Now, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$

$\therefore \quad y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$

In the first quadrant, $y > 0$

$\therefore \quad y = \frac{b}{a}\sqrt{a^2 - x^2}$

$\therefore \quad I = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$

$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} sin^{-1} \frac{x}{a}\right]_0^a$

$= \frac{b}{a}\left[\left(\frac{a}{2} \times 0 + \frac{a^2}{2} sin^{-1} 1\right) - (0 + 0)\right]$

$= \frac{b}{a}\left[\frac{a^2}{2} sin^{-1} 1\right] = \frac{b}{a}\left[\frac{a^2}{2} \cdot \frac{\pi}{2}\right] = \frac{\pi ab}{4}$

$\therefore$ Required area $= 4 \times \frac{\pi ab}{4} = \pi ab$

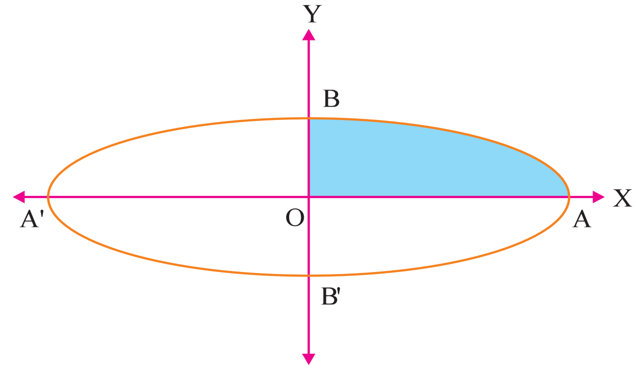


Figure 4.14

Remain : If we consider $x^2 + y^2 = r^2$ in this question then we get well known formula πr^2 for area of a circle.

Exercise 4.1

1. Find the area bounded by the parabola $y = x^2 + 2$, X-axis and the lines $x = 1$ and $x = 2$.

2. Find the area bounded by the parabola $y = x^2 - 4$, the X-axis and the lines $x = -1$ and $x = 2$.

3. What is the area bounded by the parabola $y = x^2$ and the lines $x = -2$ and $x = 1$?

4. Find the area of the region bounded by the curve $y = \sqrt{x-1}$, the Y-axis and the lines $y = 1$ and $y = 5$.

5. Find the area bounded by the X-axis the parabola $y = -x^2 + 4$.

6. Find the area bounded by the curve $y = 9 - x^2$ and the X-axis.

7. Find the area enclosed by the circle $x^2 + y^2 = a^2$.

8. Find the area of the region bounded by the parabola $y = x^2$ and the line $y = 4$.

*

4.3 Area Between Two Curves

In this section, we will find the area of the region bounded by a line and a circle, a line and a parabola, a line and an ellipse, a circle and a parabola, two circles etc.

Let us try to get intuitive idea of how area between two intersecting curves may be obtained. As discussed earlier, area of the region bounded by $y = f_1(x)$, $x = a$, $x = b$

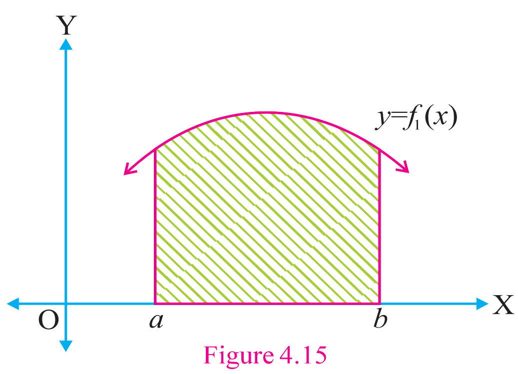


Figure 4.15

and X-axis is given by $A_1 = |I_1|$ where $I_1 = \int_a^b f_1(x)\,dx$.

Here, $I_1 \geq 0$ as we have assumed that $f_1(x) \geq 0$. (See figure 4.15)

As shown in figure 4.16 area of the region bounded by $y = f_2(x)$, $x = a$, $x = b$ and X-axis is given by $A_2 = |I_2|$ where $I_2 = \int_a^b f_2(x)\,dx$.

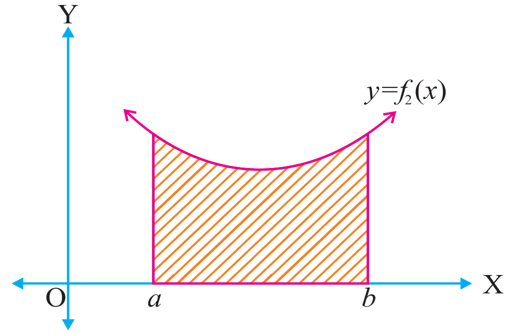


Figure 4.16

Since $f_2(x) \geq 0$ we have $I_2 \geq 0$.

If two curves $y = f_1(x)$ and $y = f_2(x)$ intersect each other at only two points for which their x-coordinates are a and b $(a \neq b)$, then the area enclosed by them is given by

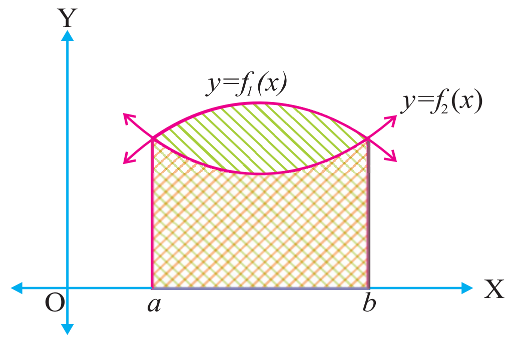


Figure 4.17

$$A = |I|$$

where $I = I_1 - I_2 = \int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx$

$$= \int_a^b (f_1(x) - f_2(x))\,dx$$

If two curves $x = g_1(y)$ and $x = g_2(y)$ intersect each other at only two points for which their y-coordinates are c and d $(c \neq d)$ then the area enclosed by them is given by $A = |I|$.

where $I = \int_c^d (g_1(y) - g_2(y))\,dy$.

Here we have assumed that $g_1(y) \geq 0$, $g_2(y) \geq 0$.

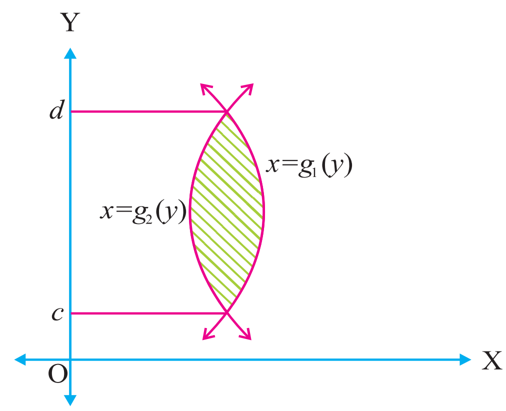


Figure 4.18

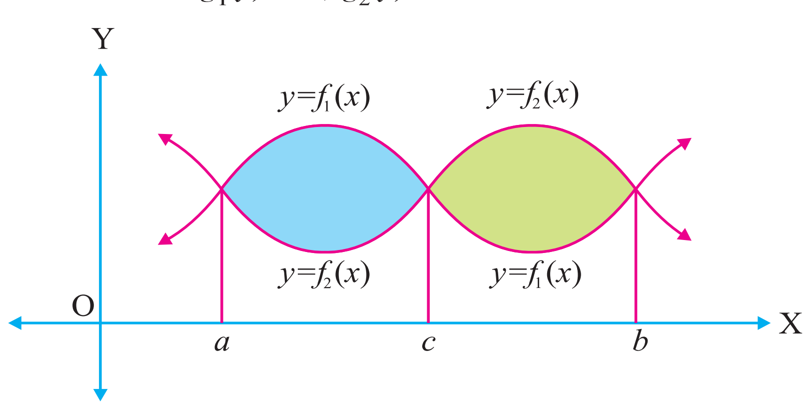


Figure 4.19

If the curves intersect once within the region being considered then as shown in the figure 4.19, the interval of integration will have to be split up. Suppose we wish to find the area between the curves

$y = f_1(x)$ and $y = f_2(x)$ and the lines $x = a$ and $x = b$. Suppose that the curves intersect each other at some point c between a and b then $A = |I_1| + |I_2|$.

where $I_1 = \int_a^c (f_1(x) - f_2(x))\, dx$, $I_2 = \int_c^b (f_1(x) - f_2(x))\, dx$

Example 5 : Find the smaller of the two areas enclosed between the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$.

Solution : The given line is $\frac{x}{a} + \frac{y}{b} = 1$ ... **(i)**

and the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... **(ii)**

Clearly, the line intersects the ellipse at A(a, 0) and B(0, b). The required area is shown as in the figure 4.20 as coloured region.

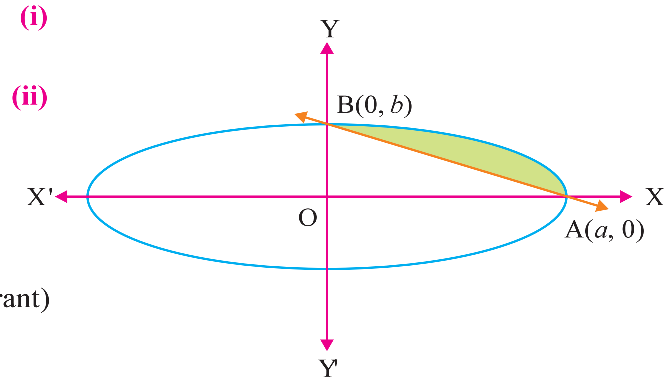


Figure 4.20

For the ellipse $y = \frac{b}{a}\sqrt{a^2 - x^2}$ (First quadrant)

Now, area of $\Delta AOB = \frac{1}{2} OA \cdot OB$

$$= \frac{1}{2}ab \quad \textbf{(iii)}$$

Also, area enclosed by the ellipse in the first quadrant is

$$\int_0^a y\,dx = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\left[\frac{a^2}{2}\sin^{-1} 1\right] = \frac{\pi ab}{4} \quad \textbf{(iv)}$$

$\therefore$ By (iii) and (iv)

Required area $= \left|\frac{\pi ab}{4} - \frac{1}{2}ab\right| = \left|\frac{(\pi - 2)ab}{4}\right| = \frac{(\pi - 2)ab}{4}$ as $\pi > 2$.

Second Method : Required area $= |I|$

where $I = \int_0^a (f_1(x) - f_2(x))\, dx$, where $f_1(x) = \frac{b}{a}\sqrt{a^2 - x^2}$ and $f_2(x) = b\left(1 - \frac{x}{a}\right)$

$$= \int_0^a \left[\frac{b}{a}\sqrt{a^2 - x^2} - b\left(1 - \frac{x}{a}\right)\right] dx$$

$$= \left[\frac{b}{a}\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right) - b\left(x - \frac{x^2}{2a}\right)\right]_0^a$$

$$= \left[\frac{b}{a}\left(0 + \frac{a^2}{2}\sin^{-1} 1\right) - b\left(a - \frac{a}{2}\right)\right] - (0)$$

$$= \frac{\pi ab}{4} - \frac{ab}{2}$$

$$= \frac{(\pi - 2)ab}{4}$$

$\therefore$ Required area $= \left| \frac{(\pi - 2)ab}{4} \right| = \frac{(\pi - 2)ab}{4}$ as $\pi > 2$.

Example 6 : Using integration, find the area of the region bounded by the circle $x^2 + y^2 = 4$, line $x - y\sqrt{3} = 0$ and X-axis in the first quadrant.

Solution : Here the given curves are $x^2 + y^2 = 4$ and $x - y\sqrt{3} = 0$.

Substitute $y = \frac{x}{\sqrt{3}}$ in $x^2 + y^2 = 4$.

$$x^2 + \frac{x^2}{3} = 4$$

$\therefore \quad 4x^2 = 12$

$\therefore \quad x = \pm \sqrt{3}$

In the first quadrant $x = \sqrt{3}$ and so $y = \frac{x}{\sqrt{3}} = 1$.

$\therefore$ In the first quadrant the point of intersection of the line and the circle is $P(\sqrt{3}, 1)$.

$\overline{PM} \perp$ X-axis and $M(\sqrt{3}, 0)$ is the foot of the perpendicular.

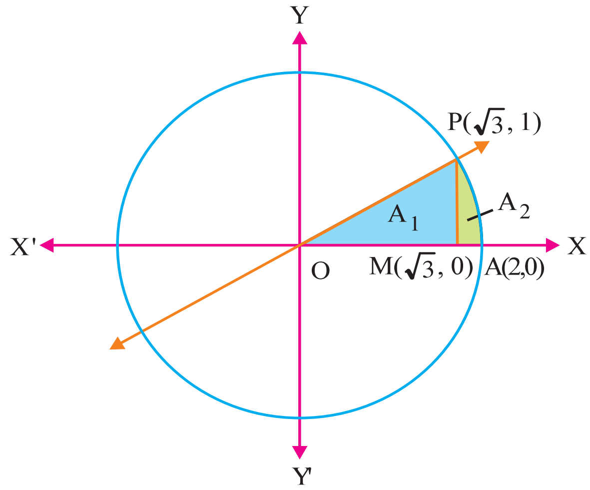


Figure 4.21

Now, area of the sector OPA.

= area of ΔOPM + Area of the region bounded by the circle $x^2 + y^2 = 4$, X-axis and the lines $x = \sqrt{3}$ and $x = 2$.

$\therefore$ Required area $= A_1 + A_2$

$A_1 = $ Area of ΔOPM

$= \frac{1}{2} OM \times PM$

$= \frac{1}{2}\sqrt{3} \times 1 = \frac{\sqrt{3}}{2}$ **(i)**

$A_2 = |I|$

where $I = \int_{\sqrt{3}}^{2} y\,dx = \int_{\sqrt{3}}^{2} \sqrt{4 - x^2}\,dx$ **($y > 0$ in the first quadrant)**

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2} sin^{-1} \frac{x}{2} \right]_{\sqrt{3}}^{2}$$

$$= \left(0 + 2sin^{-1} 1\right) - \left(\frac{\sqrt{3}}{2} + 2sin^{-1} \frac{\sqrt{3}}{2}\right)$$

$$= \pi - \frac{\sqrt{3}}{2} - \frac{2\pi}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$\therefore \quad A_2 = \left| \frac{\pi}{3} - \frac{\sqrt{3}}{2} \right| = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$ **(ii)**

$\left[\frac{\pi}{3} > 1 \text{ as } \pi > 3 \text{ and } \sqrt{3} < 2 \text{ so } \frac{\sqrt{3}}{2} < 1. \text{ So, } \frac{\pi}{3} - \frac{\sqrt{3}}{2} > 0 \right]$

$\therefore$ Required area $= \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{2} = \frac{\pi}{3}$

Second Method : Required area A = | I |

where $\mathrm{I} = \int_0^1 (g_1(y) - g_2(y))\, dy$, where $g_1(y) = \sqrt{4-y^2}$ and $g_2(y) = \sqrt{3}y$

$$= \int_0^1 \left(\sqrt{4-y^2} - \sqrt{3}y\right) dy$$

$$= \left[\frac{y}{2}\sqrt{4-y^2} + \frac{4}{2}\, sin^{-1}\frac{y}{2} - \frac{\sqrt{3}}{2}y^2\right]_0^1$$

$$= \frac{\sqrt{3}}{2} + 2sin^{-1}\frac{1}{2} - \frac{\sqrt{3}}{2} = 2 \cdot \frac{\pi}{6} = \frac{\pi}{3}$$

$\therefore$ Required area $= \frac{\pi}{3}$

> **Note :** $y = \frac{x}{\sqrt{3}}$ means $y = mx$, where $m = tan\theta = \frac{1}{\sqrt{3}}$ and $\theta = m\angle\mathrm{POM}$.
>
> So $m\angle\mathrm{POM} = \frac{\pi}{6}$.
>
> $\therefore$ Area of sector $= \frac{1}{2}r^2\theta = \frac{1}{2} \cdot 4 \cdot \frac{\pi}{6} = \frac{\pi}{3}$
>
> We may feel that it is easy to find area using geometry than using calculus. But we have to use integration to derive formula $\frac{1}{2}r^2\theta$ for area of a sector.

Example 7 : Find the area of the region bounded by the parabola $y = x^2$ and the rays $y = |x|$.

Solution : Consider the curves $y = x^2$ **(i)**

and $y = |x|$ **(ii)**

The two curves intersect where $x^2 = |x|$

$\therefore$ $|x|^2 - |x| = 0$ **($x^2 = |x|^2$)**

$\therefore$ $|x|(|x| - 1) = 0$

$\therefore$ $x = 0$ or $x = \pm 1$

For $x = 0$, $y = 0$

For $x = \pm 1$, $y = 1$

Hence, the two curves intersect at the points (−1, 1), (0, 0) and (1, 1).

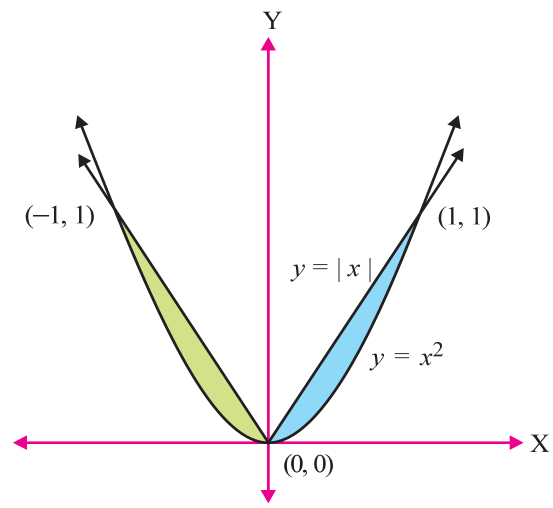


Figure 4.22

We have to find area of the region enclosed between given curves and is shown as coloured region in the figure 4.22.

As both the curves are symmetrical about Y-axis,

required area A = 2(area of the region in the first quadrant)

$= 2|\mathrm{I}|$ where $\mathrm{I} = \int_0^1 (f_1(x) - f_2(x))\, dx$, where $f_1(x) = |x|$ and $f_2(x) = x^2$

$$\mathrm{I} = \int_0^1 (|x| - x^2)\, dx$$

$$= \int_0^1 (x - x^2)\, dx \qquad \mathbf{(|x| = x \text{ in } [0, 1])}$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{6}$$

$\therefore$ Required area A $= 2 \times \frac{1}{6} = \frac{1}{3}$

Example 8 : Find the area of the region bounded by the circle $x^2 + y^2 = \frac{9}{4}$ and the parabola $x^2 = 4y$.

Solution : Given circle has equation $x^2 + y^2 = \frac{9}{4}$. **(i)**

and parabola has equation $x^2 = 4y$ **(ii)**

The two curves intersect at points where $4y = \frac{9}{4} - y^2$ **(each x^2)**

$\therefore$ $16y = 9 - 4y^2$

$\therefore$ $4y^2 + 16y - 9 = 0$

$\therefore$ $(2y - 1)(2y + 9) = 0$

$\therefore$ $y = \frac{1}{2}$ or $-\frac{9}{2}$

But $y \nless 0$, (Why ?) therefore the two curves intersect when $y = \frac{1}{2}$.

$\therefore$ $x^2 = 4y = 4 \times \frac{1}{2} = 2$

$\therefore$ $x = \pm\sqrt{2}$

$\therefore$ The two curves intersect at $\left(-\sqrt{2}, \frac{1}{2}\right)$ and $\left(\sqrt{2}, \frac{1}{2}\right)$.

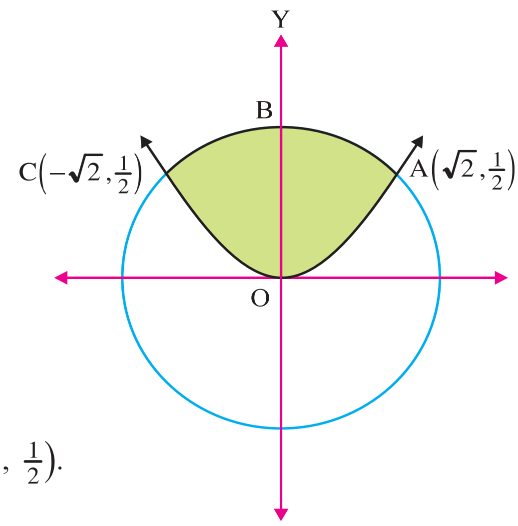


Figure 4.23

Since both the curves are symmetrical about Y-axis,

required area $= 2$(Area of region OABO)

$= 2 \,|\, \text{I} \,|$

where I $= \int_0^{\sqrt{2}} (f_1(x) - f_2(x))\, dx$, where $f_1(x) = \sqrt{\frac{9}{4} - x^2}$ and $f_2(x) = \frac{x^2}{4}$.

$$= \int_0^{\sqrt{2}} \left(\sqrt{\frac{9}{4} - x^2} - \frac{x^2}{4}\right) dx$$

$$= \left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{\left(\frac{3}{2}\right)^2}{2} sin^{-1} \frac{x}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^{\sqrt{2}}$$

$$= \left[\frac{\sqrt{2}}{2}\sqrt{\frac{9}{4} - 2} + \frac{9}{8} sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{2\sqrt{2}}{12}\right]$$

$$= \left[\frac{\sqrt{2}}{4} + \frac{9}{8} sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{\sqrt{2}}{6}\right]$$

$$= \frac{\sqrt{2}}{12} + \frac{9}{8} sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$\therefore$ Required area A $= 2\left[\frac{\sqrt{2}}{12} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

Example 9 : Using integration, find the area of the triangular region whose vertices are (4, 1), (6, 6) and (8, 4).

Solution : Let A(4, 1), B(6, 6) and C(8, 4) be the vertices of a triangle ABC. (See figure 4.24)

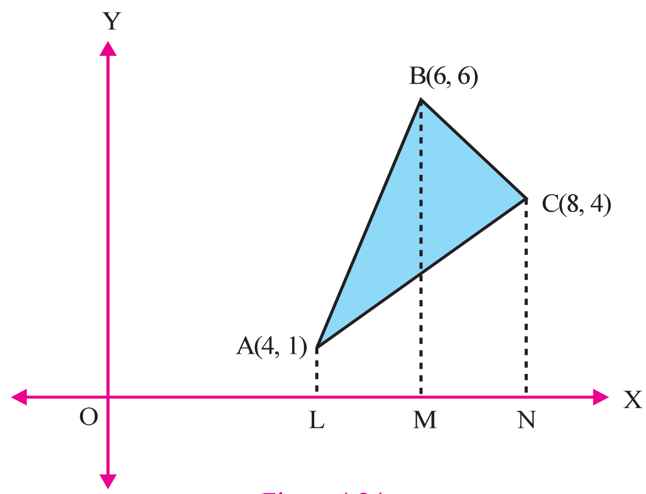


Figure 4.24

The equation of $\overleftrightarrow{AB}$ is $\frac{y-1}{6-1} = \frac{x-4}{6-4}$

$\therefore$ $y - 1 = \frac{5}{2}(x - 4)$

$\therefore$ $y - 1 = \frac{5}{2}x - 10$

$\therefore$ $y = \frac{5}{2}x - 9$

Similarly, the equation of $\overleftrightarrow{BC}$ is $y = -x + 12$ and the equation of $\overleftrightarrow{AC}$ is $y = \frac{3}{4}x - 2$

Let L, M, N be the feet of perpendiculars from A, B, C to X-axis respectively.

Now, area of ΔABC = area of region ALMB + area of region BMNC − area of region ALNC.

$$= |I_1| + |I_2| - |I_3|$$

$$= \left|\int_4^6 \left(\frac{5}{2}x - 9\right) dx\right| + \left|\int_6^8 (-x + 12)\, dx\right| - \left|\int_4^8 \left(\frac{3}{4}x - 2\right) dx\right|$$

$$= \left|\left[\frac{5x^2}{4} - 9x\right]_4^6\right| + \left|\left[-\frac{x^2}{2} + 12x\right]_6^8\right| - \left|\left[\frac{3x^2}{8} - 2x\right]_4^8\right|$$

$$= \left|\left[\left(\frac{5}{4}(36) - 54\right) - \left(\frac{5}{4}(16) - 36\right)\right]\right| + \left|\left[\left(-\frac{64}{2} + 96\right) - \left(-\frac{36}{2} + 72\right)\right]\right|$$

$$- \left|\left[\left(\frac{3}{8}(64) - 16\right) - \left(\frac{3}{8}(16) - 8\right)\right]\right|$$

$= |(-9 + 16)| + |(64 - 54)| - |(8 + 2)|$

$= 7 + 10 - 10$

$\therefore$ Required area $= 7$

Note : Area of the triangle $\Delta = \frac{1}{2}|D|$

where $D = \begin{vmatrix} 4 & 1 & 1 \\ 6 & 6 & 1 \\ 8 & 4 & 1 \end{vmatrix}$

$= 4(2) - 1(-2) + 1(-24) = -14$

$\therefore \quad \Delta = \frac{1}{2}|-14| = 7$

Example 10 : Find the area of the region bounded by the circle $x^2 + y^2 - 2ax = 0$ and the parabola $y^2 = ax$, $a > 0$ in the first quadrant.

