\alpha = \frac{\sin\theta}{n}, \quad \text{so,} \quad \cos\alpha = \sqrt{\frac{n^2 - \sin^2\theta}{n^2}} \qquad (B)$$

Differentiating Eqn.(A)

$$n\cos\alpha\, d\alpha = \cos\theta\, d\theta \quad \text{or,} \quad \frac{d\alpha}{d\theta} = \frac{\cos\theta}{n\cos\alpha} \qquad (4)$$

Using (4) in (3), we get

$$h' = \frac{h\cos^3\theta}{n\cos^3\alpha}$$

Hence $$h' = \frac{h\cos^3\theta}{n\left(\frac{n^2-\sin^2\theta}{n^2}\right)^{3/2}} = \frac{n^2 h\cos^3\theta}{\left(n^2-\sin^2\theta\right)^{3/2}} \text{ [Using Eqn.(B)]}$$

(5)

Q.19. Demonstrate that in a prism with small refracting angle θ the shaft of light deviates through the angle $\alpha \simeq (n-1)\theta$ regardless of the angle of incidence, provided that the latter is also small.

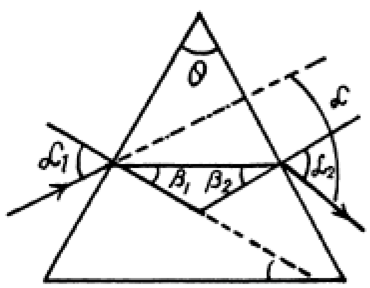
Ans. The figure shows the passage of a monochromatic ray through the given prism, placed in air medium.

From the figure, we have

$$\theta = \beta_1 + \beta_2 \qquad \text{(A)}$$

and $$\alpha = (\alpha_1 + \alpha_2) - (\beta_1 + \beta_2)$$

$$\alpha = (\alpha_1 + \alpha_2) - \theta \qquad (1)$$



From the Snell's law

$$\sin\alpha_1 = n\sin\beta_1$$

or $$\alpha_1 = n\beta_1 \text{ (for small angles)} \qquad (2)$$

and $$\sin\alpha_2 = n\sin\beta_2$$

or, $$\alpha_2 = n\beta_2 \text{ (for small angles)} \qquad (3)$$

From Eqns (1), (2) and (3), we get

$$\alpha = n(\beta_1 + \beta_2) - \theta$$

So, $\alpha = n(\theta) - \theta = (n-1)\theta$ [Using Eqn.A]

Q.20. A shaft of light passes through a prism with refracting angle θ and refractive index n. Let a be the diffraction angle of the shaft. Demonstrate that if the shaft of light passes through the prism symmetrically,
(a) the angle a is the least;
(b) the relationship between the angles α and θ is defined by Eq. (5.1e).

Ans. (a) In the general case, for the passage of a monochromatic ray through a prism as shown in the figure of the soln. of 5.19,

$$\alpha = (\alpha_1 + \alpha_2) - \theta \qquad (1)$$

And from the Snell's law,

$$\left.\begin{aligned} &\sin\alpha_1 = n\sin\beta_1 \quad \text{or} \quad \alpha_1 = \sin^{-1}(n\sin\beta_1) \\ &\text{Similarly} \quad \alpha_2 = \sin^{-1}(n\sin\beta_2) = \sin^{-1}[n\sin(\theta-\beta_1)] \ \ (\text{As } \theta = \beta_1+\beta_2) \end{aligned}\right] \quad (2)$$

Using (2) in (1)

$$\alpha = \left[\sin^{-1}(n\sin\beta_1) + \sin^{-1}\left(n\sin(\theta-\beta_1)\right)\right] - \theta$$

For α to be minimum, $\frac{d\alpha}{d\beta_1} = 0$

or,
$$\frac{n\cos\beta_1}{\sqrt{1-n^2\sin^2\beta_1}} - \frac{n\cos(\theta-\beta_1)}{\sqrt{1-n^2\sin^2(\theta-\beta_1)}} = 0$$

or,
$$\frac{\cos^2\beta_1}{(1-n^2\sin^2\beta_1)} = \frac{\cos^2(\theta-\beta_1)}{1-n^2\sin^2(\theta-\beta_1)}$$

or,
$$\cos^2\beta_1\left(1-n^2\sin^2(\theta-\beta_1)\right) = \cos^2(\theta-\beta_1)\,(1-n^2\sin^2\beta_1)$$

or,
$$(1-\sin^2\beta_1)\left(1-n^2\sin^2(\theta-\beta_1)\right) = \left(1-\sin^2(\theta-\beta_1)\right)(1-n^2\sin^2\beta_1)$$

or,
$$1-n^2\sin^2(\theta-\beta_1) - \sin^2\beta_1 + \sin^2\beta_1\, n^2\sin^2(\theta-\beta_1)$$
$$= 1-n^2\sin^2\beta_1 - \sin^2(\theta-\beta_1) + \sin^2\beta_1\, n^2\sin^2(\theta-\beta_1)$$

or,
$$\sin^2(\theta-\beta_1) - n^2\sin^2(\theta-\beta_1) = \sin^2\beta_1(1-n^2)$$

or,
$$\sin^2(\theta-\beta_1)(1-n^2) = \sin^2\beta_1(1-n^2)$$

or,
$$\theta-\beta_1 = \beta_1 \quad \text{or} \quad \beta_1 = \theta/2$$

But
$$\beta_1+\beta_2 = \theta, \quad \text{so,} \quad \beta_2 = \theta/2 = \beta_1$$

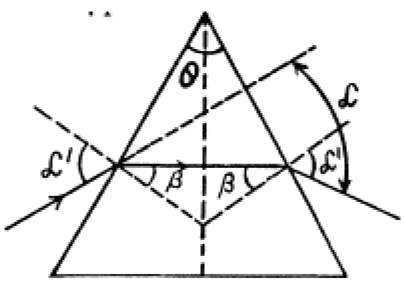
which is the case of symmetric passage of ray.
In the case of symmetric passage of ray

$\alpha_1 = \alpha_2 = \alpha'$ (say)

and $\beta_1 = \beta_2 = \beta = \theta/2$

Thus the total deviation

$\alpha = (\alpha_1 + \alpha_2) - \theta$



$\alpha = 2\alpha' - \theta$ or $\alpha' = \frac{\alpha + \theta}{2}$ (1)

But from the Snell's law $\sin \alpha = n \sin \beta$

So, $\sin \frac{\alpha + \theta}{2} = n \sin \frac{\theta}{2}$

Photometry and Geometrical Optics (Part - 2)

Q.21. The least deflection angle of a certain glass prism is equal to its refracting angle. Find the latter.

Ans. In this case we have

$$\sin\frac{\alpha+\theta}{2} = n\sin\frac{\theta}{2} \text{ (see soln. of 5.20)}$$

In our problem $\alpha = \theta$

So, $\sin\theta = n\sin(\theta/2)$ or $2\sin(\theta/2)\cos(\theta/2) = n\sin(\theta/2)$

Hence $\cos(\theta/2) = \frac{n}{2}$ or $\theta = 2\cos^{-1}(n/2) = 83°$, where $n = 1{\cdot}5$

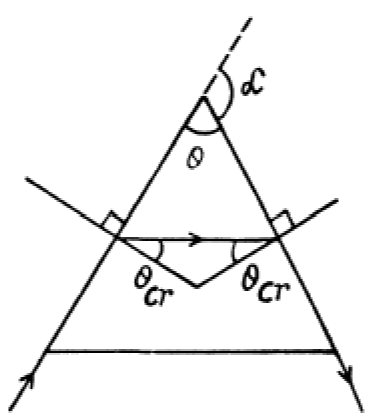
Q.22. Find the minimum and maximum deflection angles for a light ray passing through a glass prism with refracting angle θ = 60°.

Ans. In the case of minimum deviation

$$\sin\frac{\alpha+\theta}{2} = n\sin\frac{\theta}{2}$$

So, $\alpha = 2\sin^{-1}\left\{n\sin\frac{\theta}{2}\right\} - \theta = 37°$, for $n = 1{\cdot}5$

Passage of ray for grazing incidence and grazing imergence is the condition for maximum deviation (Fig.). From Fig.



$$\alpha = \pi - \theta = \pi - 2\,\theta_{cr}$$

(where θ_{cr} is the critical angle)

So, $\alpha = \pi - 2\sin(1/n) = 58°$,

for $n = 1{\cdot}5$ = R.I. of glass.

Q.23. A trihedral prism with refracting angle 60° provides the least deflection angle 37° in air. Find the least deflection angle of that prism in water.

Ans. The least deflection angle is given by the formula,

$\delta = 2\alpha - \theta$, where α is the angle of incidence at first surface and θ is the prism angle.

Also from Snell's law, $n_1 \sin\alpha = n_2 \sin(\theta/2)$, as the angle of refraction at first surface is equal to half the angle of prism for least deflection

so, $$\sin\alpha = \frac{n_2}{n_1}\sin(\theta/2) = \frac{1{\cdot}5}{1{\cdot}33}\sin 30° = {\cdot}5639$$

or, $$\alpha = \sin^{-1}({\cdot}5639) = 34{\cdot}3259°$$

Substituting in the above (1), we get, $\delta = 8{\cdot}65°$

Q.24. A light ray composed of two monochromatic components passes through a trihedral prism with refracting angle θ = 60°. Find the angle Δα between the components of the ray after its passage through the prism if their respective indices of refraction are equal to 1.515 and 1.520. The prism is oriented to provide the least deflection angle.

