Chapter 9

Electromagnetic Waves

9.1 Waves in One Dimension

9.1.1 The Wave Equation

What is a "wave?" I don't think I can give you an entirely satisfactory answer—the concept is intrinsically somewhat vague—but here's a start: A wave is a *disturbance of a continuous medium that propagates with a fixed shape at constant velocity.* Immediately I must add qualifiers: In the presence of absorption, the wave will diminish in size as it moves; if the medium is dispersive different frequencies travel at different speeds; in two or three dimensions, as the wave spreads out its amplitude will decrease; and of course *standing* waves don't propagate at all. But these are refinements; let's start with the simple case: fixed shape, constant speed (Fig. 9.1).

How would you represent such an object mathematically? In the figure I have drawn the wave at two different times, once at $t = 0$, and again at some later time t—each point on the wave form simply shifts to the right by an amount vt, where v is the velocity. Maybe the wave is generated by shaking one end of a taut string; $f(z, t)$ represents the displacement of the string at the point z, at time t. Given the *initial* shape of the string, $g(z) \equiv f(z, 0)$,

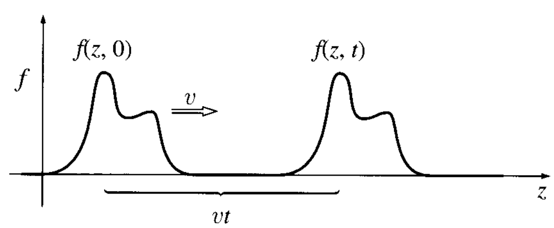


Figure 9.1

what is the subsequent form, $f(z, t)$? Evidently, the displacement at point z, at the later time t, is the same as the displacement a distance vt to the left (i.e. at $z - vt$), back at time $t = 0$:

$$f(z, t) = f(z - vt, 0) = g(z - vt). \tag{9.1}$$

That statement captures (mathematically) the essence of wave motion. It tells us that the function $f(z, t)$, which *might* have depended on z and t in *any* old way, in *fact* depends on them only in the very special combination $z - vt$; when that is true, the function $f(z, t)$ represents a wave of fixed shape traveling in the z direction at speed v. For example, if A and b are constants (with the appropriate units),

$$f_1(z, t) = Ae^{-b(z-vt)^2}, \quad f_2(z, t) = A\sin[b(z - vt)], \quad f_3(z, t) = \frac{A}{b(z - vt)^2 + 1}$$

all represent waves (with different shapes, of course), but

$$f_4(z, t) = Ae^{-b(bz^2+vt)}, \quad \text{and} \quad f_5(z, t) = A\sin(bz)\cos(bvt)^3,$$

do *not*.

Why does a stretched string support wave motion? Actually, it follows from Newton's second law. Imagine a very long string under tension T. If it is displaced from equilibrium, the net transverse force on the segment between z and $z + \Delta z$ (Fig. 9.2) is

$$\Delta F = T\sin\theta' - T\sin\theta,$$

where θ' is the angle the string makes with the z-direction at point $z + \Delta z$, and θ is the corresponding angle at point z. Provided that the distortion of the string is not too great, these angles are small (the figure is exaggerated, obviously), and we can replace the sine by the tangent:

$$\Delta F \cong T(\tan\theta' - \tan\theta) = T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \cong T\frac{\partial^2 f}{\partial z^2}\Delta z.$$

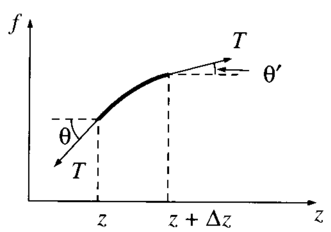


Figure 9.2

If the mass per unit length is μ, Newton's second law says

$$\Delta F = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2},$$

and therefore

$$\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2}.$$

Evidently, small disturbances on the string satisfy

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2},} \tag{9.2}$$

where v (which, as we'll soon see, represents the speed of propagation) is

$$v = \sqrt{\frac{T}{\mu}}. \tag{9.3}$$

Equation 9.2 is known as the (classical) **wave equation**, because it admits as solutions all functions of the form

$$f(z, t) = g(z - vt), \tag{9.4}$$

(that is, all functions that depend on the variables z and t in the special combination $u \equiv z - vt$), and we have just learned that such functions represent waves propagating in the z direction with speed v. For Eq. 9.4 means

$$\frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du}, \quad \frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du},$$

and

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}\frac{\partial u}{\partial z} = \frac{d^2 g}{du^2},$$

$$\frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{dg}{du}\right) = -v\frac{d^2 g}{du^2}\frac{\partial u}{\partial t} = v^2\frac{d^2 g}{du^2},$$

so

$$\frac{d^2 g}{du^2} = \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}. \quad \text{qed}$$

Note that $g(u)$ can be *any* (differentiable) *function whatever.* If the disturbance propagates without changing its shape, then it satisfies the wave equation.

But functions of the form $g(z - vt)$ are not the *only* solutions. The wave equation involves the *square* of v, so we can generate another class of solutions by simply changing the sign of the velocity:

$$f(z, t) = h(z + vt). \tag{9.5}$$

This, of course, represents a wave propagating in the *negative* z direction, and it is certainly reasonable (on physical grounds) that such solutions would be allowed. What is perhaps surprising is that the *most general* solution to the wave equation is the sum of a wave to the right and a wave to the left:

$$f(z,t) = g(z - vt) + h(z + vt). \tag{9.6}$$

(Notice that the wave equation is **linear**: The sum of any two solutions is itself a solution.) *Every* solution to the wave equation can be expressed in this form.

Like the simple harmonic oscillator equation, the wave equation is ubiquitous in physics. If something is vibrating, the oscillator equation is almost certainly responsible (at least, for small amplitudes), and if something is waving (whether the context is mechanics or acoustics, optics or oceanography), the wave equation (perhaps with some decoration) is bound to be involved.

Problem 9.1 By explicit differentiation, check that the functions f_1, f_2, and f_3 in the text satisfy the wave equation. Show that f_4 and f_5 do *not*.

Problem 9.2 Show that the **standing wave** $f(z,t) = A\sin(kz)\cos(kvt)$ satisfies the wave equation, and express it as the sum of a wave traveling to the left and a wave traveling to the right (Eq. 9.6).

9.1.2 Sinusoidal Waves

(i) Terminology. Of all possible wave forms, the sinusoidal one

$$f(z,t) = A\cos[k(z - vt) + \delta] \tag{9.7}$$

is (for good reason) the most familiar. Figure 9.3 shows this function at time $t = 0$. A is the **amplitude** of the wave (it is positive, and represents the maximum displacement from equilibrium). The argument of the cosine is called the **phase**, and δ is the **phase constant** (obviously, you can add any integer multiple of 2π to δ without changing $f(z,t)$; ordinarily, one uses a value in the range $0 \le \delta < 2\pi$). Notice that at $z = vt - \delta/k$, the phase is zero; let's call this the "central maximum." If $\delta = 0$, the central maximum passes the origin at time $t = 0$; more generally, δ/k is the distance by which the central maximum (and

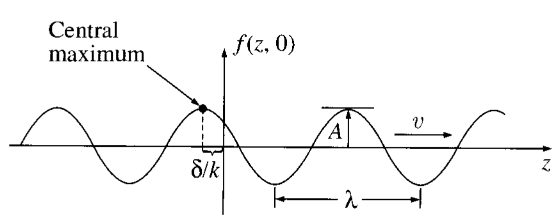


Figure 9.3

therefore the entire wave) is "delayed." Finally, k is the **wave number**; it is related to the **wavelength** λ by the equation

$$\lambda = \frac{2\pi}{k}, \tag{9.8}$$

for when z advances by $2\pi/k$, the cosine executes one complete cycle.

As time passes, the entire wave train proceeds to the right, at speed v. At any fixed point z, the string vibrates up and down, undergoing one full cycle in a **period**

$$T = \frac{2\pi}{kv}. \tag{9.9}$$

The **frequency** ν (number of oscillations per unit time) is

$$\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda}. \tag{9.10}$$

For our purposes, a more convenient unit is the **angular frequency** ω, so-called because in the analogous case of uniform circular motion it represents the number of radians swept out per unit time:

$$\omega = 2\pi\nu = kv. \tag{9.11}$$

Ordinarily, it's nicer to write sinusoidal waves (Eq. 9.7) in terms of ω, rather than v:

$$f(z,t) = A\cos(kz - \omega t + \delta). \tag{9.12}$$

A sinusoidal oscillation of wave number k and (angular) frequency ω traveling to the *left* would be written

$$f(z,t) = A\cos(kz + \omega t - \delta). \tag{9.13}$$

The sign of the phase constant is chosen for consistency with our previous convention that δ/k shall represent the distance by which the wave is "delayed" (since the wave is now moving to the *left*, a delay means a shift to the *right*). At $t = 0$, the wave looks like Fig. 9.4. Because the cosine is an *even* function, we can just as well write Eq. 9.13 thus:

$$f(z,t) = A\cos(-kz - \omega t + \delta). \tag{9.14}$$

Comparison with Eq. 9.12 reveals that, in effect, *we could simply switch the sign of k* to produce a wave with the same amplitude, phase constant, frequency, and wavelength, traveling in the opposite direction.

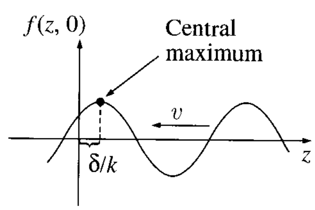


Figure 9.4

(ii) Complex notation. In view of **Euler's formula,**

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{9.15}$$

the sinusoidal wave (Eq. 9.12) can be written

$$f(z,t) = \text{Re}[Ae^{i(kz-\omega t+\delta)}], \tag{9.16}$$

where Re(ξ) denotes the real part of the complex number ξ. This invites us to introduce the **complex wave function**

$$\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)}, \tag{9.17}$$

with the **complex amplitude** $\tilde{A} \equiv Ae^{i\delta}$ absorbing the phase constant. The *actual* wave function is the real part of $\tilde{f}$:

$$f(z,t) = \text{Re}[\tilde{f}(z,t)]. \tag{9.18}$$

If you know $\tilde{f}$, it is a simple matter to find f; the *advantage* of the complex notation is that exponentials are much easier to manipulate than sines and cosines.

Example 9.1

Suppose you want to combine two sinusoidal waves:

$$f_3 = f_1 + f_2 = \text{Re}(\tilde{f}_1) + \text{Re}(\tilde{f}_2) = \text{Re}(\tilde{f}_1 + \tilde{f}_2) = \text{Re}(\tilde{f}_3),$$

with $\tilde{f}_3 = \tilde{f}_1 + \tilde{f}_2$. You simply add the corresponding *complex* wave functions, and then take the real part. In particular, if they have the same frequency and wave number,

$$\tilde{f}_3 = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)} = \tilde{A}_3 e^{i(kz-\omega t)},$$

where

$$\tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2, \quad \text{or} \quad A_3 e^{i\delta_3} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2}; \tag{9.19}$$

evidently, you just add the (complex) amplitudes. The combined wave still has the same frequency and wavelength,

$$f_3(z,t) = A_3 \cos(kz - \omega t + \delta_3),$$

and you can easily figure out A_3 and δ_3 from Eq. 9.19 (Prob. 9.3). Try doing this *without* using the complex notation—you will find yourself looking up trig identities and slogging through nasty algebra.

(iii) Linear combinations of sinusoidal waves. Although the sinusoidal function 9.17 is a very special wave form, the fact is that *any* wave can be expressed as a linear combination of sinusoidal ones:

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{i(kz-\omega t)}\,dk. \tag{9.20}$$

Here ω is a function of k (Eq. 9.11), and I have allowed k to run through negative values in order to include waves going in both directions.[1]

[1]This does not mean that λ and ω are negative—wavelength and frequency are *always* positive. If we allow negative wave numbers, then Eqs. 9.8 and 9.11 should really be written $\lambda = 2\pi/|k|$ and $\omega = |k|v$.

The formula for $\tilde{A}(k)$, in terms of the initial conditions $f(z, 0)$ and $\dot{f}(z, 0)$, can be obtained from the theory of Fourier transforms (see Prob. 9.32), but the details are not relevant to my purpose here. The *point* is that any wave can be written as a linear combination of sinusoidal waves, and therefore if you know how sinusoidal waves behave, you know in principle how *any* wave behaves. So from now on we shall confine our attention to sinusoidal waves.

Problem 9.3 Use Eq. 9.19 to determine A_3 and δ_3 in terms of A_1, A_2, δ_1, and δ_2.

Problem 9.4 Obtain Eq. 9.20 directly from the wave equation, by separation of variables.

9.1.3 Boundary Conditions: Reflection and Transmission

So far I have assumed the string is infinitely long—or at any rate long enough that we don't need to worry about what happens to a wave when it reaches the end. As a matter of fact, what happens depends a lot on how the string is *attached* at the end—that is, on the specific boundary conditions to which the wave is subject. Suppose, for instance, that the string is simply tied onto a *second* string. The tension T is the same for both, but the mass per unit length μ presumably is not, and hence the wave velocities v_1 and v_2 are different (remember, $v = \sqrt{T/\mu}$). Let's say, for convenience, that the knot occurs at $z = 0$. The **incident** wave

$$\tilde{f}_I(z, t) = \tilde{A}_I e^{i(k_1 z - \omega t)}, \quad (z < 0), \tag{9.21}$$

coming in from the left, gives rise to a **reflected** wave

$$\tilde{f}_R(z, t) = \tilde{A}_R e^{i(-k_1 z - \omega t)}, \quad (z < 0), \tag{9.22}$$

traveling *back* along string 1 (hence the minus sign in front of k_1), in addition to a **transmitted** wave

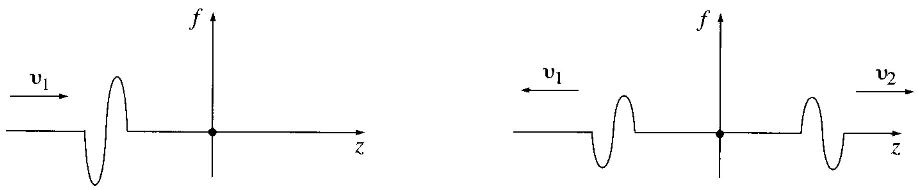
$$\tilde{f}_T(z, t) = \tilde{A}_T e^{i(k_2 z - \omega t)}, \quad (z > 0), \tag{9.23}$$

which continues on to the right in string 2.