Solution : The equation $x^2 + y^2 - 2ax = 0$ can be written as $(x - a)^2 + y^2 = a^2$ which represents a circle whose centre is $(a, 0)$ and radius is a. $y^2 = ax$ is a parabola whose vertex is $(0, 0)$ and its axis is X-axis.

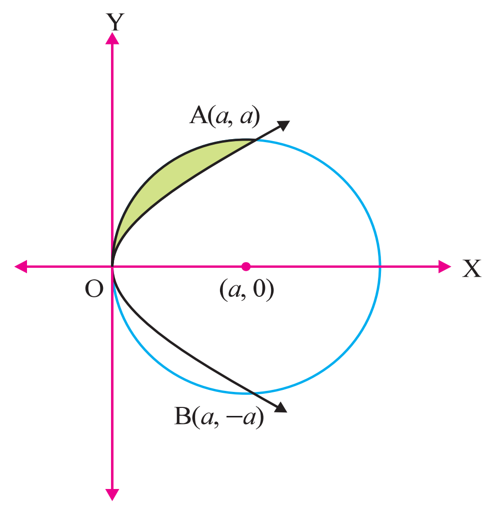


Figure 4.25

Substituting $y^2 = ax$ in $x^2 + y^2 - 2ax = 0$,

$x^2 + ax - 2ax = 0$

$\therefore \quad x^2 - ax = 0$

$\therefore \quad x(x - a) = 0$

$\therefore \quad x = 0$ or $x = a$

Since $y^2 = ax$,

$y = 0$ or $y = \pm a$

$\therefore$ Both the curves intersect at O(0, 0), A(a, a) and B(a, $-a$)

$\therefore \quad x^2 + y^2 = 2ax$ gives $y = \sqrt{2ax - x^2}$, $y^2 = ax$ gives $y = \sqrt{ax}$ **(as $y \geq 0$)**

Required area $= |I|$

where $I = \int_0^a (f_1(x) - f_2(x))\, dx$, where $f_1(x) = \sqrt{2ax - x^2}$ and $f_2(x) = \sqrt{ax}$.

$$= \int_0^a \left(\sqrt{2ax - x^2} - \sqrt{ax}\right) dx \quad \textbf{(First quadrant)}$$

$$= \int_0^a \left(\sqrt{a^2 - (x-a)^2} - \sqrt{a}\sqrt{x}\right) dx$$

$$= \left[\left(\frac{x-a}{2}\right)\sqrt{a^2 - (x-a)^2} + \frac{a^2}{2} sin^{-1}\left(\frac{x-a}{a}\right) - \sqrt{a}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$$

$$= \left[-\frac{2}{3}\sqrt{a}\cdot a^{\frac{3}{2}} - \frac{a^2}{2}\, sin^{-1}(-1)\right]$$

$$\text{I} = -\frac{2}{3}a^2 + \frac{a^2\pi}{4} = \left(\frac{3\pi - 8}{12}\right)a^2$$

$\therefore$ Required area $= \left(\frac{3\pi - 8}{12}\right)a^2$

Example 11 : Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 3$ and $x = 0$.

Solution : Here $y = x^2 + 2$

$\therefore$ $x^2 = y - 2$, which is a parabola whose vertex is (0, 2) and it opens upwards.

Let us draw a graph of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 3$ and $x = 0$.

Required area A = | I |

where $\text{I} = \int_0^3 (f_1(x) - f_2(x))\, dx$,

where $f_1(x) = x^2 + 2$ and $f_2(x) = x$.

$$= \int_0^3 (x^2 + 2 - x)\, dx$$

$$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^3$$

$$= 9 + 6 - \frac{9}{2}$$

$$= \frac{21}{2}$$

$\therefore$ $\text{A} = \frac{21}{2}$

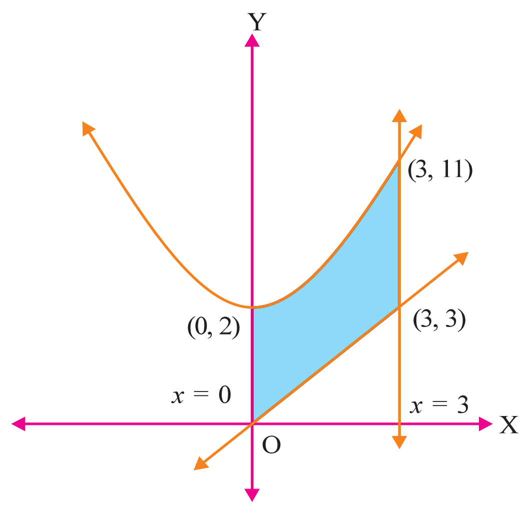


Figure 4.26

Example 12 : Find the area of the region bounded by the curves $y = 4 - x^2$, $x = 0$, $x = 3$ and X-axis.

Solution : Here $y = 4 - x^2$

So $x^2 = 4 - y$

$\therefore$ $x^2 = -(y - 4)$, which is the equation of a parabola. Its vertex is (0, 4) and opens downwards. To find its point of intersection with X-axis, let us take $y = 0$.

$\therefore$ $4 - x^2 = 0$

$\therefore$ $x = \pm 2$

$\therefore$ The points of intersection of the curve with X-axis are (2, 0) and (−2, 0).

Here, the limits of the region bounded by the curve and the X-axis are $x = 0$ and $x = 3$. The curve intersects X-axis at (2, 0) between (0, 0) and (3, 0).

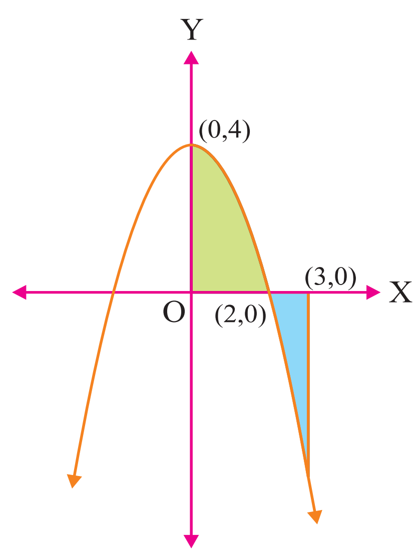


Figure 4.27

So, A = $|I_1| + |I_2|$

where $I_1 = \int_0^2 y\,dx, \quad I_2 = \int_2^3 y\,dx$

$$I_1 = \int_0^2 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

$$I_2 = \int_2^3 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_2^3 = (12 - 9) - \left(8 - \frac{8}{3}\right)$$

$$= 3 - \frac{16}{3} = -\frac{7}{3}$$

$\therefore$ Required area A = $\left|\frac{16}{3}\right| + \left|-\frac{7}{3}\right| = \frac{16}{3} + \frac{7}{3} = \frac{23}{3}$

Example 13 : Find the area bounded by the curve $y = cosx$ between $x = 0$ and $x = 2\pi$.

Solution :

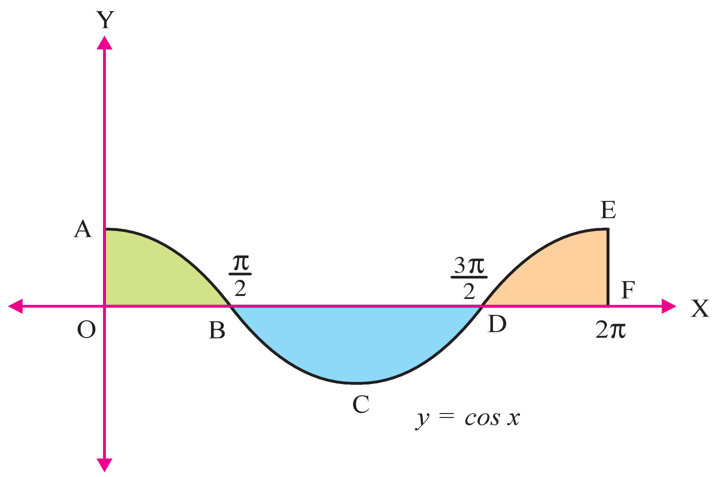


Figure 4.28

From the figure 4.28, the required area = area of the region OABO + area of the region BCDB + area of the region DEFD

$$\therefore \text{ Required area } = \left|\int_0^{\frac{\pi}{2}} cosx\;dx\right| + \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} cosx\;dx\right| + \left|\int_{\frac{3\pi}{2}}^{2\pi} cosx\;dx\right|$$

$$= \left|[sin\,x]_0^{\frac{\pi}{2}}\right| + \left|[sin\,x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right| + \left|[sin\,x]_{\frac{3\pi}{2}}^{2\pi}\right|$$

$$= |(1 - 0)| + |(-1 - 1)| + |(0 - 1)|$$

$$= 1 + 2 + 1 = 4$$

Example 14 : Determine the area of the region enclosed by $y = sinx$, $y = cosx$, $x = \frac{\pi}{2}$ and the Y-axis.

Solution : First let us draw the graph of the region.

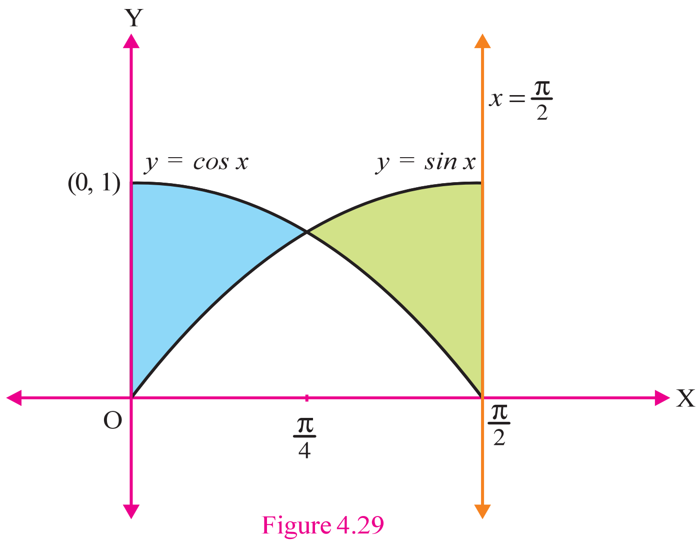


Figure 4.29

Now, from the figure it is clearly seen that we have a situation where we will need to evaluate two integrals to get the area. The point of intersection of $y = sinx$ and $y = cosx$ will be where $sinx = cosx$ in $\left[0, \frac{\pi}{2}\right]$.

This gives $x = \frac{\pi}{4}$. **(Why ?)**

The required area A = $|I_1| + |I_2|$

where $I_1 = \int_0^{\frac{\pi}{4}} (f_1(x) - f_2(x))\, dx$, where $f_1(x) = cosx$ and $f_2(x) = sinx$.

$$= \int_0^{\frac{\pi}{4}} (cosx - sinx)\, dx$$

$$= [sinx + cosx]_0^{\frac{\pi}{4}}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1)\right] = \sqrt{2} - 1 \qquad \textbf{(i)}$$

$$I_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (f_1(x) - f_2(x))\, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (cosx - sinx)\, dx$$

$$= [sinx + cosx]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[(1 + 0) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right]$$

$$= 1 - \sqrt{2} < 0 \qquad \textbf{(ii)}$$

$\therefore \quad |I_2| = \sqrt{2} - 1$

From (i) and (ii) required area A = $|I_1| + |I_2| = \sqrt{2} - 1 + \sqrt{2} - 1 = 2(\sqrt{2} - 1)$

Exercise 4.2

1. Find the area of the region enclosed by parabola $4y = 3x^2$ and the line $2y = 3x + 12$.
2. Find the area of the region bounded by curves $y = 2x - x^2$ and the line $y = -x$.
3. Find the area of the region bounded by the curves $f(x) = \cos\pi x$ and X-axis where $x \in [0, 2]$.
4. Find the area of the region bounded by the curves $f(x) = 4 - x^2$ and $g(x) = x^2 - 4$.
5. Find the area of the region bounded by the curves $y = x$, $y = 1$ and $y = \frac{x^2}{4}$ lying in the first quadrant.
6. Find the area of the region enclosed by the curves $y = x^2 + 5x$ and $y = 3 - x^2$ and bounded by $x = -2$ and $x = 0$.
7. Find the area bounded by the curves $y = x^2$, $y = 2 - x$ and above the line $y = 1$.
8. Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$.
9. Using integration, find the area of the triangular region whose sides lie along the lines $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.
10. Using integration, find the area of the triangular region formed by $(-1, 1)$, $(0, 5)$ and $(3, 2)$.
11. Find the area of the region in the first quadrant enclosed by the X-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.
12. Find the area of the region bounded by $y = 5 - x^2$, $x = 2$, $x = 3$ and X-axis.

*

Region Represented by Inequalities :

Consider $\{(x, y) \mid 0 \le y \le x^2\}$.

As shown in the figure 4.30, if we consider any point $P(x, y)$ on $\overline{AB}$, then $y \ge 0$ and $y \le x^2$.

So if B is any point on the parabola and A is on X-axis such that $\overline{AB} \perp$ X-axis then any point $P(x, y) \in \overline{AB}$ will satisfy $0 \le y \le x^2$.

Now, consider $\{(x, y) \mid 0 \le y \le x^2, 0 \le y \le x + 2, x \ge 0\}$

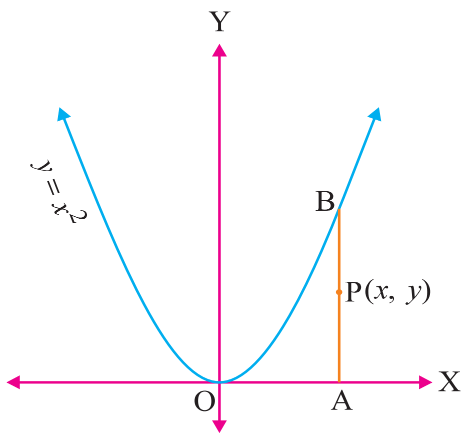


Figure 4.30

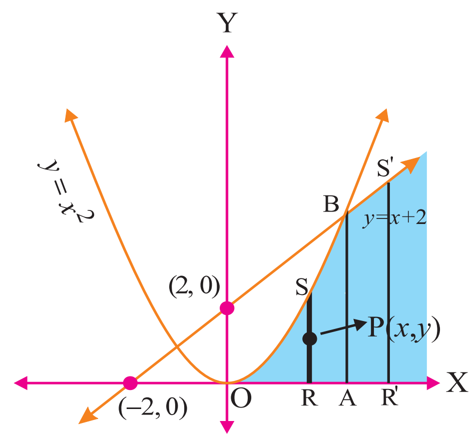


Figure 4.31

As shown in the figure 4.31, if we consider any point $P(x, y)$ on $\overline{RS}$, then $y \ge 0$, $y \le x^2$ and $y \le x + 2$. Similarly for any point on $\overline{R'S'}$ also conditions satisfied.

All such points P form a set satisfying given conditions. The region represented by the given set is coloured in the figure 4.31.

Miscellaneous Examples :

Example 15 : Find the area of the region : $\{(x, y) \mid 0 \le y \le x^2, 0 \le y \le x + 2, 0 \le x \le 3\}$.

Solution : Let us first sketch the region whose area is to be found out.

We have $0 \le y \le x^2$ **(i)**

$0 \le y \le x + 2$ **(ii)**

$0 \le x \le 3$ **(iii)**

Draw the curve $y = x^2$, a parabola with origin as vertex.

The line $y = x + 2$ intersects the parabola $y = x^2$, where $x + 2 = x^2$

$\therefore \quad x^2 - x - 2 = 0$

$\therefore \quad (x - 2)(x + 1) = 0$

$\therefore \quad x = 2, -1$

For $x = 2$, $y = 4$ and for $x = -1$, $y = 1$

The points of intersection of $y = x^2$ and $y = x + 2$ are P(2, 4) and M(−1, 1).

Since $0 \le x \le 3$ the above region is as shown as coloured region OPQRSO in figure 4.32.

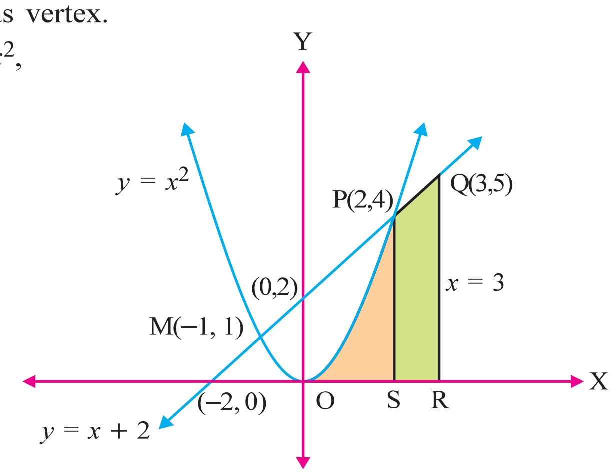


Figure 4.32

The required area A = area of region OPSO + area of the region SPQRS

The area of the region OPSO is bounded by the curve $y = x^2$, $x = 0$, $x = 2$ and X-axis.

The area of the region SPQRS is bounded by $y = x + 2$, $x = 2$, $x = 3$ and X-axis.

$$\therefore \quad \text{Required area} = \int_0^2 x^2 dx + \int_2^3 (x + 2)\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{x^2}{2} + 2x\right]_2^3$$

$$= \left(\frac{8}{3} - 0\right) + \left(\frac{9}{2} + 6\right) - (2 + 4)$$

$$= \frac{43}{6}$$

Example 16 : Find the area of the region enclosed by two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.

Solution : Here, $x^2 + y^2 = 1$

$\therefore \quad y^2 = 1 - x^2$

$(x - 1)^2 + y^2 = 1$

$\therefore \quad y^2 = 1 - (x - 1)^2$

For points of intersection, $1 - x^2 = 1 - (x - 1)^2$

$\therefore \quad -x^2 = -x^2 + 2x - 1$

$\therefore \quad x = \frac{1}{2}$

$$\therefore \quad y = \pm\sqrt{1 - x^2} = \pm\sqrt{1 - \frac{1}{4}} = \pm\frac{\sqrt{3}}{2}$$

Hence the circles intersect at the points $A\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $B\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$.

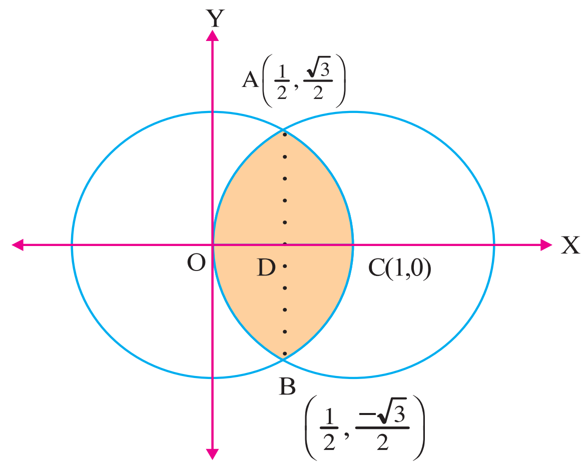


Figure 4.33

Required area = area of the region OACBO.

Since both the circles are symmetric about X-axis, the required area,

= 2(area of the region OACDO)

= 2[area of the region OADO + area of the region DACD)

The area of the region OADO is bounded by the circle $(x - 1)^2 + y^2 = 1$

i.e., $y = \sqrt{1-(x-1)^2}$ (first quadrant), $x = 0$, $x = \frac{1}{2}$ and X-axis, while the area of the region DACD is bounded by the circle $x^2 + y^2 = 1$. i.e. $y = \sqrt{1-x^2}$, $x = \frac{1}{2}$, $x = 1$ and X-axis.

The required area is sum of the two areas. **(Why not $|I_1| + |I_2|$?)**

$$\text{Required area} = 2\left[\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}\, dx + \int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\, dx\right]$$

$$= 2\left[\frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + \frac{1}{2}sin^{-1}(x-1)\right]_0^{\frac{1}{2}} + 2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}sin^{-1}x\right]_{\frac{1}{2}}^{1}$$

$$= 2\left[\frac{1}{2}\left(-\frac{1}{2}\right)\frac{\sqrt{3}}{2} + \frac{1}{2}sin^{-1}\left(-\frac{1}{2}\right) - 0 - \frac{1}{2}sin^{-1}(-1)\right] +$$

$$2\left[0 + \frac{1}{2}sin^{-1} 1 - \frac{1}{4}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}sin^{-1}\frac{1}{2}\right]$$

$$= 2\left(-\frac{\sqrt{3}}{8} - \frac{\pi}{12} + \frac{\pi}{4}\right) + 2\left(\frac{\pi}{4} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}\right)$$

$$= 2\left(-\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2}\right) = 2\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]$$

Second Method :

Required area = | I |,

$$I = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} (g_1(y) - g_2(y))\, dy$$

where $g_1(y) = \sqrt{1-y^2}$ and $g_2(y) = 1 - \sqrt{1-y^2}$ **(Why ?)**

$$I = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left[\sqrt{1-y^2} - \left(1 - \sqrt{1-y^2}\right)\right] dy$$

$$= 2\int_{0}^{\frac{\sqrt{3}}{2}} \left(2\sqrt{1-y^2} - 1\right) dy$$

$$= 4\int_{0}^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-y^2} - \frac{1}{2}\right) dy$$

$$= 4\left[\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1}y - \frac{y}{2}\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= 4\left[\frac{\sqrt{3}}{4}\sqrt{1-\frac{3}{4}} + \frac{1}{2}\sin^{-1}\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\right]$$

$$= 4\left[\frac{\sqrt{3}}{4}\cdot\frac{1}{2} + \frac{1}{2}\times\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] = 2\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]$$

$\therefore$ Required area $= 2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$

Note : From figure 4.34, $OM = \frac{1}{2}$, $AM = \frac{\sqrt{3}}{2}$

Therefore $m\angle AOM = \frac{\pi}{3}$

$\therefore$ Area of sector OAC $= \frac{1}{2}(1)^2\frac{\pi}{3} = \frac{\pi}{6}$

$\therefore$ Area of $\Delta AOM = \frac{1}{2}\times\frac{\sqrt{3}}{2}\times\frac{1}{2} = \frac{\sqrt{3}}{8}$

$\therefore$ $A_2 = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$

Similarly, $A_1 = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$

$\therefore$ Required area $= 2\left[\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) + \left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)\right]$

$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

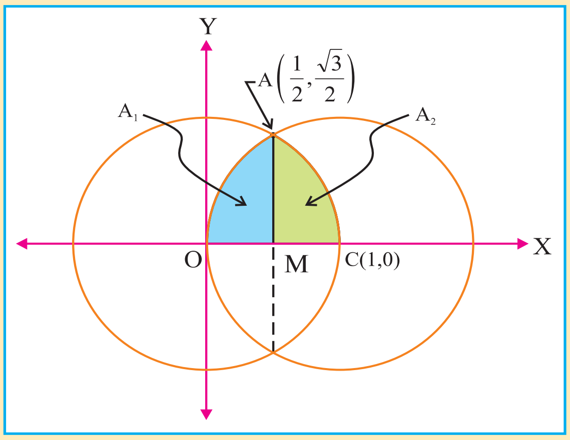


Figure 4.34

Exercise 4

1. Find the area of the region bounded by the curve $y = x^2 - x - 6$ and the X-axis.
2. Find the area of the region bounded by the Y-axis, the line $y = 3$ and the curve $y = x^2 + 2$ in the first quadrant.
3. Calculate the area bounded by the curve $y = (x - 1)(x - 2)$ and the X-axis.
4. Find the area of the region bounded by the circle $x^2 + y^2 = 3$, line $x - y\sqrt{3} = 0$ and the X-axis in the first quadrant.
5. Determine the area enclosed between the two curves $y^2 = x + 1$ and $y^2 = -x + 1$.

6. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

7. Find the area lying in the first quadrant enclosed by X-axis, the circle $x^2 + y^2 = 8x$ and parabola $y^2 = 4x$.

8. Find the area of the region bounded by the line $y = 3x + 2$, the X-axis and the lines $x = -1$ and $x = 1$.

9. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three congruent parts.