Ans. From the Cauchy's formula, and also experimentally the R.I. of a medium depends upon the wavelength of the mochromatic ray i.e. $n = f(\lambda)$ In the case of least deviation of a monochromatic ray the passage a prism, we have:

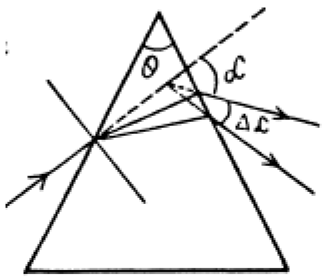
$$n \sin\frac{\theta}{2} = \sin\frac{\alpha + \theta}{2} \quad (1)$$

The above equation tales us that we have n = n (α) , so we may write

$$\Delta n = \frac{dn}{d\alpha}\Delta\alpha \quad (2)$$

From Eqn. (1)

$$dn \sin\frac{\theta}{2} = \frac{1}{2}\cos\frac{\alpha + \theta}{2}\,d\alpha$$



Or

$$\frac{dn}{d\alpha} = \frac{\cos\frac{\alpha+\theta}{2}}{2\sin\frac{\theta}{2}} \qquad (3)$$

From Eqn. (2) and (3)

$$\Delta n = \frac{\cos\frac{\alpha+\theta}{2}}{2\sin\frac{\theta}{2}}\Delta\alpha$$

$$\text{or,} \qquad \Delta n = \frac{\sqrt{1-\sin^2\left(\frac{\alpha+\theta}{2}\right)}}{2\sin\frac{\theta}{2}}\Delta\alpha = \frac{\sqrt{1-n^2\sin^2\frac{\theta}{2}}}{2\sin\frac{\theta}{2}}\Delta\alpha \text{ (Using Eqn. 1.)}$$

Thus

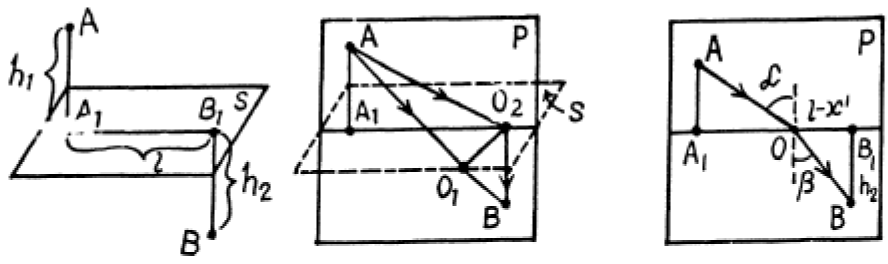
$$\Delta\alpha = \frac{2\sin\frac{\theta}{2}}{\sqrt{1-n^2\sin^2\frac{\theta}{2}}}\Delta n = 0\cdot 44$$

Q.25. Using Fermat's principle derive the laws of deflection and refraction of light on the plane interface between two media.

Ans. Fermat’s principle: “The actual path of propagation of light (trajectory of a light ray) is the path which can be followed by light with in the lest time, in comparison with all other hypothetical paths between the same two points. ” “Above statement is the original wordings of Fermat (A famous French scientist of 17th century)”

Deduction of the law of refraction from Fermat’s principle: Let the plane S be the interface between medium 1 and medium 2 with the refractive indices $n_1 = c/v_1$ and $n_2 = c/v_2$ Fig.(a). Assume, as usual, that $n_1 < n_2$. Two points are g iv en - one above the plane S (point A), the other under plane S (point B). The various distances are:

$AA_1 = h_1,\ BB_1 = h_2,\ A_1B_1 = l.$ We must find the path from A to B which can be covered by light faster than it can cover any other hypothetical path. Clearly, this path must consist of two straight lines, viz, AO in medium 1 and OB in medium 2; the point O in the plane S has to be found.
First of all, it follows from Fermat's priniciple that the point O must lie on the intersection of S and a plane P, which is perpendicular to S and passes through A and B.



(a) (b) (c)

Indeed, let us assume that this point does not lie in the plane P; let this be point 0 1 in Fig. (b). Drop the perpendicular O_2 from O_1 onto P . Since $AO_2 < AO_1$ and $BO_2 < BO_1$ it is clear that the time required to traverse AO_2B is less than that needed to cover the path AO_1B.

Thus, using Fermat's principle, we see that the first law of refraction is observed : the incident and the refracted rays lie in the same plane as the perpendicular to the interface at the point

where the ray is refracted. This plane is the plane P in Fig. (b); it is called the plane of incidence.

Now let us consider light rays in the plane of incidence Fig. (c). Designate
A_1O as x and $OB_1 = l - x$. The time it takes a ray to travel from A to O and then from O to B is

$$T = \frac{AO}{v_1} + \frac{OB}{v_2} = \frac{\sqrt{h_1^2 + x^2}}{v_1} + \frac{\sqrt{h_2^2 + (l-x)^2}}{v_2} \qquad (1)$$

The time depends on the value of x. According to fermat's principle, the value of x must minimize the time T. A t this value of x the derivative dT/dx equals zero:

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{h_1^2 + x^2}} - \frac{l-x}{v_2\sqrt{h_2^2 + (l-x)^2}} = 0. \qquad (2)$$

Now,

$$\frac{x}{\sqrt{h_1^2 + x^2}} = \sin\alpha, \text{ and } \frac{l - x}{\sqrt{h_2^2 + (l-x)^2}} = \sin\beta,$$

Consequently,

$$\frac{\sin\alpha}{v_1} - \frac{\sin\beta}{v_2} = 0, \quad \text{or} \quad \frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2}$$

So,

$$\frac{\sin\alpha}{\sin\beta} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1}$$

Note: Fermat himself could not use Eqn. 2. as mathematical analysis was developed later by Newton and Leibniz. To deduce the law of the refraction of light, Fermat used his own maximum and minimum method of calculus, which, in fact, corresponded to the subsequently developed method of finding the minimum (maximum) of a function by differentiating it and equating the derivative to zero.

Q.26. By means of plotting find: (a) the path of a light ray after reflection from a concave and convex spherical mirrors (see Fig. 5.4, where F is the focal point, 00' is the optical axis);

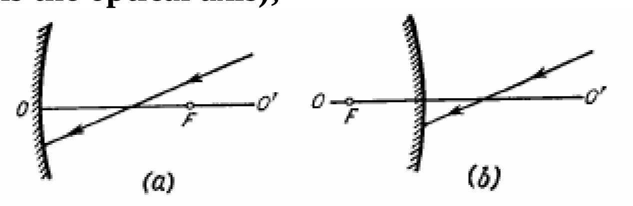


Fig. 5.4.

(b) The positions of the mirror and its focal point in the cases illustrated in Fig. 5.5, where P and P' are the conjugate points.

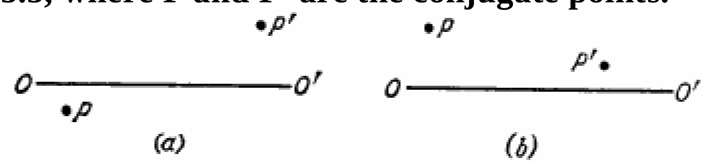
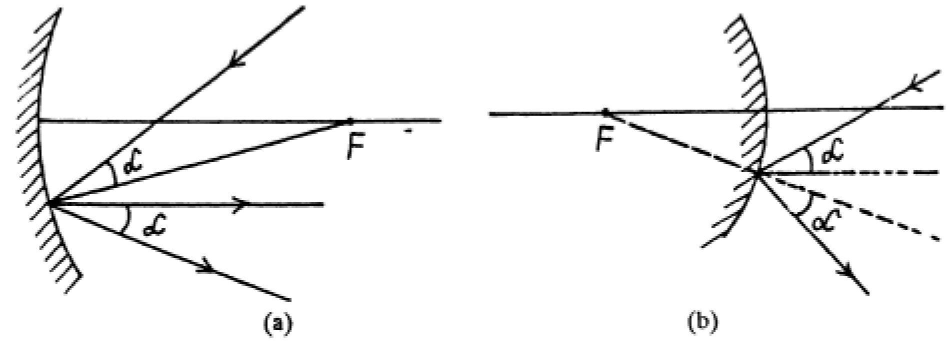


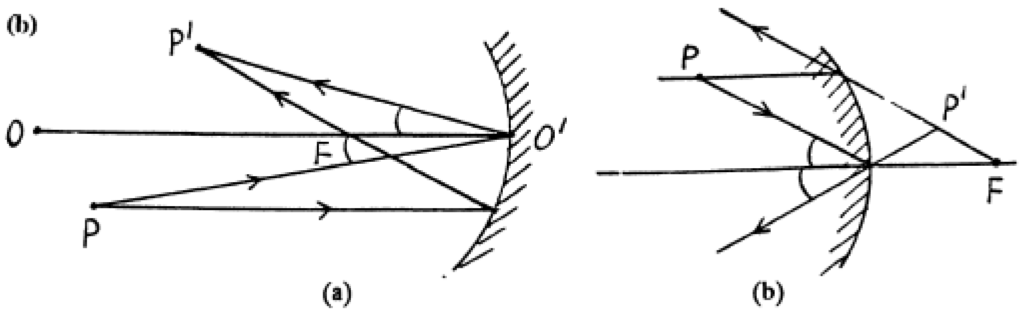
Fig. 5.5.

Ans. (a) Look for a point O' on the axis such that O' F and O' P make equal angles with O' O.

This determines the position of the mirror. Draw a ray from P parallel to the axis. This must on reflection pass through F. The intersection of the reflected ray with principal axis determines the focus.



(b) Suppose P is the object and F is the image. Then the mirror is convex because the image is virtual, erect & dim finished. Look for a point X (between P & F) on the axis such that PX and F'X make equal angle with the axis.



Q.27. Determine the focal length of a concave mirror if:
(a) with the distance between an object and its image being equal to l = 15 cm, the transverse magnification β = —2.0;
(b) in a certain position of the object the transverse magnification is β_1 = —0.50 and in another position displaced with respect to the former by a distance l = 5.0 cm the transverse magnification β_2 = = —0.25.

Ans. (a) From the mirror formula,

$$\frac{1}{s'}+\frac{1}{s}=\frac{1}{f} \text{ we get } f=\frac{s' s}{s'+s} \qquad (1)$$

In accordance with the problem $s - s' = l \quad \frac{s'}{s} = \beta,$

From these two relations, we get : $s = \frac{l}{1-\beta}, \quad s' = -\frac{l\beta}{1+\beta}$

Substituting it in the Eqn. (1),

$$f = \frac{\beta\left(\frac{l}{1-\beta}\right)^2}{l\left(\frac{1-\beta}{1-\beta}\right)} = \frac{l\beta}{(1-\beta^2)} = -10 \text{ cm}$$

(b) Again we have, $\frac{1}{s'} + \frac{1}{s} = \frac{1}{f}$ or, $\frac{s}{s'} + 1 = \frac{s}{f}$

or, $\frac{1}{\beta_1} = \frac{s}{f} - 1 = \frac{s-f}{f}$

or, $\beta_1 = \frac{f}{s-f}$ (2)

Now, it is clear from the above equation, that for smaller p, s must be large, so the object is displaced away from the mirror in second position.

i.e. $\beta_2 = \frac{f}{s+l-f}$ (3)

Eliminating 5 from the Eqn. (2) and (3), we get,

$$f = \frac{l\beta_1\beta_2}{(\beta_2 - \beta_1)} = -2{\cdot}5 \text{ cm}.$$

Q.28. A point source with luminous intensity I_0 = 100 cd is positioned at a distance s = 20.0 cm from the crest of a concave mirror with focal length f = 25.0 cm. Find the luminous intensity of the reflected ray if the reflection coefficient of the mirror is p = 0.80.