The incident wave $f_I(z, t)$ is a sinusoidal oscillation that extends (in principle) all the way back to $z = -\infty$, and has been doing so for all of history. The same goes for f_R and f_T (except that the latter, of course, extends to $z = +\infty$). *All parts of the system are oscillating at the same frequency* ω (a frequency determined by the person at $z = -\infty$, who is shaking the string in the first place). Since the wave velocities are different in the two strings, however, the wavelengths and wave numbers are also different:

$$\frac{\lambda_1}{\lambda_2} = \frac{k_2}{k_1} = \frac{v_1}{v_2}. \tag{9.24}$$

Of course, this situation is pretty artificial—what's more, with incident and reflected waves of infinite extent traveling on the same piece of string, it's going to be hard for a spectator to tell them apart. You might therefore prefer to consider an incident wave of *finite* extent—say, the pulse shown in Fig. 9.5. You can work out the details for yourself, if you like (Prob. 9.5). The *trouble* with this approach is that no *finite* pulse is truly sinusoidal. The waves in Fig. 9.5 may *look* like sine functions, but they're *not*: they're little *pieces* of sines, joined onto an entirely *different* function (namely, zero). Like any other waves, they can be built up as *linear combinations* of true sinusoidal functions (Eq. 9.20), but only by putting together a whole range of frequencies and wavelengths. If you want a *single* incident frequency (as we shall in the electromagnetic case), you must let your waves extend to infinity. In practice, if you use a very *long* pulse with many oscillations, it will be *close* to the ideal of a single frequency.



(a) Incident pulse (b) Reflected and transmitted pulses

Figure 9.5

For a sinusoidal incident wave, then, the net disturbance of the string is:

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)}, & \text{for } z < 0, \\ \tilde{A}_T e^{i(k_2 z - \omega t)}, & \text{for } z > 0. \end{cases} \tag{9.25}$$

At the join ($z = 0$), the displacement just slightly to the left ($z = 0^-$) must equal the displacement slightly to the right ($z = 0^+$), or else there would be a break between the two strings. Mathematically, $f(z, t)$ is *continuous* at $z = 0$:

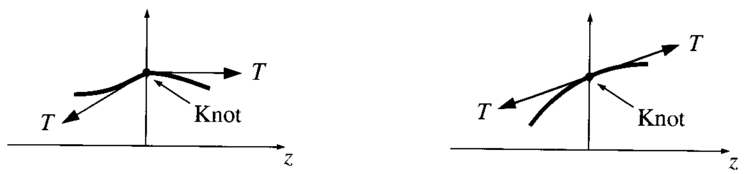
$$f(0^-, t) = f(0^+, t). \tag{9.26}$$

If the knot itself is of negligible mass, then the *derivative* of f must *also* be continuous:

$$\left.\frac{\partial f}{\partial z}\right|_{0^-} = \left.\frac{\partial f}{\partial z}\right|_{0^+}. \tag{9.27}$$

Otherwise there would be a net force on the knot, and therefore an infinite acceleration (Fig. 9.6). These boundary conditions apply directly to the *real* wave function $f(z, t)$. But since the imaginary part of $\tilde{f}$ differs from the real part only in the replacement of cosine by sine (Eq. 9.15), it follows that the complex wave function $\tilde{f}(z, t)$ obeys the same rules:

$$\tilde{f}(0^-, t) = \tilde{f}(0^+, t), \quad \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+}. \tag{9.28}$$



(a) Discontinuous slope; force on knot (a) Continuous slope; no force on knot

Figure 9.6

When applied to Eq. 9.25, these boundary conditions determine the outgoing amplitudes ($\tilde{A}_R$ and $\tilde{A}_T$) in terms of the incoming one ($\tilde{A}_I$):

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T, \quad k_1(\tilde{A}_I - \tilde{A}_R) = k_2\tilde{A}_T,$$

from which it follows that

$$\tilde{A}_R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)\tilde{A}_I, \quad \tilde{A}_T = \left(\frac{2k_1}{k_1 + k_2}\right)\tilde{A}_I. \tag{9.29}$$

Or, in terms of the velocities (Eq. 9.24):

$$\tilde{A}_R = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{A}_I, \quad \tilde{A}_T = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{A}_I. \tag{9.30}$$

The *real* amplitudes and phases, then, are related by

$$A_R e^{i\delta_R} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)A_I e^{i\delta_I}, \quad A_T e^{i\delta_T} = \left(\frac{2v_2}{v_2 + v_1}\right)A_I e^{i\delta_I}. \tag{9.31}$$

If the second string is *lighter* than the first ($\mu_2 < \mu_1$, so that $v_2 > v_1$), all three waves have the same phase angle ($\delta_R = \delta_T = \delta_I$), and the outgoing amplitudes are

$$A_R = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)A_I, \quad A_T = \left(\frac{2v_2}{v_2 + v_1}\right)A_I. \tag{9.32}$$

If the second string is *heavier* than the first ($v_2 < v_1$) the reflected wave is out of phase by 180° ($\delta_R + \pi = \delta_T = \delta_I$). In other words, since

$$\cos(-k_1 z - \omega t + \delta_I - \pi) = -\cos(-k_1 z - \omega t + \delta_I),$$

the reflected wave is "upside down." The amplitudes in this case are

$$A_R = \left(\frac{v_1 - v_2}{v_2 + v_1}\right)A_I \text{ and } A_T = \left(\frac{2v_2}{v_2 + v_1}\right)A_I. \tag{9.33}$$

In particular, if the second string is *infinitely* massive—or, what amounts to the same thing, if the first string is simply *nailed down* at the end—then

$$A_R = A_I \text{ and } A_T = 0.$$

Naturally, in this case there is *no* transmitted wave—*all* of it reflects back.

! **Problem 9.5** Suppose you send an incident wave of specified shape, $g_I(z - v_1 t)$, down string number 1. It gives rise to a reflected wave, $h_R(z + v_1 t)$, and a transmitted wave, $g_T(z - v_2 t)$. By imposing the boundary conditions 9.26 and 9.27, find h_R and g_T.

Problem 9.6

(a) Formulate an appropriate boundary condition, to replace Eq. 9.27, for the case of two strings under tension T joined by a knot of mass m.

(b) Find the amplitude and phase of the reflected and transmitted waves for the case where the knot has a mass m and the second string is massless.

! **Problem 9.7** Suppose string 2 is embedded in a viscous medium (such as molasses), which imposes a drag force that is proportional to its (transverse) speed:

$$\Delta F_{\text{drag}} = -\gamma \frac{\partial f}{\partial t} \Delta z.$$

(a) Derive the modified wave equation describing the motion of the string.

(b) Solve this equation, assuming the string oscillates at the incident frequency ω. That is, look for solutions of the form $\tilde{f}(z, t) = e^{i\omega t} \tilde{F}(z)$.

(c) Show that the waves are **attenuated** (that is, their amplitude decreases with increasing z). Find the characteristic penetration distance, at which the amplitude is $1/e$ of its original value, in terms of γ, T, μ, and ω.

(d) If a wave of amplitude A_I, phase $\delta_I = 0$, and frequency ω is incident from the left (string 1), find the reflected wave's amplitude and phase.

9.1.4 Polarization

The waves that travel down a string when you shake it are called **transverse**, because the displacement is perpendicular to the direction of propagation. If the string is reasonably elastic, it is also possible to stimulate *compression* waves, by giving the string little tugs. Compression waves are hard to see on a string, but if you try it with a slinky they're quite noticeable (Fig. 9.7). These waves are called **longitudinal**, because the displacement from equilibrium is along the direction of propagation. Sound waves, which are nothing but compression waves in air, are longitudinal; electromagnetic waves, as we shall see, are transverse.

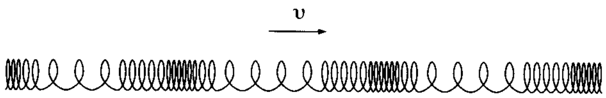


Figure 9.7

Now there are, of course, *two* dimensions perpendicular to any given line of propagation. Accordingly, transverse waves occur in two independent states of **polarization**: you can shake the string up-and-down ("vertical" polarization—Fig. 9.8a),

$$\tilde{\mathbf{f}}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{x}}, \tag{9.34}$$

or left-and-right ("horizontal" polarization—Fig. 9.8b),

$$\tilde{\mathbf{f}}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{y}}, \tag{9.35}$$

or along any other direction in the xy plane (Fig. 9.8c):

$$\tilde{\mathbf{f}}(z,t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{n}}. \tag{9.36}$$

The **polarization vector $\hat{\mathbf{n}}$** defines the plane of vibration.[2] Because the waves are transverse, $\hat{\mathbf{n}}$ is perpendicular to the direction of propagation:

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{z}} = 0. \tag{9.37}$$

In terms of the **polarization angle** θ,

$$\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}. \tag{9.38}$$

Thus, the wave pictured in Fig. 9.8c can be considered a superposition of two waves—one horizontally polarized, the other vertically:

$$\tilde{\mathbf{f}}(z,t) = (\tilde{A}\cos\theta)e^{i(kz-\omega t)}\,\hat{\mathbf{x}} + (\tilde{A}\sin\theta)e^{i(kz-\omega t)}\,\hat{\mathbf{y}}. \tag{9.39}$$

Problem 9.8 Equation 9.36 describes the most general **linearly** polarized wave on a string. Linear (or "plane") polarization (so called because the displacement is parallel to a fixed vector $\hat{\mathbf{n}}$) results from the combination of horizontally and vertically polarized waves of the *same phase* (Eq. 9.39). If the two components are of equal amplitude, but *out of phase* by 90° (say, $\delta_v = 0$, $\delta_h = 90°$), the result is a *circularly* polarized wave. In that case:

(a) At a fixed point z, show that the string moves in a circle about the z axis. Does it go *clockwise* or *counterclockwise*, as you look down the axis toward the origin? How would you construct a wave circling the *other* way? (In optics, the clockwise case is called **right circular polarization**, and the counterclockwise, **left circular polarization**.)

(b) Sketch the string at time $t = 0$.

(c) How would you shake the string in order to produce a circularly polarized wave?

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[2]Notice that you can always switch the *sign* of $\hat{\mathbf{n}}$, provided you simultaneously advance the phase constant by 180°, since both operations change the sign of the wave.

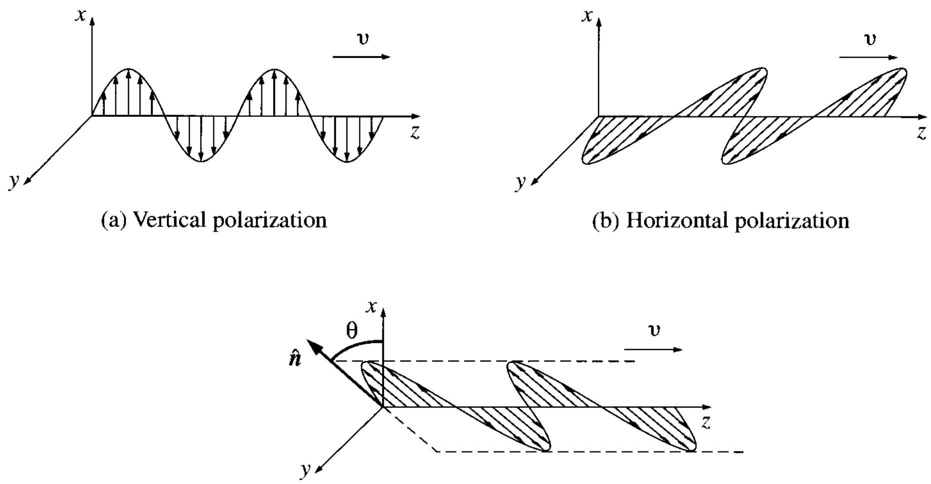

Figure 9.8

9.2 Electromagnetic Waves in Vacuum

9.2.1 The Wave Equation for E and B

In regions of space where there is no charge or current, Maxwell's equations read

$$\left.\begin{array}{llll}\text{(i)} & \nabla\cdot\mathbf{E}=0, & \text{(iii)} & \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t},\\[2ex] \text{(ii)} & \nabla\cdot\mathbf{B}=0, & \text{(iv)} & \nabla\times\mathbf{B}=\mu_0\epsilon_0\dfrac{\partial\mathbf{E}}{\partial t}.\end{array}\right\} \tag{9.40}$$

They constitute a set of coupled, first-order, partial differential equations for **E** and **B**. They can be *de*coupled by applying the curl to (iii) and (iv):

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{E}) &= \nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}=\nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right)\\ &= -\frac{\partial}{\partial t}(\nabla\times\mathbf{B})=-\mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2},\\ \nabla\times(\nabla\times\mathbf{B}) &= \nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}=\nabla\times\left(\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\\ &= \mu_0\epsilon_0\frac{\partial}{\partial t}(\nabla\times\mathbf{E})=-\mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}.\end{aligned}$$

Or, since $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$,

$$\boxed{\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.} \tag{9.41}$$

We now have *separate* equations for **E** and **B**, but they are of *second* order; that's the price you pay for decoupling them.

In vacuum, then, each Cartesian component of **E** and **B** satisfies the **three-dimensional wave equation,**

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}.$$

(This is the same as Eq. 9.2, except that $\partial^2 f / \partial z^2$ is replaced by its natural generalization, $\nabla^2 f$.) So Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, traveling at a speed

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \text{ m/s}, \tag{9.42}$$

which happens to be precisely the velocity of light, c. The implication is astounding: Perhaps light *is* an electromagnetic wave.[3] Of course, this conclusion does not surprise anyone today, but imagine what a revelation it was in Maxwell's time! Remember how ϵ_0 and μ_0 came into the theory in the first place: they were constants in Coulomb's law and the Biot-Savart law, respectively. You measure them in experiments involving charged pith balls, batteries, and wires—experiments having nothing whatever to do with light. And yet, according to Maxwell's theory you can calculate c from these two numbers. Notice the crucial role played by Maxwell's contribution to Ampère's law ($\mu_0 \epsilon_0 \partial \mathbf{E} / \partial t$); without it, the wave equation would not emerge, and there would be no electromagnetic theory of light.

9.2.2 Monochromatic Plane Waves

For reasons discussed in Sect. 9.1.2, we may confine our attention to sinusoidal waves of frequency ω. Since different frequencies in the visible range correspond to different *colors*, such waves are called **monochromatic** (Table 9.1). Suppose, moreover, that the waves are traveling in the z direction and have no x or y dependence; these are called **plane waves,**[4] because the fields are uniform over every plane perpendicular to the direction of propagation (Fig. 9.9). We are interested, then, in fields of the form

$$\tilde{\mathbf{E}}(z, t) = \tilde{\mathbf{E}}_0 e^{i(kz - \omega t)}, \quad \tilde{\mathbf{B}}(z, t) = \tilde{\mathbf{B}}_0 e^{i(kz - \omega t)}, \tag{9.43}$$

---

[3] As Maxwell himself put it, "We can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena." See Ivan Tolstoy, *James Clerk Maxwell, A Biography* (Chicago: University of Chicago Press, 1983).

[4] For a discussion of *spherical* waves, at this level, see J. R. Reitz, F. J. Milford, and R. W. Christy, *Foundations of Electromagnetic Theory*, 3rd ed., Sect. 17-5 (Reading, MA: Addison-Wesley, 1979). Or work Prob. 9.33. Of course, over small enough regions *any* wave is essentially plane, as long as the wavelength is much less than the radius of the curvature of the wave front.