10. Find the area of the region $\{(x, y) \mid 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$.

11. Find the area of the region bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 4x$.

12. Find the area of the region enclosed by $y^2 = 8x$ and $x + y = 0$.

13. Using integration, find the area of the region bounded by the curve $|x| + |y| = 1$.

14. Using integration, find the area of the given region : $\{(x, y) \mid |x - 1| \le y \le \sqrt{5 - x^2}\}$.

15. Find the area of the region enclosed by the parabola $y^2 = x$ and the line $x + y = 2$.

16. Find the area of the region bounded by $y = x^2 + 1$, $y = x$, $x = 0$ and $y = 2$.

17. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The area enclosed by $y = x$, $y = 1$, $y = 3$ and the Y-axis is .... ☐

(a) 2 (b) $\frac{9}{2}$ (c) 4 (d) $\frac{3}{2}$

(2) The area enclosed by the curve $y = 2x - x^2$ and the X-axis is .... ☐

(a) $\frac{8}{5}$ (b) 2 (c) 8 (d) $\frac{4}{3}$

(3) The area enclosed by $y = cosx$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and the X-axis is .... ☐

(a) 1 (b) 4 (c) 2 (d) π

(4) The area bounded by the curve $y = sinx$, $\pi \le x \le 2\pi$ and the X-axis is .... ☐

(a) π (b) 2 (c) -2 (d) 0

(5) The area enclosed by $y = x^2$, the X-axis and the line $x = 4$ is divided into two congruent halves by the line $x = a$. The value of a is .... ☐

(a) 2 (b) $2^{\frac{4}{3}}$ (c) $2^{\frac{5}{3}}$ (d) 4

(6) The area of the region bounded by the lines $x = 2y + 3$, $y = 1$, $y = -1$ and Y-axis is .... ☐

(a) 4 (b) $\frac{3}{2}$ (c) 6 (d) 8

(7) The area bounded by the parabola $y^2 = 4ax$ and its latus rectum is .... ☐

(a) $\frac{4}{3}a^2$ (b) $\frac{8}{3}a^2$ (c) $\frac{16}{3}a^2$ (d) $\frac{32}{3}a^2$

(8) Area bounded by the curve $y = 2x^2$, the X-axis and the line $x = 1$ is ....

(a) 2 (b) 1 (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

(9) The area bounded by the curve $y = x|x|$, X-axis and the lines $x = -1$ and $x = 1$ is ......

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

(10) The area bounded by the curves $y = cosx$, $y = sinx$, Y-axis and $0 \le x \le \frac{\pi}{4}$ is ....

(a) $2(\sqrt{2} - 1)$ (b) $\sqrt{2} - 1$ (c) $\sqrt{2} + 1$ (d) $\sqrt{2}$

(11) Area bounded by the line $y = 3 - x$ and the X-axis on the interval [0, 3] is ....

(a) $\frac{9}{2}$ (b) 4 (c) 5 (d) $\frac{11}{2}$

(12) Area bounded by the curves $y = x^2$ and $x = y^2$ is ....

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{12}$ (d) 1

(13) Area bounded by the curve $y = sinx$ bounded by $x = 0$ and $x = 2\pi$ is ....

(a) 1 (b) 2 (c) 3 (d) 4

(14) The area bounded by the curve $y = 3\,cosx$, $0 \le x \le \frac{\pi}{2}$, $y = 0$ is ....

(a) 3 (b) 1 (c) $\frac{3}{2}$ (d) $\frac{1}{2}$

(15) The area under the curve $y = cos^2x$ between $x = 0$ and $x = \pi$ is ....

(a) π (b) $\frac{\pi}{2}$ (c) 2π (d) 2

(16) The area under the curve $y = 2\sqrt{x}$ bounded by the lines $x = 0$ and $x = 1$ is ....

(a) $\frac{4}{3}$ (b) $\frac{2}{3}$ (c) 1 (d) $\frac{8}{3}$

(17) The area bounded by $y = 2x - x^2$ and X-axis is ....

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) 1 (d) $\frac{4}{3}$

(18) The area bounded by the curve $y = 3x$, X-axis and the lines $x = 1$, $x = 3$ is ....

(a) 3 (b) 6 (c) 12 (d) 36

(19) The area bounded by the curve $y = |x - 5|$, X-axis and the lines $x = 0$, $x = 1$ is ....

(a) $\frac{9}{2}$ (b) $\frac{7}{2}$ (c) 9 (d) 5

(20) The area of the region between the curve $y^2 = 4x$ and the line $x = 3$ is ....

(a) $4\sqrt{3}$ (b) $8\sqrt{3}$ (c) $16\sqrt{3}$ (d) $5\sqrt{3}$

Summary

We have studied the following points in this chapter :

1. The area A of the region bounded by the curve $y = f(x)$, X-axis and the lines $x = a$, $x = b$ is given by $A = |I|$, where $I = \int_a^b f(x)\,dx$.

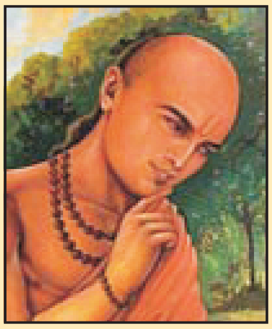
2. The area A of the region bounded by the curve $x = g(y)$, Y-axis and the lines $y = c$, $y = d$ is given by $A = |I|$, where $I = \int_c^d g(y)\,dy$.

3. If the graph of $y = f(x)$ intersects X-axis at $(c, 0)$ only and $a < c < b$, then the area of the region bounded by $y = f(x)$, $x = a$, $x = b$ and X-axis is given by $A = |I_1| + |I_2|$, where $I_1 = \int_a^c f(x)\,dx$, $I_2 = \int_c^b f(x)\,dx$.

4. If the two curves $y = f_1(x)$ and $y = f_2(x)$ intersect each other at only two points for $x = a$ and $x = b$ $(a \neq b)$, then the area enclosed by them is given by $A = |I|$, where $I = \int_a^b (f_1(x) - f_2(x))\,dx$.

5. If the two curves $x = g_1(y)$ and $x = g_2(y)$ intersect each other at only two points for $y = c$ and $y = d$ $(c \neq d)$, then the area enclosed by them is given by $A = |I|$, where $I = \int_c^d (g_1(y) - g_2(y))\,dy$.

BHASKARACHARYA



He was born in a village of Mysore district.

He was the first to give that any number divided by 0 gives infinity.

He has written a lot about zero, surds, permutation and combination.

He wrote, "The hundredth part of the circumference of a circle seems to be straight. Our earth is a big sphere and that's why it appears to be flat."

He gave the formulae like $sin(A \pm B) = sinA \cdot cosB \pm cosA \cdot sinB$

Differential Equations 5

Mathematics is the art of giving the same name to different things.
– Jules Henri

5.1 Introduction

If y is a function of x, then we denote it as $y = f(x)$. Here x is called **an independent variable** and y is called **a dependent variable**. We have already learnt various methods to find $\frac{dy}{dx}$ or $f'(x)$. Also we know how to find f using indefinite integration when we are given an equation like $f'(x) = g(x)$ (Primitive) i.e. $\frac{dy}{dx} = g(x)$

Here the equation $\frac{dy}{dx} = g(x)$ contains the variable x and derivative of y *w.r.t.* x. This type of an equation is known as **a differential equation**. We will give a formal definition later.

Differential equations play an important role in the solution of problems of Physics, Chemistry, Biology, Engineering etc. Here we will study the basic concepts of differential equations, the solution of a first order - first degree differential equation and also simple applications of differential equations.

Note : If the function $y = f(x)$ is a differentiable function of x, then its first order derivative is denoted by $\frac{dy}{dx}$, y_1, y' or $f'(x)$. If $f'(x)$ is also a differentiable function of x, then the second order derivative of the function $y = f(x)$ is denoted by $\frac{d^2y}{dx^2}$, y_2, y'' or $f''(x)$. Similarly we may get third order, fourth order derivatives of the function $y = f(x)$ etc. In general nth order derivative of the function $y = f(x)$ is denoted by the symbols $\frac{d^ny}{dx^n}$, y_n, $y^{(n)}$ or $f^{(n)}(x)$. Here, $y_n = \frac{d}{dx}(y_{n-1})$.

5.2 Differential Equation

An equation containing an independent variable and a dependent variable and the derivatives of the dependent variable with respect to the independent variable is called an ordinary differential equation.

If x is an independent variable, y is a dependent variable depending upon x i.e. $y = f(x)$ or $G(x, y) = 0$ and the derivatives of y *w.r.t.* x are $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$ then the functional equation $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}\right) = 0$ is called an ordinary differential equation (Derivatives must occur in this equation)

For instance, (1) $\dfrac{dy}{dx} + y\cos x = \sin x$ (2) $\dfrac{d^2y}{dx^2} = 2x$

(3) $\dfrac{dy}{dx} + y = x^2$ (4) $2y = x\dfrac{dy}{dx} + \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

(5) $2x^2\left(\dfrac{d^2y}{dx^2}\right)^3 + 5y\dfrac{dy}{dx} = 2xy$ (6) $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 5\dfrac{d^2y}{dx^2}$

(7) $e^{\frac{dy}{dx}} + \dfrac{dy}{dx} = ky$ (8) $\log\left|\dfrac{dy}{dx}\right| = kx$

5.3 Order and Degree of a Differential Equation

Order of the highest order derivative of the dependent variable with respect to the independent variable occurring in a given differential equation is called the order of differential equation.

(1) $\dfrac{dy}{dx} + y\cos x = \sin x$

The order of the highest order derivative is 1. So it is a differential equation of order 1.

(2) $2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = e^x$

The order of the highest order derivative is 2. So it is a differential equation of order 2.

(3) $\left(\dfrac{dy}{dx}\right)^2 + 6y + x = 0.$

The order of the highest order derivative is 1. So it is a differential equation of order 1.

(4) $\dfrac{d^4y}{dx^4} - 6\left(\dfrac{dy}{dx}\right)^6 - 4y = 0.$

The order of the highest order derivative is 4. So it is a differential equation of order 4.

(5) $\dfrac{d^2y}{dx^2} = \sqrt{\dfrac{dy}{dx}+5}.$

The order of the highest order derivative is 2. So it is a differential equation of order 2.

Degree of a Differential Equation :

When a differential equation is in a polynomial form in derivatives, the highest power of the highest order derivative occurring in the differential equation is called the degree of the differential equation.

Obviously to obtain the degree of a differential equation, we should make the equation free from radicals and fractional powers.

The degree of a differential equation is a positive integer.

(1) $\left(\dfrac{dy}{dx}\right)^2 + 2y = \sin x.$

In this equation the highest power of the highest order derivative is 2. So the degree of the differential equation is 2.

(2) $\dfrac{d^3y}{dx^3} + 7\left(\dfrac{dy}{dx}\right)^4 - 4y = 0$

In this equation the highest power of the highest order derivative is 1. So its degree is 1. (Why not 4?)

(3) $x = y\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Convert this equation in a polynomial form in derivatives.

We get, $(y^2 - 1)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + x^2 - 1 = 0$

In this equation, the power of highest order derivative is 2. So the differential equation has degree 2.

Note : To determine the degree, the differential equation has to be expressed in a polynomial form. If the differential equation cannot be expressed in a polynomial form in the derivatives, the degree of the differential equation is not defined.

For example,

(1) $x\frac{dy}{dx} + sin\left(\frac{dy}{dx}\right) = 0$ is a given differential equation. Its order is 1 and degree is not defined because the equation is not in a polynomial form in derivatives.

(2) $\frac{d^2y}{dx^2} = \log\left(\frac{dy}{dx}\right) + y$, the order of the equation is 2 and the degree is not defined because we cannot express this equation in a polynomial form in derivatives.

Example 1 : Obtain the order and degree (if possible) of the following differential equation :

(1) $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + y = x^2$

(2) $\frac{d^2y}{dx^2} = \sqrt[3]{1+\left(\frac{dy}{dx}\right)^2}$

(3) $xe^{\frac{dy}{dx}} + \frac{dy}{dx} + 2 = 0$

(4) $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + xy = 0$

(5) $\left(\frac{d^2y}{dx^2}\right)^3 = siny + 3x$

Solution : (1) The highest order derivative is $\frac{d^3y}{dx^3}$ and its power is 1.

$\therefore$ The differential equation has order 3 and degree 1.

(2) $\frac{d^2y}{dx^2} = \sqrt[3]{1+\left(\frac{dy}{dx}\right)^2}$

To make it radical free, we cube both the sides.

$\therefore \left(\frac{d^2y}{dx^2}\right)^3 = 1 + \left(\frac{dy}{dx}\right)^2$

This differential equation has order 2 and degree 3.

(3) The highest order derivatives is $\frac{dy}{dx}$. Hence the differential equation has order 1. But we can not express the differential equation in a polynomial form in derivatives. So the degree is not defined.

(4) The highest order derivative is $\frac{d^2y}{dx^2}$ and its power is 1, so the differential equation has order 2 and degree 1.

(5) The highest order derivative is $\frac{d^2y}{dx^2}$ and its power is 3, so the differential equation has order 2 and degree 3.

Exercise 5.1

Obtain the order and degree (if possible) of the following differential equations :

1. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2$

2. $x + \left(\frac{dy}{dx}\right)^2 = \sqrt{1+y}$

3. $\frac{d^2y}{dx^2} + sin\left(\frac{dy}{dx}\right) + y = 0$

4. $y^{\frac{dy}{dx}} = x$

5. $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^4 + x \log y = 0$

6. $\sqrt[3]{\frac{d^2y}{dx^2}} = \sqrt{\frac{dy}{dx}}$

7. $\left(\frac{dy}{dx}\right) + \frac{x}{\left(\frac{dy}{dx}\right)} = 0$

8. $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 = 0$

9. $\frac{d^2y}{dx^2} = 3sin\, 3x$

10. $x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^5 - 5y = 0$

*

5.4 Formation of a Differential Equation

Now let us try to understand a family of curves. Consider the equation $x^2 + y^2 = r^2$ **(i)** and assign different values to r.

If $r = 1$, then $x^2 + y^2 = 1$

If $r = 2$, then $x^2 + y^2 = 4$

If $r = 3$, then $x^2 + y^2 = 9$

If $r = 4$, then $x^2 + y^2 = 16$

From the above equations, it is clear that equation (i) represents **a family of concentric circles** having center at origin and having different radii.

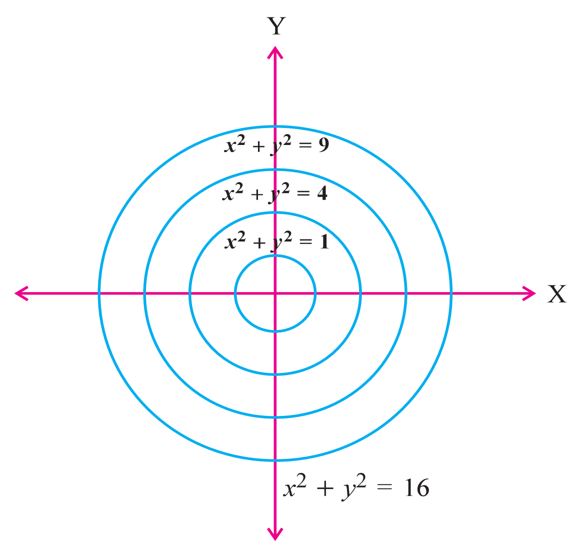


Figure 5.1

Now we are interested to find the differential equation which is satisfied by each member of the family irrespective of radius. The above equation has one arbitrary constant. i.e. r. We should find an equation which is free from r.

Differentiate $x^2 + y^2 = r^2$ *w.r.t.* x

So $2x + 2y\frac{dy}{dx} = 0$

$x + y\frac{dy}{dx} = 0$

This is the required differential equation satisfied by all the members of the family of concentric circles $x^2 + y^2 = r^2$ and note that it does not contain arbitrary constant r.

Example 2 : Obtain the differential equation of the family of parabolas having vertex at origin and having Y-axis as axis.

Solution : We know that the equation of the family of parabolas having vertex at origin and axis along positive direction of Y-axis is $x^2 = 4by$.

Let S(0, b) be the focus of one of these parabolas where b is an arbitrary constant.

Now differentiating both the sides of the equation $x^2 = 4by$ *w.r.t.* x we get,

$\therefore \quad 2x = 4b\dfrac{dy}{dx}$

$\therefore \quad 2xy = 4by\dfrac{dy}{dx}$

But $4by = x^2$

$\therefore \quad x^2\dfrac{dy}{dx} = 2xy \quad$ or $\quad x^2\dfrac{dy}{dx} - 2xy = 0$

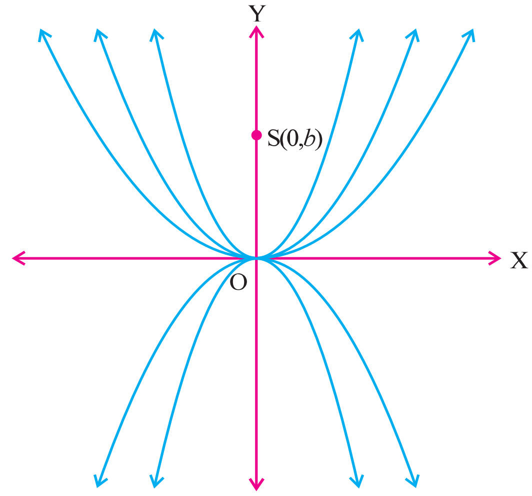


Figure 5.2

$\therefore \quad x\dfrac{dy}{dx} = 2y \qquad$ **($x \neq 0$)**

This is the differential equation of the given family of parabolas.

Note : If $x = 0$, then $y = 0$, since $x^2 = 4by$.

$\therefore \quad (0, 0)$ also satisfies $x\dfrac{dy}{dx} = y$.

Example 3 : Obtain the differential equation of family of all the parallel lines represented by $y = 2x + c$ having slope 2. (c is an arbitrary constant).

Solution : $y = 2x + c$ is the given equation of line where c is an arbitrary constant.

For distinct values of c we get different lines. All the lines are parallel to each other.

So, $y = 2x + c$, (c abitrary constant) is a family of parallel lines.

Now we shall find an equation not containing the arbitrary constant and which is satisfied by all such members of the family of parallel lines.

Hence differentiating $y = 2x + c$ with respect to x.

$$\frac{dy}{dx} = 2$$

This equation not containing arbitrary constant represents the differential equation of family of lines.

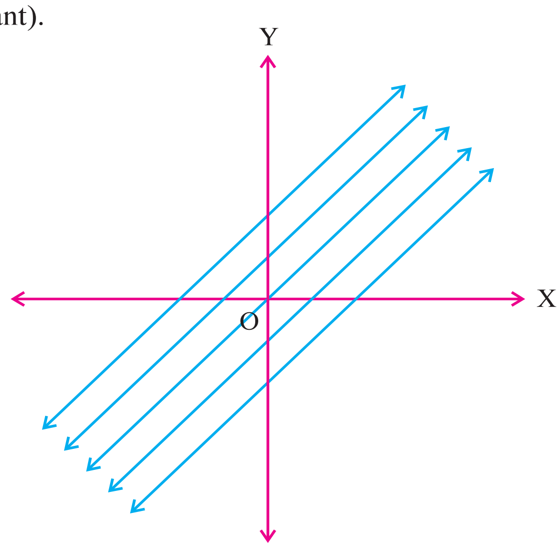


Figure 5.3

Example 4 : Obtain the differential equation of the family of curves $y = a\,sin(x + b)$, (a and b are arbitrary constants).

Solution : $y = a\,sin(x + b)$ is a given family curves.

Differentiating *w.r.t.* x, $\dfrac{dy}{dx} = a\,cos(x + b)$

Again differentiating *w.r.t.* x,

$$\frac{d^2y}{dx^2} = -a\,sin(x + b)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -y \quad \text{or} \quad \frac{d^2y}{dx^2} + y = 0$$ is the differential equation representing the given family.

From examples 2 and 3, we can say that the differential equation of a family of curves having one arbitray constant is of order one. From example 4, we can say that the differential equation of a family of curves having two arbitrary constants is of order two. From these examples let us understand the formation of a differential equation as under.

(a) If the family of curves has only one arbitrary constant c, then it can be represented by the equation $f(x, y, c) = 0$. Differentiating above equation *w.r.t.* x, we get a new functional relation showing relation among x, y, y' and c. Let this functional relation be $g(x, y, y', c) = 0$

Now eliminating c from the equations $f(x, y, c) = 0$ and $g(x, y, y', c) = 0$, we get an equation $F(x, y, y') = 0$ representing differential equation of the family $f(x, y, c) = 0$.

(b) If the family of curves has two arbitrary constants c_1 and c_2, then it can be represented by the equation $f(x, y, c_1, c_2) = 0$.

Differentiating *w.r.t.* x, we get a new functional relation showing relation among x, y, y', c_1 and c_2. Let this functional relation be the equation $g(x, y, y', c_1, c_2) = 0$ relating x, y, y', c_1 and c_2. But both arbitrary constants c_1 and c_2 can not be eliminated from only these two equations. Differentiating equation $g(x, y, y', c_1, c_2) = 0$ again *w.r.t.* x,

the equation $h(x, y, y', y'', c_1, c_2) = 0$ is obtained relating x, y, y', y'', c_1 and c_2.

Now eliminating arbitrary constants c_1 and c_2 from $f(x, y, c_1, c_2) = 0$ and $g(x, y, y', c_1, c_2) = 0$ and $h(x, y, y', y'', c_1, c_2) = 0$ we get an equation $F(x, y, y', y'') = 0$ which represents the differential equation of given family $f(x, y, c_1, c_2) = 0$.

In short differentiating n times, the functional relation $f(x, y, c_1, c_2, ..., c_n) = 0$ containing n arbitrary constants, we get $(n + 1)$ equations including given equation.

Eliminating $c_1, c_2, ..., c_n$; we get the differential equation of the given family. Remember that, if the number of arbitrary constants is n, then the order of the differential equation so obtained is also n.

Example 5 : Obtain the differential equation representing the family of ellipses having focii on X-axis or Y-axis and centre at the origin.

Solution : We have the equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ where, a and b are arbitrary constant. **$(a \neq b)$ (i)**

This equation represents a family of ellipses.

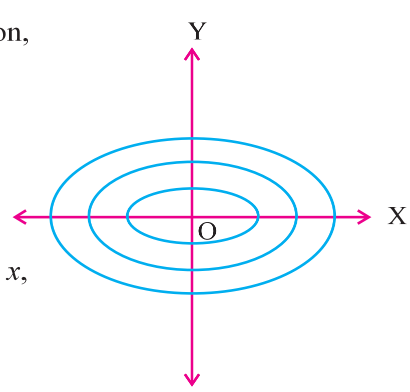


Figure 5.4(a)

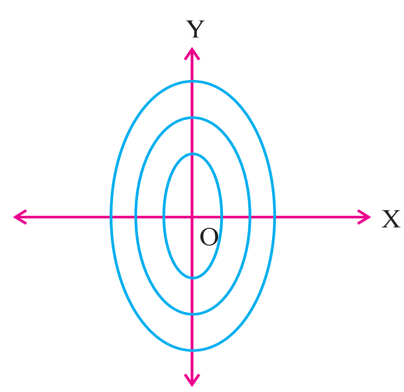


Figure 5.4(b)

Differentiating equation (i) *w.r.t.* x,

We get $$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\therefore \quad y\frac{dy}{dx} = -\frac{b^2}{a^2}x \qquad \textbf{(ii)}$$

Differentiating both the sides of equation (ii) *w.r.t.* x,

We get, $\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}$

Multiply by x on both sides

$$x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}x$$

$$\therefore \quad x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} = y\frac{dy}{dx} \qquad \textbf{(using (ii))}$$

$$\therefore \quad x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0$$

This is the required differential equation representing the family of ellipses.

Note : There are two arbitrary constants. So we have differentiated twice. The differential equation is of order 2.

Example 6 : Find the differential equation of the family of circles having centre on X-axis and radius 1 unit.

Solution :

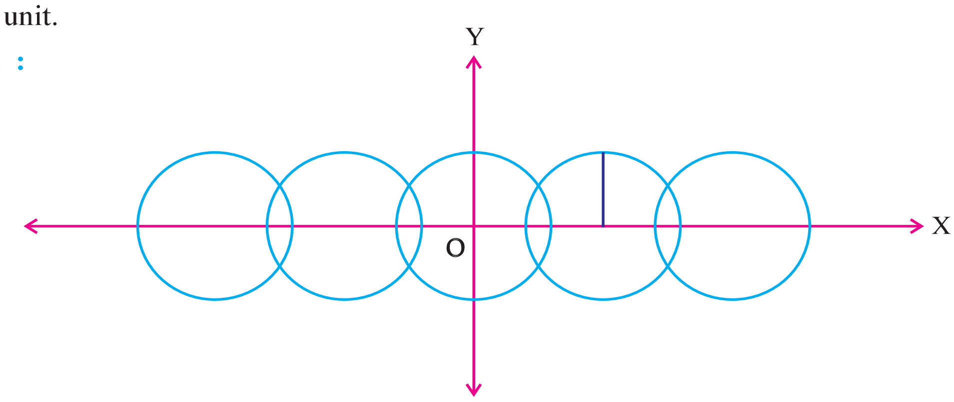


Figure 5.4

Here the centres of the circles in the family are on X-axis. Let the centre of a circle be $(a, 0)$, $(a \in R)$ and let these circles have radius 1.