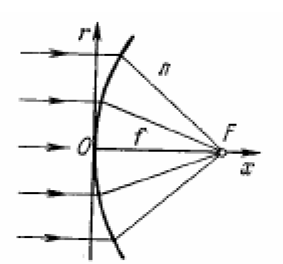


Fig. 5.6.

Ans. For a concave mirror as usual

$$\frac{1}{s'} + \frac{1}{s} = \frac{1}{f} \text{ so } s' = \frac{sf}{s-f}$$

(In coordinate convention s = - s is negative $\& \ f = -|f|$ is also negative.)

If A is the area of the mirror (assumed small) and the object is on the principal axis, then the light incident on the mirror per second is $I_0 \frac{A}{s^2}$.

This follows from the definition of luminous intensity as light emitted per second per unit solid angle in a given direction and the fact that $\frac{A}{s^2}$ is the solid angle subtended by the mirror at the source. of this a fraction p is reflected so if I is the luminous intensity of the image,

then $I \frac{A}{s'^2} = \rho I_0 \frac{A}{s^2}$

Hence

$$I = \rho I_0 \left(\frac{|f|}{|f| - s}\right)^2$$

(Because our convention makes f - ve for a concave mirror, we have to write $|f|$)

Substitution gives $I = 2.0 \times 10^3$ cd.

Q.29. Proceeding from Fermat's principle derive the refraction formula for paraxial rays on a spherical boundary surface of radius R between media with refractive indices n and n'.

Ans. For O_1 to be the image, the optical paths of all rays OAO_1 must be equal upto terms of leading order in h Thus

$$n_1 OA + n_2 AO_1 = \text{constant}$$

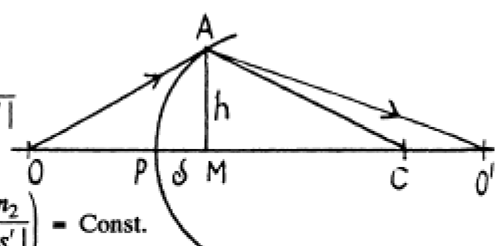
But, $OP = |s|$, $O_1 P = |s'|$ and so

$$OA = \sqrt{h^2 + (|s| + \delta)^2} \cong |s| + \delta + \frac{h^2}{2|s|}$$

$$O_1 A = \sqrt{h^2 + (|s'| - \delta)^2} \cong |s'| - \delta + \frac{h^2}{2|s'|}$$

(neglecting products $h^2 \delta$). Then

$$n_1 |s| + n_2 |s'| + n_1 \delta - n_2 \delta + \frac{h^2}{2}\left(\frac{n_1}{|s|} + \frac{n_2}{|s'|}\right) = \text{Const.}$$



Now $(r - \delta)^2 + h^2 = r^2$

or $h^2 = 2\dot{r}\delta$ or $\delta = \frac{h^2}{2r}$

Here $r = CP$.

Hence $n_1|s| + n_2|s'| + \frac{h^2}{2}\left\{\frac{n_1 - n_2}{r} + \frac{n_1}{|s|} + \frac{n_2}{|s'|}\right\} = \text{Constant}$

Since this must hold for all h, we have

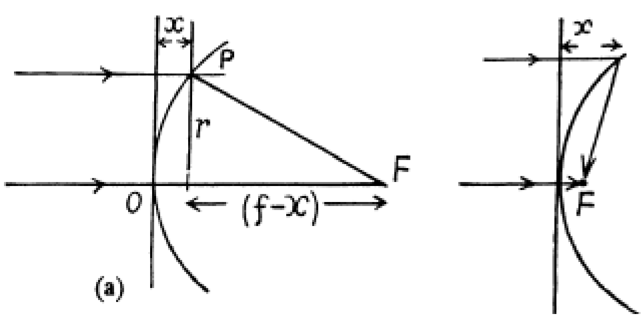
$$\frac{n_2}{|s'|} + \frac{n_1}{|s|} = \frac{n_2 - n_1}{r}$$

From our sign convention, $s' > 0,\ s < 0$ so we get

$$\frac{n_2}{s'} - \frac{n_1}{s} = \frac{n_2 - n_1}{r}.$$

Q.30. A parallel beam of light falls from vacuum on a surface enclosing a medium with refractive index n (Fig. 5.6). Find the shape of that surface, x (r), if the beam is brought into focus at the point F at a distance f from the crest O. What is the maximum radius of a beam that can still be focussed?

Ans. All rays focusing at a point must have traversed the same optical path. Thus



$$x + n\sqrt{r^2 + (f-x)^2} = nf \quad \text{or} \quad (nf - x)^2 = n^2 r^2 + n^2 (f-x)^2$$

or, $n^2 r^2 = (nf - x)^2 - [n(f-x)]^2 = (nf - x + nf - nx)(nf - x - nf + nx)$

$= x(n-1)(2nf - (n+1)x)$

$= 2n(n-1)fx - (n+1)(n-1)x^2$

Thus, $(n+1)(n-1)x^2 - 2n(n-1)fx + n^2r^2 = 0$

so, $$x = \frac{n(n-1)f \pm \sqrt{n^2(n-1)^2f^2 - n^2r^2(n+1)(n-1)}}{(n+1)(n-1)}$$

$$= \frac{nf}{n+1}\left[1 \pm \sqrt{1 - \frac{n+1}{n-1}\frac{r^2}{f^2}}\right]$$

Ray must move forward so $x < f$, for + sign $x > f$ for small r, so –sign.

(Also $x \to 0$ as $r \to 0$)

($x > f$ means ray turning back in the direction of incidence. (see Fig.)

Hence $$x = \frac{nf}{n+1}\left[1 - \sqrt{1 - \frac{n+1}{n-1}\frac{r^2}{f^2}}\right]$$

For the maximum value of r,

$$\sqrt{1 - \frac{n+1}{n-1}\frac{r^2}{f^2}} = 0 \qquad \text{(A)}$$

Because the expression under the radical sign must be non-negative, which gives the maximum value of r.

Hence from Eqn. (A), $r_{max} = f\sqrt{(n-1)/(n+1)}$

Q.31. A point source is located at a distance of 20 cm from the front surface of a symmetrical glass biconvex lens. The lens is 5.0 cm thick and the curvature radius of its surfaces is 5.0 cm. How far beyond the rear surface of this lens is the image of the source formed?

Ans. As the given lense has significant thickness, the thin lense, formula cannote be used.

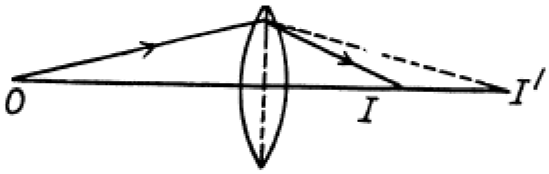
For refraction at the front surface from the formula

$$\frac{n'}{s'} - \frac{n}{s} = \frac{n' - n}{R}$$

$$\frac{1 \cdot 5}{s'} - \frac{1}{-20} = \frac{1 \cdot 5 - 1}{5}$$

On simplifying we get , s' = 30 cm.

Thus the image l' produced by the front surface behaves as a virtual source for the rear surface at distance 25 cm from it, because the thickness of the lense is 5 cm. Again from the refraction formula at cerve surface



$$\frac{n'}{s'} - \frac{n}{s} = \frac{n' - n}{R}$$

$$\frac{1}{s'} - \frac{1 \cdot 5}{25} = \frac{1 - 1 \cdot 5}{-5}$$

On simplifying, $s' = +6 \cdot 25$ cm

Thus we get a real image l at a distance 6- 25 cm beyond the rear surface (Fig.).

Q.32. An object is placed in front of convex surface of a glass piano-convex lens of thickness d = 9.0 cm. The image of that object is formed on the plane surface of the lens serving as'a screen. Find:
(a) the transverse magnification if the curvature radius of the lens's convex surface is R = 2.5 cm;
(b) the image illuminance if the luminance of the object is L = 7700cd/m^2 and the entrance aperture diameter of the lens is D = 5.0 mm; losses of light are negligible.

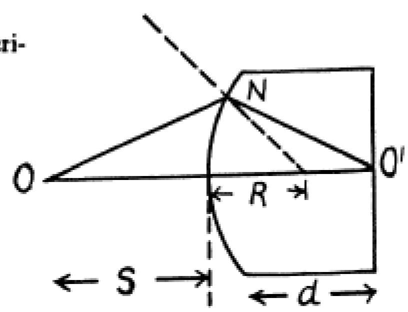
Ans. (a) The formation of the image of a source 5, placed at a distance u from the pole of the convex surface of plano-convex lens of thickness d is shown in the figure.

On applying the formula for refraction through spherical surface, we get

$$\frac{n}{s'} - \frac{1}{s} = (n-1)/R \text{ , (here } n_2 = n \text{ and } n_1 = 1)$$

or, $\frac{n}{d} - \frac{1}{s} = (n-1)/R$ or, $\frac{1}{s} = \frac{n}{d} - \frac{(n-1)}{R}$

or, $\frac{s'}{s} = s'\left\{\frac{n}{d} - \frac{(n-1)}{R}\right\}$



But in this case optical path of the light, corresponding to the distance v in the medium is v/n, so the magnification produced will be,

$$\beta = \frac{s'}{n\,s} = \frac{s'}{n}\left\{\frac{n}{d} - \frac{(n-1)}{R}\right\} = \frac{d}{n}\left\{\frac{n}{d} - \frac{(n-1)}{R}\right\} = 1 - \frac{d\,(n-1)}{nR}$$

Substituting the values, we get magnification β = - 0-20.
(b) If the transverse area of the object is A (assumed small), the area of the image is $\beta^2 A$.

We shall assume that $\frac{\pi D^2}{4} > A$. Then light falling on the Jens is $LA\,\frac{\pi D^2/4}{s^2}$ from the definition of luminance (See Eqn. (5.1c) of the book; here

$\cos\theta \approx 1$ if $D^2 << s^2$ and $d\Omega = \frac{\pi D^2/4}{s^2}$). Then the illum inance of the image is

$$L A \frac{\pi D^2/4}{s^2} \Big/ \beta^2 A = L n^2 \pi D^2/4d^2$$

Substitution gives 42 lx.

Q.33. Find the optical power and the focal lengths
(a) of a thin glass lens in liquid with refractive index n_o = 1.7 if its optical power in air is $\Phi_0 = -5.0$ D;
(b) of a thin symmetrical biconvex glass lens, with air on one side and water on the other side, if the optical power of that lens in air is $\Phi_0 = +10$ D.