The Electromagnetic Spectrum		
Frequency (Hz)	Type	Wavelength (m)
10^{22}		10^{-13}
10^{21}	gamma rays	10^{-12}
10^{20}		10^{-11}
10^{19}		10^{-10}
10^{18}	x rays	10^{-9}
10^{17}		10^{-8}
10^{16}	ultraviolet	10^{-7}
10^{15}	visible	10^{-6}
10^{14}	infrared	10^{-5}
10^{13}		10^{-4}
10^{12}		10^{-3}
10^{11}		10^{-2}
10^{10}	microwave	10^{-1}
10^{9}		1
10^{8}	TV, FM	10
10^{7}		10^{2}
10^{6}	AM	10^{3}
10^{5}		10^{4}
10^{4}	RF	10^{5}
10^{3}		10^{6}

The Visible Range		
Frequency (Hz)	Color	Wavelength (m)
1.0×10^{15}	near ultraviolet	3.0×10^{-7}
7.5×10^{14}	shortest visible blue	4.0×10^{-7}
6.5×10^{14}	blue	4.6×10^{-7}
5.6×10^{14}	green	5.4×10^{-7}
5.1×10^{14}	yellow	5.9×10^{-7}
4.9×10^{14}	orange	6.1×10^{-7}
3.9×10^{14}	longest visible red	7.6×10^{-7}
3.0×10^{14}	near infrared	1.0×10^{-6}

Table 9.1

where $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are the (complex) amplitudes (the *physical* fields, of course, are the real parts of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$).

Now, the wave equations for **E** and **B** (Eq. 9.41) were derived from Maxwell's equations. However, whereas every solution to Maxwell's equations (in empty space) must obey the wave equation, the converse is *not* true; Maxwell's equations impose extra constraints on

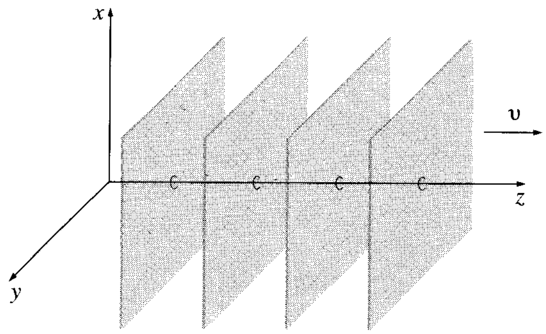

Figure 9.9

$\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$. In particular, since $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$, it follows[5] that

$$(\tilde{E}_0)_z = (\tilde{B}_0)_z = 0. \tag{9.44}$$

That is, *electromagnetic waves are transverse*: the electric and magnetic fields are perpendicular to the direction of propagation. Moreover, Faraday's law, $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, implies a relation between the electric and magnetic amplitudes, to wit:

$$-k(\tilde{E}_0)_y = \omega(\tilde{B}_0)_x, \quad k(\tilde{E}_0)_x = \omega(\tilde{B}_0)_y, \tag{9.45}$$

or, more compactly:

$$\tilde{\mathbf{B}}_0 = \frac{k}{\omega}(\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_0). \tag{9.46}$$

Evidently, $\mathbf{E}$ and $\mathbf{B}$ are *in phase* and *mutually perpendicular;* their (real) amplitudes are related by

$$B_0 = \frac{k}{\omega}E_0 = \frac{1}{c}E_0. \tag{9.47}$$

The fourth of Maxwell's equations, $\nabla \times \mathbf{B} = \mu_0\epsilon_0(\partial \mathbf{E}/\partial t)$, does not yield an independent condition; it simply reproduces Eq. 9.45.

Example 9.2

If $\mathbf{E}$ points in the x direction, then $\mathbf{B}$ points in the y direction (Eq. 9.46):

$$\tilde{\mathbf{E}}(z, t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}(z, t) = \frac{1}{c}\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}\hat{\mathbf{y}},$$

or (taking the real part)

$$\boxed{\mathbf{E}(z, t) = E_0 \cos(kz - \omega t + \delta)\,\hat{\mathbf{x}}, \quad \mathbf{B}(z, t) = \frac{1}{c}E_0 \cos(kz - \omega t + \delta)\,\hat{\mathbf{y}}.} \tag{9.48}$$

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[5]Because the real part of $\tilde{\mathbf{E}}$ differs from the imaginary part only in the replacement of sine by cosine, if the former obeys Maxwell's equations, so does the latter, and hence $\tilde{\mathbf{E}}$ as well.

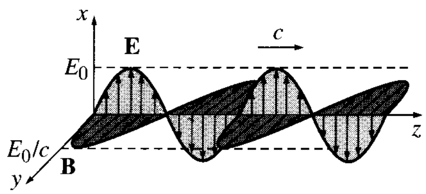

Figure 9.10

This is the paradigm for a monochromatic plane wave (see Fig. 9.10). The wave as a whole is said to be polarized in the x direction (by convention, we use the direction of $\mathbf{E}$ to specify the polarization of an electromagnetic wave).

There is nothing special about the z direction, of course—we can easily generalize to monochromatic plane waves traveling in an arbitrary direction. The notation is facilitated by the introduction of the **propagation** (or **wave**) **vector**, $\mathbf{k}$, pointing in the direction of propagation, whose magnitude is the wave number k. The scalar product $\mathbf{k}\cdot\mathbf{r}$ is the appropriate generalization of kz (Fig. 9.11), so

$$\boxed{\begin{aligned} \tilde{\mathbf{E}}(\mathbf{r},t) &= \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{n}}, \\ \tilde{\mathbf{B}}(\mathbf{r},t) &= \frac{1}{c}\tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\hat{\mathbf{k}}\times\hat{\mathbf{n}}) = \frac{1}{c}\hat{\mathbf{k}}\times\tilde{\mathbf{E}}, \end{aligned}} \tag{9.49}$$

where $\hat{\mathbf{n}}$ is the polarization vector. Because $\mathbf{E}$ is transverse,

$$\hat{\mathbf{n}}\cdot\hat{\mathbf{k}} = 0. \tag{9.50}$$

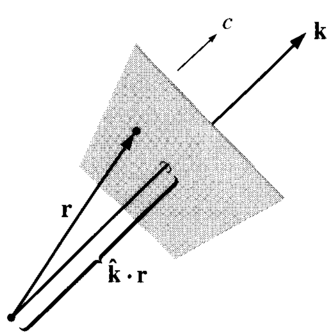

Figure 9.11

(The transversality of **B** follows automatically from Eq. 9.49.) The actual (real) electric and magnetic fields in a monochromatic plane wave with propagation vector **k** and polarization $\hat{\mathbf{n}}$ are

$$\mathbf{E}(\mathbf{r}, t) = E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)\, \hat{\mathbf{n}}, \tag{9.51}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta)(\hat{\mathbf{k}} \times \hat{\mathbf{n}}). \tag{9.52}$$

Problem 9.9 Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0, frequency ω, and phase angle zero that is (a) traveling in the negative x direction and polarized in the z direction; (b) traveling in the direction from the origin to the point (1, 1, 1), with polarization parallel to the xz plane. In each case, sketch the wave, and give the explicit Cartesian components of **k** and $\hat{\mathbf{n}}$.

9.2.3 Energy and Momentum in Electromagnetic Waves

According to Eq. 8.13, the energy per unit volume stored in electromagnetic fields is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right). \tag{9.53}$$

In the case of a monochromatic plane wave (Eq. 9.48)

$$B^2 = \frac{1}{c^2} E^2 = \mu_0 \epsilon_0 E^2, \tag{9.54}$$

so the *electric and magnetic contributions are equal*:

$$u = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta). \tag{9.55}$$

As the wave travels, it carries this energy along with it. The energy flux density (energy per unit area, per unit time) transported by the fields is given by the Poynting vector (Eq. 8.10):

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}). \tag{9.56}$$

For monochromatic plane waves propagating in the z direction,

$$\mathbf{S} = c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\, \hat{\mathbf{z}} = cu\, \hat{\mathbf{z}}. \tag{9.57}$$

Notice that **S** is the energy density (u) times the velocity of the waves ($c\,\hat{\mathbf{z}}$)—as it *should* be. For in a time Δt, a length $c\,\Delta t$ passes through area A (Fig. 9.12), carrying with it an energy $uAc\,\Delta t$. The energy per unit time, per unit area, transported by the wave is therefore uc.

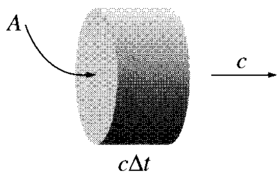

Figure 9.12

Electromagnetic fields not only carry *energy*, they also carry *momentum*. In fact, we found in Eq. 8.30 that the momentum density stored in the fields is

$$\boldsymbol{\wp} = \frac{1}{c^2}\mathbf{S}. \tag{9.58}$$

For monochromatic plane waves, then,

$$\boldsymbol{\wp} = \frac{1}{c}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{\mathbf{z}} = \frac{1}{c}u\,\hat{\mathbf{z}}. \tag{9.59}$$

In the case of *light*, the wavelength is so short ($\sim 5 \times 10^{-7}$ m), and the period so brief ($\sim 10^{-15}$ s), that any macroscopic measurement will encompass many cycles. Typically, therefore, we're not interested in the fluctuating cosine-squared term in the energy and momentum densities; all we want is the *average* value. Now, the average of cosine-squared over a complete cycle[6] is $\frac{1}{2}$, so

$$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2, \tag{9.60}$$

$$\langle \mathbf{S} \rangle = \frac{1}{2}c\epsilon_0 E_0^2\,\hat{\mathbf{z}}, \tag{9.61}$$

$$\langle \boldsymbol{\wp} \rangle = \frac{1}{2c}\epsilon_0 E_0^2\,\hat{\mathbf{z}}. \tag{9.62}$$

I use brackets, $\langle\ \rangle$, to denote the (time) average over a complete cycle (or *many* cycles, if you prefer). The average power per unit area transported by an electromagnetic wave is called the **intensity**:

$$I \equiv \langle S \rangle = \frac{1}{2}c\epsilon_0 E_0^2. \tag{9.63}$$

[6]There is a cute trick for doing this in your head: $\sin^2\theta + \cos^2\theta = 1$, and over a complete cycle the average of $\sin^2\theta$ is equal to the average of $\cos^2\theta$, so $\langle \sin^2 \rangle = \langle \cos^2 \rangle = 1/2$. More formally,

$$\frac{1}{T}\int_0^T \cos^2(kz - 2\pi t/T + \delta)\,dt = 1/2.$$

When light falls on a perfect absorber it delivers its momentum to the surface. In a time Δt the momentum transfer is (Fig. 9.12) $\Delta \mathbf{p} = \langle \wp \rangle Ac\,\Delta t$, so the **radiation pressure** (average force per unit area) is

$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}. \tag{9.64}$$

(On a perfect *reflector* the pressure is *twice* as great, because the momentum switches direction, instead of simply being absorbed.) We can account for this pressure qualitatively, as follows: The electric field (Eq. 9.48) drives charges in the x direction, and the magnetic field then exerts on them a force ($q\mathbf{v}\times\mathbf{B}$) in the z direction. The net force on all the charges in the surface produces the pressure.

Problem 9.10 The intensity of sunlight hitting the earth is about 1300 W/m^2. If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?

Problem 9.11 In the complex notation there is a clever device for finding the time average of a product. Suppose $f(\mathbf{r}, t) = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_a)$ and $g(\mathbf{r}, t) = B\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_b)$. Show that $\langle fg \rangle = (1/2)\text{Re}(\tilde{f}\tilde{g}^*)$, where the star denotes complex conjugation. [Note that this only works if the two waves have the same $\mathbf{k}$ and ω, but they need not have the same amplitude or phase.] For example

$$\langle u \rangle = \frac{1}{4}\text{Re}(\epsilon_0 \tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*) \quad \text{and} \quad \langle \mathbf{S} \rangle = \frac{1}{2\mu_0}\text{Re}(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*).$$

Problem 9.12 Find all elements of the Maxwell stress tensor for a monochromatic plane wave traveling in the z direction and linearly polarized in the x direction (Eq. 9.48). Does your answer make sense? (Remember that $\overleftrightarrow{\mathbf{T}}$ represents the momentum flux density.) How is the momentum flux density related to the energy density, in this case?

9.3 Electromagnetic Waves in Matter

9.3.1 Propagation in Linear Media

Inside matter, but in regions where there is no *free* charge or *free* current, Maxwell's equations become

$$\left.\begin{array}{llll} \text{(i)} & \nabla\cdot\mathbf{D} = 0, & \text{(iii)} & \nabla\times\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla\cdot\mathbf{B} = 0, & \text{(iv)} & \nabla\times\mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t}. \end{array}\right\} \tag{9.65}$$

If the medium is *linear*,

$$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}, \tag{9.66}$$

and *homogeneous* (so ϵ and μ do not vary from point to point), Maxwell's equations reduce to

$$\left.\begin{array}{llll} \text{(i)} & \nabla\cdot\mathbf{E}=0, & \text{(iii)} & \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla\cdot\mathbf{B}=0, & \text{(iv)} & \nabla\times\mathbf{B}=\mu\epsilon\dfrac{\partial\mathbf{E}}{\partial t}, \end{array}\right\} \tag{9.67}$$

which (remarkably) differ from the vacuum analogs (Eqs. 9.40) only in the replacement of $\mu_0\epsilon_0$ by $\mu\epsilon$.[7] Evidently electromagnetic waves propagate through a linear homogeneous medium at a speed

$$v=\frac{1}{\sqrt{\epsilon\mu}}=\frac{c}{n}, \tag{9.68}$$

where

$$n\equiv\sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \tag{9.69}$$

is the **index of refraction** of the material. For most materials, μ is very close to μ_0, so

$$n\cong\sqrt{\epsilon_r}, \tag{9.70}$$

where ϵ_r is the dielectric constant (Eq. 4.34). Since ϵ_r is almost always greater than 1, light travels *more slowly* through matter—a fact that is well known from optics.

All of our previous results carry over, with the simple transcription $\epsilon_0\to\epsilon$, $\mu_0\to\mu$, and hence $c\to v$ (see Prob. 8.15). The energy density is[8]

$$u=\frac{1}{2}\left(\epsilon E^2+\frac{1}{\mu}B^2\right), \tag{9.71}$$

and the Poynting vector is

$$\mathbf{S}=\frac{1}{\mu}(\mathbf{E}\times\mathbf{B}). \tag{9.72}$$

For monochromatic plane waves the frequency and wave number are related by $\omega=kv$ (Eq. 9.11), the amplitude of **B** is $1/v$ times the amplitude of **E** (Eq. 9.47), and the intensity is

$$I=\frac{1}{2}\epsilon v E_0^2. \tag{9.73}$$

[7]This observation is mathematically pretty trivial, but the physical implications are astonishing: As the wave passes through, the fields busily polarize and magnetize all the molecules, and the resulting (oscillating) dipoles create their own electric and magnetic fields. These combine with the original fields in such a way as to create a *single* wave with the same frequency but a different speed. This extraordinary conspiracy is responsible for the phenomenon of **transparency**. It is a distinctly *non*trivial consequence of the *linearity* of the medium. For further discussion see M. B. James and D. J. Griffiths, *Am. J. Phys.* **60**, 309 (1992).