$\therefore$ The equation of this family of circles is

$$(x - a)^2 + y^2 = 1 \qquad \textbf{(i)}$$

Differentiating *w.r.t.* x,

$$\therefore \quad 2(x - a) + 2y\frac{dy}{dx} = 0$$

$$\therefore \quad (x - a) + y\frac{dy}{dx} = 0$$

$$\therefore \quad (x - a) = -y\frac{dy}{dx} \qquad \textbf{(ii)}$$

To remove the arbitrary constant a, substitute the value of $(x - a)$ in equation (i),

$$\left(-y\frac{dy}{dx}\right)^2 + y^2 = 1$$

$\therefore \quad y^2\left(\frac{dy}{dx}\right)^2 + y^2 - 1 = 0$ is the differential equation of the given family of circles.

Note : There is only one arbitrary constant. So we have differentiated only once. We get first order differential equation.

5.5 Solution of a Differential Equation

The solution of a differential equation is a function $y = f(x)$ or functions obtained from functional relation $f(x, y) = 0$ which independent of derivatives and shows relation between variables and satisfies the given differential equation along with all its derivatives.

If for a function $y = f(x)$, defined on some interval, there exist derivatives of f upto order n and if the function f and its derivatives together satisfy the given differential equation, then this function $y = f(x)$ is called a solution of given differential equation.

In order that a function $y = f(x)$ is a solution of a given differential equation it is necessary that some conditions regarding domain and continuity of functions are satisfied. In other words if solution of a differential equation can be obtained, we discuss how to obtain the solution under some favourable conditions. We will not discuss the existence of a solution of a differentiable equation. We will study some methods to obtain the solution, when it exists and we will not mention the conditions or circumstances under which the solution exists.

Solution of a differential equation :

$y = 2x + c$ is a solution of $\frac{dy}{dx} = 2$. (Example 3) because $y = 2x + c$, satisfies the differential equation $\frac{dy}{dx} = 2$.

Let us see another example.

$y = sinx$, $x \in$ R is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$

because differentiating $y = sinx$ *w.r.t.* x, $\frac{dy}{dx} = cosx$

$$\therefore \quad \frac{d^2y}{dx^2} = -sinx = -y$$

$$\therefore \quad \frac{d^2y}{dx^2} + y = 0$$

Now $y = cosx$, $x \in$ R is also a solution of $\frac{d^2y}{dx^2} + y = 0$.

Here $y = cosx$

Differentiating *w.r.t.* x

$$\frac{dy}{dx} = -sin\,x$$

$$\therefore \quad \frac{d^2y}{dx^2} = -cosx = -y$$

$$\therefore \quad \frac{d^2y}{dx^2} + y = 0$$

From the above examples, we say that in general there can be more than one solution of a differential equation.

General and Particular Solution :

The general solution of a differential equation is a function $y = f(x, c_1, c_2,..., c_n)$ or $f(x, y, c_1, c_2,..., c_n) = 0$ with arbitrary constants whose number is equal to the order of the differential equation.

In general, there are n arbitrary constants in the solution of the differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^ny}{dx^n}\right) = 0.$$

This solution is denoted by $G(x, y, c_1, c_2, ..., c_n) = 0$ where $c_1, c_2, ..., c_n$ are arbitrary constants.

If we can find definite values of the arbitrary constants occurring in the general solution of the differential equation under some conditions on the given variables x, y and derivaties $\frac{dy}{dx}, \frac{d^2y}{dx^2}$,... etc, then the solution of the differential equation with definite values of arbitrary constants is called a particular solution and the given conditions are called initial conditions or boundary conditions.

If a solution other than general solution of a differential equation cannot be obtained as a particular solution from the general solution, then such a solution of the differential equation is called a singular solution.

Example 7 : Verify that the function $y = A\cos x + B\sin x$, where A and B are arbitrary constants, is a general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Solution : Here $y = A\cos x + B\sin x$ is the given function.

Differentiating both sides of the equation *w.r.t.* x,

we get, $\frac{dy}{dx} = -A\sin x + B\cos x$

$\therefore \quad \frac{d^2y}{dx^2} = -A\cos x - B\sin x$

$\therefore \quad \frac{d^2y}{dx^2} = -(A\cos x + B\sin x)$

$\therefore \quad \frac{d^2y}{dx^2} = -y$

$\therefore \quad \frac{d^2y}{dx^2} + y = 0$

Therefore, the given function $y = A\cos x + B\sin x$ is the general solution of the given differential equation $\frac{d^2y}{dx^2} + y = 0$, because there are two arbitrary constants in this solution of the differential equation.

Example 8 : Verify that $y = cx + \frac{1}{c}$ is a solution of the differential equation $y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + 1$, where c is an arbitrary constant.

Solution : Here $y = cx + \frac{1}{c}$ (c is an arbitrary constant)

Differentiating *w.r.t.* x, $\frac{dy}{dx} = c$

Substituting $c = \frac{dy}{dx}$ in the equation $y = cx + \frac{1}{c}$,

we get, $y = \left(\frac{dy}{dx}\right)x + \frac{1}{\left(\frac{dy}{dx}\right)}$

$\therefore \quad y\left(\frac{dy}{dx}\right) = \left(\frac{dy}{dx}\right)^2 x + 1$

Therefore, the function $cx + \frac{1}{c}$ is a solution of the given differential equation.

Example 9 : Verify $y = cx^4$ is a solution of the differential equation $x\frac{dy}{dx} - 4y = 0$, where c is an arbitrary constant.

Solution : Here given relation is $y = cx^4$ **(i)**

Differentiating (i) *w.r.t.* x,

we get $\frac{dy}{dx} = 4cx^3$ **(ii)**

$$\therefore \quad x\frac{dy}{dx} - 4y = x(4cx^3) - 4cx^4$$
$$= 4cx^4 - 4cx^4$$
$$= 0$$

Hence, $y = cx^4$ is a solution of $x\frac{dy}{dx} - 4y = 0$.

Example 10 : Verify that $y = ax + a^2$ (a is an arbitrary constant) is the general solution of the differential equation $\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) = y$. Find a particular solution, when $a = 3$. Also show that a singular solution of this differential equation is $x^2 + 4y = 0$.

Solution : Here $y = ax + a^2$ (a is an arbitrary constant)

$$\therefore \quad \frac{dy}{dx} = a$$

Substituting $a = \frac{dy}{dx}$ in $y = ax + a^2$, we get the given differential equation

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx}$$

Because of presence of one arbitary constant $y = ax + a^2$ is the general solution of $\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) = y$.

Now substitute $a = 3$ in the general solution.

We get $y = 3x + 9$, which is a particular solution of the given differential equation.

Now consider $x^2 + 4y = 0$

$$\therefore \quad 4y = -x^2$$

$$\therefore \quad 4\frac{dy}{dx} = -2x$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{2}$$

Substituting this value of $\frac{dy}{dx}$ in the given differential equation, we get,

$\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) = \frac{x^2}{4} + x\left(-\frac{x}{2}\right) = -\frac{x^2}{4} = y$, which shows that $x^2 + 4y = 0$ satisfies given differential equation.

Thus $x^2 + 4y = 0$ satisfies the given differential equation. This is a solution of the differential equation. But this solution cannot be obtained by substituting any value of a in the general solution. Hence this solution is a singular solution of the differential equation.

Note : General solution represents a family of lines. A singular solution $x^2 + 4y = 0$ represents a parabola.

Exercise 5.2

1. Find the differential equation of all the circles which touch the coordinate axes in the first quadrant.

2. Obtain the differential equation representing family of lines $y = mx + c$ (m and c are arbitrary constant)

3. Form the differential equation representing family of curves $y^2 = m(a^2 - x^2)$ (m and a are arbitrary constants).

4. Find the differential equation of the family of all the circles touching X-axis at the origin.

5. Show that the differential equation $\frac{dy}{dx} + 2xy = 4x^3$ has the solution $y = 2(x^2 - 1) + ce^{-x^2}$, where c is an arbitrary constant.

6. Verify that $y^2 = 4b(x + b)$ is a solution of the differential equation $y\left[1 - \left(\frac{dy}{dx}\right)^2\right] = 2x\frac{dy}{dx}$.

7. Prove $y = a\, cos(\log x) + b\, sin(\log x)$ is a solution of the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, where a and b are arbitrary constants.

8. Verify that differential equation $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$ has solution $y = a\, cos^{-1}x + b$. (where a and b are arbitrary constants.)

9. Find the differential equation of the following family of curves, where a and b are arbitrary constants :

(1) $\frac{x}{a} + \frac{y}{b} = 1$ (2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (3) $(y - b)^2 = 4(x - a)$ (4) $y = \left(ax + \frac{b}{x}\right)$

(5) $y = ax^3$ (6) $y = e^{2x}(a + bx)$ (7) $y^2 = a(b^2 - x^2)$

10. Verify that $y = 5sin4x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 16y = 0$.

11. Show that $Ax^2 + By^2 = 1$ is the general solution of the differential equation

$x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = y\left(\frac{dy}{dx}\right)$. (A, B are arbitrary constants)

12. Show that $y = \frac{a}{x} + b$ is a solution of $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0$.

*

5.6 Solution of Differential Equation of First Order and First Degree :

A first order and first degree differential equation is represented by $\frac{dy}{dx} = F(x, y)$, $x \in I$ (I is any interval). If we let $F(x, y) = \frac{-f(x, y)}{g(x, y)}$

$f(x, y)dx + g(x, y)dy = 0$ is also another form of first order and first degree differential equation. The first order and first degree differential equation may not be always solvable but we will discuss particular forms of these equations which have a general solution.

Now we shall discuss some methods to solve a first order and first degree differential equation.

(1) Method of Variables Separable : In the differential equation $f(x, y)dx + g(x, y)dy = 0$ of first order and first degree, if $f(x, y)$ is a function $p(x)$ of x only and $g(x, y)$ is a function $q(y)$ of y only, then the general form of first order and first degree differential equation is $p(x)dx + q(y)dy = 0$. Such an equation is said to be in variable-separable form.

Now $\int p(x)dx + \int q(y)dy = c$ (c is an arbitrary constant) is the general solution.

> **Note :** In the general solution of a differential equation, we can take arbitrary constant in a form according to our convenience.

Example 11 : Solve the differential equation, $x(1 + y^2)dx - y(1 + x^2)dy = 0$.

Solution : Here $x(1 + y^2)dx = y(1 + x^2)dy$

$\therefore \quad \frac{x}{1+x^2}\,dx = \frac{y}{1+y^2}\,dy$ **(Variables Separable form)**

$\therefore \quad \frac{2x}{1+x^2}\,dx = \frac{2y}{1+y^2}\,dy$

Integrating on both the sides,

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{2y}{1+y^2}\,dy$$

$\therefore \quad \log|1 + x^2| = \log|1 + y^2| + \log c$ (Instead of c, let $\log c$ be the arbitrary constant, $c > 0$)

$\therefore \quad \log\left(\frac{1+x^2}{1+y^2}\right) = \log c \quad (c > 0)$ **($1 + x^2 > 0$, $1 + y^2 > 0$)**

$\therefore \quad \frac{1+x^2}{1+y^2} = c$

$\therefore \quad (1 + x^2) = c(1 + y^2)$

This is the general solution and c is an arbitrary positive constant.

Example 12 : Solve the differential equation $(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$

Solution : Here $(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$

$\therefore \quad dy = \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$ **(Variables Separable)**

Integrating on both the sides,

$$\int dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

$y = \log|e^x + e^{-x}| + c$

which is the required general solution of the given equation.

We may write $y = \log(e^x + e^{-x}) + c$ as $e^x + e^{-x} > 0$.

Example 13 : Find the particular solution of the differential equation $\frac{dy}{dx} = y \tan x$ given that $y = 1$ when $x = 0$. $(y \neq 0)$

Solution : $\frac{dy}{dx} = y \tan x$

$\therefore \quad \frac{1}{y}\, dy = \tan x \; dx$ **(i)**

Integrating on both sides of equation (i),

we get, $\int \frac{1}{y}\, dy = \int \tan x \; dx$

$\therefore \quad \log |y| = \log |\sec x| + \log |c|$ **(log | c | arbitrary constant)**

$\therefore \quad \log |y| = \log |c \sec x|$

$\therefore \quad y = c \sec x$ **(ii)**

This is the general solution.

Substituting $y = 1$ and $x = 0$ in equation (ii), we get value of arbitrary constant c which gives a particular solution

$1 = \sec 0 \cdot c$

$1 = 1 \cdot c$

$c = 1$

$\therefore \quad y = \sec x$ is the required particular solution.

Note : Sometimes if y is a function of x, we express it as $y = y(x)$. Thus if $y(x) = x^2$, $y(1) = 1$, $y(2) = 4$ etc. Find $y(2)$ means find $y(x)$, when $x = 2$. In this example we can say $y(0) = 1$.

Example 14 : Solve the differential equation $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

Solution : Here we have $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

$\therefore \quad \frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^2}{e^y}$

$\therefore \quad \frac{dy}{dx} = \frac{e^x + x^2}{e^y}$

$\therefore \quad e^y\, dy = (e^x + x^2)\, dx$

Integrating on both the sides,

$\int e^y\, dy = \int (e^x + x^2)\, dx$

$\therefore \quad e^y = e^x + \frac{x^3}{3} + c$ **(c arbitrary constant)**

is the general solution of the given differential equation.

Example 15 : Solve : $\frac{dy}{dx} = (x + y)^2$

Solution : This differential equation cannot be expressed in the form $p(x)\, dx + q(y)\, dy = 0$. So at first sight this differential equation does not seem to be of variables separable form. But we can transform it into that form.

Here $\frac{dy}{dx} = (x + y)^2$

Substitute $x + y = z$ in the equation.

$\therefore \quad 1 + \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{dz}{dx} - 1$

So the equation will become

$\therefore \quad \frac{dz}{dx} - 1 = z^2$

$\therefore \quad \frac{dz}{dx} = 1 + z^2$

$\therefore \quad \frac{dz}{1+z^2} = dx$ **(Variables Separable form)**

Integrating on both the sides

$$\int \frac{dz}{1+z^2} = \int dx$$

$\therefore \quad tan^{-1}z = x + c$ **(c arbitrary constant)**

$\therefore \quad tan^{-1}(x + y) = x + c$ is the general solution.

Example 16 : Solve $cos(x - y)dy = dx$

Solution : Here $\frac{dy}{dx} = \frac{1}{cos(x-y)}$ **(i)**

Substituting $x - y = t$, **(ii)**

$$1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$\therefore \quad \frac{dy}{dx} = 1 - \frac{dt}{dx}$ **(iii)**

From (i), (ii) and (iii)

$$1 - \frac{dt}{dx} = \frac{1}{cos\,t}$$

$\therefore \quad 1 - \frac{1}{cos\,t} = \frac{dt}{dx}$

$\therefore \quad \frac{cos\,t - 1}{cos\,t} = \frac{dt}{dx}$

$\therefore \quad \frac{-(1 - cos\,t)}{cos\,t} = \frac{dt}{dx}$

$\therefore \quad -dx = \frac{cos\,t}{1 - cos\,t}dt$

Integrating on both the sides,

$\therefore \quad -\int dx = \int \frac{cos\,t}{1 - cos\,t} \times \frac{1 + cos\,t}{1 + cos\,t}\,dt$

$\therefore \quad -\int dx = \int \frac{cos\,t + cos^2 t}{sin^2 t}\,dt$

$\therefore \quad -\int dx = \int cosec\,t \cdot cot\,t\,dt + \int cot^2 t\,dt$

$\therefore \quad -\int dx = \int cosec\,t \cdot cot\,t\,dt + \int (cosec^2 t - 1)\,dt$

$\therefore \quad -x + c = -cosec\, t - cot\, t - t$

$\therefore \quad -x + c = -cosec(x - y) - cot(x - y) - (x - y)$

$\therefore \quad cosec(x - y) + cot(x - y) + c = y$

Exercise 5.3

1. Solve the following differential equations. Also find particular solution where initial conditions are given :

(1) $xy(y + 1)\, dy = (x^2 + 1)\, dx$

(2) $y(1 + e^x)\, dy = (y + 1)\, e^x\, dx$

(3) $\frac{dy}{dx} = -tanx\ tany$

(4) $\frac{dy}{dx} - y\ tanx = -y\ sec^2x$

(5) $(e^y + 1)\, cosx\ dx + e^y\ sin\, x\ dy = 0$

(6) $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

(7) $y \log y\ dx - x\ dy = 0$

(8) $\frac{dy}{dx} = -4xy^2;\ y(0) = 1$

(9) $x\, dy = (2x^2 + 1)\, dx \quad (x \neq 0);\ y(1) = 1$

(10) $xy \frac{dy}{dx} = y + 2;\quad y(2) = 0$

(11) $\frac{dy}{dx} = 2e^x y^3;\ y(0) = \frac{1}{2}$

(12) $x \frac{dy}{dx} + cot\ y = 0;\quad y(\sqrt{2}) = \frac{\pi}{4}$

(13) $e^{\frac{dy}{dx}} = x + 1;\quad y(0) = 3,\ x > -1$

(14) $sin\left(\frac{dy}{dx}\right) = a$ when $x = 0,\ y = 1,\ (a \in R)$

(15) $\frac{dy}{dx} = y\ tanx,\ y(0) = 1$

(16) $(x + 1)^2 \frac{dy}{dx} = xe^x$

2. Solve the following differential equations :

(1) $\frac{dy}{dx} = sin(x + y)$

(2) $\frac{dy}{dx} = \frac{(x - y) + 3}{2(x - y) + 5}$

(3) $(x + y + 1) \frac{dy}{dx} = 1$

(4) $\frac{dy}{dx} = e^{x + y}$

(5) $(x + y)^2 \frac{dy}{dx} = a^2$

*

5.7 Homogeneous Differential Equations :

Let $f(x,\ y) = 3x^2 + 2xy + y^2$

$$= x^2 \left(3 + 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2\right)$$

$$= x^2\ \phi\left(\frac{y}{x}\right)$$

$$\therefore \quad f(x,\ y) = x^2\ \phi\left(\frac{y}{x}\right)$$

Here we have expressed $f(x,\ y)$ in the form of $x^n\ \phi\left(\frac{y}{x}\right)$. **If a two variable function $f(x, y)$ can be written as $f(x,\ y) = x^n\ \phi\left(\frac{y}{x}\right)$ form, then the function $f(x,\ y)$ is called a homogeneous function of degree n.**

Now let us see a method to solve a differential equation of first order and first degree.

In place of x and y substitute λx and λy respectively in $f(x, y)$. (where $\lambda \neq 0$ is constant)

We get, $f(\lambda x, \lambda y) = 3(\lambda x)^2 + 2(\lambda x)(\lambda y) + (\lambda y)^2$

$$= 3\lambda^2 x^2 + 2\lambda^2 xy + \lambda^2 y^2$$

$$= \lambda^2 (3x^2 + 2xy + y^2)$$

$$= \lambda^2 f(x, y)$$

Here we have expressed the relation in the form $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. Such a function $f(x, y)$ is called a homogeneous function of degree n and λ is a non-zero constant.

$f(x, y) = tanx + tany$. This type of function cannot be written in the form $f(x, y) = x^n \phi\left(\frac{y}{x}\right)$. So it is not a homogeneous function.

Homogeneous Differential Equation : If in a differential equation $f(x, y)\, dx + g(x, y)\, dy = 0$, $f(x, y)$ and $g(x, y)$ are homogeneous functions with same degree, then this differential equation is called homogeneous differential equation.

> **Note :** $\phi\left(\frac{y}{x}\right)$ type of functions are always homogeneous.

Solution of homogeneous Differential Equation :

Let the homogeneous differential equation $f(x, y)\, dx + g(x, y)\, dy = 0$ be in the form of $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$.

Let $\frac{y}{x} = v$, so $y = vx$

Differentiating *w.r.t.* 'x',

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore \quad v + x\frac{dv}{dx} = \phi(v)$ $\qquad \left(\frac{dy}{dx} = \phi\left(\frac{y}{x}\right) = \phi(v)\right)$

$\therefore \quad x\frac{dv}{dx} = \phi(v) - v$

$\therefore \quad \frac{dv}{\phi(v) - v} = \frac{dx}{x}$ $\qquad$ **(Variables Separable form)**

Integrating on both the sides, we get,

$$\int \frac{dv}{\phi(v) - v} = \int \frac{1}{x}\, dx$$

$\therefore \quad \int \frac{dv}{\phi(v) - v} = \log|x| + c$ $\qquad$ **($x \neq 0$)**

This is the general solution of a homogeneous differential equation and c is an arbitrary constant.

Example 17 : Solve $\frac{dy}{dx} + \frac{y(x + y)}{x^2} = 0$

Solution : $\frac{dy}{dx} = -\frac{y(x+y)}{x^2} = -\left[\frac{y}{x} + \left(\frac{y}{x}\right)^2\right]$ **(i)**

Let $\frac{y}{x} = v$

$\therefore \quad y = vx$ **(ii)**

So, $\frac{dy}{dx} = v + x\frac{dv}{dx}$ **(iii)**

$\therefore \quad v + x\frac{dv}{dx} = -v - v^2$ **(using (i), (ii) and (iii))**

$\therefore \quad x\frac{dv}{dx} = -(2v + v^2)$

$\therefore \quad \frac{dv}{2v+v^2} = -\frac{dx}{x}$ **(Variables Separable form)**

$\therefore \quad \int \frac{1}{v(v+2)}\,dv = \int -\frac{1}{x}\,dx$ **(Integrating both the sides)**

$\therefore \quad \frac{1}{2}\int \frac{v+2-v}{(v+2)v}\,dv = -\int \frac{1}{x}\,dx$

$\therefore \quad \frac{1}{2}\int \frac{1}{v}\,dv - \frac{1}{2}\int \frac{1}{v+2}\,dv = -\int \frac{1}{x}\,dx$

$\therefore \quad \frac{1}{2}\log|v| - \frac{1}{2}\log|v+2| = -\log|x| + \frac{1}{2}\log|c|$ **(c is an arbitrary constant)**

$\therefore \quad \log|v| - \log|v+2| = -2\log|x| + \log|c|$

$\therefore \quad \log\left|\frac{v}{v+2}\right| = \log\left|\frac{c}{x^2}\right|$

$\therefore \quad \log\left|\frac{y}{y+2x}\right| = \log\left|\frac{c}{x^2}\right|$

$x^2y = c(2x + y)$ $\left(v = \frac{y}{x}\right)$

This is the general solution.

Example 18 : Solve $x^2\frac{dy}{dx} = x^2 + xy + y^2$.

Solution : $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$

$\therefore \quad \frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$ **(i)**

Let $\frac{y}{x} = v$, so $y = vx$ **(ii)**

$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$ **(iii)**

From equations (i), (ii) and (iii),

$v + x\frac{dv}{dx} = 1 + v + v^2$

$\therefore \quad x\frac{dv}{dx} = 1 + v^2$

$\therefore \quad \dfrac{dv}{1+v^2} = \dfrac{dx}{x}$ **($x \neq 0$) (Variables Separable form)**

Integrating both the sides, we get,

$$\int \frac{1}{1+v^2}\, dv = \int \frac{1}{x}\, dx$$

$tan^{-1} v = \log |x| + \log |c|$ **(c arbitrary constant)**

$$tan^{-1} v = \log |xc|$$

$tan^{-1}\left(\dfrac{y}{x}\right) = \log |xc|$ is the general solution of the given differential equation.

Example 19 : Solve $x \sin\left(\dfrac{y}{x}\right) \dfrac{dy}{dx} + x - y \sin\left(\dfrac{y}{x}\right) = 0$. Find the particular solution, if the initial condition is $y(1) = \dfrac{\pi}{2}$.

Solution : Here $x \sin\left(\dfrac{y}{x}\right) \dfrac{dy}{dx} + x - y \sin\left(\dfrac{y}{x}\right) = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{y \sin\left(\frac{y}{x}\right) - x}{x \sin\left(\frac{y}{x}\right)}$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{y}{x} \sin\left(\frac{y}{x}\right) - 1}{\sin\left(\frac{y}{x}\right)}$ **(i)**

Let $\dfrac{y}{x} = v$ **(ii)**

So, $y = vx$

$\therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ **(iii)**

From equations (i), (ii) and (iii),

$$v + x\frac{dv}{dx} = \frac{v \sin v - 1}{\sin v}$$

$$\therefore \quad v + x\frac{dv}{dx} = v - \frac{1}{\sin v}$$

$$\therefore \quad x\frac{dv}{dx} = -\frac{1}{\sin v}$$

$$\therefore \quad \sin v \, dv = -\frac{dx}{x}$$

Integrating both the sides,

$$\int \sin v \, dv = -\int \frac{dx}{x}$$

$$\therefore \quad -\cos v = -\log |x| - \log |c|$$

$$\therefore \quad \cos\left(\frac{y}{x}\right) = \log |x| + \log |c|$$

$\therefore \quad \cos\dfrac{y}{x} = \log |cx|$ **(iv)**

This is the general solution.