Ans. (a) Optical power of a thin lens of R.I. n in a medium with R I . n_0 is given by :

$$\Phi = (n - n_0)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \text{(A)}$$

From Eqn.(A), when the lens is placed in air :

$$\Phi_0 = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (1)$$

Similarly from Eqn.(A), when the lens is placed in liquid :

$$\Phi = (n - n_0)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (2)$$

Thus from Eqns (1) and (2)

$$\Phi = \frac{n - n_0}{n - 1}\Phi_0 = 2D$$

The second focal length, is given by

$f' = \frac{n'}{\Phi}$, where n is the R.I. of the medium in which it is placed.

$$f' = \frac{n_0}{\Phi} = 85 \text{ cm}$$

(b) Optical power of a thin lens of RI. n placed in a medium of RI. n_o is given by :

$$\Phi = (n - n_0)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \text{(A)}$$

For a biconvex lens placed in air medium from Eqn. (A)

$$\Phi_0 = (n-1)\left(\frac{1}{R} - \frac{1}{-R}\right) = \frac{2(n-1)}{R} \qquad (1)$$

where R is the radius of each curve surface of the lens
Optical power of a spherical refractive surface is given by :

$$\Phi = \frac{n' - n}{R} \qquad \text{(B)}$$

For the rear surface of the lens which divides air and glass medium

$$\Phi_0 = \frac{n-1}{R} \text{ (Here } n \text{ is the R.I. (2) of glass)}$$

Similarly for the front surface which divides wati nd glass medium

$$\Phi_l = \frac{n - n}{-R} = \frac{n - n_0}{R} \qquad (3)$$

Hence the optical power of the given optical system

$$\Phi = \Phi_a + \Phi_l = \frac{n-1}{R} + \frac{n - n_0}{R} = \frac{2n - n_0 - 1}{R} \qquad (4)$$

From Eqns (1) and (4)

$$\frac{\Phi}{\Phi_0} = \frac{2n - n_0 - 1}{2(n-1)} \quad \text{So} \quad \Phi = \frac{(2n - n_0 - 1)}{2(n-1)}\Phi_0$$

Focal length in air, $f = \frac{1}{\Phi} = 15 \text{ cm}$

and focal length in water $= \frac{n_0}{\Phi} = 20 \text{ cm}$ for $n_0 = \frac{4}{3}$.

Q.34. By means of plotting find: (a) the path of a ray of light beyond thin converging and diverging lenses (Fig. 5.7, where 00' is the optical axis, F and F' are the front and rear focal points);

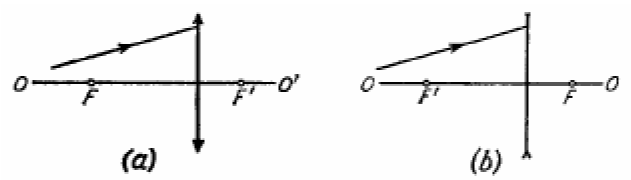


Fig. 5.7.

(b) the position of a thin lens and its focal points if the position of the optical axis 00' and the positions of the cojugate points P, P' (see Fig. 5.5) are known; the media on both sides of the lenses are identical; (c) the path of ray 2 beyond the converging and diverging lenses (Fig. 5.8) if the path of ray I and the positions of the lens and of its

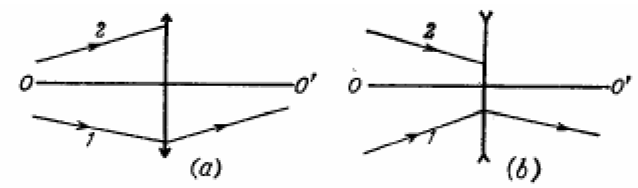
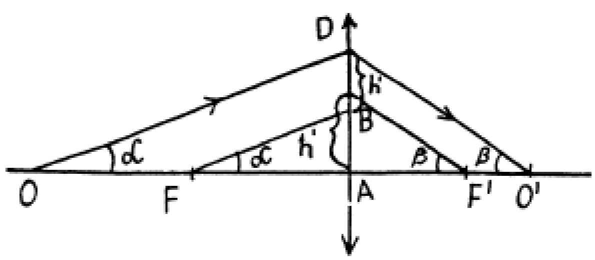


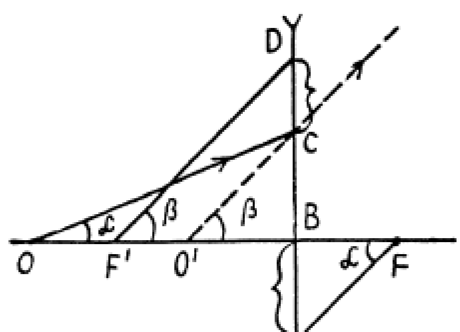
Fig. 5.8.

optical axis 00' are all known; the media on both sides of the lenses are identical.

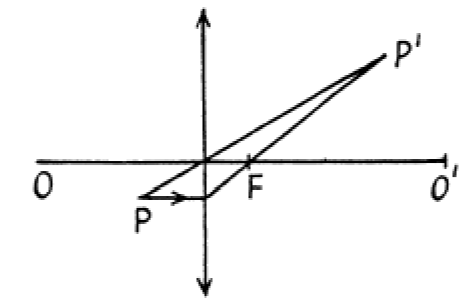
Ans. (a) Clearly the media on the sides are different. The front focus F is the position of the object (virtual or real) for which the image is formated at infinity. The rear focus F' is the position of the image (virtual or real) of the object at infinity, (a) Figures 5.7 (a) & (b). This geometrical construction ensines that the second of the equations (5.1g) is obeyed



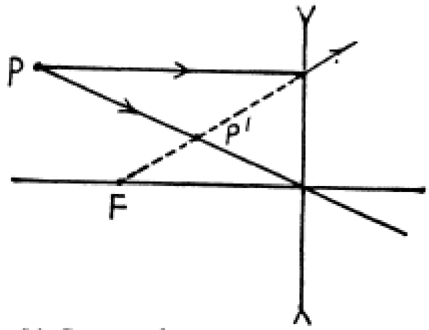
(a) Convex lens
(b) Figure 5.5 (a) & (b) with lens



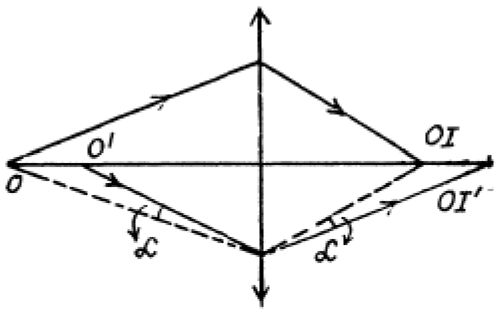
(b) Concave lens



(a) Convex lens
(*P* is the object)



(b) Concave lens



(c) Figure (5.8) (a) & (b).
Clearly, the important case is that when the rays (1) & (2) are not symmetric about the principal axis, otherwise the figure can be completed by reflection in the principal axis. Knowing one path we know the path of all rays connecting the two points. For a different object. We proceed as shown below, we use the fact that a ray incident at a given height above the optic centre suffers a definite deviation.
The concave lens can be discussed similarly.

Q.35. A thin converging lens with focal length f = 25 cm projects the image of an object on a screen removed from the lens by a distance 1=5.0 m. Then the screen was drawn closer to the lens by a distance Δl = 18 cm. By what distance should the object be shifted for its image to become sharp again?

Ans. Since the image is formed on the screen, it is real, so for a conversing lens object is in the incident side.

Let S_1 and S_2 be the magnitudes of the object distance in the first and second case respectively. We have the lens formula

$$\frac{1}{s'} - \frac{1}{s} = \frac{1}{f} \qquad (1)$$

In the first case from Eqn. (1)

$$\frac{1}{(+l)} - \frac{1}{(-s_1)} = \frac{1}{f} \quad \text{or,} \quad s_1 = \frac{f(l)}{l-f} = 26\cdot 31 \text{ cm}.$$

Similarly from Eqn.(l) in the second case

$$\frac{1}{(l-\Delta l)} - \frac{1}{(-s_2)} = \frac{1}{f} \quad \text{or,} \; s_2 = \frac{lf}{(l-\Delta l)-f} = 26\cdot 36 \text{ cm}.$$

Thus the sought distance $\Delta x = s_2 - s_1 = 0\cdot 5 \text{ mm} \approx \Delta l f^2/(l-f^2)$

Q.36. A source of light is located at a distance l = 90 cm from a screen. A thin converging lens provides the sharp image of the source when placed between the source of light and the screen at two positions. Determine the focal length of the lens if (a) the distance between the two positions of the lens is Δl = = 30 cm; (b) the transverse dimensions of the image at one position of the lens are η = 4.0 greater than those at the other position.

Ans. The distance between the object and the image is l. Let x = distance between the object and the lens. Then, since the image is real, we have in our convention, $u = -x, \; v = l - x$

so $$\frac{1}{x} + \frac{1}{l-x} = \frac{1}{f}$$

or $$x(l-x) = lf \quad \text{or } x^2 - xl + lf = 0$$

Solving we get the roots

$$x = \frac{1}{2}\left[l \pm \sqrt{l^2 - 4lf}\right]$$

(We must have / > 4f for real roots.)

(a) If the distance between the two positions of the lens is Δl, then clearly

$$\Delta l = x_2 - x_1 = \text{difference between roots} = \sqrt{l^2 - 4lf}$$

so $$f = \frac{l^2 - \Delta l^2}{4l} = 20 \text{ cm}.$$

(b) The two roots are conjugate in the sense that if one gives the object distance the other gives the corresponding image distance (in both cases). Thus the magnifications are

$$-\frac{l+\sqrt{l^2-4lf}}{l-\sqrt{l^2-4lf}} \text{ (enlarged) and } -\frac{l-\sqrt{l^2-4lf}}{l+\sqrt{l^2-4lf}} \text{ (diminished).}$$

The ratio of these magnification being η we have

$$\frac{l-\sqrt{l^2-4lf}}{l-\sqrt{l^2-4lf}} = \sqrt{\eta} \qquad \text{or} \qquad \frac{\sqrt{l^2-4lf}}{l} = \frac{\sqrt{\eta}-1}{\sqrt{\eta}+1}$$

or
$$1-\frac{4f}{l} = \left(\frac{\sqrt{\eta}-1}{\sqrt{\eta}+1}\right)^2 = 1-4\frac{\sqrt{\eta}}{(1+\sqrt{\eta})^2}$$

Hence
$$f = l\frac{\sqrt{\eta}}{(1+\sqrt{\eta})^2} = 20 \text{ cm.}$$

Q.37. A thin converging lens is placed between an object and a screen whose positions are fixed. There are two positions of the lens at which the sharp image of the object is formed on the screen. Find the transverse dimension of the object if at one position of the lens the image dimension equals h' = 2.0 mm and at the other, h" = 4.5 mm.