[8]Refer to Sect. 4.4.3 for the precise *meaning* of "energy density," in the context of linear media.

The interesting question is this: What happens when a wave passes from one transparent medium into another—air to water, say, or glass to plastic? As in the case of waves on a string, we expect to get a reflected wave and a transmitted wave. The details depend on the exact nature of the electrodynamic boundary conditions, which we derived in Chapter 7 (Eq. 7.64):

$$\left.\begin{array}{ll} \text{(i)}\ \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, & \text{(iii)}\ \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}, \\ \text{(ii)}\ B_1^{\perp} = B_2^{\perp}, & \text{(iv)}\ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel}. \end{array}\right\} \tag{9.74}$$

These equations relate the electric and magnetic fields just to the left and just to the right of the interface between two linear media. In the following sections we use them to deduce the laws governing reflection and refraction of electromagnetic waves.

9.3.2 Reflection and Transmission at Normal Incidence

Suppose the xy plane forms the boundary between two linear media. A plane wave of frequency ω, traveling in the z direction and polarized in the x direction, approaches the interface from the left (Fig. 9.13):

$$\left.\begin{array}{l} \tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{x}}, \\ \tilde{\mathbf{B}}_I(z,t) = \dfrac{1}{v_1}\tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{y}}. \end{array}\right\} \tag{9.75}$$

It gives rise to a reflected wave

$$\left.\begin{array}{l} \tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{x}}, \\ \tilde{\mathbf{B}}_R(z,t) = -\dfrac{1}{v_1}\tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{y}}, \end{array}\right\} \tag{9.76}$$

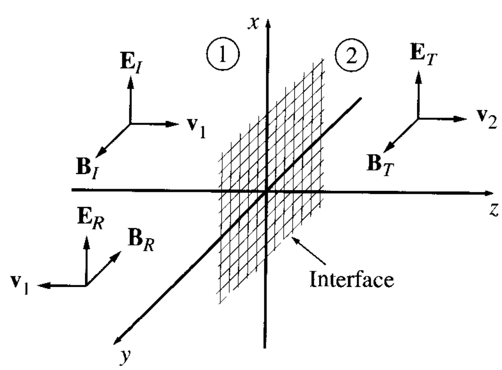

Figure 9.13

which travels back to the left in medium (1), and a transmitted wave

$$\left.\begin{aligned}\tilde{\mathbf{E}}_T(z,t) &= \tilde{E}_{0_T} e^{i(k_2 z - \omega t)}\,\hat{\mathbf{x}},\\ \tilde{\mathbf{B}}_T(z,t) &= \frac{1}{v_2}\tilde{E}_{0_T} e^{i(k_2 z - \omega t)}\,\hat{\mathbf{y}},\end{aligned}\right\} \tag{9.77}$$

which continues on the the right in medium (2). Note the minus sign in $\tilde{\mathbf{B}}_R$, as required by Eq. 9.49—or, if you prefer, by the fact that the Poynting vector aims in the direction of propagation.

At $z = 0$, the combined fields on the left, $\tilde{\mathbf{E}}_I + \tilde{\mathbf{E}}_R$ and $\tilde{\mathbf{B}}_I + \tilde{\mathbf{B}}_R$, must join the fields on the right, $\tilde{\mathbf{E}}_T$ and $\tilde{\mathbf{B}}_T$, in accordance with the boundary conditions 9.74. In this case there are no components perpendicular to the surface, so (i) and (ii) are trivial. However, (iii) requires that

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}, \tag{9.78}$$

while (iv) says

$$\frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0_I} - \frac{1}{v_1}\tilde{E}_{0_R}\right) = \frac{1}{\mu_2}\left(\frac{1}{v_2}\tilde{E}_{0_T}\right), \tag{9.79}$$

or

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \beta \tilde{E}_{0_T}, \tag{9.80}$$

where

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}. \tag{9.81}$$

Equations 9.78 and 9.80 are easily solved for the outgoing amplitudes, in terms of the incident amplitude:

$$\tilde{E}_{0_R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0_I}. \tag{9.82}$$

These results are strikingly similar to the ones for waves on a string. Indeed, if the permittivities μ are close to their values in vacuum (as, remember, they *are* for most media), then $\beta = v_1/v_2$, and we have

$$\tilde{E}_{0_R} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{E}_{0_I}, \tag{9.83}$$

which are *identical* to Eqs. 9.30. In that case, as before, the reflected wave is *in phase* (right side up) if $v_2 > v_1$ and *out of phase* (upside down) if $v_2 < v_1$; the real amplitudes are related by

$$E_{0_R} = \left|\frac{v_2 - v_1}{v_2 + v_1}\right| E_{0_I}, \quad E_{0_T} = \left(\frac{2v_2}{v_2 + v_1}\right) E_{0_I}, \tag{9.84}$$

or, in terms of the indices of refraction,

$$E_{0_R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0_I}, \quad E_{0_T} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{0_I}. \tag{9.85}$$

What fraction of the incident energy is reflected, and what fraction is transmitted? According to Eq. 9.73, the intensity (average power per unit area) is

$$I = \frac{1}{2}\epsilon v E_0^2.$$

If (again) $\mu_1 = \mu_2 = \mu_0$, then the ratio of the reflected intensity to the incident intensity is

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2, \tag{9.86}$$

whereas the ratio of the transmitted intensity to the incident intensity is

$$T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}. \tag{9.87}$$

R is called the **reflection coefficient** and T the **transmission coefficient**; they measure the fraction of the incident energy that is reflected and transmitted, respectively. Notice that

$$R + T = 1, \tag{9.88}$$

as conservation of energy, of course, requires. For instance, when light passes from air ($n_1 = 1$) into glass ($n_2 = 1.5$), $R = 0.04$ and $T = 0.96$. Not surprisingly, most of the light is transmitted.

Problem 9.13 Calculate the *exact* reflection and transmission coefficients, *without* assuming $\mu_1 = \mu_2 = \mu_0$. Confirm that $R + T = 1$.

Problem 9.14 In writing Eqs. 9.76 and 9.77, I tacitly assumed that the reflected and transmitted waves have the same *polarization* as the incident wave—along the x direction. Prove that this *must* be so. [*Hint:* Let the polarization vectors of the transmitted and reflected waves be

$$\hat{\mathbf{n}}_T = \cos\theta_T\,\hat{\mathbf{x}} + \sin\theta_T\,\hat{\mathbf{y}}, \quad \hat{\mathbf{n}}_R = \cos\theta_R\,\hat{\mathbf{x}} + \sin\theta_R\,\hat{\mathbf{y}},$$

and prove from the boundary conditions that $\theta_T = \theta_R = 0$.]

9.3.3 Reflection and Transmission at Oblique Incidence

In the last section I treated reflection and transmission at *normal* incidence—that is, when the incoming wave hits the interface head-on. We now turn to the more general case of *oblique* incidence, in which the incoming wave meets the boundary at an arbitrary angle θ_I (Fig. 9.14). Of course, normal incidence is really just a special case of oblique incidence, with $\theta_I = 0$, but I wanted to treat it separately, as a kind of warm-up, because the algebra is now going to get a little heavy.

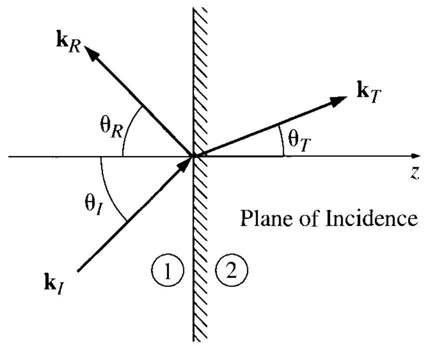

Figure 9.14

Suppose, then, that a monochromatic plane wave

$$\tilde{\mathbf{E}}_I(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)}, \quad \tilde{\mathbf{B}}_I(\mathbf{r}, t) = \frac{1}{v_1}(\hat{\mathbf{k}}_I \times \tilde{\mathbf{E}}_I) \tag{9.89}$$

approaches from the left, giving rise to a reflected wave,

$$\tilde{\mathbf{E}}_R(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)}, \quad \tilde{\mathbf{B}}_R(\mathbf{r}, t) = \frac{1}{v_1}(\hat{\mathbf{k}}_R \times \tilde{\mathbf{E}}_R), \tag{9.90}$$

and a transmitted wave

$$\tilde{\mathbf{E}}_T(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}, \quad \tilde{\mathbf{B}}_T(\mathbf{r}, t) = \frac{1}{v_2}(\hat{\mathbf{k}}_T \times \tilde{\mathbf{E}}_T). \tag{9.91}$$

All three waves have the same *frequency* ω—that is determined once and for all at the source (the flashlight, or whatever, that produces the incident beam). The three wave numbers are related by Eq. 9.11:

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega, \quad \text{or} \quad k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T. \tag{9.92}$$

The combined fields in medium (1), $\tilde{\mathbf{E}}_I + \tilde{\mathbf{E}}_R$ and $\tilde{\mathbf{B}}_I + \tilde{\mathbf{B}}_R$, must now be joined to the fields $\tilde{\mathbf{E}}_T$ and $\tilde{\mathbf{B}}_T$ in medium (2), using the boundary conditions 9.74. These all share the generic structure

$$(\ )e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} + (\ )e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} = (\ )e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}, \quad \text{at } z = 0. \tag{9.93}$$

I'll fill in the parentheses in a moment; for now, the important thing to notice is that the x, y, and t dependence is confined to the exponents. *Because the boundary conditions must hold at all points on the plane, and for all times, these exponential factors must be equal.* Otherwise, a slight change in x, say, would destroy the equality (see Prob. 9.15). Of course, the time factors are *already* equal (in fact, you could regard this as an independent confirmation that the transmitted and reflected frequencies must match the incident one). As for the spatial terms, evidently

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r}, \quad \text{when } z = 0, \tag{9.94}$$

or, more explicitly,

$$x(k_I)_x + y(k_I)_y = x(k_R)_x + y(k_R)_y = x(k_T)_x + y(k_T)_y, \tag{9.95}$$

for all x and all y.

But Eq. 9.95 can *only* hold if the components are separately equal, for if $x = 0$, we get

$$(k_I)_y = (k_R)_y = (k_T)_y, \tag{9.96}$$

while $y = 0$ gives

$$(k_I)_x = (k_R)_x = (k_T)_x. \tag{9.97}$$

We may as well orient our axes so that $\mathbf{k}_I$ lies in the xz plane (i.e. $(k_I)_y = 0$); according to Eq. 9.96, so too will $\mathbf{k}_R$ and $\mathbf{k}_T$. *Conclusion:*

First Law: The incident, reflected, and transmitted wave vectors form a plane (called the **plane of incidence**), which also includes the normal to the surface (here, the z axis).

Meanwhile, Eq. 9.97 implies that

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T, \tag{9.98}$$

where θ_I is the **angle of incidence**, θ_R is the **angle of reflection**, and θ_T is the angle of transmission, more commonly known as the **angle of refraction**, all of them measured with respect to the normal (Fig. 9.14). In view of Eq. 9.92, then,

Second Law: The angle of incidence is equal to the angle of reflection,

$$\theta_I = \theta_R. \tag{9.99}$$

This is the **law of reflection**.

As for the transmitted angle,

Third Law:

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}. \tag{9.100}$$

This is the **law of refraction**, or **Snell's law**.

These are the three fundamental laws of geometrical optics. It is remarkable how little actual *electrodynamics* went into them: we have yet to invoke any *specific* boundary conditions—all we used was their generic form (Eq. 9.93). Therefore, any *other* waves (water waves, for instance, or sound waves) can be expected to obey the same "optical" laws when they pass from one medium into another.

Now that we have taken care of the exponential factors—they cancel, given Eq. 9.94—the boundary conditions 9.74 become:

$$\left.\begin{aligned} &\text{(i)}\ \ \epsilon_1(\tilde{\mathbf{E}}_{0_I}+\tilde{\mathbf{E}}_{0_R})_z=\epsilon_2(\tilde{\mathbf{E}}_{0_T})_z \\ &\text{(ii)}\ \ (\tilde{\mathbf{B}}_{0_I}+\tilde{\mathbf{B}}_{0_R})_z=(\tilde{\mathbf{B}}_{0_T})_z \\ &\text{(iii)}\ \ (\tilde{\mathbf{E}}_{0_I}+\tilde{\mathbf{E}}_{0_R})_{x,y}=(\tilde{\mathbf{E}}_{0_T})_{x,y} \\ &\text{(iv)}\ \ \frac{1}{\mu_1}(\tilde{\mathbf{B}}_{0_I}+\tilde{\mathbf{B}}_{0_R})_{x,y}=\frac{1}{\mu_2}(\tilde{\mathbf{B}}_{0_T})_{x,y} \end{aligned}\right\} \tag{9.101}$$

where $\tilde{\mathbf{B}}_0=(1/v)\hat{\mathbf{k}}\times\tilde{\mathbf{E}}_0$ in each case. (The last two represent *pairs* of equations, one for the x-component and one for the y-component.)