Now we are given $y(1) = \frac{\pi}{2}$ i.e. when $x = 1$ and $y = \frac{\pi}{2}$

So, from equation (iv),

$$\cos \frac{\pi}{2} = \log |c|$$

$$\therefore \quad \log |c| = 0$$

$$\therefore \quad |c| = 1$$

$\therefore \quad \cos\left(\frac{y}{x}\right) = \log |x| \; (x \neq 0)$ is the required particular solution.

Example 20 : Solve $\left[x \sin^2\left(\frac{y}{x}\right) - y\right] dx + x\, dy = 0$. Find the particular solution, if the initial condition is $y(1) = \frac{\pi}{4}$.

Solution : Here $\left[x \sin^2\left(\frac{y}{x}\right) - y\right] dx + x\, dy = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x} - \sin^2 \frac{y}{x} \qquad \textbf{(i)}$$

Let $\frac{y}{x} = v$, so $y = vx$ **(ii)**

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \textbf{(iii)}$$

From equations (i), (ii) and (iii) we get

$$v + x\frac{dv}{dx} = v - \sin^2 v$$

$$x\frac{dv}{dx} = -\sin^2 v$$

$$\therefore \quad \frac{1}{\sin^2 v}\, dv = -\frac{dx}{x} \qquad \textbf{(Variables Separable form)}$$

Integrating both the sides

$$\int \operatorname{cosec}^2 v \; dv = -\int \frac{1}{x}\, dx$$

$$-\cot v = -\log |x| - \log |c|$$

$\cot\left(\frac{y}{x}\right) = \log |cx|$ which is general solution.

Now we are given $y(1) = \frac{\pi}{4}$ i.e. when $x = 1$ $y = \frac{\pi}{4}$

$$\cot \frac{\pi}{4} = \log |c|$$

$$\therefore \quad \log |c| = 1$$

$$\therefore \quad |c| = e$$

$\therefore \quad cot \frac{y}{x} = \log |ex| = \log |x| + \log e$ **$(x \neq 0)$**

$= \log |x| + 1$

This is the required particular solution.

Example 21 : Solve $2xy + y^2 - 2x^2 \frac{dy}{dx} = 0$. Also find the particular solution for $y(1) = 2$.

Solution : Here $2xy + y^2 - 2x^2 \frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = \frac{y}{x} + \frac{1}{2}\left(\frac{y}{x}\right)^2$ **(i)**

Let $\frac{y}{x} = v$ **(ii)**

$\therefore \quad y = vx$

$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$ **(iii)**

From equations (i), (ii) and (iii)

$v + x\frac{dv}{dx} = v + \frac{1}{2}v^2$

$x\frac{dv}{dx} = \frac{1}{2}v^2$

$\frac{2}{v^2}\,dv = \frac{dx}{x}$ **(Variables Separable form)**

Integrating both the sides,

$2\int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$

$-\frac{2}{v} = \log |x| + c$

$-\frac{2x}{y} = \log |x| + c$ is the general solution.

Now $y(1) = 2$. So if $x = 1$, $y = 2$

$\therefore \quad -\frac{2}{2} = \log |1| + c$

$\therefore \quad c = -1$

$-\frac{2x}{y} = \log |x| - 1$

$y = \frac{2x}{1 - \log |x|}$ **$(x \neq 0, x \neq e)$**

Exercise 5.4

1. Solve the following differential equations :

(1) $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$

(2) $\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y = \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\frac{dy}{dx}$

(3) $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$

(4) $y\, e^{\frac{x}{y}}\, dx = (x\, e^{\frac{x}{y}} + y^2)\, dy$

(5) $x\sin\left(\frac{y}{x}\right)\frac{dy}{dx} = y\sin\left(\frac{y}{x}\right) + x$

(6) $y + 2ye^{\frac{x}{y}}\frac{dx}{dy} = 2xe^{\frac{x}{y}}$

(7) $x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$

(8) $(1 + e^{\frac{x}{y}})\, dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

(9) $x\frac{dy}{dx} = x + y$

(10) $y\, dx + x\log\left(\frac{y}{x}\right) dy = 2x\, dy$

(11) $(xe^{\frac{y}{x}} \frac{y}{x} + y)\, dx = x\, dy$

(12) $\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$

(13) $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

2. Find the particular solution of the given differential equations under given initial condition :

(1) $(x^2 + y^2)\, dx + xy\, dy = 0;\ y(1) = 1$

(2) $x\, e^{\frac{y}{x}} - y + x\frac{dy}{dx} = 0;\ y(e) = 0$

(3) $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\frac{y}{x} = 0;\ y(1) = 0$

(4) $(x^2 - 2y^2)\, dx + 2xy\, dy = 0;\ y(1) = 1$

(5) $2xy + y^2 - 2x^2\frac{dy}{dx} = 0;\ y(1) = 2$

(6) $(x^2 + 3xy + y^2)\, dx - x^2\, dy = 0;\ y(1) = 0$

*

5.8 Linear Differential Equation :

If P(x) and Q(x) are functions of variable x, then the differential equation $\frac{dy}{dx}$ + P(x) y = Q(x) is called a Linear Differential Equation.

For example, (1) $\frac{dy}{dx} + xy = \cos x$ P(x) = x, Q(x) = $\cos x$

(2) $\frac{dy}{dx} - \frac{y}{x} = e^x$ P(x) = $-\frac{1}{x}$, Q(x) = e^x

(3) $x\frac{dy}{dx} + 2y = x^3$ P(x) = $\frac{2}{x}$, Q(x) = x^2

(4) $\frac{dy}{dx} + y = x$ P(x) = 1, Q(x) = x

Method of solving a linear differential equation :

Let $\frac{dy}{dx} + P(x)\, y = Q(x)$ be a given linear differential equation.

If we multiply both the sides by $e^{\int P(x)\, dx}$, we get $\frac{dy}{dx} e^{\int P(x)\, dx} + y\, e^{\int P(x)\, dx} \cdot P(x) = Q(x)\, e^{\int P(x)\, dx}$

$\therefore\ \frac{d}{dx}\left[ye^{\int P(x)\, dx}\right] = Q(x)\, e^{\int P(x)\, dx}$

Integrating *w.r.t.* x, we get

$$ye^{\int P(x)\, dx} = \int \left[Q(x)\, e^{\int P(x)\, dx}\right] dx$$

Note : **Here the linear differential equation is multiplied on both the sides by $e^{\int P(x)\,dx}$ to make it easily integrable. So $e^{\int P(x)\,dx}$ is called an Integrating Factor - I.F.**

The first order linear differential equation is $\frac{dy}{dx} + P(x)y = Q(x)$.

If we multiply both the sides by $h(x)$, a function of x, we get

$$h(x)\frac{dy}{dx} + h(x)\ P(x)y = h(x)Q(x) \qquad \textbf{(i)}$$

Choose a function $h(x)$ in such a way that $h(x)Q(x)$ becomes a derivative of $y\,h(x)$.

$$\therefore \quad h(x)\frac{dy}{dx} + h(x)\ P(x)y = \frac{d}{dx}\,yh(x)$$

$$\therefore \quad h(x)\frac{dy}{dx} + h(x)\ P(x)y = h(x)\frac{dy}{dx} + yh'(x)$$

$$\therefore \quad h(x)\cdot P(x)y = yh'(x)$$

$$\therefore \quad h(x)\cdot P(x) = h'(x)$$

$$\therefore \quad P(x) = \frac{h'(x)}{h(x)}$$

Integrating both the sides with respect to x,

$$\therefore \quad \int P(x)\,dx = \int \frac{1}{h(x)}\ h'(x)\,dx$$

$$\therefore \quad \int P(x)\,dx = \log\,|(h(x))|$$

$$\therefore \quad h(x) = e^{\int P(x)\,dx}$$

In the equation (i) substitute the value of $h(x)$,

$$\therefore \quad e^{\int P(x)\,dx}\frac{dy}{dx} + e^{\int P(x)\,dx}\ P(x)y = e^{\int P(x)\,dx}\ Q(x)$$

$$\therefore \quad \frac{d}{dx}\left(e^{\int P(x)\,dx}\,y\right) = e^{\int P(x)\,dx}\,Q(x)$$

$$\therefore \quad e^{\int P(x)\,dx}\,y = \int e^{\int P(x)\,dx}\,Q(x)\ dx.$$

In this way we get the solution of a linear differential equation.

The function $h(x) = e^{\int P(x)\,dx}$ is an Integrating Factor.

Example 22 : Solve $\frac{dy}{dx} + \frac{y}{x} = x^2$. The given linear differential equation of the type $\frac{dy}{dx} + P(x)\,y = Q(x)$.

Solution : The given differential equation is linear.

Here $P(x) = \frac{1}{x}$, $Q(x) = x^2$

$$\therefore \quad \text{I.F.} = e^{\int P(x)\,dx}$$

$$= e^{\int \frac{1}{x}\,dx}$$

$$= e^{\log x}$$

$$= x$$

We can take I.F. as x because if we multiply both sides of the differential equation by x, then there will no change.

Multiply by x on both the sides.

$$x\frac{dy}{dx} + y = x^3$$

$$\therefore \quad \frac{d}{dx}(xy) = x^3$$

$$\therefore \quad xy = \int x^3\, dx$$

$$\therefore \quad xy = \frac{x^4}{4} + c$$ **(*c* is an arbitrary constant)**

This is the general solution of the given differential equation.

Example 23 : Solve $\frac{dy}{dx} + y\sec x = \tan x$.

Solution : $\frac{dy}{dx} + y\sec x = \tan x$.

This is a linear differential equation.

Here, $P(x) = \sec x$, $Q(x) = \tan x$

$$\therefore \quad \text{I.F.} = e^{\int P(x)\, dx}$$

$$= e^{\int \sec x\, dx}$$

$$= e^{\log|\sec x + \tan x|}$$

$$= |\sec x + \tan x|$$

We can take I.F. $= \sec x + \tan x$

Multiply both the sides of given equation by I.F., we get,

$$(\sec x + \tan x)\frac{dy}{dx} + \sec x\,(\sec x + \tan x)\, y = \tan x\,(\sec x + \tan x)$$

$$\frac{d}{dx}[y(\sec x + \tan x)] = \tan x\,(\sec x + \tan x)$$

$$\therefore \quad y(\sec x + \tan x) = \int \tan x\,(\sec x + \tan x)\, dx$$

$$\therefore \quad y(\sec x + \tan x) = \int \sec x \tan x\, dx + \int \tan^2 x\, dx$$

$$\therefore \quad y(\sec x + \tan x) = \int \sec x \tan x\, dx + \int (\sec^2 x - 1)\, dx$$

$$\therefore \quad y(\sec x + \tan x) = \sec x + \tan x - x + c$$ **(*c* is an arbitrary constant)**

is the general solution.

Example 24 : Solve $\frac{dy}{dx} = y\tan x + e^x$

Solution : $\frac{dy}{dx} = y\tan x + e^x$ is a linear differential equation in the form $\frac{dy}{dx} + P(x)y = Q(x)$.

Here $P(x) = -\tan x$ and $Q(x) = e^x$

Now, I.F. $= e^{\int P(x)\, dx}$

$$= e^{\int -\tan x\, dx}$$

$$= e^{-\log|\sec x|}$$

$$= e^{\log|\cos x|}$$

$$= |\cos x|$$

We can take I.F. $= \cos x$

$\therefore$ General solution of this linear equation is,

$y \cos x = \int e^x \cos x \, dx$ $\left(y e^{\int P(x)\,dx} = \int Q(x) \, e^{\int P(x)\,dx} \, dx \right)$

$\therefore$ $y \cos x = \frac{e^x}{2} (\cos x + \sin x) + c$ is the general solution. **(c arbitrary constant)**

Example 25 : Solve $\frac{dy}{dx} + \frac{y}{x} = \log x$

Solution : This is a linear differential equation in the form $\frac{dy}{dx} + P(x)y = Q(x)$.

Here $P(x) = \frac{1}{x}$ and $Q(x) = \log x$

$$\text{I.F.} = e^{\int P(x)\,dx}$$
$$= e^{\int \frac{1}{x} dx}$$
$$= e^{\log |x|}$$
$$= x$$

We can take I.F. $= x$

According to the general solution,

$$y e^{\int P dx} = \int Q(x) \cdot e^{\int P(x)\,dx} \, dx$$

$$yx = \int x \log x \, dx$$

$\therefore$ $yx = \log x \int x \, dx - \int \left(\frac{d}{dx} (\log x) \int x \, dx \right) dx$

$\therefore$ $yx = \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2} \, dx$

$\therefore$ $yx = \frac{x^2}{2} \log x - \frac{1}{4} x^2 + c$ **(c is an arbitrary constant)**

This is the general solution.

Exercise 5.5

Solve the following differential equations :

1. $\frac{dy}{dx} + 2y = \sin x$

2. $x \frac{dy}{dx} - y = (1 + x) e^{-x}$

3. $x \frac{dy}{dx} = x + y$

4. $\frac{dy}{dx} - \frac{2xy}{1 + x^2} = x^2 + 1$

5. $\frac{dy}{dx} = x + y$

6. $\frac{dy}{dx} + \frac{2y}{x} = e^x$

7. $4 \frac{dy}{dx} + 8y = 5e^{-3x}$

8. $(1 + x^2) \frac{dy}{dx} + 2xy - 4x^2 = 0$

9. $(1 + y^2) \, dx = (\tan^{-1} y - x) \, dy$

10. $x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x, \; x > 0$

11. $\sin^2 x \frac{dy}{dx} + y = \cot x$

12. $y \, dx - (x + 2y^2) \, dy = 0$

*

5.9 Applications of differention Equations :

As we know the study of differential equations began in order to solve the problems that originated from different branches of mathematics, physics, biological sciences etc.

(1) Physics (RL circuit) : Let us consider RL circuit. This circuit contains resistor (R) and Inductor (L). So it is known as RL circuit. At $t = 0$, the switch is closed and current does not pass throuch the circuit. When switch is on, the current passes through the circuit. As per the electricity law, when voltage across a resistor of resistance R is equal to Ri, the voltage across an inductor is given by $L\frac{di}{dt}$, where i is the current.

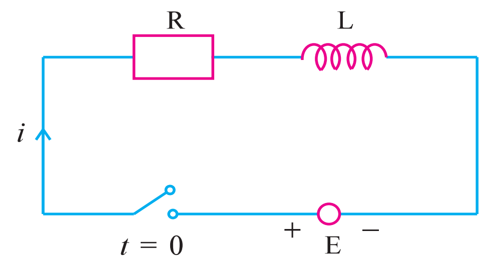


Figure 5.5

Example 26 : The equation of electromotive force (*e.m.f.*) is $E = Ri + L\frac{di}{dt}$, where R is resistance, L is the self inductance and i is electric current. Find the equation relating time (t) and electric current (i).

Solution : The given equation can be written as $L\frac{di}{dt} = E - Ri$

$$\therefore \quad \frac{1}{E - Ri}\, di = \frac{1}{L}\, dt$$

$$\therefore \quad \frac{-R}{E - Ri}\, di = \frac{-R}{L}\, dt \qquad \textbf{(Variable Separable form)}$$

Now integrating both the sides,

$$\int \frac{-R}{E - Ri}\, di = \int \frac{-R}{L}\, dt$$

$$\therefore \quad \log(E - Ri) = \frac{-R}{L}t + \log c$$

$$\therefore \quad \log\frac{(E - Ri)}{c} = \frac{-R}{L}t$$

$$\therefore \quad E - Ri = ce^{\frac{-R}{L}t}$$

$$Ri = E - ce^{\frac{-R}{L}t}$$

$$\therefore \quad i = \frac{E}{R} - \frac{ce^{-\frac{R}{L}t}}{R} \text{ is the required equation.}$$

Another Method :

Given equation is $L\frac{di}{dt} = E - Ri$

$$\therefore \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

This is a linear differential equation. I.F. $= e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$

Multiplying both the sides by I.F., $e^{\frac{R}{L}t}\frac{di}{dt} + e^{\frac{R}{L}t}\frac{R}{L}i = \frac{E}{L}e^{\frac{R}{L}t}$

$$\therefore \quad \frac{d}{dt}\left(e^{\frac{R}{L}t}i\right) = \frac{E}{L}e^{\frac{R}{L}t}$$

Integrating both the sides *w.r.t.* t,

$$e^{\frac{R}{L}t} \cdot i = \int \frac{E}{L} e^{\frac{R}{L}t} dt$$

$$\therefore \quad e^{\frac{R}{L}t} \cdot i = \frac{\frac{E}{L} e^{\frac{R}{L}t}}{\frac{R}{L}} - \frac{c}{R} \qquad \left(-\frac{c}{R} \text{ arbitrary constant}\right)$$

$$\therefore \quad e^{\frac{R}{L}t} \cdot i = \frac{E}{R} e^{\frac{R}{L}t} - \frac{c}{R}$$

$$\therefore \quad i = \frac{E}{R} - \frac{c}{R} e^{-\frac{R}{L}t}$$

This is the general solution.

(2) Application in Geometry :

$y = f(x)$ is a given curve.

If $y = f(x)$ is differentiable at (x_0, y_0) then, slope of the tangent at the point (x_0, y_0) is given by $m = \left(\frac{dy}{dx}\right)_{(x_0,\, y_0)}$

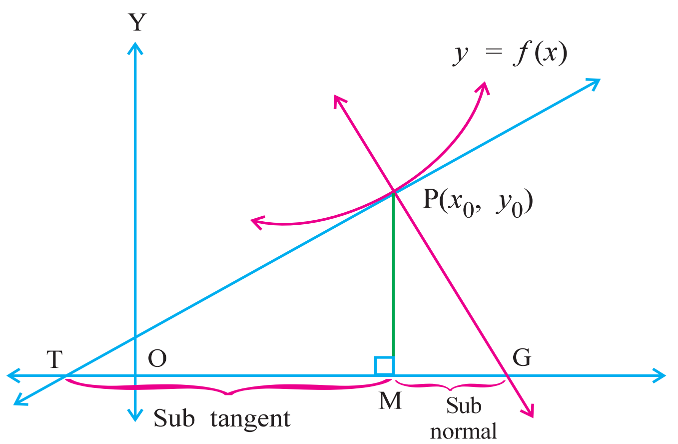


Figure 5.6

(1) The equation of the tangent to the curve at point (x_0, y_0) is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0,\, y_0)} (x - x_0)$$

(2) The equation of the normal to the curve at point (x_0, y_0) is

$$y - y_0 = -\left(\frac{dx}{dy}\right)_{(x_0,\, y_0)} (x - x_0) \qquad \left(\frac{dy}{dx} \neq 0\right)$$

Let $M(x_0, 0)$ be the foot of perpendicular from $P(x_0, y_0)$ on the X-axis. Suppose tangent at P intersects X-axis at T, then $\overline{TM}$ is called the subtangent.

$$\text{Length of subtangent TM} = \left| \frac{y_0}{\left(\frac{dy}{dx}\right)_{(x_0,\, y_0)}} \right|$$

Suppose the normal at P intersects X-axis at G, then $\overline{MG}$ is called the subnormal.

$$\text{Length of subnormal MG} = \left| y_0 \left(\frac{dy}{dx}\right)_{(x_0,\, y_0)} \right|$$

Example 27 : The slope of the tangent to the curve at any point is reciprocal of the y-coordinate of that point $(y \neq 0)$ and the curve passes through $(-1, 2)$. Find the equation of the curve.

Solution : Let $P(x, y)$ be any point on the curve.

Slope of the tangent to the curve at the point $P(x, y)$ is $\frac{dy}{dx}$.

But the slope of the tangent to the curve at point $P(x, y) = \frac{1}{y}$.

$\therefore \quad \frac{dy}{dx} = \frac{1}{y}$

$\therefore \quad ydy = dx$

Integrating both the sides,

$$\int y\ dy = \int dx$$

$$\frac{y^2}{2} = x + \frac{c}{2} \qquad \textbf{(c is an arbitrary constant)}$$

$\therefore \quad y^2 = 2x + c,$

It passes through $(-1, 2)$

$\therefore \quad 4 = -2 + c$

$\therefore \quad c = 6$

$\therefore \quad y^2 = 2x + 6$ is the equation of the curve.

(3) Exponential Growth :

Let $p(t)$ be a quantity which increases with time t. Suppose at time $t = 0$, $p(t) = p_0$.

So the rate of increase of the quantity is proportional to the given quantity $p(t)$.

i.e. $\frac{d\ p(t)}{dt} \propto p(t)$

$$\frac{d\ p(t)}{dt} = kp(t) \qquad \textbf{(k > 0)}$$

$$\frac{1}{P(t)} \frac{d\ p(t)}{dt} = k$$

Integrating both the sides, we get

$$\int \frac{d\ p(t)}{dt} = \int k\ dt$$

$$\log p(t) = kt + \log c$$

$\therefore \quad \log p(t) - \log c = kt$

$\therefore \quad \log \frac{p(t)}{c} = kt$

$\therefore \quad p(t) = ce^{kt}$, where c is an arbitrary constant.

Suppose at $t = 0$, $p(t) = p_0$.

Then $p(0) = ce^0$

$\therefore \quad c = p(0)$

$\therefore \quad p(t) = p(0)e^{kt}$

Using this solution, we can find the growth of quantity $p(t)$ at any time t.

Example 28 : The population of a city increases at the rate of 2 % per year. How many years will it take to double the population ?

Solution : Let the p_0 be the population at present and after t years suppose it will be $p(t)$.

Now population increases at the rate of 2 %.

So, $\frac{dp}{dt} = \frac{2}{100} p$

$$\int \frac{dp}{p} = \frac{1}{50} \int dt$$

$\therefore$ $\log p = \frac{1}{50} t + \log c$

$\therefore$ $p = ce^{\frac{1}{50} t}$

At $t = 0$, $p = p_0$

So $p_0 = ce^0$

$\therefore$ $c = p_0$

$\therefore$ $p = p_0 e^{\frac{1}{50} t}$

Now if the population doubles, then $p = 2p_0$.

$\therefore$ $2p_0 = p_0 e^{\frac{1}{50} t}$

$\therefore$ $\log_e 2 = \frac{1}{50} t$

$\therefore$ $t = 50 \log_e 2 = 34.65 \cong 35$ years

(4) Exponential Decay :

Let $m(t)$ be the mass of a product which decreases with time t.

The rate of decrease is proportional to the given mass m.

So, $\frac{dm}{dt} = -km$ **($k > 0$)**

Using the above method, we can find the decay.

Example 29 : A certain radioactive material has a half life of 2000 years. (This is called half life period of the substance.) Find the time required for a given amount to become one tenth of its original mass.

Solution : Let initial mass of the material be m_0 grams.

If the mass of the material is m grams after time t, then from the rate of decay we have,

$\frac{dm}{dt} = -km$ **($k > 0$)**

$\frac{dm}{m} = -k\ dt$

$\therefore$ $\int \frac{dm}{m} = \int -k\ dt$

$\therefore$ $\log m = -kt + \log c$

$\therefore$ $m = ce^{-kt}$

Now when $t = 0$, $m = m_0$

$m_0 = ce^0$

$\therefore$ $c = m_0$

$\therefore \quad m = m_0 e^{-kt}$ **(i)**

At $t = 2000$ years, $m = \frac{m_0}{2}$

So, $\frac{m_0}{2} = m_0 e^{-k\,2000}$

$\therefore \quad \frac{1}{2} = e^{-k\,2000}$

$\therefore \quad -k2000 = -\log 2$

$\therefore \quad k = \frac{\log 2}{2000}$

Now at some time t, m will be $\frac{m_0}{10}$,

From equation (i),

$\therefore \quad \frac{m_0}{10} = m_0 e^{-kt}$

$\therefore \quad -kt = \log \frac{1}{10}$

$\therefore \quad -kt = -\log 10$

$\therefore \quad kt = \log 10$

$\therefore \quad t = \frac{1}{k} \log 10 = \frac{2000}{\log_e 2} \cdot \log 10 \simeq 6644$ years

(5) Newton's Law of Cooling :

The rate of change of temperature of a body is proportional to the difference between the temperature of the body itself and that of the surroundings.