Ans. We know from the previous problem that the two magnifications are reciprocals of each other (B' β" = 1). If h is the size of the object then h' = β' h and
$h'' = \beta'' h$

Hence $h = \sqrt{h' h''}$.

Q.38. A thin converging lens with aperture ratio D : f = 1: 3.5 (D is the lens diameter, f is its focal length) provides the image of a sufficiently distant object on a photographic plate. The object luminance is L = 260 cd/m^2. The losses of light in the lens amount to α = 0.10. Find the illuminance of the image.

Ans. Refer to problem 5.32 (b). If A is the area of the object, then provided the angular diameter of the object at the lens is much smaller than other relevant angles like $\frac{D}{f}$ we calculate the light falling on the lens as $LA\frac{\pi D^2}{4s^2}$

Where u^2 is the object distance squared. If β is the transverse magnification $\left(\beta = \frac{s'}{u}\right)$ then the area of the image is β^{2}A. Hence the illuminance of the image (also taking account of the light lost in the lens)

$$E = (1-\alpha)LA\frac{\pi D^2}{4s^2}\frac{1}{\beta^2 A} = \frac{(1-\alpha)\pi D^2 L}{4f^2}$$

Since s' = f for a distant object Substitution gives E = 15 lx.

Q.39. How does the luminance of a real image depend on diameter D of a thin converging lens if that image is observed (a) directly; (b) on a white screen backscattering according to Lambert's law?

Ans. (a) If s = object distance, s' = average distance, L = luminance of the sounce, ΔS = area of the source assumed to be a plane surface held normal to the principal axis, then we find for the flux $\Delta\Phi$ incident on the lens

$$\Delta\Phi = \int L\,\Delta S\cos\theta\,d\Omega$$

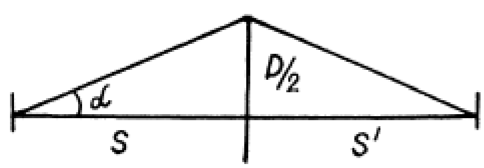
$$= L\,\Delta S\int_0^{\infty}\cos\theta\,2\pi\sin\theta\,d\theta = L\,\Delta S\,\pi\sin^2\alpha = L\,\Delta S\,\frac{\pi D^2}{4s^2}$$

Here we are assuming D « s, and ignoring the variation of L since a is small

Then if L' is the luminance of the image, and $\Delta S' = \left(\frac{S'}{S}\right)^2\Delta S$ is the area of the image then similarly

$$L'\,\Delta S'\,\frac{D^2}{4s'^2}\pi = L'\,\Delta S\,\frac{D^2}{4s^2}\pi = L\,\Delta S\,\frac{D^2}{4s^2}\pi$$

or $L' = L$ irrespective of D.



(b) In this case the image on the white screen from a Lambert source. Then if its luminance is L_0 its luminosity will be the πL_0 and $\pi L_0\frac{s'^2}{s^2}\Delta S = L\,\Delta S\,\frac{D^2}{4s^2}\pi$

or $L_0 \propto D^2$

since s' depends on f, s but not on D.

Q.40. There are two thin symmetrical lenses: one is converging, with refractive index $n_1 = 1.70$, and the other is diverging with refractive index $n_2 = 1.51$. Both lenses have the same curvature radius of their surfaces equal to R = 10 cm. The lenses were put close together and submerged into water. What is the focal length of this system in water?

Ans. Focal length of the converging lens, when it is submerged in water of R.I. n_0 (say):

$$\frac{1}{f_1} = \left(\frac{n_1}{n_0} - 1\right)\left(\frac{1}{R} - \frac{1}{R}\right), \frac{2(n_1 - n_0)}{n_0 R} \qquad (1)$$

Similarly, the focal length of diverging lens in water.

$$\frac{1}{f_2} = \left(\frac{n_2}{n_0} - 1\right)\left(\frac{1}{-R} - \frac{1}{R}\right) \frac{-2(n_2 - n_0)}{n_0 R} \qquad (2)$$

Now, when they are put together in the water, the focal length of the system,

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

$$= \frac{2(n_1 - n_2)}{n_0 R} - \frac{2(n_2 - n_0)}{n_0 R} = \frac{2(n_1 - n_2)}{n_0 R}$$

Or, $f = \dfrac{-n_0 R}{2(n_1 - n_2)} = 35 \text{ cm}$

Photometry and Geometrical Optics (Part - 3)

Q.41. Determine the focal length of a concave spherical mirror which is manufactured in the form of a thin symmetric biconvex glass lens one of whose surfaces is silvered. The curvature radius of the lens surface is R = 40 cm.

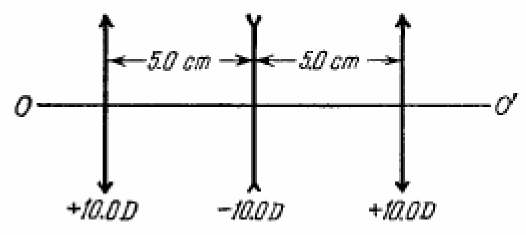
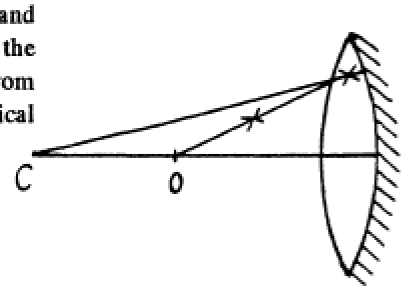


Fig. 5.9.

Ans.

C is the centre of curvature of the silvered surface and O is the effective centre of the equivalent mirror in the sence that an object at O forms a coincident image. From the figure, using the formula for refraction at a spherical surface, we have



$$\frac{n}{-R} - \frac{1}{2f} = \frac{n-1}{R} \quad \text{or} \quad f = \frac{-R}{2(2n-1)}$$

(In our convention f is $-ve$).

Substitution gives $f = -10$ cm.

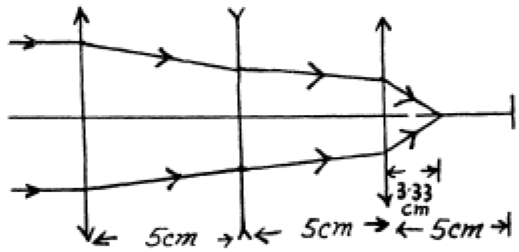
Q.42. Figure 5.9 illustrates an aligned system consisting of three thin lenses. The system is located in air. Determine:

(a) the position of the point of convergence of a parallel ray incoming from the left after passing through the system;

(b) the distance between the first lens and a point lying on the axis to the left of the system, at which that point and its image are located symmetrically with respect to the lens system.

Ans. (a) Path of a ray, as it passes through the lens system is as shown below.

Focal length of all the three lenses,



(a)

$f = \frac{1}{10} m = 10$ cm, , n e g le c t in g their signs.

Applying lens formula for the first lens, considering a ray coming from infinity,

$$\frac{1}{s'} - \frac{1}{\infty} = \frac{1}{f} \quad \text{or,} \quad s' = f = 10 \text{ cm},$$

and so the position of the image is 5 cm to the right of the second lens, when only the firstone is present, but the ray again gets refracted while passing through the second, so

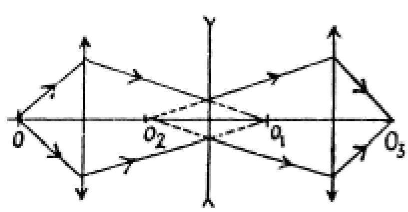
$$\frac{1}{s'} - \frac{1}{5} = \frac{1}{f} = \frac{1}{-10}$$

or, s' = 10 cm, which is now 5 cm left to the third lens so for this lens,

$$\frac{1}{s''} - \frac{1}{5} = \frac{1}{10} \quad \text{or} \quad \frac{1}{s''} = \frac{3}{10}$$

Or, $s'' = 10/3 = 3{\cdot}33$ cm. from the last lens.

(b) This means that if the object is x cm to be left of the first lens on the axis OO' then the image is x on to the right of the 3rd (last) lens. Call the lenses 1, 2, 3 from the left and let to be the object, O_1 its image by the first lens, O_2 the image of O_1 by the 2nd lens and O_3, the image of O_2 by the third lens.



(b)

O_1 and O_2 must be symmetrically located with respect to the lens L_2 and since this lens is concave, O_1must be at a distance $2|f_2|$ to be the right of L_2 and O_2 must be $2|f_2|$ to be the left of L_2, One can check that this satisfies lens equation for the third lens Ls

$$u = -(2|f_2| + 5) = -25 \text{ cm.}$$

$$s' = x, \quad f_3 = 10 \text{ cm.}$$

Hence $\frac{1}{x} + \frac{1}{25} = \frac{1}{10}$ so $x = 16.67$ cm.

Q.43. A Galilean telescope of 10-fold magnification has the length of 45 cm when adjusted to infinity. Determine:
(a) the focal lengths of the telescope's objective and ocular;
(b) by what distance the ocular should be displaced to adjust the telescope to the distance of 50 m.

Ans. (a) Angular magnification for Galilean telescope in normal adjustment is given as.

$$\Gamma = f_o/f_e$$

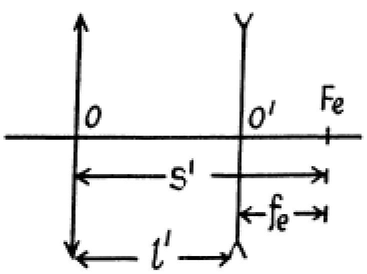
Or, $10 = f_o/f_e \quad \text{or} \quad f_o = 10 f_e$ (1)

The length of the telescope in this case.

$$l = f_o - f_e = 45 \text{ cm. given,}$$

So, using (1), we get,

$$f_e = +5 \text{ and } f_o = +50 \text{ cm.}$$



(b) Using lens formula for the objective,

$$\frac{1}{s'_o} - \frac{1}{s_o} = \frac{1}{f_o} \text{ or, } s'_o = \frac{s_o f_o}{s_o + f_o} = 50{\cdot}5 \text{ cm}$$

From the figure, it is clear that,

$s'_o = l' + f_e$, where l' is the new tube length.

or, $l' = v_o - f_e = 50{\cdot}5 - 5 = 45{\cdot}5$ cm.

So, the displacement of ocular is,

$$\Delta l = l' - l = 45{\cdot}5 - 45 = 0{\cdot}5 \text{ cm}$$

Q.44. Find the magnification of a Kepler Ian telescope adjusted to infinity if the mounting of the objective has a diameter D and the image of that mounting formed by the telescope's ocular has a diameter d.