Suppose that the polarization of the incident wave is *parallel* to the plane of incidence (the xz plane in Fig. 9.15); it follows (see Prob. 9.14) that the reflected and transmitted waves are also polarized in this plane. (I shall leave it for you to analyze the case of polarization *perpendicular* to the plane of incidence; see Prob. 9.16.) Then (i) reads

$$\epsilon_1(-\tilde{E}_{0_I}\sin\theta_I+\tilde{E}_{0_R}\sin\theta_R)=\epsilon_2(-\tilde{E}_{0_T}\sin\theta_T); \tag{9.102}$$

(ii) adds nothing ($0=0$), since the magnetic fields have no z components; (iii) becomes

$$\tilde{E}_{0_I}\cos\theta_I+\tilde{E}_{0_R}\cos\theta_R=\tilde{E}_{0_T}\cos\theta_T; \tag{9.103}$$

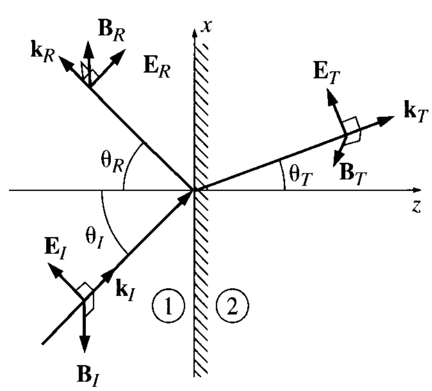

Figure 9.15

and (iv) says

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0_I} - \tilde{E}_{0_R}) = \frac{1}{\mu_2 v_2}\tilde{E}_{0_T}. \tag{9.104}$$

Given the laws of reflection and refraction, Eqs. 9.102 and 9.104 both reduce to

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \beta \tilde{E}_{0_T}, \tag{9.105}$$

where (as before)

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}, \tag{9.106}$$

and Eq. 9.103 says

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \alpha \tilde{E}_{0_T}, \tag{9.107}$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}. \tag{9.108}$$

Solving Eqs. 9.105 and 9.107 for the reflected and transmitted amplitudes, we obtain

$$\boxed{\tilde{E}_{0_R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{\alpha + \beta}\right)\tilde{E}_{0_I}.} \tag{9.109}$$

These are known as **Fresnel's equations**, for the case of polarization in the plane of incidence. (There are two other Fresnel equations, giving the reflected and transmitted amplitudes when the polarization is *perpendicular* to the plane of incidence—see Prob. 9.16.) Notice that the transmitted wave is always *in phase* with the incident one; the reflected wave is either in phase ("right side up"), if $\alpha > \beta$, or 180° out of phase ("upside down"), if $\alpha < \beta$.[9]

The amplitudes of the transmitted and reflected waves depend on the angle of incidence, because α is a function of θ_I:

$$\alpha = \frac{\sqrt{1 - \sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1 - [(n_1/n_2)\sin\theta_I]^2}}{\cos\theta_I}. \tag{9.110}$$

In the case of normal incidence ($\theta_I = 0$), $\alpha = 1$, and we recover Eq. 9.82. At grazing incidence ($\theta_I = 90°$), α diverges, and the wave is totally reflected (a fact that is painfully familiar to anyone who has driven at night on a wet road). Interestingly, there is an intermediate angle, θ_B (called **Brewster's angle**), at which the reflected wave is completely extinguished.[10] According to Eq. 9.109, this occurs when $\alpha = \beta$, or

$$\sin^2\theta_B = \frac{1 - \beta^2}{(n_1/n_2)^2 - \beta^2}. \tag{9.111}$$

[9]There is an unavoidable ambiguity in the phase of the reflected wave, since (as I mentioned in footnote 2) changing the sign of the polarization vector is equivalent to a 180° phase shift. The convention I adopted in Fig. 9.15, with $\mathbf{E}_R$ positive "upward," is consistent with some, but not all, of the standard optics texts.

[10]Because waves polarized *perpendicular* to the plane of incidence exhibit no corresponding quenching of the reflected component, an arbitrary beam incident at Brewster's angle yields a reflected beam that is *totally* polarized parallel to the interface. That's why Polaroid glasses, with the transmission axis vertical, help to reduce glare off a horizontal surface.

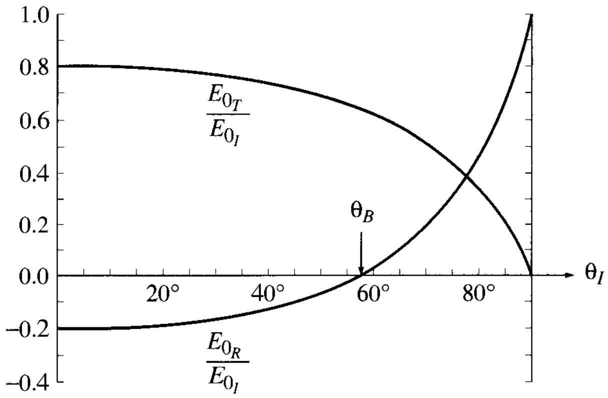

Figure 9.16

For the typical case $\mu_1 \cong \mu_2$, so $\beta \cong n_2/n_1$, $\sin^2\theta_B \cong \beta^2/(1+\beta^2)$, and hence

$$\tan\theta_B \cong \frac{n_2}{n_1}. \tag{9.112}$$

Figure 9.16 shows a plot of the transmitted and reflected amplitudes as functions of θ_I, for light incident on glass ($n_2 = 1.5$) from air ($n_1 = 1$). (On the graph, a *negative* number indicates that the wave is 180° out of phase with the incident beam—the amplitude itself is the absolute value.)

The power per unit area striking the interface is $\mathbf{S}\cdot\hat{\mathbf{z}}$. Thus the incident intensity is

$$I_I = \frac{1}{2}\epsilon_1 v_1 E_{0_I}^2 \cos\theta_I, \tag{9.113}$$

while the reflected and transmitted intensities are

$$I_R = \frac{1}{2}\epsilon_1 v_1 E_{0_R}^2 \cos\theta_R, \quad \text{and} \quad I_T = \frac{1}{2}\epsilon_2 v_2 E_{0_T}^2 \cos\theta_T. \tag{9.114}$$

(The cosines are there because I am talking about the average power per unit area of *interface*, and the interface is at an angle to the wave front.) The reflection and transmission coefficients for waves polarized parallel to the plane of incidence are

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2, \tag{9.115}$$

$$T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2. \tag{9.116}$$

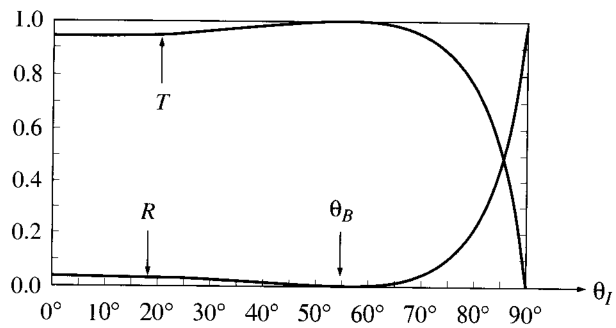

Figure 9.17

They are plotted as functions of the angle of incidence in Fig. 9.17 (for the air/glass interface). R is the fraction of the incident energy that is reflected—naturally, it goes to zero at Brewster's angle; T is the fraction transmitted—it goes to 1 at θ_B. Note that $R + T = 1$, as required by conservation of energy: the energy per unit time *reaching* a particular patch of area on the surface is equal to the energy per unit time *leaving* the patch.

Problem 9.15 Suppose $Ae^{iax} + Be^{ibx} = Ce^{icx}$, for some nonzero constants A, B, C, a, b, c, and for all x. Prove that $a = b = c$ and $A + B = C$.

! **Problem 9.16** Analyze the case of polarization *perpendicular* to the plane of incidence (i.e. electric fields in the y direction, in Fig. 9.15). Impose the boundary conditions 9.101, and obtain the Fresnel equations for $\tilde{E}_{0_R}$ and $\tilde{E}_{0_T}$. Sketch $(\tilde{E}_{0_R}/\tilde{E}_{0_I})$ and $(\tilde{E}_{0_T}/\tilde{E}_{0_I})$ as functions of θ_I, for the case $\beta = n_2/n_1 = 1.5$. (Note that for this β the reflected wave is *always* 180° out of phase.) Show that there is no Brewster's angle for *any* n_1 and n_2: $\tilde{E}_{0_R}$ is *never* zero (unless, of course, $n_1 = n_2$ and $\mu_1 = \mu_2$, in which case the two media are optically indistinguishable). Confirm that your Fresnel equations reduce to the proper forms at normal incidence. Compute the reflection and transmission coefficients, and check that they add up to 1.

Problem 9.17 The index of refraction of diamond is 2.42. Construct the graph analogous to Fig. 9.16 for the air/diamond interface. (Assume $\mu_1 = \mu_2 = \mu_0$.) In particular, calculate (a) the amplitudes at normal incidence, (b) Brewster's angle, and (c) the "crossover" angle, at which the reflected and transmitted amplitudes are equal.

9.4 Absorption and Dispersion

9.4.1 Electromagnetic Waves in Conductors

In Sect. 9.3 I stipulated that the free charge density ρ_f and the free current density $\mathbf{J}_f$ are zero, and everything that followed was predicated on that assumption. Such a restriction is perfectly reasonable when you're talking about wave propagation through a vacuum or through insulating materials such as glass or (pure) water. But in the case of conductors we do not independently control the flow of charge, and in general $\mathbf{J}_f$ is certainly *not* zero. In fact, according to Ohm's law, the (free) current density in a conductor is proportional to the electric field:

$$\mathbf{J}_f = \sigma\mathbf{E}. \tag{9.117}$$

With this, Maxwell's equations for linear media assume the form

$$\left.\begin{array}{ll} \text{(i)}\ \nabla\cdot\mathbf{E} = \frac{1}{\epsilon}\rho_f, & \text{(iii)}\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \\ \text{(ii)}\ \nabla\cdot\mathbf{B} = 0, & \text{(iv)}\ \nabla\times\mathbf{B} = \mu\sigma\mathbf{E} + \mu\epsilon\frac{\partial\mathbf{E}}{\partial t}. \end{array}\right\} \tag{9.118}$$

Now the continuity equation for free charge,

$$\nabla\cdot\mathbf{J}_f = -\frac{\partial\rho_f}{\partial t}, \tag{9.119}$$

together with Ohm's law and Gauss's law (i), gives

$$\frac{\partial\rho_f}{\partial t} = -\sigma(\nabla\cdot\mathbf{E}) = -\frac{\sigma}{\epsilon}\rho_f$$

for a homogeneous linear medium, from which it follows that

$$\rho_f(t) = e^{-(\sigma/\epsilon)t}\rho_f(0). \tag{9.120}$$

Thus any initial free charge density $\rho_f(0)$ dissipates in a characteristic time $\tau \equiv \epsilon/\sigma$. This reflects the familiar fact that if you put some free charge on a conductor, it will flow out to the edges. The time constant τ affords a measure of how "good" a conductor is: For a "perfect" conductor $\sigma = \infty$ and $\tau = 0$; for a "good" conductor, τ is much less than the other relevant times in the problem (in oscillatory systems, that means $\tau \ll 1/\omega$); for a "poor" conductor, τ is *greater* than the characteristic times in the problem ($\tau \gg 1/\omega$).[11] At present we're not interested in this transient behavior—we'll wait for any accumulated free charge to disappear. From then on $\rho_f = 0$, and we have

$$\left.\begin{array}{ll} \text{(i)}\ \nabla\cdot\mathbf{E} = 0, & \text{(iii)}\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \\ \text{(ii)}\ \nabla\cdot\mathbf{B} = 0, & \text{(iv)}\ \nabla\times\mathbf{B} = \mu\epsilon\frac{\partial\mathbf{E}}{\partial t} + \mu\sigma\mathbf{E}. \end{array}\right\} \tag{9.121}$$

[11] N. Ashby, *Am. J. Phys.* **43**, 553 (1975), points out that for good conductors τ is absurdly short (10^{-19} s, for copper, whereas the time between collisions is $\tau_c = 10^{-14}$ s). The problem is that Ohm's law itself breaks down on time scales shorter than τ_c; actually, the time it takes free charge to dissipate in a good conductor is of order τ_c, not τ. Moreover, H. C. Ohanian, *Am. J. Phys.* **51**, 1020 (1983), shows that it takes even longer for the fields and currents to equilibrate. But none of this is relevant to our present purpose; the free charge density in a conductor does *eventually* dissipate, and exactly how long the process takes is beside the point.

These differ from the corresponding equations for *non*conducting media (9.67) only in the addition of the last term in (iv).

Applying the curl to (iii) and (iv), as before, we obtain modified wave equations for **E** and **B**:

$$\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{E}}{\partial t}, \quad \nabla^2\mathbf{B} = \mu\epsilon\frac{\partial^2\mathbf{B}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{B}}{\partial t}. \tag{9.122}$$

These equations still admit plane-wave solutions,

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z-\omega t)}, \tag{9.123}$$

but this time the "wave number" $\tilde{k}$ is complex:

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega, \tag{9.124}$$

as you can easily check by plugging Eq. 9.123 into Eq. 9.122. Taking the square root,

$$\tilde{k} = k + i\kappa, \tag{9.125}$$

where

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}, \quad \kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}. \tag{9.126}$$

The imaginary part of $\tilde{k}$ results in an attenuation of the wave (decreasing amplitude with increasing z):

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)}. \tag{9.127}$$

The distance it takes to reduce the amplitude by a factor of $1/e$ (about a third) is called the **skin depth**:

$$d \equiv \frac{1}{\kappa}; \tag{9.128}$$

it is a measure of how far the wave penetrates into the conductor. Meanwhile, the real part of $\tilde{k}$ determines the wavelength, the propagation speed, and the index of refraction, in the usual way:

$$\lambda = \frac{2\pi}{k}, \quad v = \frac{\omega}{k}, \quad n = \frac{ck}{\omega}. \tag{9.129}$$

The attenuated plane waves (Eq. 9.127) satisfy the modified wave equation (9.122) for *any* $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$. But Maxwell's equations (9.121) impose further constraints, which serve to determine the relative amplitudes, phases, and polarizations of **E** and **B**. As before, (i) and (ii) rule out any z components: the fields are *transverse*. We may as well orient our axes so that **E** is polarized along the x direction:

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{x}}. \tag{9.130}$$

Then (iii) gives

$$\tilde{\mathbf{B}}(z,t) = \frac{\tilde{k}}{\omega}\tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{y}}. \tag{9.131}$$

(Equation (iv) says the same thing.) Once again, the electric and magnetic fields are mutually perpendicular.

Like any complex number, $\tilde{k}$ can be expressed in terms of its modulus and phase:

$$\tilde{k} = K e^{i\phi}, \tag{9.132}$$

where

$$K \equiv |\tilde{k}| = \sqrt{k^2+\kappa^2} = \omega\sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} \tag{9.133}$$

and

$$\phi \equiv \tan^{-1}(\kappa/k). \tag{9.134}$$

According to Eq. 9.130 and 9.131, the complex amplitudes $\tilde{E}_0 = E_0 e^{i\delta_E}$ and $\tilde{B}_0 = B_0 e^{i\delta_B}$ are related by

$$B_0 e^{i\delta_B} = \frac{K e^{i\phi}}{\omega} E_0 e^{i\delta_E}. \tag{9.135}$$

Evidently the electric and magnetic fields are no longer in phase; in fact,

$$\delta_B - \delta_E = \phi; \tag{9.136}$$

the magnetic field *lags behind* the electric field. Meanwhile, the (real) amplitudes of **E** and **B** are related by

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}. \tag{9.137}$$

The (real) electric and magnetic fields are, finally,

$$\left.\begin{aligned} \mathbf{E}(z,t) &= E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\,\hat{\mathbf{x}}, \\ \mathbf{B}(z,t) &= B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{y}}. \end{aligned}\right\} \tag{9.138}$$

These fields are shown in Fig. 9.18.

Problem 9.18

(a) Suppose you imbedded some free charge in a piece of glass. About how long would it take for the charge to flow to the surface?

(b) Silver is an excellent conductor, but it's expensive. Suppose you were designing a microwave experiment to operate at a frequency of 10^{10} Hz. How thick would you make the silver coatings?

(c) Find the wavelength and propagation speed in copper for radio waves at 1 MHz. Compare the corresponding values in air (or vacuum).