Let S be the constant temperature of surroundings. Let T be the temperature of the body at any time t. Then,

$$\frac{dT}{dt} \propto (T - S)$$

$\therefore \quad \frac{dT}{dt} = -k(T - S)$ **($k > 0$ is a constant)**

$\therefore \quad \frac{1}{T - S}\, dT = -kdt$

Integrating both the sides,

$$\log |T - S| = -kt + \log c$$

$\therefore \quad \log \left|\frac{T - S}{c}\right| = -kt$

$$T - S = ce^{-kt}$$

Example 30 : The temperature of a body in a room is 80° F. After five minutes the temperature of the body becomes 60° F. After another 5 minutes the temperature becomes 50° F. What is the temperature of surroundings ?

Solution : Let T be the temperature of the body at any time t.

Let S be the constant temperature of the surroundings. (i.e. room temperature)

Then by Newton's law of cooling.

$$\frac{dT}{dt} \propto (T - S)$$

$\therefore \quad \frac{dT}{dt} = -k(T - S)$ **($k > 0$ is a constant as temperature decreases in time interval)**

$\therefore \quad \frac{dT}{T - S} = -kT$

$\therefore \quad \int \frac{dT}{T - S} = \int -k \, dt$

$\therefore \quad \log (T - S) = -kt + c$ **(i)**

Now at $t = 0$, T = 80° F

$\therefore \quad \log (80 - S) = c$

From equation (i), we get

$\log (T - S) = -kt + \log (80 - S)$

Also at $t = 5$, T = 60° F

$\therefore \quad \log (60 - S) = -5k + \log (80 - S)$ **(ii)**

Also at $t = 10$, T = 50° F

$\therefore \quad \log (50 - S) = -10k + \log (80 - S)$ **(iii)**

From equations (ii) and (iii), we get

$\therefore \quad \frac{1}{5} \log \left(\frac{60 - S}{80 - S}\right) = -k = \frac{1}{10} \log \left(\frac{50 - S}{80 - S}\right)$

$\therefore \quad 2 \log \left(\frac{60 - S}{80 - S}\right) = \log \left(\frac{50 - S}{80 - S}\right)$

$\therefore \quad \left(\frac{60 - S}{80 - S}\right)^2 = \left(\frac{50 - S}{80 - S}\right)$

$\therefore \quad (60 - S)^2 = (80 - S)(50 - S)$

$\therefore \quad 3600 - 120S + S^2 = 4000 - 130S + S^2$

$\therefore \quad 10S = 400$

$\therefore \quad S = 40°$ F

Hence, temperature of the room is 40° F.

Example 31 : Saptesh has a fixed deposit of ₹ 10,000 in a bank. Principal amount increases continuously at the rate of 7 % per year. In how many years will it get doubled ?

Solution : Let P be the amount at any time t.

According to the given conditions,

$\frac{dP}{dt} = \frac{7P}{100}$

$\therefore \quad \frac{dp}{P} = \frac{7}{100} dt$ **(Variables Separable form)**

Integrating both the sides,

$\int \frac{dp}{P} = \int \frac{7}{100} dt$

$\therefore \quad \log P = \frac{7}{100} t + \log c$

$\therefore \quad P = ce^{\frac{7t}{100}}$

At $t = 0$, P = ₹ 10000

$10000 = ce^{0}$

$\therefore \quad c = 10000$

$\therefore \quad P = 10000\, e^{\frac{7t}{100}}$ **(i)**

Let t be the time to double the investment.

After time t, P $= 2 \times$ principal

$= 2 \times 10000$

= ₹ 20000

From equation (i),

$\therefore \quad 20000 = 10000\, e^{\frac{7t}{100}}$

$\therefore \quad 2 = e^{\frac{7t}{100}}$

$\therefore \quad \log_e 2 = \frac{7}{100} t$

$\therefore \quad t = \frac{100}{7} \log_e 2$ which is approximately 9.9 years.

Exercise 5.6

1. If the X intercept of the tangent to a curve at any point is four times its y-coordinate, then find the equation of the curve.

2. In an experiment of culture of bacteria in a laboratory, the rate of increase of bacteria is proportional to the number of bacteria present at that time. If in one hour the number of bacteria gets doubled, then

(1) What is the number of bacteria at the end of 4 hours ?

(2) If the number of bacteria is 24,000 at the end of 3 hours. Find the number of bacteria in the beginning.

3. A curve passes through $(3, -4)$. Slope of tangent at any point (x, y) is $\frac{2y}{x}$. Find the equation of the curve.

4. The increase in the principal amount kept at the compound interest in a bank is proportional to the product of the principal amount and annual rate of interest.

(1) Annual rate of interest in a bank is 5 %. How many years will it take to double the principal amount ?

(2) At what annual rate of interest, the principal amount will double in 10 years ?

5. Rate of decay of a radioactive body is proportional to its mass present at that time. After a decay of one year the mass of the body is 100 grams and after two years it is 80 grams. Find the initial mass of the body.

6. If the length of the subnormal of a curve is constant and if it passes through the origin, then find its equation.

7. Find the equation of the curve passing through the point (1, 2), given that at any point (x, y) on the curve, if the product of the slope of its tangent and y-coordinate of the point is equal to the x-coordinate of the point.

Exercise 5

1. Verify that the function $y = cx + \frac{a}{c}$ is the general solution of the differential equation,

$y = x\left(\frac{dy}{dx}\right) + a\left(\frac{dx}{dy}\right)$ (c is an arbitrary constant).

2. Show that the solution of the differential equation $\frac{dy}{dx} = 1 + xy^2 + x + y^2$, $y(0) = 0$ is

$y = tan\left(x + \frac{x^2}{2}\right)$.

3. Show that $y = e^{-x} + ax + b$ is a solution of the differential equation $e^x \frac{d^2y}{dx^2} - 1 = 0$.

4. Verify that the function $y = ae^{2x} + be^{-x}$ is a solution of the differential equation

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$.

5. Find the differential equation for the family of the curves represented by $y^2 = a(b + x)(b - x)$.

(a, b arbitrary constant)

6. Solve :

(1) $\frac{dy}{dx} = cos\,(x + y) + sin\,(x + y)$

(2) $\frac{dy}{dx} + \frac{4xy}{x^2+1} = \frac{1}{(x^2+1)^3}$

(3) $2ye^{\frac{x}{y}}\,dx + (y - 2xe^{\frac{x}{y}})\,dy = 0$

(4) $xy\frac{dy}{dx} = x^2 - y^2$

(5) $(x^2 - y^2)\,dx + 2xy\,dy = 0$ $\quad y(1) = 1$

(6) $cos^2x\frac{dy}{dx} + y = tan\,x$

7. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The order of a differential equation whose general solution is $y = $ A$sinx$ + B$cosx$ is ...... (A, B are arbitrary constants.) ☐

(a) 4 (b) 2 (c) 0 (d) 3

(2) The order and degree of $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 + y = 0$ are ...... respectively. ☐

(a) 3, 2 (b) 2, 3 (c) 3, not defined (d) 2, 3

(3) $y' + y = \frac{5}{y'}$ has degree ....

(a) 1 (b) 2 (c) not defined (d) −1

(4) The differential equation $\frac{dy}{dx} = -\frac{x+y}{1+x^2}$ is ....

(a) of variable separable form (b) homogeneous

(c) linear (d) of second order

(5) $f(x, y) = \frac{x^3 - y^3}{x + y}$ is a homogeneous function of degree ....

(a) 1 (b) 2 (c) 3 (d) not defined

(6) An integrating factor of differential equation $\frac{dy}{dx} = \frac{1}{x+y+2}$ is ....

(a) e^x (b) e^{x+y+2} (c) e^{-y} (d) $\log |x + y + 2|$

(7) The differential equation of the family of rectangular hyperbolas is ....

(a) $y_2 = 0$ (b) $xy + y_2 = 0$ (c) $yy_1 = x$ (d) $xy_1 + y = 0$

(8) The order and the degree of the differential equation $\frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} + xy = sinx$, are ...... respectively.

(a) 1, 1 (b) 2, 1 (c) 3, 2 (d) 2, not defined

(9) Which of the following function is a solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$?

(a) $y = 4x$ (b) $y = 4$ (c) $y = 2x^2 + 4$ (d) $y = 2x - 4$

(10) Solution of the differential equation $x\frac{dy}{dx} + y = 0$ is ....

(a) $e^{xy} = c$ (b) $y = cx$ (c) $x = cy$ (d) $e^x y = c$

(11) The solution of the differential equation $\frac{dy}{dx} + \frac{2y}{x} = 0$ with $y(1) = 1$ is given by ......

(a) $y = \frac{1}{x}$ (b) $y = \frac{1}{x^2}$ (c) $x = \frac{1}{y^2}$ (d) $x^2 = \frac{1}{y^2}$

(12) The number of arbitrary constants in the general solution of differential equation of second order is ......

(a) 1 (b) 0 (c) 2 (d) 4

(13) The number of arbitrary constants in the particular solution of a differential equation of fourth order is ......

(a) 4 (b) 2 (c) 1 (d) 0

(14) The differential equation $\frac{dy}{dx} = e^{x+y}$ has solution ....

(a) $e^x + e^{-y} = c$ (b) $e^x + e^y = c$ (c) $e^{-x} + e^y = c$ (d) $e^{-x} + e^{-y} = c$

(15) The degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^3\right]^{\frac{2}{3}} = x\left(\frac{d^2y}{dx^2}\right)$ is ....

(a) 3 (b) 2 (c) 6 (d) 1

(16) The solution of the differential equation $2x\frac{dy}{dx} - y = 0$; $y(1) = 2$ represents ....

(a) straight line (b) parabola (c) circle (d) ellipse

Summary

We have studied the following points in this chapter :

1. An equation involving independent variable (x), dependent variable (y) and derivatives of the dependent variable *w.r.t.* independent variable is known as a differential equation.
2. Order of the highest order derivative occuring in the given differential equation is called the order of the differential equation.
3. If the differential equation is in a polynomial form in derivatives, then the highest power of the highest order derivative occurring in the differential equation is called the degree of the equation.
4. Solution of a differential equation of order n is a function which satisfies the differential equation. The solution which contain n arbitrary constants is called the general solution and the solution free from all arbitrary constants is called a particular solution.
5. Variables separable method is used to solve the differential equation in which variables can be separated completely.
6. If a two variable function $f(x, y)$ can be written as $f(x, y) = x^n \phi\left(\frac{y}{x}\right)$ form, then the function $f(x, y)$ is called homogeneous function having degree n.
7. P(x) and Q(x) are functions of variable x, then the differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ is called linear differential equation.
8. Applications of differential equations.

6 VECTOR ALGEBRA

> **Mathematics knows no races or geographic boundaries;**
> **for mathematics, the cultural world is one country.**
>
> **– Jules Henri**

6.1 Introduction

In everyday conversation, when we talk of a quantity, we generally discuss a scalar quantity which has only magnitude. If we say that we drove through a distance of 50 *km*, we talk about the distance travelled. Here we do not bother in which direction we have travelled. 50 *km* is a scalar quantity. Now, if we drive towards our home, then simply to say driving 50 *km* is not enough, but we have to say that we should drive 50 *km* South to reach our home. This information provides not just magnitude but also the direction of the quantity. This quantity is a vector quantity.

The latin word **vector** means 'Carrier'. Vector 'carries' magnitude as the distance between two points (i.e. distance between initial point and terminal point) and also the direction from the first point to the last point (i.e. from initial point to terminal point). Most of the basic algebraic operations like addition, subtraction, multiplication and division are reflected equally well in vector-operations as addition, subtraction and multiplication by a scalar. Vector addition also follows the algebraic properties of R like commutativity, associativity.

Vector is a very important concept in the study of Physics. Many physical quantities like velocity, acceleration, force acting on an object etc. are described by vectors. Many physical quantities do not represent distance but are still represented by vectors and so it helps a lot to understand the concepts of Physics.

Generally, gravity, electrostatic force, magnetic force, electromagnetic force or mechanical force are studied in physics. Physicists had found by scientific experiments that these forces in general conditions act in a linear (vector) way and their resultant forces are also the result of the addition of vectors, e.g. **Coulomb's law of electrostatics**. So vector space and its algebraic operations etc are developed to study these forces.

Vectors are denoted by small arrow (→) or bar (–) sign above the letter or bold letters in print form. In Mathematics, Physics and Engineering, we frequently come across scalar quantities such as length, distance, speed, time, mass etc and also vector quantities like, displacement, velocity, acceleration, force, weight etc.

We have already studied in std. XI about vector space R^2 as well as R^3 and some operations on vectors like addition of vectors, multiplication of a vector by a scalar and their properties, magnitude of a vector, a unit vector etc. These concepts are needed for further study. So in this chapter, we shall summarise them and consolidate by solving some examples.

6.2 Vector as an Element of a Vector Space

$R^2 = \{(x, y) \mid x \in R, y \in R\}$

$R^3 = \{(x, y, z) \mid x \in R, y \in R, z \in R\}$

The sets R^2 and R^3 under operations of addition and multiplication by a scalar given on page 192 are called vector spaces over R.

The elements of R^2 and R^3 as vector space are denoted by $\bar{x}$, $\bar{y}$, $\bar{z}$ etc. $\bar{x}$, $\bar{y}$, $\bar{z}$ are called vectors. Elements of R are called scalars.

Equality of Vectors :

$(x_1, y_1, z_1) = (x_2, y_2, z_2) \Leftrightarrow x_1 = x_2, y_1 = y_2$ and $z_1 = z_2$.

Addition of Vectors :

$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

Multiplication of a Vector by a Scalar :

$k(x_1, y_1, z_1) = (kx_1, ky_1, kz_1), \quad \forall k \in R$

Properties of Addition of Elements of R^3 and Multiplication by a Scalar

(1) **Closure property :** $\forall \bar{x}, \bar{y} \in R^3, \bar{x} + \bar{y} \in R^3$

(2) **Commutative law of addition :** $\bar{x} + \bar{y} = \bar{y} + \bar{x}; \quad \forall \bar{x}, \bar{y} \in R^3$

(3) **Associative law of addition :** $(\bar{x} + \bar{y}) + \bar{z} = \bar{x} + (\bar{y} + \bar{z}); \quad \forall \bar{x}, \bar{y}, \bar{z} \in R^3$

(4) **Existence of additive identity :** There exists a vector $\bar{0} \in R^3$ such that $\bar{x} + \bar{0} = \bar{0} + \bar{x} = \bar{x}, \forall \bar{x} \in R^3$, $\bar{0}$ is called zero vector or null-vector. $\bar{0} = (0, 0, 0)$

(5) **Existence of additive inverse :** For every $\bar{x} \in R^3$, there exists a vector, $-\bar{x} \in R^3$ such that $\bar{x} + (-\bar{x}) = (-\bar{x}) + \bar{x} = \bar{0}$. This vector $-\bar{x}$ is called additive inverse vector of $\bar{x}$ or negation of $\bar{x}$.

(6) $\forall k \in R$ and $\bar{x} \in R^3, \quad k\bar{x} \in R^3$.

(7) $\forall k \in R, k(\bar{x} + \bar{y}) = k\bar{x} + k\bar{y}; \quad \forall \bar{x}, \bar{y} \in R^3$

(8) $\forall k, l \in R, (k + l)\bar{x} = k\bar{x} + l\bar{x}; \quad \forall \bar{x} \in R^3$

(9) $\forall l, k \in R, (kl)\bar{x} = k(l\bar{x}); \forall \bar{x} \in R^3$

(10) $1\bar{x} = \bar{x}, \forall \bar{x} \in R^3$

The above rules are also true for the elements of R^2.

Some Basic Concepts

Magnitude of a Vector : If $\bar{x} = (x_1, x_2, x_3)$, then magnitude of $\bar{x}$, denoted by $|\bar{x}|$ is defined as $|\bar{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. If $\bar{x} = (x_1, x_2)$, then $|\bar{x}| = \sqrt{x_1^2 + x_2^2}$.

For example, if $\bar{x} = (1, 2, -2)$, then $|\bar{x}| = \sqrt{(1)^2 + (2)^2 + (-2)^2} = 3$.

Some obvious results : ($\bar{x} \in R^2$ or R^3)

(1) $|\bar{x}| \geq 0$

(2) $|\bar{x}| = 0 \Leftrightarrow \bar{x} = \bar{0}$

(3) $|k\bar{x}| = |k|\,|\bar{x}|, k \in R$

Unit Vector : If $|\bar{x}| = 1$, then $\bar{x}$ is called a unit vector. A unit vector is denoted by $\hat{x}$.

For example, if $\bar{x} = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, then $|\bar{x}| = 1$ and hence $\bar{x}$ is a unit vector.

$\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$, $\hat{k} = (0, 0, 1)$ are unit vectors in the positive direction of X-axis, Y-axis and Z-axis respectively.

6.3 Direction of vectors

Let $\bar{x}$ and $\bar{y}$ be non-zero vectors of R^2 or R^3 and $k \in R$.

If (i) $\bar{x} = k\bar{y}$, $k > 0$, then $\bar{x}$ and $\bar{y}$ are vectors having same direction.

(ii) $\bar{x} = k\bar{y}$, $k < 0$, then $\bar{x}$ and $\bar{y}$ are vectors having opposite directions.

(iii) $\bar{x} \neq k\bar{y}$, for any $k \in R$, then $\bar{x}$ and $\bar{y}$ are vectors having different directions.

If directions of non-zero vectors $\bar{x}$ and $\bar{y}$ are same or opposite, they are called collinear vectors.

$\therefore$ If $\bar{x} = k\bar{y}$ then and only then $\bar{x}$ and $\bar{y}$ are collinear. ($\bar{x} \neq \bar{0}, \bar{y} \neq \bar{0}$)

Notation : Let $\bar{x} = (x_1, x_2, x_3)$. Direction of $\bar{x}$ is denoted by $<x_1, x_2, x_3>$ and direction opposite, to the direction of $\bar{x}$ is denoted by $-<x_1, x_2, x_3>$.

It follows from the definition that,

(i) $<x_1, x_2, x_3> = <kx_1, kx_2, kx_3>$, if $k > 0$.

(ii) $-<x_1, x_2, x_3> = <kx_1, kx_2, kx_3>$, if $k < 0$.

We also denote direction of $\bar{x}$ as (kx_1, kx_2, kx_3), $k \in R - \{0\}$

We accept the following theorems without proving them.

Theorem 6.1 : Non-zero vectors $\bar{x}$ and $\bar{y}$ are equal if and only if $|\bar{x}| = |\bar{y}|$ and $\bar{x}$ and $\bar{y}$ have the same direction.

Theorem 6.2 : If $\bar{x} \neq \bar{0}$ then there is a unique unit vector in the direction of $\bar{x}$.

Unit Vector in the Direction of a Given Vector : If $\bar{x}$ is any non-zero vector, then $\frac{1}{|\bar{x}|}\bar{x}$ is a unit vector in the direction of $\bar{x}$ and it is denoted by $\hat{x}$.

$\bar{y} = \frac{k\bar{x}}{|\bar{x}|}$, $k > 0$ has same direction as $\bar{x}$ and has magnitude k.

$\bar{y} = \frac{-k\bar{x}}{|\bar{x}|}$, $k > 0$ is in direction opposite to the direction of $\bar{x}$ and has magnitude k.

Example 1 : Find the vector of magnitude 10 in the direction opposite to the direction of $\bar{x} = (3, 0, -4)$.

Solution : $|\bar{x}| = \sqrt{9+0+16} = 5$

$\therefore$ The vector of magnitude 10 in the direction opposite to the direction of $\bar{x}$ is

$\frac{-10}{|\bar{x}|}\bar{x} = \frac{-10}{5}(3, 0, -4) = (-6, 0, 8)$.

Right Hand Thumb Rule : Let O be a fixed point in space and take three mutually perpendicular lines through O. These are taken as X-axis, Y-axis and Z-axis. Normally, X-axis and Y-axis are so arranged that they are in a horizontal plane. Z-axis is perpendicular to both X-axis and Y-axis. **The positive directions of these axes follow the Right Hand Thumb rule**, that is, if you curl the fingers of your right hand around the Z-axis in the direction of counter clockwise $\frac{\pi}{2}$ rotation from the positive X-axis to the positive Y-axis, then your thumb points in the positive direction of positive Z-axis.

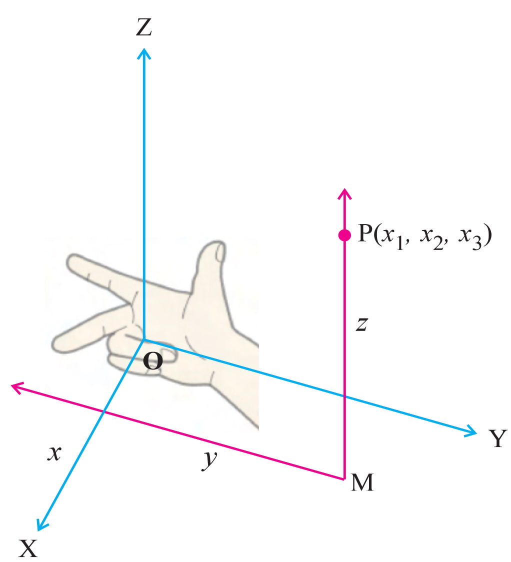


Figure 6.1

6.4 Position Vector

Let $\bar{x} = (x_1, x_2, x_3) \in R^3$ be a vector and a point P in space having coordinates (x_1, x_2, x_3). The directed line-segment $\overline{OP}$ with initial point O and terminal point P is called the position vector of the point P and it is denoted as $\overrightarrow{OP}$. Thus the position vector of P is $\bar{x} = (x_1, x_2, x_3)$, i.e. $\overrightarrow{OP} = (x_1, x_2, x_3)$. If the position vector of a point is $\bar{x}$, then $\overrightarrow{OP} = \bar{x}$ is the the geometrical representation of the vector.

If $A(x_1, x_2, x_3)$ and $B(y_1, y_2, y_3)$ are two distinct points in R^3, the vector joining the points A and B with initial point A is $\overrightarrow{AB}$.

Theorem 6.3 : (1) Every vector of R^2 can be uniquely expressed as linear combination of $\hat{i}$ and $\hat{j}$.

Proof : Suppose $\overline{x} = (x_1, x_2) \in R^2$.

Then $\overline{x} = (x_1, x_2) = (x_1, 0) + (0, x_2)$

$= x_1(1, 0) + x_2(0, 1)$

$= x_1\hat{i} + x_2\hat{j}$

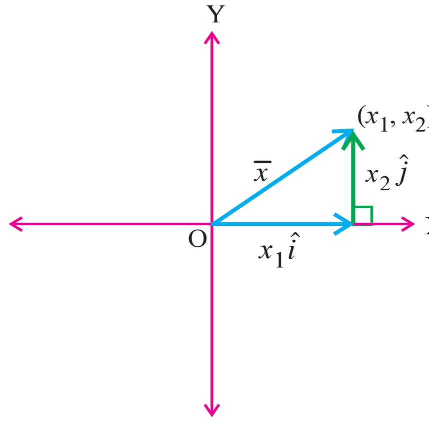


Figure 6.2

Thus, $\overline{x}$ is **a linear combination** of $\hat{i}$ and $\hat{j}$. Now, suppose $\overline{x}$ can be expressed as a linear combination of $\hat{i}$ and $\hat{j}$ as $\overline{x} = p\hat{i} + q\hat{j}$ also.

Then $(x_1, x_2) = \overline{x} = p\hat{i} + q\hat{j}$

$= p(1, 0) + q(0, 1)$

$= (p, 0) + (0, q)$

$= (p, q)$

$\therefore \quad x_1 = p$ and $x_2 = q$

$p\hat{i} + q\hat{j}$ and $x_1\hat{i} + x_2\hat{j}$ are same.

Thus $\overline{x} = x_1\hat{i} + x_2\hat{j}$ is a unique linear combination of $\hat{i}$ and $\hat{j}$.

(2) Every vector in R^3 can be uniquely expressed as a linear combination of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Proof : Suppose $\overline{x} = (x_1, x_2, x_3) \in R^3$.

Then $\overline{x} = (x_1, x_2, x_3) = (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3)$

$= x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1)$

$= x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$

If $\overline{x} = p\hat{i} + q\hat{j} + r\hat{k}$, then we can prove $x_1 = p$, $x_2 = q$ and $x_3 = r$ as before.

Thus, $\overline{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ is unique linear combination of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Geometric Representation :

Let $\overrightarrow{OP} = (x_1, x_2, x_3)$.

Let L be the foot of perpendicular from P to XY plane (figure 6.3). So $L(x_1, x_2, 0)$.

Then $\overrightarrow{LP} = \overrightarrow{OC} = x_3\hat{k}$. Similarly, M and N are the feet of perpendiculars from P to YZ and ZX plane respectively. So $M(0, x_2, x_3)$ and $N(x_1, 0, x_3)$

and so $\overrightarrow{MP} = \overrightarrow{OA} = x_1\hat{i}$ and $\overrightarrow{NP} = \overrightarrow{OB} = x_2\hat{j}$.