Ans. In the Kepler Ian telescope, in normal adjustment, the distance between the objective and eyepiece is $f_0 + f_e$. The image of the mounting produced by the eyepiece is formed at a distance v to the right where

$$\frac{1}{s'} - \frac{1}{s} = \frac{1}{f_e}$$

But $s = -(f_0 + f_e),$

So $\frac{1}{s'} = \frac{1}{f_e} - \frac{1}{f_0 + f_e} = \frac{f_0}{f_e(f_0 + f_e)}$

The linear magnification produced by the eyepiece of the mounting is, in magnitude,

$$|\beta| = \left|\frac{s'}{s}\right| = \frac{f_e}{f_0}$$

This equals $\frac{d}{D}$ according to the problem so

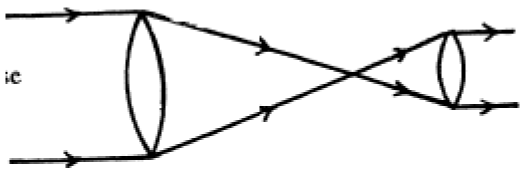
$$\Gamma = \frac{f_e}{f_e} = \frac{D}{d}.$$

Q.45. On passing through a telescope a flux of light increases its intensity $\eta = 4.0.10^4$ times. Find the angular dimension of a distant object if its image formed by that telescope has an angular dimension $\psi' = 2.0°$.

Ans. It is clear from the figure that a parallel beam of light, originally of intensity I_0 has, on emerging from the telescope, an intensity.

$$I = I_0 \left(\frac{f_0}{f_e}\right)^2$$

because it is concentrated over a section whose diameter is f_e/f_o of the diameter of the cross section of the incident beam.



Thus $$\eta = \left(\frac{f_0}{f_e}\right)^2$$

So $$\Gamma = \frac{f_0}{f_e} = \sqrt{\eta}$$

Now $$\Gamma = \frac{\tan\Psi'}{\tan\Psi} = \frac{\Psi'}{\Psi}$$

Hence $\Psi = \Psi'/\sqrt{\eta} = 0.6'$ on substitution.

Q.46. A Kepler Ian telescope with magnification Γ = 15 was submerged into water which filled up the inside of the telescope. To make the system work as a telescope again within the former dimensions, the objective was replaced. What has the magnification of the telescope become equal to? The refractive index of the glass of which the ocular is made is equal to n == 1.50.

Ans. When a glass lens is immersed in water its focal length increases approximately four times. We check this as follows as :

$$\frac{1}{f_a'} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f_w} = \left(\frac{n}{n_0} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{\frac{n}{n_0} - 1}{n-1} \cdot \frac{1}{f_a} = \frac{n - n_0}{n_0(n-1)}\frac{1}{f_a}$$

Now back to the problem. Originally in air

$$\Gamma = \frac{f_0}{f_e} = 15 \quad \text{so} \quad l = f_0 + f_e = f_e(\Gamma + 1)$$

In water, $$f_e' = \frac{n_0(n-1)}{n - n_0} f_e$$

and the focal length of the replaced objective is given by the condition

$$f_0' + f_e' = l = (\Gamma + 1) f_e$$

or

$$f_0' = (\Gamma + 1) f_e - f_e'$$

Hence

$$\Gamma' = \frac{f_0'}{f_e'} = (\Gamma + 1) \frac{n - n_0}{n_0 (n - 1)} - 1$$

Substitution gives ($n = 1.5$, $n_0 = 1.33$), $\Gamma' = 3.09$

Q.47. At what magnification Γ of a telescope with a diameter of the objective D = 6.0 cm is the illuminance of the image of an objection the retina not less than without the telescope? The pupil diameter is assumed to be equal to d_0 = 3.0 mm. The losses of light in the telescope are negligible.

Ans. If L is the luminance of the object, A is its area, s = distance of the object then light falling on the objective is

$$\frac{L \pi D^2}{4 s^2} A$$

The area of the image formed by the telescope (assuming that the image coincides with

The object) is Γ^2 A and the area of the final image on the retina is

$$= \left(\frac{f}{s}\right)^2 \Gamma^2 A$$

Where f = focal length of the eye lens. Thus the illuminance of the image on the retin

(when the object is observed through the telescope) is

$$\frac{L \pi D^2 A}{4 u^2 \left(\frac{f}{s}\right)^2 \Gamma^2 A} = \frac{L \pi D^2}{4 f^2 \Gamma^2}$$

When the object is viewed directly, the illuminance is, similarly, $\frac{L \pi d_0^2}{4 f^2}$

We want $\frac{L \pi D^2}{4 f^2 \Gamma^2} \geq \frac{L \pi d_0^2}{4 f^2}$

So, $\Gamma \leq \frac{D}{d_0} = 20$ on substitution of the values.

Q.48. The optical powers of the objective and the ocular of a microscope are equal to 100 and 20 D respectively. The microscope magnification is equal to 50. What will the magnification of the microscope be when the distance between the objective and the ocular is increased by 2.0 cm?

Ans. Obviously, $f_o = +1\text{ cm}$ and $f_e = +5\text{ cm}$

Now, we know that, magnification of a microscope,

$$\Gamma = \left(\frac{s'_o}{f_o} - 1\right)\frac{D}{f_e},$$ for distinct vision

Or, $$50 = \left(\frac{s'_o}{1} - 1\right)\frac{25}{5} \quad \text{or,} \quad v_o = 11\text{ cm.}$$

Since distance between objective and ocular has increased by 2 cm, hence it will cause the increase of tube length by 2cm.

so, $s'_o = s'_o + 2 = 13$

And hence, : $$\Gamma' = \left(\frac{s'_o}{f_o} - 1\right)\frac{D}{f_e} = 60$$

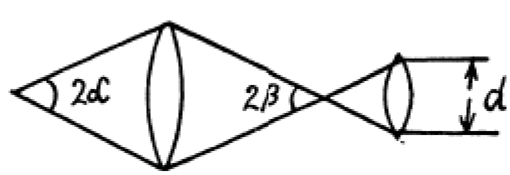
Q.49. A microscope has a numerical aperture sin α = 0.12, where a is the aperture angle subtended by the entrance pupil of the microscope. Assuming the diameter of an eye's pupil to be equal to d_o = = 4.0 mm, determine the microscope magnification at which
(a) the diameter of the beam of light coming from the microscope is equal to the diameter of the eye's pupil;
(b) the illuminance of the image on the retina is independent of magnification (consider the case when the beam of light passing through the system "microscope-eye" is bounded by the mounting of the objective).

Ans. It is implied in the problem that final image of the object is at infinity (otherwise light coming out of the eyepiece will not have a definite diameter).

(a) We see that $s'_0 \, 2\beta = |s_0| \, 2\alpha$, then

$$\beta = \frac{|s_0|}{s'_0}\alpha$$

Then, from the figure



$$d = 2f_e\beta = 2f_e\alpha \Big/ \frac{s'_0}{|s_0|}$$

But when the final image is at infinity, the magnification Γ in a microscope is given by

$$\Gamma = \frac{s'_0}{|s_0|}\cdot\frac{l}{f_e}\ (l = \text{least distance of distinct vision})\ \text{So}\ d = 2l\alpha/\Gamma$$

$$\text{So } d = d_0 \text{ when } \Gamma = \Gamma_0 = \frac{2l\alpha}{d_0} = 15 \text{ on putting the values.}$$

(b) If Γ is the magnification produced by the microscope, then the area of the image produced on the retina (when we observe an object through a microscope) is:

$$\Gamma^2\left(\frac{f}{s}\right)^2 A$$

Where u = distance of the image produced by the microscope from the eye lens, f = focal length of the eye lens and A = area of the object If φ = luminous flux reaching the objective from the object and d = d_0 so that the entire flux is admitted into the eye), then

the illuminance of the final image on the retina $= \dfrac{\Phi}{\Gamma^2\,(f/s)^2 A}$

But if $d \geq d_0^2$ then only a fraction $(d_0 \mid d)^2$ of light is admitted into the eye and the illuminance becomes

$$\frac{\Phi}{A\left(\frac{f}{s}\right)^2\Gamma^2}\left(\frac{d_0}{d}\right)^2 = \frac{\Phi\, d_0^2}{A\left(\frac{f}{s}\right)^2(2l\alpha)^2}$$

Independent of Γ The condition for this is then

$$d \geq d_0 \quad \text{or} \quad \Gamma \leq \Gamma_0 = 15.$$

Q.50. Find the positions of the principal planes, the focal and nodal points of a thin biconvex symmetric glass lens with curvature radius of its surfaces equal to R = 7.50 cm. There is air on one side of the lens and water on the other.

Ans. The primary and secondary focal length of a thick lens are given as,

$$f = -(n/\Phi)\{1-(d/n')\Phi_2\}$$

And $f' = +(n''/\Phi)\{1-(d/n')\Phi_1\}$, where φ is the lens power n , n' and n" are the refractive indices of first medium, lens material and the second medium beyond the lens. Φ_1 and Φ_2 are the powers of first and second spherical surface of the lens.

Here, $n = 1$, for lens, $n' = n$, for air

and $n'' = n_0$, for water.

So,

$$\left.\begin{aligned} f &= -1/\Phi_1 \\ \text{and } f' &= +n_0/\Phi \end{aligned}\right\}, \text{ as } d \approx 0, \quad (1)$$

Now, power of a thin lens,

$$\Phi = \Phi_1 + \Phi_2,$$

where,
$$\Phi_1 = \frac{(n-1)}{R}$$

and
$$\Phi_2 = \frac{(n_0 - n)}{-R}$$

So,
$$\Phi = (2n - n_0 - 1)/R \quad (2)$$

From equations (1) and (2), we get

$$f = \frac{-R}{(2n - n - 1)} = -11.2 \text{ cm}$$

and
$$f' = \frac{n_0 R}{(2n - n_0 - 1)} = +14.9 \text{ cm.}$$

Since the distance between the primary principal point and primary nodal point is given as,

$$x = f'\{(n'' - n)/n''\}$$

So, in this case,

$$x = (n_0/\Phi)(n_0 - 1)/n_0 = (n_0 - 1)/\Phi$$
$$= \frac{n_0}{\Phi} - \frac{1}{\Phi} = f' + f = 3{\cdot}7 \text{ cm.}$$

Q.51. By means of plotting find the positions of focal points and principal planes of aligned optical systems illustrated in Fig. 5.10:
(a) a telephoto lens, that is a combination of a converging and a diverging thin lenses (f_1 = 1.5 a, f_2 = —1.5 a);

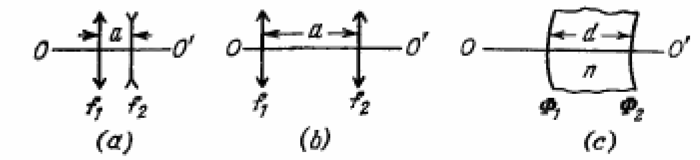


Fig. 5.10.