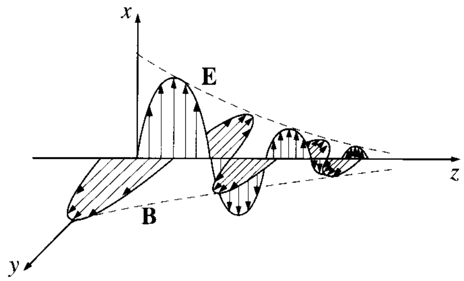

Figure 9.18

Problem 9.19

(a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\epsilon$) is $(2/\sigma)\sqrt{\epsilon/\mu}$ (independent of frequency). Find the skin depth (in meters) for (pure) water.

(b) Show that the skin depth in a good conductor ($\sigma \gg \omega\epsilon$) is $\lambda/2\pi$ (where λ is the wavelength *in the conductor*). Find the skin depth (in nanometers) for a typical metal ($\sigma \approx 10^7\,(\Omega\ \mathrm{m})^{-1}$) in the visible range ($\omega \approx 10^{15}$/s), assuming $\epsilon \approx \epsilon_0$ and $\mu \approx \mu_0$. Why are metals opaque?

(c) Show that in a good conductor the magnetic field lags the electric field by 45°, and find the ratio of their amplitudes. For a numerical example, use the "typical metal" in part (b).

Problem 9.20

(a) Calculate the (time averaged) energy density of an electromagnetic plane wave in a conducting medium (Eq. 9.138). Show that the magnetic contribution always dominates. [*Answer:* $(k^2/2\mu\omega^2)E_0^2 e^{-2\kappa z}$]

(b) Show that the intensity is $(k/2\mu\omega)E_0^2 e^{-2\kappa z}$.

9.4.2 Reflection at a Conducting Surface

The boundary conditions we used to analyze reflection and refraction at an interface between two dielectrics do not hold in the presence of free charges and currents. Instead, we have the more general relations (7.63):

$$\left.\begin{array}{ll} \text{(i)}\ \ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, & \text{(iii)}\ \ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0, \\[2ex] \text{(ii)}\ \ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}, \end{array}\right\} \qquad (9.139)$$

where σ_f (not to be confused with conductivity) is the free surface charge, $\mathbf{K}_f$ the free surface current, and $\hat{\mathbf{n}}$ (not to be confused with the polarization of the wave) is a unit vector perpendicular to the surface, pointing from medium (2) into medium (1). For ohmic conductors ($\mathbf{J}_f = \sigma\mathbf{E}$) there can be no free surface current, since this would require an infinite electric field at the boundary.

Suppose now that the xy plane forms the boundary between a nonconducting linear medium (1) and a conductor (2). A monochromatic plane wave, traveling in the z direction and polarized in the x direction, approaches from the left, as in Fig. 9.13:

$$\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{y}}. \tag{9.140}$$

This incident wave gives rise to a reflected wave,

$$\tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{y}}, \tag{9.141}$$

propagating back to the left in medium (1), and a transmitted wave

$$\tilde{\mathbf{E}}_T(z,t) = \tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_T(z,t) = \frac{\tilde{k}_2}{\omega}\tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)}\,\hat{\mathbf{y}}, \tag{9.142}$$

which is attenuated as it penetrates into the conductor.

At $z = 0$, the combined wave in medium (1) must join the wave in medium (2), pursuant to the boundary conditions 9.139. Since $E^\perp = 0$ on both sides, boundary condition (i) yields $\sigma_f = 0$. Since $B^\perp = 0$, (ii) is automatically satisfied. Meanwhile, (iii) gives

$$\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}, \tag{9.143}$$

and (iv) (with $\mathbf{K}_f = 0$) says

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0_I} - \tilde{E}_{0_R}) - \frac{\tilde{k}_2}{\mu_2\omega}\tilde{E}_{0_T} = 0, \tag{9.144}$$

or

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \tilde{\beta}\tilde{E}_{0_T}, \tag{9.145}$$

where

$$\tilde{\beta} \equiv \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2. \tag{9.146}$$

It follows that

$$\tilde{E}_{0_R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0_I}. \tag{9.147}$$

These results are formally identical to the ones that apply at the boundary between *non*conductors (Eq. 9.82), but the resemblance is deceptive since $\tilde{\beta}$ is now a complex number.

For a *perfect* conductor ($\sigma = \infty$), $k_2 = \infty$ (Eq. 9.126), so $\tilde{\beta}$ is infinite, and

$$\tilde{E}_{0_R} = -\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = 0. \tag{9.148}$$

In this case the wave is totally reflected, with a 180° phase shift. (That's why excellent conductors make good mirrors. In practice, you paint a thin coating of silver onto the back of a pane of glass—the glass has nothing to do with the *reflection*; it's just there to support the silver and to keep it from tarnishing. Since the skin depth in silver at optical frequencies is on the order of 100 Å, you don't need a very thick layer.)

Problem 9.21 Calculate the reflection coefficient for light at an air-to-silver interface ($\mu_1 = \mu_2 = \mu_0$, $\epsilon_1 = \epsilon_0$, $\sigma = 6 \times 10^7 (\Omega \cdot \text{m})^{-1}$), at optical frequencies ($\omega = 4 \times 10^{15}/\text{s}$).

9.4.3 The Frequency Dependence of Permittivity

In the preceding sections, we have seen that the propagation of electromagnetic waves through matter is governed by three properties of the material, which we took to be constants: the permittivity ϵ, the permeability μ, and the conductivity σ. Actually, each of these parameters depends to some extent on the frequency of the waves you are considering. Indeed, if the permittivity were *truly* constant, then the index of refraction in a transparent medium, $n \cong \sqrt{\epsilon_r}$, would also be constant. But it is well known from optics that n is a function of wavelength (Fig. 9.19 shows the graph for a typical glass). A prism or a raindrop bends blue light more sharply than red, and spreads white light out into a rainbow of colors. This phenomenon is called **dispersion**. By extension, whenever the speed of a wave depends on its frequency, the supporting medium is called **dispersive**.[12]

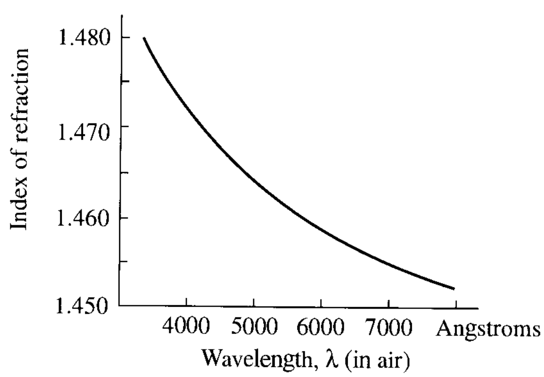

Figure 9.19

[12]Conductors, incidentally, are dispersive: see Eqs. 9.126 and 9.129.

Because waves of different frequency travel at different speeds in a dispersive medium, a wave form that incorporates a range of frequencies will change shape as it propagates. A sharply peaked wave typically flattens out, and whereas each sinusoidal component travels at the ordinary **wave** (or **phase**) **velocity**,

$$v = \frac{\omega}{k}, \tag{9.149}$$

the packet as a whole (the "envelope") travels at the so-called **group velocity**[13]

$$v_g = \frac{d\omega}{dk}. \tag{9.150}$$

[You can demonstrate this by dropping a rock into the nearest pond and watching the waves that form: While the disturbance as a whole spreads out in a circle, moving at speed v_g, the ripples that go to make it up will be seen to travel *twice* as fast ($v = 2v_g$ in this case). They appear at the back end of the packet, growing as they move forward to the center, then shrinking again and fading away at the front (Fig. 9.20).] We shall not concern ourselves with these matters—I'll stick to monochromatic waves, for which the problem does not arise. But I should just mention that the *energy* carried by a wave packet in a dispersive medium ordinarily travels at the *group* velocity, not the phase velocity. Don't be too alarmed, therefore, if in some circumstances v comes out greater than c.[14]

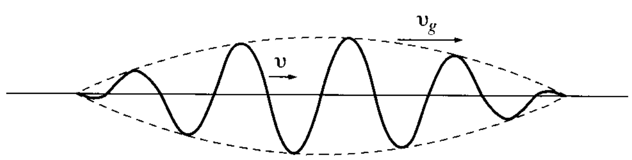

Figure 9.20

My purpose in this section is to account for the frequency dependence of ϵ in nonconductors, using a simplified model for the behavior of electrons in dielectrics. Like all classical models of atomic-scale phenomena, it is at best an approximation to the truth; nevertheless, it does yield qualitatively satisfactory results, and it provides a plausible mechanism for dispersion in transparent media.

The electrons in a nonconductor are bound to specific molecules. The actual binding forces can be quite complicated, but we shall picture each electron as attached to the end of an imaginary spring, with force constant k_{spring} (Fig. 9.21):

$$F_{\text{binding}} = -k_{\text{spring}}x = -m\omega_0^2 x, \tag{9.151}$$

[13]See A. P. French, *Vibrations and Waves*, p. 230 (New York: W. W. Norton & Co., 1971), or F. S. Crawford, Jr., *Waves*, Sect. 6.2 (New York: McGraw-Hill, 1968).

[14]Even the group velocity can exceed c in special cases—see P. C. Peters, *Am. J. Phys.* **56**, 129 (1988). Incidentally, if *two* different "speeds of light" are not enough to satisfy you, check out S. C. Bloch, *Am. J. Phys.* **45**, 538 (1977), in which no fewer than *eight* distinct velocities are identified!

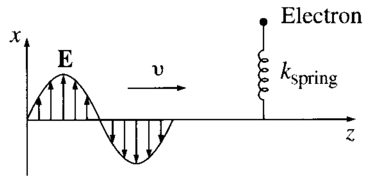

Figure 9.21

where x is displacement from equilibrium, m is the electron's mass, and ω_0 is the natural oscillation frequency, $\sqrt{k_{spring}/m}$. [If this strikes you as an implausible model, look back at Ex. 4.1, where we were led to a force of precisely this form. As a matter of fact, practically *any* binding force can be approximated this way for sufficiently small displacements from equilibrium, as you can see by expanding the potential energy in a Taylor series about the equilibrium point:

$$U(x) = U(0) + xU'(0) + \frac{1}{2}x^2U''(0) + \cdots .$$

The first term is a constant, with no dynamical significance (you can always adjust the zero of potential energy so that $U(0) = 0$). The second term automatically vanishes, since $dU/dx = -F$, and by the nature of an equilibrium the force at that point is zero. The third term is precisely the potential energy of a spring with force constant $k_{spring} = d^2U/dx^2\big|_0$ (the second derivative is positive, for a point of stable equilibrium). As long as the displacements are small, the higher terms in the series can be neglected. Geometrically, all I am saying is that virtually *any* function can be fit near a minimum by a suitable parabola.]

Meanwhile, there will presumably be some damping force on the electron:

$$F_{damping} = -m\gamma\frac{dx}{dt}. \tag{9.152}$$

[Again I have chosen the simplest possible form; the damping must be opposite in direction to the velocity, and making it *proportional* to the velocity is the easiest way to accomplish this. The *cause* of the damping does not concern us here—among other things, an oscillating charge radiates, and the radiation siphons off energy. We will calculate this "radiation damping" in Chapter 11.]

In the presence of an electromagnetic wave of frequency ω, polarized in the x direction (Fig. 9.21), the electron is subject to a driving force

$$F_{driving} = qE = qE_0\cos(\omega t), \tag{9.153}$$

where q is the charge of the electron and E_0 is the amplitude of the wave at the point z where the electron is situated. (Since we're only interested in one point, I have reset the clock so that the maximum E occurs there at $t = 0$.) Putting all this into Newton's second law gives

$$m\frac{d^2x}{dt^2} = F_{tot} = F_{binding} + F_{damping} + F_{driving},$$

or

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = qE_0\cos(\omega t). \tag{9.154}$$

Our model, then, describes the electron as a damped harmonic oscillator, driven at frequency ω. (I assume that the much more massive nuclei remain at rest.)

Equation 9.154 is easier to handle if we regard it as the real part of a *complex* equation:

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x} = \frac{q}{m}E_0 e^{-i\omega t}. \tag{9.155}$$

In the steady state, the system oscillates at the driving frequency:

$$\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}. \tag{9.156}$$

Inserting this into Eq. 9.155, we obtain

$$\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0. \tag{9.157}$$

The dipole moment is the real part of

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0 e^{-i\omega t}. \tag{9.158}$$

The imaginary term in the denominator means that p is *out of phase* with E—lagging behind by an angle $\tan^{-1}[\gamma\omega/(\omega_0^2 - \omega^2)]$ that is very small when $\omega \ll \omega_0$ and rises to π when $\omega \gg \omega_0$.

In general, differently situated electrons within a given molecule experience different natural frequencies and damping coefficients. Let's say there are f_j electrons with frequency ω_j and damping γ_j in each molecule. If there are N molecules per unit volume, the polarization $\mathbf{P}$ is given by[15] the real part of

$$\tilde{\mathbf{P}} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\tilde{\mathbf{E}}. \tag{9.159}$$

Now, I defined the electric susceptibility as the proportionality constant between $\mathbf{P}$ and $\mathbf{E}$ (specifically, $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$). In the present case $\mathbf{P}$ is *not* proportional to $\mathbf{E}$ (this is not, strictly speaking, a linear medium) because of the difference in phase. However, the *complex* polarization $\tilde{\mathbf{P}}$ *is* proportional to the *complex* field $\tilde{\mathbf{E}}$, and this suggests that we introduce a **complex susceptibility**, $\tilde{\chi}_e$:

$$\tilde{\mathbf{P}} = \epsilon_0\tilde{\chi}_e\tilde{\mathbf{E}}. \tag{9.160}$$

[15]This applies directly to the case of a dilute gas; for denser materials the theory is modified slightly, in accordance with the Clausius-Mossotti equation (Prob. 4.38). By the way, don't confuse the "polarization" of a medium, $\mathbf{P}$, with the "polarization" of a *wave*—same *word*, but two completely unrelated meanings.

All of the manipulations we went through before carry over, on the understanding that the physical polarization is the real part of $\tilde{\mathbf{P}}$, just as the physical field is the real part of $\tilde{\mathbf{E}}$. In particular, the proportionality between $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{E}}$ is the **complex permittivity** $\tilde{\epsilon} = \epsilon_0(1+\tilde{\chi}_e)$, and the **complex dielectric constant** (in this model) is

$$\tilde{\epsilon}_r = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}. \tag{9.161}$$

Ordinarily, the imaginary term is negligible; however, when ω is very close to one of the resonant frequencies (ω_j) it plays an important role, as we shall see.