$\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are bound vectors corresponding to free vectors $\overrightarrow{MP}$, $\overrightarrow{NP}$, $\overrightarrow{LP}$ respectively.

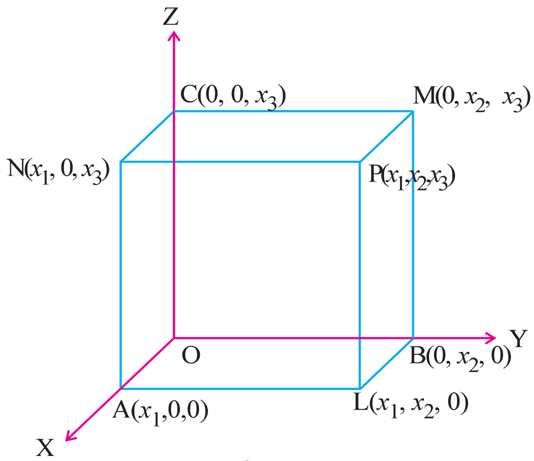


Figure 6.3

[The coordinates of A, B and C are A(x_1, 0, 0), B(0, x_2, 0) and C(0, 0, x_3).]

Now, $\overrightarrow{OL} = \overrightarrow{OA} + \overrightarrow{AL} = \overrightarrow{OA} + \overrightarrow{OB} = x_1\hat{i} + x_2\hat{j}$ $(\overrightarrow{OB} = \overrightarrow{AL})$

[The coordinates of L are (x_1, x_2, 0). Similarly coordinates of M and N are (0, x_2, x_3) and (x_1, 0, x_3) respectively.]

$\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$. The form $\overrightarrow{OP} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ of a vector is also called **component form**. Here x_1, x_2 and x_3 are the scalar components of $\overrightarrow{OP}$, while $x_1\hat{i}$, $x_2\hat{j}$ and $x_3\hat{k}$ are the **vector components** of $\overrightarrow{OP}$.

Note : (1) Distance of P(x_1, x_2, x_3) from XY plane is PL = $|x_3|$. Similarly, distance of P from YZ plane = PM = $|x_1|$ and distance from ZX plane = PN = $|x_2|$.

(2) Distance of P(x_1, x_2, x_3) from X-axis = AP = $\sqrt{x_2^2 + x_3^2}$. Similarly distance from Y-axis = BP = $\sqrt{x_3^2 + x_1^2}$ and distance from Z-axis = CP = $\sqrt{x_1^2 + x_2^2}$.

(3) Distance of P(x_1, x_2, x_3) from origin = OP = $\sqrt{x_1^2 + x_2^2 + x_3^2}$.

6.5 Triangle Law of Vector Addition

A particle is displaced from A to B and the displacement is represented by $\overrightarrow{AB}$ and the displacement from B to C is represented by $\overrightarrow{BC}$ as shown in figure 6.4. The displacement of the particle from A to C is given by the vector $\overrightarrow{AC}$. The result $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ is called the Triangle Law of Vector Addition.

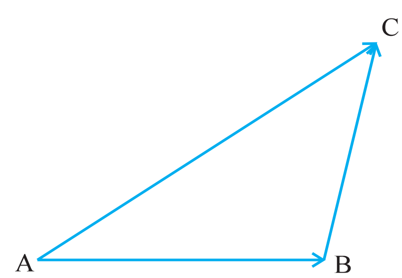


Figure 6.4

Let A, B, C have position vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively.

$$\overrightarrow{AB} + \overrightarrow{BC} = (\bar{b} - \bar{a}) + (\bar{c} - \bar{b})$$
$$= \bar{c} - \bar{a} = \overrightarrow{AC}$$

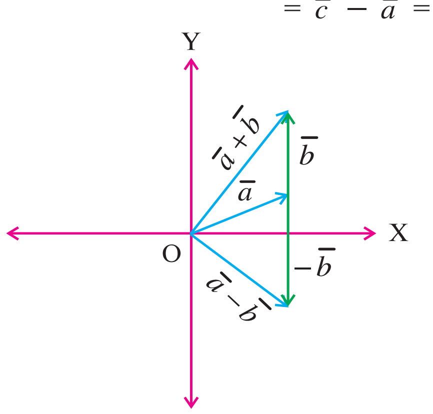


Figure 6.5

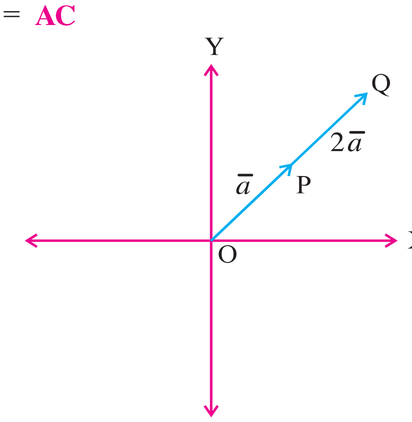


Figure 6.6

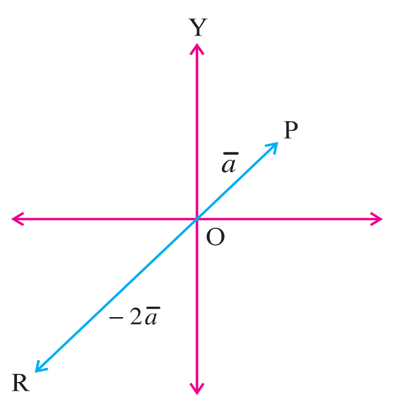


Figure 6.7

If $\bar{a}$ and $\bar{b}$ are two non-zero vectors, then the operations of addition and subtraction of vectors $\bar{a}$ and $\bar{b}$ in R^2 are shown in figure 6.5. Figures 6.6 and 6.7 illustrate scalar multiplication of vector in R^2. Here $\overrightarrow{OP} = \bar{a}$, $\overrightarrow{OQ} = 2\bar{a}$ and $\overrightarrow{OR} = -2\bar{a}$.

Parallelogram Law for Vector Addition :

Let $\overrightarrow{OA} = \bar{a}$ and $\overrightarrow{OB} = \bar{b}$ be two distinct vectors. We construct parallelogram OACB (figure 6.8). The vector along the diagonal from their common initial point $\overrightarrow{OC}$ represents the sum of vectors $\bar{a}$ and $\bar{b}$. Thus $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$. This law is known as the parallelogram law for vector addition.

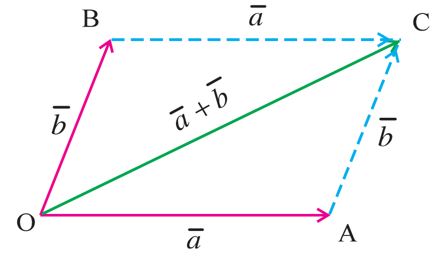


Figure 6.8

> **Note :** $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$ $\quad (\overrightarrow{OB} = \overrightarrow{AC})$
>
> $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$

Properties of Vector Addition (Geometrically) :

Property 1 : For any two vectors $\bar{x}$ and $\bar{y}$, $\bar{x} + \bar{y} = \bar{y} + \bar{x}$ (Commutative property)

Let $\overrightarrow{AB} = \bar{x}$ and $\overrightarrow{AD} = \bar{y}$. We complete the parallelogram ABCD.

$\therefore$ Obviously, $\overrightarrow{BC} = \bar{y}$ and $\overrightarrow{DC} = \bar{x}$ **(By theorem 6.1)**

Now, applying triangle law for ΔABC,

we get $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \bar{x} + \bar{y}$.

Similarly, for ΔADC, $\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$

$\therefore \bar{y} + \bar{x} = \overrightarrow{AC}$.

Thus, $\bar{x} + \bar{y} = \bar{y} + \bar{x}$.

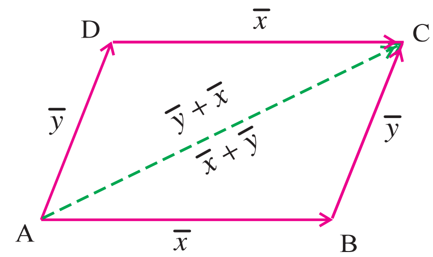
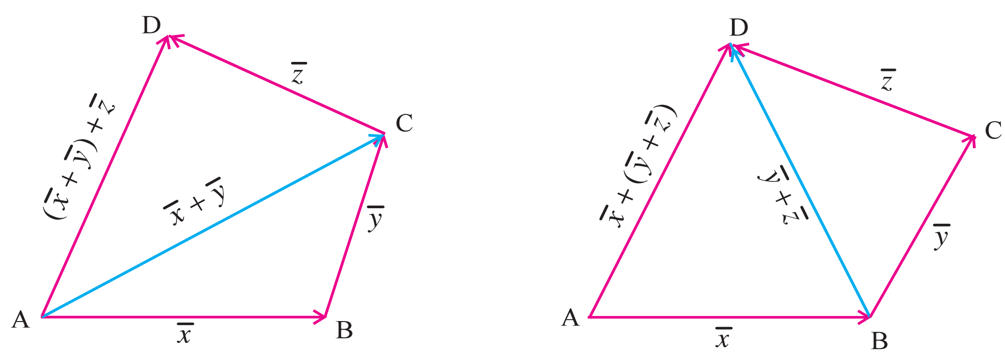


Figure 6.9

Property 2 : For vectors $\bar{x}, \bar{y}, \bar{z}$, $(\bar{x} + \bar{y}) + \bar{z} = \bar{x} + (\bar{y} + \bar{z})$ (Associative property)



(a) Figure 6.10 (b)

Let $\overrightarrow{AB} = \bar{x}$, $\overrightarrow{BC} = \bar{y}$, $\overrightarrow{CD} = \bar{z}$. Using triangle law of addition,

From figure 6.10(a)

From ΔABC,

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\therefore \bar{x} + \bar{y} = \overrightarrow{AC}$.

From ΔACD,

$\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

$\therefore (\bar{x} + \bar{y}) + \bar{z} = \overrightarrow{AD}$.

From figure 6.10(b)

From ΔBCD,

$\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$

$\therefore \bar{y} + \bar{z} = \overrightarrow{BD}$.

From ΔABD,

$\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

$\therefore \bar{x} + (\bar{y} + \bar{z}) = \overrightarrow{AD}$.

Thus, $(\bar{x} + \bar{y}) + \bar{z} = \bar{x} + (\bar{y} + \bar{z})$.

Example 2 : Find the vector having initial point (3, 2, −1) and terminal point (4, −2, 0) and its magnitude.

Solution : A(3, 2, −1) is the initial point and B(4, −2, 0) is the terminal point of $\overrightarrow{AB}$.

$\therefore \quad \overrightarrow{AB}$ = Position vector of B − Position vector of A

= (4, −2, 0) − (3, 2, −1)

= (1, −4, 1)

Magnitude of $\overrightarrow{AB} = |\overrightarrow{AB}| = \sqrt{(1)^2 + (-4)^2 + (1)^2}$

$\therefore \quad AB = \sqrt{18}$

$= 3\sqrt{2}$

Exercise 6.1

1. Find the magnitude of the following vectors :

(1) $(2, 3, \sqrt{3})$ **(2)** $3\hat{i} - 4\hat{k}$ **(3)** $\hat{i} + \hat{j} - 4\hat{k}$

2. Find the unit vector in the direction of $2\hat{i} - 2\hat{j} + \hat{k}$.

3. Find the vector of magnitude $2\sqrt{17}$ in the direction of (3, −2, −2).

4. Find the vector of magnitude 20 in the direction opposite to the direction of vector $-3\hat{i} + 2\sqrt{3}\hat{j} - 2\hat{k}$.

5. For vectors $\overline{x} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\overline{y} = 2\hat{i} + \hat{j}$, find the unit vector in the direction of $\overline{x} + 2\overline{y}$.

6. Find the scalar and vector components of the vector with initial point (−2, 1, 0) and terminal point (1, −5, 7).

7. If the position vector of a point P is (4, 5, −3), then find the distance of P, **(i)** from ZX plane **(ii)** from Y-axis and **(iii)** from the origin.

*

6.6 Inner Product of Vectors in R^2 and R^3

If $\overline{x} = (x_1, x_2)$ and $\overline{y} = (y_1, y_2)$ are vectors in R^2, their inner product is defined as $x_1y_1 + x_2y_2$ and is denoted by $\overline{x} \cdot \overline{y}$.

Similarly, for $\overline{x} = (x_1, x_2, x_3)$ and $\overline{y} = (y_1, y_2, y_3)$ in R^3, $\overline{x} \cdot \overline{y} = x_1y_1 + x_2y_2 + x_3y_3$.

Here, $\overline{x}$ and $\overline{y}$ are vectors, but $\overline{x} \cdot \overline{y}$ is not a vector, it is a real number. Thus inner product of two vectors is a scalar, so the inner product is also called **Scalar Product**. This operation is known as **Scalar Multiplication.** Since notation for inner product is a dot (**.**) between the two vectors, so inner product is also called **Dot Product of Vectors**.

Note : Difference between scalar product and product by a scalar.

Scalar product is performed between two vectors and the result is a scalar quantity and product by a scalar with a vector is a vector quantity.

If $\overline{x} = (2, 3, -1)$ and $\overline{y} = (-1, 4, -2)$, then scalar product of $\overline{x}$ and $\overline{y}$ is

$\overline{x} \cdot \overline{y} = -2 + 12 + 2 = 12$ is a scalar quantity.

While product of $\overline{x} = (2, 3, -1)$ with a scalar, say 2 is $2\overline{x} = 2(2, 3, -1) = (4, 6, -2)$ is a vector quantity.

Properties of Inner Product :

Suppose $\overline{x} = (x_1, x_2, x_3)$, $\overline{y} = (y_1, y_2, y_3)$ and $\overline{z} = (z_1, z_2, z_3)$ are vectors in R^3 and $k \in R$.

(1) $\overline{x} \cdot \overline{x} \geq 0$ and $\overline{x} \cdot \overline{x} = 0 \Leftrightarrow \overline{x} = \overline{0}$.

$\overline{x} \cdot \overline{x} = (x_1, x_2, x_3) \cdot (x_1, x_2, x_3)$

$= x_1^2 + x_2^2 + x_3^2 \geq 0$ **(Property of R)**

$\overline{x} \cdot \overline{x} = 0 \Leftrightarrow x_1 = x_2 = x_3 = 0 \Leftrightarrow \overline{x} = \overline{0}$

(2) $\overline{x} \cdot \overline{x} = |\overline{x}|^2$ as $\overline{x} \cdot \overline{x} = x_1^2 + x_2^2 + x_3^2 = |\overline{x}|^2$

(3) $\overline{x} \cdot \overline{y} = \overline{y} \cdot \overline{x}$

(4) $\overline{x} \cdot (k\overline{y}) = (k\overline{x}) \cdot \overline{y} = k(\overline{x} \cdot \overline{y})$

(5) $\overline{x} \cdot (\overline{y} + \overline{z}) = \overline{x} \cdot \overline{y} + \overline{x} \cdot \overline{z}$

$\overline{x} \cdot (\overline{y} + \overline{z}) = (x_1, x_2, x_3) \cdot (y_1 + z_1, y_2 + z_2, y_3 + z_3)$

$= x_1(y_1 + z_1) + x_2(y_2 + z_2) + x_3(y_3 + z_3)$

$= x_1y_1 + x_1z_1 + x_2y_2 + x_2z_2 + x_3y_3 + x_3z_3$ **(Distributive law in R)**

$= (x_1y_1 + x_2y_2 + x_3y_3) + (x_1z_1 + x_2z_2 + x_3z_3)$

$= \overline{x} \cdot \overline{y} + \overline{x} \cdot \overline{z}$

These properties are also valid for the vectors in R^2.

Example 3 : Find $\overline{x} \cdot \overline{y}$, where $\overline{x} = (1, 2, -1)$, $\overline{y} = (-3, 4, -2)$.

Solution : $\overline{x} \cdot \overline{y} = (1, 2, -1) \cdot (-3, 4, -2)$

$= -3 + 8 + 2$

$= 7$

Example 4 : If $\overline{x} = 5\hat{i} + 4\hat{j} - 3\hat{k}$ and $\overline{y} = 2\hat{i} - \hat{j} + 2\hat{k}$, then find $(\overline{x} + 2\overline{y}) \cdot (2\overline{x} - \overline{y})$.

Solution : $\overline{x} + 2\overline{y} = (5\hat{i} + 4\hat{j} - 3\hat{k}) + 2(2\hat{i} - \hat{j} + 2\hat{k})$

$= 5\hat{i} + 4\hat{j} - 3\hat{k} + 4\hat{i} - 2\hat{j} + 4\hat{k}$

$= 9\hat{i} + 2\hat{j} + \hat{k}$

or $\overline{x} + 2\overline{y} = (5, 4, -3) + 2(2, -1, 2) = (5, 4, -3) + (4, -2, 4) = (9, 2, 1)$

$2\overline{x} - \overline{y} = 2(5\hat{i} + 4\hat{j} - 3\hat{k}) - (2\hat{i} - \hat{j} + 2\hat{k})$

$= 10\hat{i} + 8\hat{j} - 6\hat{k} - 2\hat{i} + \hat{j} - 2\hat{k}$

$= 8\hat{i} + 9\hat{j} - 8\hat{k}$

or $2\overline{x} - \overline{y} = 2(5, 4, -3) - (2, -1, 2) = (10, 8, -6) + (-2, 1, -2) = (8, 9, -8)$

Now, $(\overline{x} + 2\overline{y}) \cdot (2\overline{x} - \overline{y}) = (9\hat{i} + 2\hat{j} + \hat{k}) \cdot (8\hat{i} + 9\hat{j} - 8\hat{k})$

$= (9, 2, 1) \cdot (8, 9, -8)$

$= 72 + 18 - 8$

$= 82$

Outer Product of Vectors in R^3 :

If $\overline{x} = (x_1, x_2, x_3)$ and $\overline{y} = (y_1, y_2, y_3)$ are vectors in R^3, their outer product is denoted by $\overline{x} \times \overline{y}$ and defined as

$$\overline{x} \times \overline{y} = (x_1, x_2, x_3) \times (y_1, y_2, y_3)$$

$$= \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right)$$

$$= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$$

Here, $\overline{x}$ and $\overline{y}$ are vectors and their outer product $\overline{x} \times \overline{y}$ is also a vector. So outer product is also called **Vector Product**. The operation of obtaining outer product is known as **Vector Multiplication**. Since the notation for outer product is a cross (×) between the two vectors, outer product is also called **Cross Product**.

Properties of Outer Product :

(1) $\overline{x} \times \overline{y} = -\overline{y} \times \overline{x}$ **(Interchange of rows in a determinant)**

(2) $\overline{x} \times \overline{x} = \overline{0}$ **(Two identical rows in a determinant)**

(3) $\overline{x} \times (k\overline{y}) = (k\overline{x}) \times \overline{y} = k(\overline{x} \times \overline{y})$

(4) $\overline{x} \times (\overline{y} + \overline{z}) = \overline{x} \times \overline{y} + \overline{x} \times \overline{z}$

(5) $\overline{x} \times \overline{0} = \overline{0} \times \overline{x} = \overline{0}$

Difference Between Inner and Outer Product of Vectors :

(1) Inner product is a scalar quantity, while outer product is a vector quantity.

(2) Inner product is defined in R^2 as well as in R^3, while outer product is not defined in R^2.

(3) Inner product is commutative, while outer product is not commutative.

Note : $\overline{x} \cdot \overline{x} = |\overline{x}|^2$, but $\overline{x} \times \overline{x} = \overline{0}$.

Example 5 : Find $\overline{x} \times \overline{y}$, where $\overline{x} = (1, 3, -2)$ and $\overline{y} = (-2, 1, 5)$

Solution : $\overline{x} \times \overline{y} = \left(\begin{vmatrix} 3 & -2 \\ 1 & 5 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ -2 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} \right)$

$= (15 + 2, -(5 - 4), 1 + 6) = (17, -1, 7)$

Example 6 : If $\overline{x} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\overline{y} = 3\hat{i} - 2\hat{j} + \hat{k}$, find $|\overline{x} \times \overline{y}|$.

Solution : $\overline{x} = (2, 1, -3)$, $\overline{y} = (3, -2, 1)$

$$\overline{x} \times \overline{y} = \left(\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} \right)$$

$= (1 - 6, -(2 + 9), -4 - 3) = (-5, -11, -7)$

$\therefore \quad |\overline{x} \times \overline{y}| = \sqrt{25 + 121 + 49} = \sqrt{195}$

Box Product and Vector Triple Product :

If $\bar{x}$, $\bar{y}$ and $\bar{z}$ are vectors in R^3, the product $\bar{x} \cdot (\bar{y} \times \bar{z})$ is called the box product of $\bar{x}$, $\bar{y}$ and $\bar{z}$, it is denoted by $[\bar{x} \ \ \bar{y} \ \ \bar{z}]$.

Let $\bar{x} = (x_1, x_2, x_3)$, $\bar{y} = (y_1, y_2, y_3)$ and $\bar{z} = (z_1, z_2, z_3)$. Then

$$\bar{x} \cdot (\bar{y} \times \bar{z}) = (x_1, x_2, x_3) \cdot (y_2z_3 - y_3z_2, -(y_1z_3 - y_3z_1), y_1z_2 - y_2z_1)$$

$$\therefore \ [\bar{x} \ \ \bar{y} \ \ \bar{z}] = x_1(y_2z_3 - y_3z_2) - x_2(y_1z_3 - y_3z_1) + x_3(y_1z_2 - y_2z_1)$$

$$\therefore \ [\bar{x} \ \ \bar{y} \ \ \bar{z}] = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Properties of Box Product :

(1) $[\bar{x} \ \ \bar{y} \ \ \bar{z}] = [\bar{y} \ \ \bar{z} \ \ \bar{x}] = [\bar{z} \ \ \bar{x} \ \ \bar{y}]$

Proof : $[\bar{x} \ \ \bar{y} \ \ \bar{z}] = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

$$= -\begin{vmatrix} y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \qquad (R_{12})$$

$$= \begin{vmatrix} y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \end{vmatrix} \qquad (R_{23})$$

$$= [\bar{y} \ \ \bar{z} \ \ \bar{x}]$$

Similarly, we can prove that $[\bar{x} \ \ \bar{y} \ \ \bar{z}] = [\bar{z} \ \ \bar{x} \ \ \bar{y}]$.

(2) $[\bar{x} \ \ \bar{x} \ \ \bar{y}] = 0$, $[\bar{x} \ \ \bar{y} \ \ \bar{x}] = 0$, $[\bar{x} \ \ \bar{y} \ \ \bar{y}] = 0$

(3) $[m\bar{x} \ \ \bar{y} \ \ \bar{z}] = m[\bar{x} \ \ \bar{y} \ \ \bar{z}]$; $[\bar{x} \ \ m\bar{y} \ \ \bar{z}] = m[\bar{x} \ \ \bar{y} \ \ \bar{z}]$; $[\bar{x} \ \ \bar{y} \ \ m\bar{z}] = m[\bar{x} \ \ \bar{y} \ \ \bar{z}]$; $m \in R$

(4) $[\bar{x} \ \ \bar{y} \ \ \bar{0}] = 0$

Note : **(1)** If the vectors are changed in cyclic order, the box product remains unchanged.

(2) Interchange of any two vectors in $[\bar{x} \ \ \bar{y} \ \ \bar{z}]$ results in mere interchange of two rows in the determinant. So the value of the box product will be additive inverse, i.e. $[\bar{x} \ \ \bar{y} \ \ \bar{z}] = -[\bar{y} \ \ \bar{x} \ \ \bar{z}]$.

The product $\bar{x} \times (\bar{y} \times \bar{z})$ is called the vector triple product.

It can be proved that $\bar{x} \times (\bar{y} \times \bar{z}) = (\bar{x} \cdot \bar{z})\bar{y} - (\bar{x} \cdot \bar{y})\bar{z}$.

Similarly $(\bar{x} \times \bar{y}) \times \bar{z} = (\bar{z} \cdot \bar{x})\bar{y} - (\bar{z} \cdot \bar{y})\bar{x}$.