(b) a system of two thin converging lenses $(f_1 = 1.5\,a,\ f_2 = 0.5\,a)$;
(c) a thick convex-concave lens $(d = 4\text{ cm},\ n = 1.5,\ \Phi_1 = +50\text{ D},\ \Phi_2 = -50\text{ D})$.

Q.52. An optical system is located in air. Let 00' be its optical axis, F and F' are the front and rear focal points, H and H' are the front and rear principal planes, P and P' are the conjugate points. By means of plotting find:
(a) the positions F' and H' (Fig. 5.11a);
(b) the position of the point S' conjugate to the point S (Fig. 5.11b);

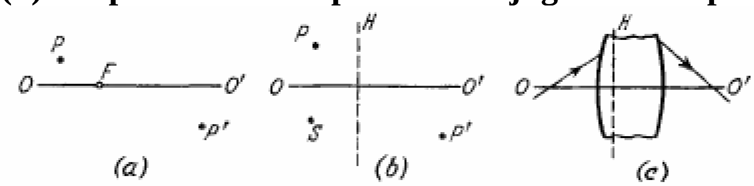
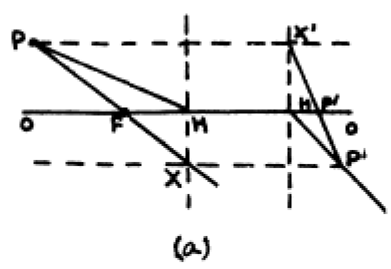


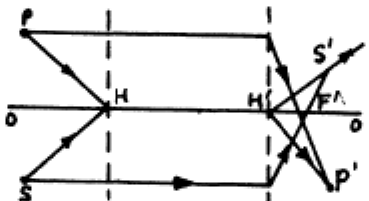
Fig. 5.11.

(c) the positions F, F', and H' (Fig. 5.11c, where the path of the ray of light is shown before and after passing through the system).

Ans. (a) Draw P' X parallel to the axis OO' and let PF interest it at X. That determines the principal point H. As the' medium on both sides of the system is the same, the principal point coincides with the nodal point Draw a ray parallel to PH through P' . That determines H' Draw a ray P X' parallel to the axis and join P'X".
That gives F' .

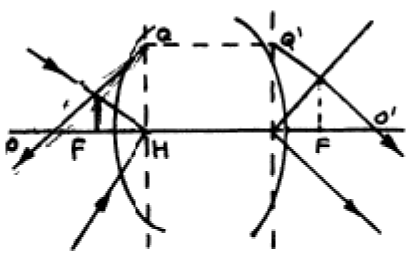


(b) We let H stand for the principal point (on the axis). Determine H" by drawing a ray P'H" passing through P' and parallel to PH. One ray (conjugate to SH) can be obtained from this. To get the other ray one needs to know F or F". This is easy because P and F are known. Finally we get S'.



(b)

(c) From the incident ray we determine Q. A line parallel to OO' through Q determines Q and hence H'.
H and H ' are then also the nodal points. A ray parallel to the incident ray through H will emeige parallel to it se lf through H '. That determines F . Similarity a ray parallel to the emergent ray through H determines F.

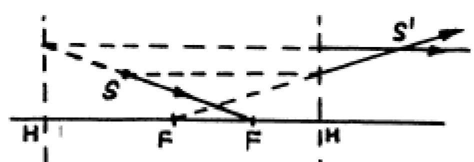


(c)

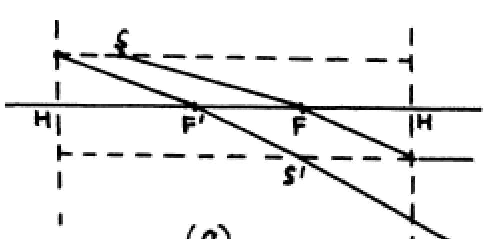
Q.53. Suppose F and F' are the front and rear focal points of an optical system, and H and H' are its front and rear principal points. By means of plotting find the position of the image S' of the point S for the following relative positions of the points S, F, F', H, and H':
(a) FSHH'F'; (b) HSF'FH'; (c) H'SF'FH; (d) F'H'SHF.

Ans. Here we do not assume that the media on the two sides of the system are the same.

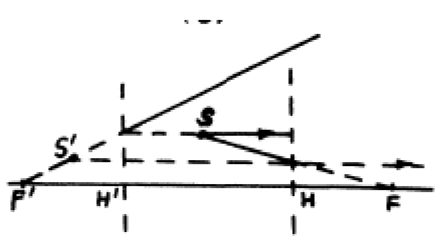
(a)



(b)



(c)



(d)

Q.54. A telephoto lens consists of two thin lenses, the front converging lens and the rear diverging lens with optical powers $\Phi_1 = = +10$ D and $\Phi_2 = -10$ D. **Find:**
(a) the focal length and the positions of principal axes of that system if the lenses are separated by a distance d = 4.0 cm;
(b) the distance d between the lenses at which the ratio of a focal length f of the system to a distance l between the converging lens and the rear principal focal point is the highest. What is this ratio equal to?

Ans. (a) Optical power of the system of combination of two lenses,
$\Phi = \Phi_1 + \Phi_2 - d\,\Phi_1\,\Phi_2$

On putting the values,

$\Phi = 4\,D$

Or, $f = \frac{1}{\Phi} = 25$ cm

Now, the position of primary principal plane with respect to the vertex of conveiging lens,

$X = \frac{d\,\Phi_2}{\Phi} = 10$ cm

Similarly, the distance of secondary principal plane with respect to the vertex of diverging lens.

$'X' = -\frac{d\,\Phi_1}{\Phi} = -10$ cm, i.e. 10 cm left to it.

(b) The distance between the rear principal focal point F' and the vertex of converging

lens,

$$l = d + \left(\frac{1}{\Phi}\right)(-d\,\Phi_1) = \frac{\Phi\,d}{\Phi} + \left(\frac{-d\,\Phi_1}{\Phi}\right)$$

and

$$f/l = \left(\frac{1}{\Phi}\right) \Big/ \frac{\Phi\,d}{\Phi} - \frac{d\,\Phi_1}{\Phi}, \quad \text{as} \quad f = \frac{1}{\Phi}$$

$$= 1/d\,\Phi - d\,\Phi_1$$

$$= 1/d\,(\Phi_1 + \Phi_2 - d\,\Phi_1\,\Phi_2) - d\,\Phi_1 = 1/d\,\Phi_2 - d^2\,\Phi_1\,\Phi_2$$

Now, if f/l is maximum for certain value of d then f/l will be minimum for the same value of d. And for minimum f/l,

$$d(l/f)/d\,d = \Phi_2 - 2\,d\,\Phi_1\,\Phi_2 = 0$$

or, $$d = \Phi_2/2\,\Phi_1\,\Phi_2$$

or, $$d = 1/2\,\Phi_1 = 5\text{ cm}$$

So, the required maximum ratio of $f/l = 4/3$.

Q.55. Calculate the positions of the principal planes and focal points of a thick convex-concave glass lens if the curvature radius of the convex surface is equal to R_1 = 10.0 cm and of the concave surface to R_2 = 5.0 cm and the lens thickness is d = 3.0 cm.

Ans. The optical power of first convex surface is,

$$\Phi = \frac{P\,(n-1)}{R_1} = 5\text{ D}, \quad \text{as } R_1 = 10\text{ cm}$$

and the optical power of second concave surface is,

$$\Phi_2 = \frac{(1-n)}{R_2} = -10\text{ D}$$

So, the optical power of the system,

$$\Phi = \Phi_1 + \Phi_2 - \frac{d}{n}\Phi_1\,\Phi_2 = -4\text{ D}$$

Now , the distance of the primary principal plane from the vertex of convex surface is given as,

$$x = \left(\frac{1}{\Phi}\right)\left(\frac{d}{n}\right)\Phi_2\text{ , here } n_1 = 1 \text{ and } n_2 = n\,.$$

$$= \frac{d\,\Phi_2}{\Phi\,n} = 5\text{ cm}$$

and the distance of secondary principal plane from the vertex of second concave surface,

$$x' = -\left(\frac{1}{\Phi}\right)\left(\frac{d}{n}\right)\Phi_1 = -\frac{d\,\Phi_1}{\Phi\,n} = 2{\cdot}5\text{ cm}$$

Q.56. An aligned optical system consists of two thin lenses with focal lengths f_1 and f_2 the distance between the lenses being equal to d. The given system has to be replaced by one thin lens which, at any position of an object, would provide the same transverse magnification as the system. What must the focal length of this

lens be equal to and in what position must it be placed with respect to the two-lens system?

Ans. The optical power of the system of two thin lenses placed in air is given as,

$$\Phi = \Phi_1 + \Phi_2 - d\,\Phi_1\,\Phi_2$$

or, $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$, where f is the equivalent focal length

So, $$\frac{1}{f} = \frac{f_2 + f_1 - d}{f_1 f_2}$$

or, $$f = \frac{f_1 f_2}{f_1 + f_2 - d} \qquad (1)$$

This equivalent focal length of the ‘system of two lenses is measured from the primary principal plane.

A s clear from the figure, the distance of the primary principal plane from the optical centre of the first is

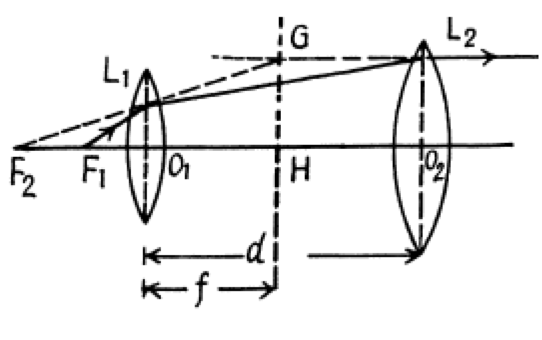
$O_1 H = x = +(n/\Phi)\,(d/n')\,\Phi_1$

$= \frac{d\,\Phi_1}{\Phi}$, as $n = n' = 1$, for air.