In a dispersive medium the wave equation for a given frequency reads

$$\nabla^2\tilde{\mathbf{E}} = \tilde{\epsilon}\mu_0\frac{\partial^2\tilde{\mathbf{E}}}{\partial t^2}; \tag{9.162}$$

it admits plane wave solutions, as before,

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}, \tag{9.163}$$

with the complex wave number

$$\tilde{k} \equiv \sqrt{\tilde{\epsilon}\mu_0}\,\omega. \tag{9.164}$$

Writing $\tilde{k}$ in terms of its real and imaginary parts,

$$\tilde{k} = k + i\kappa, \tag{9.165}$$

Eq. 9.163 becomes

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}. \tag{9.166}$$

Evidently the wave is *attenuated* (this is hardly surprising, since the damping absorbs energy). Because the intensity is proportional to E^2 (and hence to $e^{-2\kappa z}$), the quantity

$$\alpha \equiv 2\kappa \tag{9.167}$$

is called the **absorption coefficient**. Meanwhile, the wave velocity is ω/k, and the index of refraction is

$$n = \frac{ck}{\omega}. \tag{9.168}$$

I have deliberately used notation reminiscent of Sect. 9.4.1. However, in the present case k and κ have nothing to do with conductivity; rather, they are determined by the parameters of our damped harmonic oscillator. For gases, the second term in Eq. 9.161 is small, and we can approximate the square root (Eq. 9.164) by the first term in the binomial expansion, $\sqrt{1+\varepsilon} \cong 1 + \frac{1}{2}\varepsilon$. Then

$$\tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} \cong \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right], \tag{9.169}$$

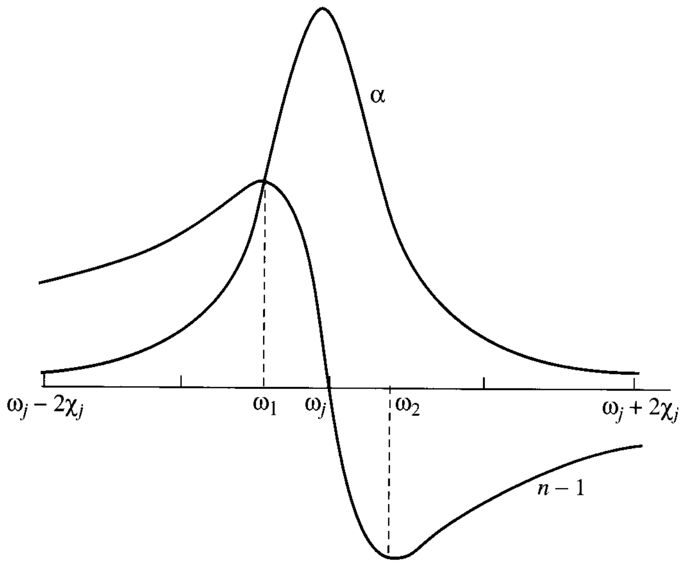

Figure 9.22

so

$$n = \frac{ck}{\omega} \cong 1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}, \tag{9.170}$$

and

$$\alpha = 2\kappa \cong \frac{Nq^2\omega^2}{m\epsilon_0 c} \sum_j \frac{f_j\gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}. \tag{9.171}$$

In Fig. 9.22 I have plotted the index of refraction and the absorption coefficient in the vicinity of one of the resonances. *Most* of the time the index of refraction *rises* gradually with increasing frequency, consistent with our experience from optics (Fig. 9.19). However, in the immediate neighborhood of a resonance the index of refraction *drops* sharply. Because this behavior is atypical, it is called **anomalous dispersion**. Notice that the region of anomalous dispersion ($\omega_1 < \omega < \omega_2$, in the figure) coincides with the region of maximum absorption; in fact, the material may be practically opaque in this frequency range. The reason is that we are now driving the electrons at their "favorite" frequency; the amplitude of their oscillation is relatively large, and a correspondingly large amount of energy is dissipated by the damping mechanism.

In Fig. 9.22, n runs below 1 above the resonance, suggesting that the wave speed exceeds c. As I mentioned earlier, this is no cause for alarm, since energy does not travel at the wave velocity but rather at the *group* velocity (see Prob. 9.25). Moreover, the graph does not include the contributions of other terms in the sum, which add a relatively constant "background" that, in some cases, keeps $n > 1$ on both sides of the resonance.

If you agree to stay away from the resonances, the damping can be ignored, and the formula for the index of refraction simplifies:

$$n = 1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2}. \tag{9.172}$$

For most substances the natural frequencies ω_j are scattered all over the spectrum in a rather chaotic fashion. But for transparent materials, the nearest significant resonances typically lie in the ultraviolet, so that $\omega < \omega_j$. In that case

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2}\left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1} \cong \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right),$$

and Eq. 9.172 takes the form

$$n = 1 + \left(\frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2}\right) + \omega^2 \left(\frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^4}\right). \tag{9.173}$$

Or, in terms of the wavelength in vacuum ($\lambda = 2\pi c/\omega$):

$$n = 1 + A\left(1 + \frac{B}{\lambda^2}\right). \tag{9.174}$$

This is known as **Cauchy's formula**; the constant A is called the **coefficient of refraction** and B is called the **coefficient of dispersion**. Cauchy's equation applies reasonably well to most gases, in the optical region.

What I have described in this section is certainly not the complete story of dispersion in nonconducting media. Nevertheless, it does indicate how the damped harmonic motion of electrons can account for the frequency dependence of the index of refraction, and it explains why n is ordinarily a slowly increasing function of ω, with occasional "anomalous" regions where it precipitously drops.

Problem 9.22

(a) Shallow water is nondispersive; the waves travel at a speed that is proportional to the square root of the depth. In deep water, however, the waves can't "feel" all the way down to the bottom—they behave as though the depth were proportional to λ. (Actually, the distinction between "shallow" and "deep" itself depends on the wavelength: If the depth is less than λ the water is "shallow"; if it is substantially greater than λ the water is "deep.") Show that the wave velocity of deep water waves is *twice* the group velocity.

(b) In quantum mechanics, a free particle of mass m traveling in the x direction is described by the wave function

$$\Psi(x, t) = Ae^{i(px - Et)/\hbar},$$

where p is the momentum, and $E = p^2/2m$ is the kinetic energy. Calculate the group velocity and the wave velocity. Which one corresponds to the classical speed of the particle? Note that the wave velocity is *half* the group velocity.

Problem 9.23 If you take the model in Ex. 4.1 at face value, what natural frequency do you get? Put in the actual numbers. Where, in the electromagnetic spectrum, does this lie, assuming the radius of the atom is 0.5 Å? Find the coefficients of refraction and dispersion and compare them with those for hydrogen at 0°C and atmospheric pressure: $A = 1.36 \times 10^{-4}$, $B = 7.7 \times 10^{-15} \mathrm{m}^2$.

Problem 9.24 Find the width of the anomalous dispersion region for the case of a single resonance at frequency ω_0. Assume $\gamma \ll \omega_0$. Show that the index of refraction assumes its maximum and minimum values at points where the absorption coefficient is at half-maximum.

Problem 9.25 Assuming negligible damping ($\gamma_j = 0$), calculate the group velocity ($v_g = d\omega/dk$) of the waves described by Eqs. 9.166 and 9.169. Show that $v_g < c$, even when $v > c$.

9.5 Guided Waves

9.5.1 Wave Guides

So far, we have dealt with plane waves of infinite extent; now we consider electromagnetic waves confined to the interior of a hollow pipe, or **wave guide** (Fig. 9.23). We'll assume the wave guide is a perfect conductor, so that $\mathbf{E} = 0$ and $\mathbf{B} = 0$ inside the material itself, and hence the boundary conditions at the inner wall are[16]

$$\left.\begin{aligned} &\text{(i)} \quad \mathbf{E}^{\parallel} = 0, \\ &\text{(ii)} \quad B^{\perp} = 0. \end{aligned}\right\} \tag{9.175}$$

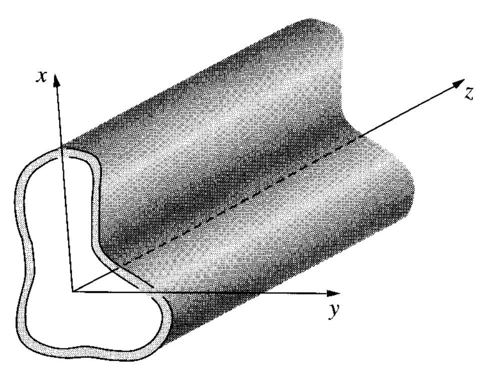

Figure 9.23

[16]See Eq. 9.139 and Prob. 7.42. In a perfect conductor $\mathbf{E} = 0$, and hence (by Faraday's law) $\partial\mathbf{B}/\partial t = 0$; assuming the magnetic field *started out* zero, then, it will *remain* so.

Free charges and currents will be induced on the surface in such a way as to enforce these constraints. We are interested in monochromatic waves that propagate down the tube, so **E** and **B** have the generic form

$$\left.\begin{aligned} &\text{(i)} \quad \tilde{\mathbf{E}}(x, y, z, t) = \tilde{\mathbf{E}}_0(x, y)e^{i(kz-\omega t)}, \\ &\text{(ii)} \quad \tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y)e^{i(kz-\omega t)}. \end{aligned}\right\} \tag{9.176}$$

(For the cases of interest k is real, so I shall dispense with the tilde.) The electric and magnetic fields must, of course, satisfy Maxwell's equations, in the interior of the wave guide:

$$\left.\begin{aligned} &\text{(i)} \ \nabla \cdot \mathbf{E} = 0, \quad \text{(iii)} \ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ &\text{(ii)} \ \nabla \cdot \mathbf{B} = 0, \quad \text{(iv)} \ \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \end{aligned}\right\} \tag{9.177}$$

The problem, then, is to find functions $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ such that the fields (9.176) obey the differential equations (9.177), subject to boundary conditions (9.175).

As we shall soon see, *confined* waves are *not* (in general) transverse; in order to fit the boundary conditions we shall have to include longitudinal components (E_z and B_z):[17]

$$\tilde{\mathbf{E}}_0 = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}} + E_z\,\hat{\mathbf{z}}, \quad \tilde{\mathbf{B}}_0 = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}, \tag{9.178}$$

where each of the components is a function of x and y. Putting this into Maxwell's equations (iii) and (iv), we obtain (Prob. 9.26a)

$$\left.\begin{aligned} &\text{(i)} \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z, \quad &&\text{(iv)} \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z, \\ &\text{(ii)} \quad \frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x, \quad &&\text{(v)} \quad \frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x, \\ &\text{(iii)} \quad ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y, \quad &&\text{(vi)} \quad ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y. \end{aligned}\right\} \tag{9.179}$$

[17] To avoid cumbersome notation I shall leave the subscript 0 and the tilde off the individual components.

Equations (ii), (iii), (v), and (vi) can be solved for E_x, E_y, B_x, and B_y:

$$\left.\begin{array}{ll}
\text{(i)} & E_x = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial E_z}{\partial x} + \omega\dfrac{\partial B_z}{\partial y}\right), \\[2ex]
\text{(ii)} & E_y = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial E_z}{\partial y} - \omega\dfrac{\partial B_z}{\partial x}\right), \\[2ex]
\text{(iii)} & B_x = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial B_z}{\partial x} - \dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial y}\right), \\[2ex]
\text{(iv)} & B_y = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial B_z}{\partial y} + \dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial x}\right).
\end{array}\right\} \tag{9.180}$$

It suffices, then, to determine the longitudinal components E_z and B_z; if we knew those, we could quickly calculate all the others, just by differentiating. Inserting Eq. 9.180 into the remaining Maxwell equations (Prob. 9.26b) yields uncoupled equations for E_z and B_z:

$$\left.\begin{array}{ll}
\text{(i)} & \left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] E_z = 0, \\[2ex]
\text{(ii)} & \left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] B_z = 0.
\end{array}\right\} \tag{9.181}$$

If $E_z = 0$ we call these **TE** ("transverse electric") **waves**; if $B_z = 0$ they are called **TM** ("transverse magnetic") **waves**; if both $E_z = 0$ and $B_z = 0$, we call them **TEM waves.**[18] It turns out that TEM waves cannot occur in a hollow wave guide.

Proof: If $E_z = 0$, Gauss's law (Eq. 9.177i) says

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0,$$

and if $B_z = 0$, Faraday's law (Eq. 9.177iii) says

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0.$$

Indeed, the vector $\tilde{\mathbf{E}}_0$ in Eq. 9.178 has zero divergence and zero curl. It can therefore be written as the gradient of a scalar potential that satisfies Laplace's equation. But the boundary condition on **E** (Eq. 9.175) requires that the surface be an equipotential, and since Laplace's equation admits no local maxima or minima (Sect. 3.1.4), this means that the potential is constant throughout, and hence the electric field is *zero*—no wave at all. qed

[18] In the case of TEM waves (including the unconfined plane waves of Sect. 9.2), $k = \omega/c$, Eqs. 9.180 are indeterminate, and you have to go back to Eqs. 9.179.

Notice that this argument applies only to a completely *empty* pipe—if you run a separate conductor down the middle, the potential at *its* surface need not be the same as on the outer wall, and hence a nontrivial potential is possible. We'll see an example of this in Sect. 9.5.3.

! **Problem 9.26**

(a) Derive Eqs. 9.179, and from these obtain Eqs. 9.180.

(b) Put Eq. 9.180 into Maxwell's equations (i) and (ii) to obtain Eq. 9.181. Check that you get the same results using (i) and (iv) of Eq. 9.179.

9.5.2 TE Waves in a Rectangular Wave Guide

Suppose we have a wave guide of rectangular shape (Fig. 9.24), with height a and width b, and we are interested in the propagation of TE waves. The problem is to solve Eq. 9.181ii, subject to the boundary condition 9.175ii. We'll do it by separation of variables. Let

$$B_z(x, y) = X(x)Y(y),$$

so that

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + [(\omega/c)^2 - k^2]XY = 0.$$

Divide by XY and note that the x- and y-dependent terms must be constant:

$$\text{(i)}\ \frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2, \quad \text{(ii)}\ \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2, \tag{9.182}$$

with

$$-k_x^2 - k_y^2 + (\omega/c)^2 - k^2 = 0. \tag{9.183}$$

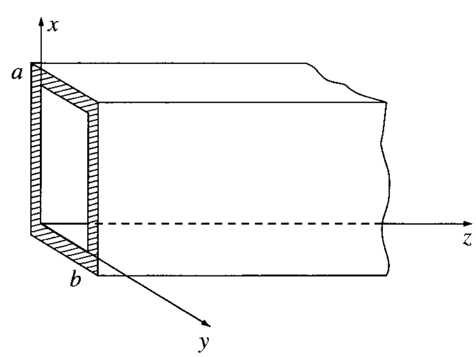

Figure 9.24

The general solution to Eq. 9.182i is

$$X(x) = A \sin(k_x x) + B \cos(k_x x).$$

But the boundary conditions require that B_x—and hence also (Eq. 9.180iii) dX/dx—vanishes at $x = 0$ and $x = a$. So $A = 0$, and

$$k_x = m\pi/a, \quad (m = 0, 1, 2, \ldots). \tag{9.184}$$

The same goes for Y, with

$$k_y = n\pi/b, \quad (n = 0, 1, 2, \ldots), \tag{9.185}$$

and we conclude that

$$B_z = B_0 \cos(m\pi x/a) \cos(n\pi y/b). \tag{9.186}$$

This solution is called the TE_{mn} mode. (The first index is conventionally associated with the *larger* dimension, so we assume $a \geq b$. By the way, at least *one* of the indices must be nonzero—see Prob. 9.27.) The wave number (k) is obtained by putting Eqs. 9.184 and 9.185 into Eq. 9.183:

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]}. \tag{9.187}$$

If

$$\omega < c\pi\sqrt{(m/a)^2 + (n/b)^2} \equiv \omega_{mn}, \tag{9.188}$$

the wave number is imaginary, and instead of a traveling wave we have exponentially attenuated fields (Eq. 9.176). For this reason ω_{mn} is called the **cutoff frequency** for the mode in question. The *lowest* cutoff frequency for a given wave guide occurs for the mode TE_{10}:

$$\omega_{10} = c\pi/a. \tag{9.189}$$

Frequencies less than this will not propagate at all.

The wave number can be written more simply in terms of the cutoff frequency:

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}. \tag{9.190}$$

The wave velocity is

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}}, \tag{9.191}$$

which is greater than c. However (see Prob. 9.29), the energy carried by the wave travels at the *group* velocity (Eq. 9.150):

$$v_g = \frac{1}{dk/d\omega} = c\sqrt{1 - (\omega_{mn}/\omega)^2} < c. \tag{9.192}$$

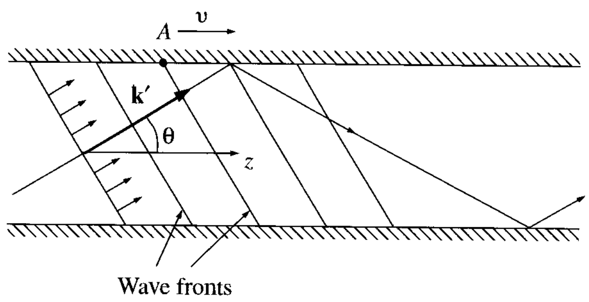

Figure 9.25

There's another way to visualize the propagation of an electromagnetic wave in a rectangular pipe, and it serves to illuminate many of these results. Consider an ordinary *plane* wave, traveling at an angle θ to the z axis, and reflecting perfectly off each conducting surface (Fig. 9.25). In the x and y directions the (multiply reflected) waves interfere to form standing wave patterns, of wavelength $\lambda_x = 2a/m$ and $\lambda_y = 2b/n$ (hence wave number $k_x = 2\pi/\lambda_x = \pi m/a$ and $k_y = \pi n/b$), respectively. Meanwhile, in the z direction there remains a traveling wave, with wave number $k_z = k$. The propagation vector for the "original" plane wave is therefore

$$\mathbf{k}' = \frac{\pi m}{a}\,\hat{\mathbf{x}} + \frac{\pi n}{b}\,\hat{\mathbf{y}} + k\,\hat{\mathbf{z}},$$

and the frequency is

$$\omega = c|\mathbf{k}'| = c\sqrt{k^2 + \pi^2[(m/a)^2 + (n/b)^2]} = \sqrt{(ck)^2 + (\omega_{mn})^2}.$$

Only certain angles will lead to one of the allowed standing wave patterns:

$$\cos\theta = \frac{k}{|\mathbf{k}'|} = \sqrt{1 - (\omega_{mn}/\omega)^2}.$$

The plane wave travels at speed c, but because it is going at an angle θ to the z axis, its net velocity down the wave guide is

$$v_g = c\cos\theta = c\sqrt{1 - (\omega_{mn}/\omega)^2}.$$

The *wave* velocity, on the other hand, is the speed of the wave fronts (A, say, in Fig. 9.25) down the pipe. Like the intersection of a line of breakers with the beach, they can move much faster than the waves themselves—in fact

$$v = \frac{c}{\cos\theta} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2)}}.$$

Problem 9.27 Show that the mode TE_{00} cannot occur in a rectangular wave guide. [*Hint:* In this case $\omega/c = k$, so Eqs. 9.180 are indeterminate, and you must go back to 9.179. Show that B_z is a constant, and hence—applying Faraday's law in integral form to a cross section—that $B_z = 0$, so this would be a TEM mode.]

Problem 9.28 Consider a rectangular wave guide with dimensions 2.28 cm × 1.01 cm. What TE modes will propagate in this wave guide, if the driving frequency is 1.70×10^{10} Hz? Suppose you wanted to excite only *one* TE mode; what range of frequencies could you use? What are the corresponding wavelengths (in open space)?

Problem 9.29 Confirm that the energy in the TE_{mn} mode travels at the group velocity. [*Hint:* Find the time averaged Poynting vector $\langle \mathbf{S} \rangle$ and the energy density $\langle u \rangle$ (use Prob. 9.11 if you wish). Integrate over the cross section of the wave guide to get the energy per unit time and per unit length carried by the wave, and take their ratio.]

Problem 9.30 Work out the theory of TM modes for a rectangular wave guide. In particular, find the longitudinal electric field, the cutoff frequencies, and the wave and group velocities. Find the ratio of the lowest TM cutoff frequency to the lowest TE cutoff frequency, for a given wave guide. [*Caution*: What is the lowest TM mode?]

9.5.3 The Coaxial Transmission Line

In Sect. 9.5.1, I showed that a *hollow* wave guide cannot support TEM waves. But a coaxial transmission line, consisting of a long straight wire of radius a, surrounded by a cylindrical conducting sheath of radius b (Fig. 9.26), *does* admit modes with $E_z = 0$ and $B_z = 0$. In this case Maxwell's equations (in the form 9.179) yield

$$k = \omega/c \tag{9.193}$$

(so the waves travel at speed c, and are nondispersive),

$$cB_y = E_x \quad \text{and} \quad cB_x = -E_y \tag{9.194}$$

(so **E** and **B** are mutually perpendicular), and (together with $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{B} = 0$):

$$\left. \begin{aligned} \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0, \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0, \\ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0, \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0. \end{aligned} \right\} \tag{9.195}$$

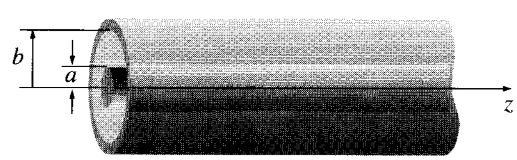

Figure 9.26

These are precisely the equations of *electrostatics* and *magnetostatics*, for empty space, in two dimensions; the solution with cylindrical symmetry can be borrowed directly from the case of an infinite line charge and an infinite straight current, respectively:

$$\mathbf{E}_0(s, \phi) = \frac{A}{s}\,\hat{\mathbf{s}}, \qquad \mathbf{B}_0(s, \phi) = \frac{A}{cs}\,\hat{\boldsymbol{\phi}}, \tag{9.196}$$

for some constant A. Substituting these into Eq. 9.176, and taking the real part:

$$\left.\begin{aligned} \mathbf{E}(s, \phi, z, t) &= \frac{A\cos(kz - \omega t)}{s}\,\hat{\mathbf{s}}, \\ \mathbf{B}(s, \phi, z, t) &= \frac{A\cos(kz - \omega t)}{cs}\,\hat{\boldsymbol{\phi}}. \end{aligned}\right\} \tag{9.197}$$

Problem 9.31

(a) Show directly that Eqs. 9.197 satisfy Maxwell's equations (9.177) and the boundary conditions 9.175.

(b) Find the charge density, $\lambda(z, t)$, and the current, $I(z, t)$, on the inner conductor.

More Problems on Chapter 9

! **Problem 9.32** The "inversion theorem" for Fourier transforms states that

$$\tilde{\phi}(z) = \int_{-\infty}^{\infty} \tilde{\Phi}(k)e^{ikz}\,dk \iff \tilde{\Phi}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\phi}(z)e^{-ikz}\,dz. \tag{9.198}$$

Use this to determine $\tilde{A}(k)$, in Eq. 9.20, in terms of $f(z, 0)$ and $\dot{f}(z, 0)$.
[*Answer:* $(1/2\pi)\int_{-\infty}^{\infty}[f(z, 0) + (i/\omega)\dot{f}(z, 0)]e^{-ikz}\,dz$]

Problem 9.33 Suppose

$$\mathbf{E}(r, \theta, \phi, t) = A\frac{\sin\theta}{r}\left[\cos(kr - \omega t) - (1/kr)\sin(kr - \omega t)\right]\hat{\boldsymbol{\phi}}, \quad \text{with } \frac{\omega}{k} = c.$$

(This is, incidentally, the simplest possible **spherical wave**. For notational convenience, let $(kr - \omega t) \equiv u$ in your calculations.)

(a) Show that $\mathbf{E}$ obeys all four of Maxwell's equations, in vacuum, and find the associated magnetic field.

(b) Calculate the Poynting vector. Average $\mathbf{S}$ over a full cycle to get the intensity vector $\mathbf{I}$. (Does it point in the expected direction? Does it fall off like r^{-2}, as it should?)

(c) Integrate $\mathbf{I} \cdot d\mathbf{a}$ over a spherical surface to determine the total power radiated. [*Answer:* $4\pi A^2/3\mu_0 c$]

! **Problem 9.34** Light of (angular) frequency ω passes from medium 1, through a slab (thickness d) of medium 2, and into medium 3 (for instance, from water through glass into air, as in Fig. 9.27). Show that the transmission coefficient for normal incidence is given by

$$T^{-1} = \frac{1}{4n_1n_3}\left[(n_1+n_3)^2 + \frac{(n_1^2-n_2^2)(n_3^2-n_2^2)}{n_2^2}\sin^2\left(\frac{n_2\omega d}{c}\right)\right]. \qquad (9.199)$$

[*Hint:* To the *left*, there is an incident wave and a reflected wave; to the *right*, there is a transmitted wave; inside the slab there is a wave going to the right and a wave going to the left. Express each of these in terms of its complex amplitude, and relate the amplitudes by imposing suitable boundary conditions at the two interfaces. All three media are linear and homogeneous; assume $\mu_1 = \mu_2 = \mu_3 = \mu_0$.]

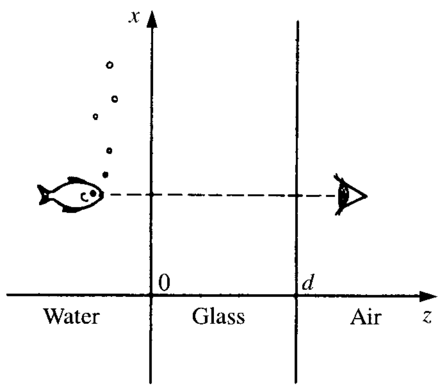

Figure 9.27

Problem 9.35 A microwave antenna radiating at 10 GHz is to be protected from the environment by a plastic shield of dielectric constant 2.5. What is the minimum thickness of this shielding that will allow perfect transmission (assuming normal incidence)? [*Hint:* use Eq. 9.199]

Problem 9.36 Light from an aquarium (Fig. 9.27) goes from water ($n = \frac{4}{3}$) through a plane of glass ($n = \frac{3}{2}$) into air ($n = 1$). Assuming it's a monochromatic plane wave and that it strikes the glass at normal incidence, find the minimum and maximum transmission coefficients (Eq. 9.199). You can see the fish clearly; how well can it see you?

! **Problem 9.37** According to Snell's law, when light passes from an optically dense medium into a less dense one ($n_1 > n_2$) the propagation vector **k** bends *away* from the normal (Fig. 9.28). In particular, if the light is incident at the **critical angle**

$$\theta_c \equiv \sin^{-1}(n_2/n_1), \qquad (9.200)$$

then $\theta_T = 90°$, and the transmitted ray just grazes the surface. If θ_I *exceeds* θ_c, there is no refracted ray at all, only a reflected one (this is the phenomenon of **total internal reflection**,

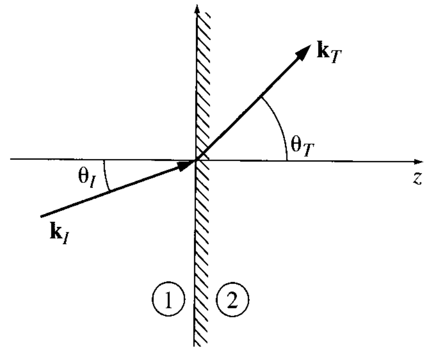

Figure 9.28

on which light pipes and fiber optics are based). But the *fields* are not zero in medium 2; what we get is a so-called **evanescent wave**, which is rapidly attenuated and transports no energy into medium 2.[19]

A quick way to construct the evanescent wave is simply to quote the results of Sect. 9.3.3. with $k_T = \omega n_2/c$ and

$$\mathbf{k}_T = k_T(\sin\theta_T\,\hat{\mathbf{x}} + \cos\theta_T\,\hat{\mathbf{z}});$$

the only change is that

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I$$

is now greater than 1, and

$$\cos\theta_T = \sqrt{1-\sin^2\theta_T} = i\sqrt{\sin^2\theta_T - 1}$$

is imaginary. (Obviously, θ_T can no longer be interpreted as an *angle*!)

(a) Show that

$$\tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{\mathbf{E}}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}, \tag{9.201}$$

where

$$\kappa \equiv \frac{\omega}{c}\sqrt{(n_1\sin\theta_I)^2 - n_2^2} \quad \text{and} \quad k \equiv \frac{\omega n_1}{c}\sin\theta_I. \tag{9.202}$$

This is a wave propagating in the x direction (*parallel* to the interface!), and attenuated in the z direction.

(b) Noting that α (Eq. 9.108) is now imaginary, use Eq. 9.109 to calculate the reflection coefficient for polarization parallel to the plane of incidence. [Notice that you get 100% reflection, which is better than at a conducting surface (see, for example, Prob. 9.21).]

(c) Do the same for polarization perpendicular to the plane of incidence (use the results of Prob. 9.16).

[19] The evanescent fields can be detected by placing a second interface a short distance to the right of the first: in a close analog to quantum mechanical **tunneling**, the wave crosses the gap and reassembles to the right. See F. Albiol, S. Navas, and M. V. Andres, *Am. J. Phys.* **61**, 165 (1993).

(d) In the case of polarization perpendicular to the plane of incidence, show that the (real) evanescent fields are

$$\left.\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= E_0 e^{-\kappa z} \cos(kx - \omega t)\,\hat{\mathbf{y}}, \\ \mathbf{B}(\mathbf{r}, t) &= \frac{E_0}{\omega} e^{-\kappa z} \left[\kappa \sin(kx - \omega t)\,\hat{\mathbf{x}} + k \cos(kx - \omega t)\,\hat{\mathbf{z}}\right]. \end{aligned}\right\} \tag{9.203}$$

(e) Check that the fields in (d) satisfy all of Maxwell's equations (9.67).

(f) For the fields in (d), construct the Poynting vector, and show that, on average, no energy is transmitted in the z direction.

! **Problem 9.38** Consider the **resonant cavity** produced by closing off the two ends of a rectangular wave guide, at $z = 0$ and at $z = d$, making a perfectly conducting empty box. Show that the resonant frequencies for both TE and TM modes are given by

$$\omega_{lmn} = c\pi\sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2}, \tag{9.204}$$

for integers l, m, and n. Find the associated electric and magnetic fields.