$= \frac{d f}{f_1}$

$= \left(\frac{d}{f_1}\right)\left(\frac{f_1 f_2}{f_1 + f_2 - d}\right)$

$= \frac{d f_2}{f_1 + f_2 - d}$



Now, if we place the equivalent lens at the primary principal plane of the lens system, it will provide the same transverse magnification as the system. So, the distance of equivalent lens from the Vertex of the first lens is,

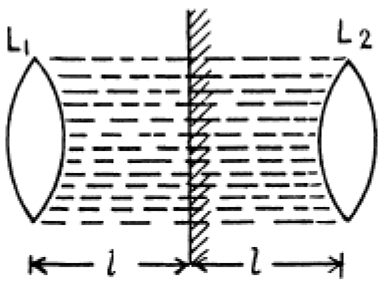
$x = \frac{d f_2}{f_1 + f_2 - d}$

Q.57. A system consists of a thin symmetrical converging glass lens with the curvature radius of its surfaces R = 38 cm and a plane mirror oriented at right angles to the optical axis of the lens. The distance between the lens and the mirror is l = 12 cm. What is the optical power of this system when the space between the lens and the mirror is filled up with water?

Ans. The plane mirror forms the image of the lens, and water, filled in the space between the two, behind the mirror, as shown in the figure.

So, the whole optical system is equivalent to two similar lenses, seperated by a distance $2l$ and thus, the power of this system,

$$\Phi = \Phi_1 + \Phi_2 - \frac{d\,\Phi_1 \Phi_2}{n_0}, \text{ where } \Phi_1 = \Phi_2 = \Phi'_1$$



= optical power of individual lens and n_0 = R.I. of water.

Now, Φ' = optical power of first convex surface + optical power of second concave surface.

$$= \frac{(n-1)}{R} + \frac{n_0 - n}{R},$$ n is the refractive index of glass.

$$\frac{(2n - n_0 - 1)}{R} \quad (1)$$

and so, the optical power of whole system,

$$\Phi = 2\Phi' - \frac{2d\,\Phi'^2}{n_0} = 3{\cdot}0 \text{ D},$$ Substituting the values.

Q.58. At what thickness will a thick convex-concave glass lens in air
(a) serve as a telescope provided the curvature radius of its convex surface is ΔR = 1.5 cm greater than that of its concave surface?

(b) have the optical power equal to —1.0 D if the curvature radii of its convex and concave surfaces are equal to 10.0 and 7.5 cm respectively?

Ans. (a) A telescope in normal adjustment is a zero power combination of lenses. Thus we require

$$\Phi = O = \Phi_1 + \Phi_2 - \frac{d}{n}\Phi_1 \Phi_2$$

But Φ_1 = Power of the convex surface = $\frac{n-1}{R_0 + \Delta R}$

Φ_2 = Power of the concave surface = $-\frac{n-1}{R_0}$

Thus, $$O = \frac{(n-1)\Delta R}{R_0(R_0+\Delta R)} + \frac{d}{n}\frac{(n-1)^2}{R_0\,(R_0+\Delta R)}$$

So $$d = \frac{n\Delta R}{n-1} = 4.5 \text{ cm. on putting the values.}$$

(b) Here, $$\Phi = -1 = \frac{.5}{.1} - \frac{.5}{.075} + \frac{d}{1.5} \times \frac{.5 \times .5}{.1 \times .075}$$

$$= 5 - \frac{20}{3} + \frac{d \times 2}{3} \times \frac{5 \times 20}{3} = -\frac{5}{3} + \frac{200\,d}{9}$$

$$= \frac{200\,d}{9} = \frac{2}{3} \text{ or } d = (3/100)\text{ m} = 3 \text{ cm.}$$

Q.59. Find the positions of the principal planes, the focal length and the sign of the optical power of a thick convex-concave glass lens
(a) whose thickness is equal to d and curvature radii of the surfaces are the same and equal to R;
(b) whose refractive surfaces are concentric and have the curvature radii R_1 and R_2 ($R_2 > R_1$).

Ans. (a) The power of the lens is (as in the previous problem)

$$\Phi = \frac{n-1}{R} - \frac{n-1}{R} - \frac{d}{n}\left(\frac{n-1}{R}\right)\left(-\frac{n-1}{R}\right) = \frac{d(n-1)^2}{nR^2} > 0.$$

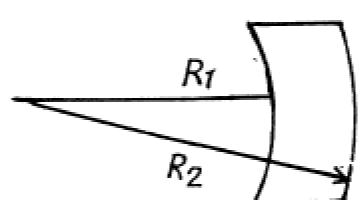
The principal planes are located on the side of the convex surface at a distance d from each other, with the front principal plane being removed from the con vex surface of the lens by a distance R /(n - 1) .

(b)

Here $$\Phi = -\frac{n-1}{R_1} + \frac{n+1}{R_2} + \frac{R_2 - R_1}{n}\frac{(n-1)^2}{R_1 R_2}$$

$$= \frac{(n-1)(R_2 - R_1)}{R_2 R_1}\left[-1 + \frac{n-1}{n}\right]$$

$$= -\frac{n-1}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) < 0$$



Q.60. A telescope system consists of two glass balls with radii R_1 = 5.0 cm and R_2 = 1.0 cm. What are the distance between the centres of the balls and the magnification of the system if the bigger ball serves as an objective?

Ans. Let the optical powers of the first and second surfaces of the ball of radius R_1 be Φ_1' and Φ_1'', then

$$\Phi_1' = (n-1)/R_1 \quad \text{and} \quad \Phi_1'' = (1-n)\big/(-R_1) = \frac{(n-1)}{R_1}$$

This ball may be treated as a thick spherical lens of thickness $2R_1$. So the optical power of this sphere is,

$$\Phi = \Phi_1' - \frac{2R_1 \Phi_1' \Phi_1''}{n} = 2(n-1)/nR_1 \qquad (1)$$

Similarly, the optical power of second ball,

$$\Phi_2 = 2(n-1)/nR_2$$

If the distance between the centres of these balls bed. Then the optical power of whole system,

$$\begin{aligned} \Phi &= \Phi_1 + \Phi_2 - d\,\Phi_1\Phi_2 \\ &= \frac{2(n-1)}{nR_1} + \frac{2(n-1)}{nR_2} - \frac{4d(n-1)^2}{n^2R_1R_2} \\ &= \frac{2(n-1)}{nR_1R_2}\left[(R_1+R_2) - \frac{2d(n-1)}{n}\right]. \end{aligned}$$

Now, since this system serves as telescope, the optical power of the system must be equal to zero.

$$(R_1+R_2) = \frac{2d(n-1)}{n}, \quad \text{as} \quad \frac{2(n-1)}{nR_1R_2} \neq 0.$$

$$d = \frac{n(R_1+R_2)}{2(n-1)} = 9\ \text{cm}.$$

Since the diameter D of the objective is $2R_1$ and that of the eye-piece is $d = 2R_2$ So, the magnification,

$$\Gamma = D/d = \frac{2R_1}{2R_2} = R_1 R_2 = 5 .$$

Q.61. Two identical thick symmetrical biconvex lenses are put close together. The thickness of each lens equals the curvature radius of its surfaces, i.e. d = R = 3.0 cm. Find the optical power of this system in air.

Ans. Optical powers of the two surfaces of the lens are

$$\Phi_1 = (n-1)/R \quad \text{and} \quad \Phi_2 = (1-n)/-R = \frac{n-1}{R}$$

So, the power of the lens of thickness d,

$$\Phi' = \Phi_1 + \Phi_2 - \frac{d\,\Phi_1 \Phi_2}{n} = \frac{n-1}{R} + \frac{n-1}{R} - \frac{d\,(n-1)^2/R^2}{n^2} = \frac{n^2-1}{nR}$$

and optical power of the combination of these two thick lenses,

$$\Phi = \Phi' + \Phi' = 2\,\Phi' = \frac{2\,(n^2-1)}{nR}$$

So, power of this system in air is, $\Phi_0 = \frac{\Phi}{n} = \frac{2\,(n^2-1)}{n^2 R} = 37\text{ D.}$

Q.62. A ray of light propagating in an isotropic medium with refractive index n varying gradually from point to point has a curvature radius ρ determined by the formula

$$\frac{1}{\rho} = \frac{\partial}{\partial N}(\ln n),$$

Where the derivative is taken with respect to the principal normal to the ray. Derive this formula, assuming that in such a medium the law of refraction n sin θ = const holds. Here θ is the angle between the ray and the direction of the vector ∇n at a given point.

Ans.

We consider a ray QPR in a medium of gradually varying refractive index n. At P, the gradient of n is a vector with the given direction while is nearly the same at neighbouring points Q, R. The arc length QR is ds. We apply Snell's formula $n \sin\theta = \text{constant}$ where θ is to be measured from the direction $\vec{\nabla} n$. The refractive indices at Q,R whose mid point is P are $\eta \pm \frac{1}{2}\,|\nabla \eta|\,d\theta\;\cos\theta$

so $$(\eta - \frac{1}{2}|\nabla n| d\theta \cos\theta)(\sin\theta + \frac{1}{2}\cos\theta\ d\theta)$$

$$= (\eta + \frac{1}{2}|\nabla n| d\theta \cos\theta)(\sin\theta - \frac{1}{2}\cos\theta\ d\theta) \quad \text{or} \quad n\cos\theta\ d\theta = |\nabla n| ds \cos\theta \sin\theta$$

(we have used here $\sin(\theta \pm \frac{1}{2}d\theta) = \sin\theta \pm \frac{1}{2}\cos\theta\ d\theta$)

Now using the definition of the radius of curvature $\frac{1}{\rho} = \frac{d\theta}{ds}$

$$\frac{1}{\rho} = \frac{1}{\eta}|\nabla n| \sin\theta$$

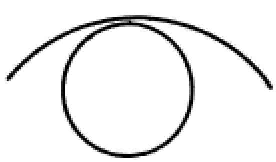
The quantity $|\nabla n| \sin\theta$ can be called $\frac{\delta n}{\delta N}$ i.e. the derivative of n along the normal N to the ray. Then $$\frac{1}{\rho} = \frac{\delta}{\delta N}\ln n.$$

Q.63. Find the curvature radius of a ray of light propagating in a horizontal direction close to the Earth's surface where the gradient of the refractive index in air is equal to approximately 3.10^{-8} m^{-1}. At what value of that gradient would the ray of light propagate all the way round the Earth?

Ans.

$$\frac{1}{\rho} = \frac{1}{n}\hat{p}\cdot\vec{\nabla} n \approx \hat{p}\cdot\nabla n \approx |\nabla n| = 3\times10^{-8}\,\text{m}^{-1}$$

(since $\hat{p} \,||\, \vec{\nabla} n$ both being vertical). So $\rho = 3{\cdot}3\times10^{7}$ m



$\rho = R = 6400$ km $= 6{\cdot}4\times10^{6}$ m

Thus $|\nabla n| = 1{\cdot}6\times10^{-7}\,\text{m}^{-1}$