We shall prove the following result :

$\bar{x} \times (\bar{y} \times \bar{z}) = (\bar{x} \cdot \bar{z})\bar{y} - (\bar{x} \cdot \bar{y})\bar{z}$

Proof : Let $\bar{x} = (x_1, x_2, x_3)$, $\bar{y} = (y_1, y_2, y_3)$, $\bar{z} = (z_1, z_2, z_3)$

Then $\bar{x} \times (\bar{y} \times \bar{z}) = (x_1, x_2, x_3) \times (y_2z_3 - y_3z_2, y_3z_1 - y_1z_3, y_1z_2 - y_2z_1)$
$= (p_1, p_2, p_3)$, say

Now, $p_1 = x_2(y_1z_2 - y_2z_1) - x_3(y_3z_1 - y_1z_3)$

$= y_1(x_2z_2 + x_3z_3) - z_1(x_2y_2 + x_3y_3)$

$= y_1(x_1z_1 + x_2z_2 + x_3z_3) - z_1(x_1y_1 + x_2y_2 + x_3y_3)$ **(Adding and subtracting $x_1y_1z_1$)**

$= y_1(\bar{x} \cdot \bar{z}) - z_1(\bar{x} \cdot \bar{y})$

Similarly $p_2 = y_2(\bar{x} \cdot \bar{z}) - z_2(\bar{x} \cdot \bar{y})$ and $p_3 = y_3(\bar{x} \cdot \bar{z}) - z_3(\bar{x} \cdot \bar{y})$

$\therefore \quad \bar{x} \times (\bar{y} \times \bar{z}) = ((\bar{x} \cdot \bar{z})y_1 - (\bar{x} \cdot \bar{y})z_1, (\bar{x} \cdot \bar{z})y_2 - (\bar{x} \cdot \bar{y})z_2, (\bar{x} \cdot \bar{z})y_3 - (\bar{x} \cdot \bar{y})z_3)$

$= (\bar{x} \cdot \bar{z})(y_1, y_2, y_3) - (\bar{x} \cdot \bar{y})(z_1, z_2, z_3)$

$= (\bar{x} \cdot \bar{z})\bar{y} - (\bar{x} \cdot \bar{y})\bar{z}$

Example 7 : Find $[\bar{x} \quad \bar{y} \quad \bar{z}]$, if $\bar{x} = (1, 2, 0)$, $\bar{y} = (3, -1, 2)$, $\bar{z} = (1, 1, 1)$.

Solution : $[\bar{x} \quad \bar{y} \quad \bar{z}] = \begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix}$

$= 1(-3) - 2(1) + 0$

$= -5$

Example 8 : Find $\bar{x} \times (\bar{y} \times \bar{z})$, if $\bar{x} = (1, 2, 3)$, $\bar{y} = (2, 3, 5)$, $\bar{z} = (1, -1, -1)$.

Solution : Method 1 : $\bar{x} \cdot \bar{z} = (1, 2, 3) \cdot (1, -1, -1) = 1 - 2 - 3 = -4$

$\bar{x} \cdot \bar{y} = (1, 2, 3) \cdot (2, 3, 5) = 2 + 6 + 15 = 23$

$\bar{x} \times (\bar{y} \times \bar{z}) = (\bar{x} \cdot \bar{z})\bar{y} - (\bar{x} \cdot \bar{y})\bar{z}$

$= -4(2, 3, 5) - 23(1, -1, -1)$

$= (-8, -12, -20) + (-23, 23, 23)$

$= (-31, 11, 3)$

Method 2 : $\bar{y} = (2, 3, 5)$ and

$\bar{z} = (1, -1, -1)$

$\therefore \quad \bar{y} \times \bar{z} = (-3 + 5, -(-2 - 5), -2 - 3) = (2, 7, -5)$

$\therefore \quad \bar{x} = (1, 2, 3)$ and

$\bar{y} \times \bar{z} = (2, 7, -5)$

$\bar{x} \times (\bar{y} \times \bar{z}) = (-10 - 21, -(-5 - 6), 7 - 4) = (-31, 11, 3)$

Example 9 : $\forall \bar{x}, \bar{y}, \bar{z} \in R^3$, prove that, $[(\bar{x} + \bar{y}) \times (\bar{y} + \bar{z})] \cdot (\bar{x} + \bar{z}) = 2[\bar{x} \quad \bar{y} \quad \bar{z}]$

Solution : L.H.S. $= [(\bar{x} + \bar{y}) \times (\bar{y} + \bar{z})] \cdot (\bar{x} + \bar{z})$

$= [\bar{x} \times \bar{y} + \bar{x} \times \bar{z} + \bar{y} \times \bar{y} + \bar{y} \times \bar{z}] \cdot (\bar{x} + \bar{z})$

$= [\bar{x} \times \bar{y} + \bar{x} \times \bar{z} + \bar{y} \times \bar{z}] \cdot (\bar{x} + \bar{z})$ $\quad (\bar{y} \times \bar{y} = \bar{0})$

$= (\bar{x} \times \bar{y}) \cdot \bar{x} + (\bar{x} \times \bar{y}) \cdot \bar{z} + (\bar{x} \times \bar{z}) \cdot \bar{x} + (\bar{x} \times \bar{z}) \cdot \bar{z} + (\bar{y} \times \bar{z}) \cdot \bar{x} + (\bar{y} \times \bar{z}) \cdot \bar{z}$

$= [\bar{x} \quad \bar{y} \quad \bar{x}] + [\bar{x} \quad \bar{y} \quad \bar{z}] + [\bar{x} \quad \bar{z} \quad \bar{x}] + [\bar{x} \quad \bar{z} \quad \bar{z}] + [\bar{y} \quad \bar{z} \quad \bar{x}] + [\bar{y} \quad \bar{z} \quad \bar{z}]$

$= 0 + [\bar{x} \quad \bar{y} \quad \bar{z}] + 0 + 0 + [\bar{x} \quad \bar{y} \quad \bar{z}] + 0$ $\quad ([\bar{y} \quad \bar{z} \quad \bar{x}] = [\bar{x} \quad \bar{y} \quad \bar{z}])$

$= 2[\bar{x} \quad \bar{y} \quad \bar{z}] =$ R.H.S.

Exercise 6.2

Find the vector or scalar as required :

1. $(2, 3, 1) \cdot (2, -1, 4)$
2. $(1, -1, 2) \times (2, 3, 1)$
3. $(2, -1, -2) \times (4, 1, 8)$
4. $|(2, 1, 3) \times (0, -4, -4)|$
5. $|(3, -4, -1) \cdot (1, 2, -2)|$
6. $(1, 1, 2) \times [(1, 2, 1) \times (2, 1, 1)]$
7. $(1, 0, 1) \cdot [(1, 1, 0) \times (1, 0, -1)]$
8. $(2, 3, 4) \cdot [(1, 1, 1) \times (3, 4, 5)]$
9. $[(1, 5, 1) \times (2, -1, 2)] \times (4, 1, -3)$
10. $|[(2, 3, 4) \cdot (-4, 3, -2)]\,(1, -1, 2)|$

*

6.7 Lagrange's Identity

If $x_1, x_2, x_3, y_1, y_2, y_3 \in R$, then

$$(x_1y_1 + x_2y_2 + x_3y_3)^2 + (x_1y_2 - x_2y_1)^2 + (x_1y_3 - x_3y_1)^2 + (x_2y_3 - x_3y_2)^2 =$$
$$(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) \quad \text{(Verify !)}$$

This identity is known as Lagrange's identity.

If we take $\overline{x} = (x_1, x_2, x_3)$ and $\overline{y} = (y_1, y_2, y_3)$, then vector form of Lagrange's identity is

$$|\overline{x} \cdot \overline{y}|^2 + |\overline{x} \times \overline{y}|^2 = |\overline{x}|^2\,|\overline{y}|^2.$$

because $\overline{x} \cdot \overline{y} = x_1y_1 + x_2y_2 + x_3y_3$, $\overline{x} \times \overline{y} = (x_2y_3 - x_3y_2, -(x_1y_3 - x_3y_1), x_1y_2 - x_2y_1)$

$|\overline{x}|^2 = x_1^2 + x_2^2 + x_3^2$ and $|\overline{y}|^2 = y_1^2 + y_2^2 + y_3^2$.

Example 10 : If $\overline{x}$ and $\overline{y}$ are unit vectors and $\overline{x} \cdot \overline{y} = 0$, then prove that $\overline{x} \times \overline{y}$ is a unit vector.

Solution : $\overline{x}$ and $\overline{y}$ are unit vectors.

$\therefore \quad |\overline{x}| = 1 = |\overline{y}|$

Using Lagrange's identity,

$|\overline{x} \times \overline{y}|^2 + |\overline{x} \cdot \overline{y}|^2 = |\overline{x}|^2\,|\overline{y}|^2.$

$\therefore \quad |\overline{x} \times \overline{y}|^2 + 0 = (1)(1)$

$\therefore \quad |\overline{x} \times \overline{y}| = 1$

$\therefore \quad \overline{x} \times \overline{y}$ is a unit vector.

Cauchy-Schwartz Inequality :

For any two vectors $\overline{x}$ and $\overline{y}$ of R^2 or R^3, $|\overline{x} \cdot \overline{y}| \le |\overline{x}|\,|\overline{y}|$.

This inequality is known as Cauchy - Schwartz inequality.

In R^3, according to the Lagrange's identity,

$|\overline{x} \times \overline{y}|^2 + |\overline{x} \cdot \overline{y}|^2 = |\overline{x}|^2\,|\overline{y}|^2.$

$\therefore \quad |\overline{x} \cdot \overline{y}|^2 \le |\overline{x}|^2\,|\overline{y}|^2 \qquad (|\overline{x} \times \overline{y}|^2 \ge 0)$

$\therefore \quad |\overline{x} \cdot \overline{y}| \le |\overline{x}|\,|\overline{y}|$

For R^2, let $\overline{x} = (x_1, x_2)$ and $\overline{y} = (y_1, y_2)$

So, $\overline{x} \cdot \overline{y} = x_1y_1 + x_2y_2$

Now, $(x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2 = (x_1^2 + x_2^2)\cdot(y_1^2 + y_2^2)$ **(Verify !)**

$\therefore \quad |x_1y_1 + x_2y_2|^2 \leq (x_1^2 + x_2^2)\cdot(y_1^2 + y_2^2)$ **($(x_1y_2 - x_2y_1)^2 \geq 0$)**

$\therefore \quad |\bar{x}\cdot\bar{y}|^2 \leq |\bar{x}|^2|\bar{y}|^2$ and hence $|\bar{x}\cdot\bar{y}| \leq |\bar{x}||\bar{y}|$.

Second Proof : This is valid for R^2 and R^3.

If $\bar{x} = \bar{0}$ or $\bar{y} = \bar{0}$, then $\bar{x}\cdot\bar{y} = 0$ and $|\bar{x}||\bar{y}| = 0$

So $|\bar{x}\cdot\bar{y}| = |\bar{x}||\bar{y}|$

Let $\bar{x} \neq \bar{0}$ and $\bar{y} \neq \bar{0}$

Suppose $|\bar{x}| = 1$ and $|\bar{y}| = 1$.

Now, $(\bar{x} - \bar{y})\cdot(\bar{x} - \bar{y}) \geq 0$

$\therefore \quad \bar{x}\cdot\bar{x} - 2\bar{x}\cdot\bar{y} + \bar{y}\cdot\bar{y} \geq 0$

$\therefore \quad |\bar{x}|^2 - 2\bar{x}\cdot\bar{y} + |\bar{y}|^2 \geq 0$

$\therefore \quad 2 - 2\bar{x}\cdot\bar{y} \geq 0$ **($|\bar{x}| = |\bar{y}| = 1$)**

Hence, $\bar{x}\cdot\bar{y} \leq 1$

Similarly, $(\bar{x} + \bar{y})\cdot(\bar{x} + \bar{y}) \geq 0$

$\therefore \quad |\bar{x}|^2 + 2\bar{x}\cdot\bar{y} + |\bar{y}|^2 \geq 0$

$\therefore \quad 2 + 2\bar{x}\cdot\bar{y} \geq 0$

$\therefore \quad -1 \leq \bar{x}\cdot\bar{y}$

Thus, $-1 \leq \bar{x}\cdot\bar{y} \leq 1$

$\therefore \quad |\bar{x}\cdot\bar{y}| \leq 1$

$\therefore \quad |\bar{x}\cdot\bar{y}| \leq |\bar{x}||\bar{y}|$ **($|\bar{x}| = 1 = |\bar{y}|$) (i)**

Finally, let $\bar{x} \neq \bar{0}$ and $\bar{y} \neq \bar{0}$, so $|\bar{x}| \neq 0$, $|\bar{y}| \neq 0$

Let $\bar{u} = \frac{\bar{x}}{|\bar{x}|}$, $\bar{v} = \frac{\bar{y}}{|\bar{y}|}$. Then $|\bar{u}| = 1 = |\bar{v}|$

So by (i), $|\bar{u}\cdot\bar{v}| \leq |\bar{u}||\bar{v}|$

$$\therefore \quad \left|\frac{\bar{x}}{|\bar{x}|}\cdot\frac{\bar{y}}{|\bar{y}|}\right| \leq \left|\frac{\bar{x}}{|\bar{x}|}\right|\left|\frac{\bar{y}}{|\bar{y}|}\right| = \frac{|\bar{x}|}{|\bar{x}|}\frac{|\bar{y}|}{|\bar{y}|} = 1$$

$\therefore \quad |\bar{x}\cdot\bar{y}| \leq |\bar{x}||\bar{y}|$

For non-zero vectors $\bar{x}$ and $\bar{y}$,

if $\bar{x}\cdot\bar{y} = |\bar{x}||\bar{y}|$, then

$$\begin{aligned} |t\bar{x} - \bar{y}|^2 &= (t\bar{x} - \bar{y})\cdot(t\bar{x} - \bar{y}) \\ &= t^2|\bar{x}|^2 - 2t\bar{x}\cdot\bar{y} + |\bar{y}|^2 \\ &= t^2|\bar{x}|^2 - 2t|\bar{x}||\bar{y}| + |\bar{y}|^2 \\ &= (t|\bar{x}| - |\bar{y}|)^2 \end{aligned}$$

($\bar{x}\cdot\bar{y} = |\bar{x}||\bar{y}|$)

Taking $t = \frac{|\bar{y}|}{|\bar{x}|}$ **($|\bar{x}| \neq 0$)**

$|t\bar{x} - \bar{y}|^2 = 0$

$\therefore \quad t\bar{x} = \bar{y}$

$\therefore \quad \bar{y} = t\bar{x}$ **($t > 0$)**

$\therefore \quad \bar{x}$ and $\bar{y}$ are in the same direction.

If $\bar{x} \cdot \bar{y} = -|\bar{x}||\bar{y}|$, then $(|t\bar{x} - \bar{y}|)^2 = (t|\bar{x}| + |\bar{y}|)^2$

Now taking $t = -\frac{|\bar{y}|}{|\bar{x}|}$, we get

$t\bar{x} - \bar{y} = \bar{0}$

$\therefore \quad \bar{y} = t\bar{x}$ **($t < 0$)**

$\therefore \quad \bar{x}$ and $\bar{y}$ are in the opposite direction.

$\therefore$ In Cauchy-Schwartz inequality, if $|\bar{x} \cdot \bar{y}| = |\bar{x}||\bar{y}|$, for non-zero vectors $\bar{x}$ and $\bar{y}$, then $\bar{x}$ and $\bar{y}$ are in the same or in the opposite direction.

Triangle Inequality :

For vectors $\bar{x}$, $\bar{y}$ in R^2 as well as in R^3, $|\bar{x} + \bar{y}| \le |\bar{x}| + |\bar{y}|$.

Proof : $|\bar{x} + \bar{y}|^2 = (\bar{x} + \bar{y}) \cdot (\bar{x} + \bar{y})$

$= \bar{x} \cdot \bar{x} + \bar{x} \cdot \bar{y} + \bar{y} \cdot \bar{x} + \bar{y} \cdot \bar{y}$

$= |\bar{x}|^2 + 2\bar{x} \cdot \bar{y} + |\bar{y}|^2$ **($\bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x}$)**

$\le |\bar{x}|^2 + 2|\bar{x} \cdot \bar{y}| + |\bar{y}|^2$ **($\forall a \in R,\ a \le |a|$)**

$\le |\bar{x}|^2 + 2|\bar{x}||\bar{y}| + |\bar{y}|^2$ **(Cauchy-Schwartz Inequality)**

$\le (|\bar{x}| + |\bar{y}|)^2$

$\therefore \quad |\bar{x} + \bar{y}| \le |\bar{x}| + |\bar{y}|$

Geometric Interpretation :

Let $P(\bar{x})$ and $Q(\bar{y})$ be two distinct points. In figure 6.11, $\square$OPRQ is a parallelogram whose sides $\overline{OP}$ and $\overline{OQ}$ represent two vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ respectively. By the parallelogram law of vector addition,

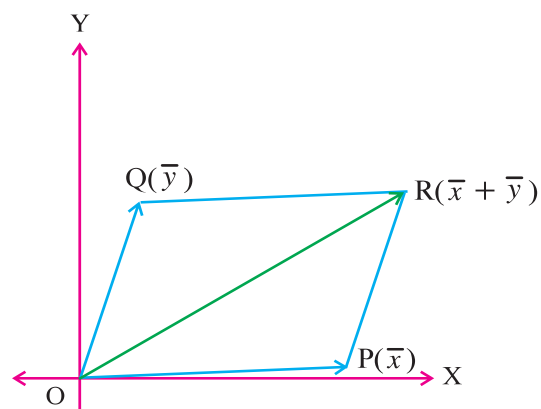


Figure 6.11

$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR}$

In ΔOPR, OP + PR > OR

$\therefore$ OP + OQ > OR **(Opposite sides of a parallelogram are congruent)**

$\therefore \quad |\bar{x}| + |\bar{y}| > |\bar{x} + \bar{y}|$

If O, P, Q are collinear and O–P–Q or O–Q–P (See figure 6.12),

then OP + OQ = OR

$\therefore \quad |\bar{x}| + |\bar{y}| = |\bar{x} + \bar{y}|$

Also, if O–P–Q or O–Q–P is not the case and O, P, Q are collinear, then OP + OQ > OR.

Thus $|\bar{x}| + |\bar{y}| > |\bar{x} + \bar{y}|$

$\therefore \quad |\bar{x} + \bar{y}| \le |\bar{x}| + |\bar{y}|$

In all cases $|\bar{x} + \bar{y}| \le |\bar{x}| + |\bar{y}|$.

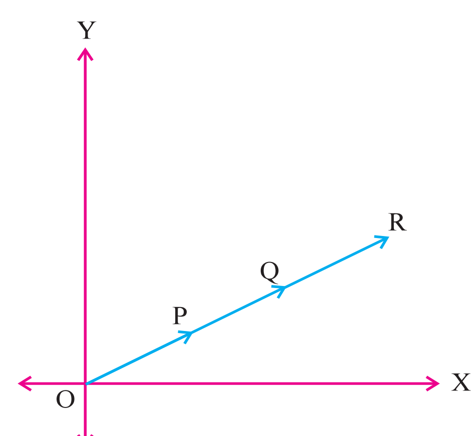


Figure 6.12

6.8 Collinear and Coplanar Vectors

We know that, if $\overline{x} \neq \overline{0}$, $\overline{y} \neq \overline{0}$ and if $\overline{x} = k\overline{y}$, $k \neq 0$ then $\overline{x}$ and $\overline{y}$ have same or opposite directions. If two vectors have same or opposite directions, then they are called collinear vectors. Free vectors equivalent to the same bound vector or a non-zero multiple of it are conventionally called parallel vectors. If the bound vectors are not collinear, their directions are different. Hence either two bound vectors are collinear or have different directions. They can not be parallel.

Theorem 6.4 : Non-zero vectors $\overline{x} = (x_1, x_2)$ and $\overline{y} = (y_1, y_2)$ of R^2 are collinear if and only if $x_1y_2 - x_2y_1 = 0$.

Proof : $\overline{x}$ and $\overline{y}$ are collinear $\Rightarrow \overline{x} = k\overline{y}$, $k \in R - \{0\}$, $\overline{x} \neq \overline{0}$, $\overline{y} \neq \overline{0}$

$$\Rightarrow (x_1, x_2) = k(y_1, y_2)$$

$$\therefore \quad x_1 = ky_1, x_2 = ky_2$$

$$\therefore \quad x_1y_2 - x_2y_1 = ky_1y_2 - ky_2y_1 = 0$$

Conversely, let $x_1y_2 - x_2y_1 = 0$

$\therefore \quad x_1y_2 = x_2y_1$

Let $y_1 \neq 0$, $y_2 \neq 0$

Then $\frac{x_1}{y_1} = \frac{x_2}{y_2} = k$, say.

If $k = 0$, then $x_1 = 0$, $x_2 = 0$. So $\overline{x} = \overline{0}$, But $\overline{x} \neq \overline{0}$. So $k \neq 0$.

$\therefore \quad \overline{x} = (x_1, x_2) = (ky_1, ky_2) = k(y_1, y_2) = k\overline{y}$, $k \in R - \{0\}$

If $y_1 = 0$ or $y_2 = 0$, (both cannot be zero as $\overline{y} \neq \overline{0}$),

let for definiteness $y_2 = 0$, $y_1 \neq 0$

$\therefore \quad x_1y_2 = 0$

$\therefore \quad x_2y_1 = 0$ **$(x_1y_2 = x_2y_1)$**

$\therefore \quad x_2 = 0$ as $y_1 \neq 0$

Let $\frac{x_1}{y_1} = k$

$\therefore \quad (x_1, x_2) = (ky_1, 0) = (ky_1, ky_2)$ **$(y_2 = 0)$**

$$= k(y_1, y_2)$$

Again $k = 0 \Rightarrow x_1 = 0$, $x_2 = 0$. So $\overline{x} = \overline{0}$, But $\overline{x} \neq \overline{0}$.

$\therefore \quad \overline{x} = k\overline{y}$, $k \in R - \{0\}$

$\therefore \quad$ If $x_1y_2 - x_2y_1 = 0$, then for $k \in R - \{0\}$, $\overline{x} = k\overline{y}$ and hence $\overline{x}$ and $\overline{y}$ are collinear.

(1) $|\overline{x} \cdot \overline{y}| = |\overline{x}||\overline{y}|$, if and only if $\overline{x} = k\overline{y}$, $k \in R - \{0\}$, $\overline{x} \neq \overline{0}$, $\overline{y} \neq \overline{0}$

Proof : Let $\overline{x} = k\overline{y}$, $k \in R - \{0\}$

$\therefore \quad |\overline{x} \cdot \overline{y}| = |(k\overline{y}) \cdot \overline{y}| = |k(\overline{y} \cdot \overline{y})|$

$$= |k| \, |\overline{y} \cdot \overline{y}|$$

$= |k| \, |\overline{y}|^2$

$= |k| \, |\overline{y}| \, |\overline{y}|$

$= |k\overline{y}| \, |\overline{y}|$

$= |\overline{x}| |\overline{y}|$

Conversely, let $|\overline{x} \cdot \overline{y}| = |\overline{x}| |\overline{y}|$.

Now, vector form of Lagrange's identity is

$|\overline{x} \times \overline{y}|^2 + |\overline{x} \cdot \overline{y}|^2 = |\overline{x}|^2 |\overline{y}|^2$

$\therefore \quad |\overline{x} \times \overline{y}|^2 = 0$ **($|\overline{x} \cdot \overline{y}| = |\overline{x}| |\overline{y}|$)**

$\therefore \quad \overline{x} \times \overline{y} = \overline{0}$

We can prove that $\overline{x} = k\overline{y} \quad k \in R - \{0\}$. **(See exercise 6)**

Thus $|\overline{x} \cdot \overline{y}| < |\overline{x}| |\overline{y}|$ if and only if $\overline{x} \neq k\overline{y}$, for any $k \in R - \{0\}$, $\overline{x} \neq \overline{0}$, $\overline{y} \neq \overline{0}$

(2) $|\overline{x} + \overline{y}| = |\overline{x}| + |\overline{y}|$, if and only if $\overline{x} = k\overline{y}$, $k > 0$, $\overline{x} \neq \overline{0}$, $\overline{y} \neq \overline{0}$

i.e. $\overline{x}$ and $\overline{y}$ have the same direction.

Proof : Let $\overline{x} = k\overline{y}$, $k > 0$.

$\therefore \quad |\overline{x} + \overline{y}| = |(k\overline{y}) + \overline{y}| = |(k + 1)\overline{y}| = |k + 1| \, |\overline{y}|$

$= (k + 1) |\overline{y}|$ **($k > 0$)**

$= k|\overline{y}| + |\overline{y}|$

$= |k| |\overline{y}| + |\overline{y}|$ **($k > 0$)**

$= |k\overline{y}| + |\overline{y}|$

$= |\overline{x}| + |\overline{y}|$

Conversely, let $|\overline{x} + \overline{y}| = |\overline{x}| + |\overline{y}|$

$|\overline{x} + \overline{y}|^2 = (|\overline{x}| + |\overline{y}|)^2$

$\therefore \quad (\overline{x} + \overline{y}) \cdot (\overline{x} + \overline{y}) = |\overline{x}|^2 + 2|\overline{x}| \, |\overline{y}| + |\overline{y}|^2$

$\therefore \quad |\overline{x}|^2 + 2\overline{x} \cdot \overline{y} + |\overline{y}|^2 = |\overline{x}|^2 + 2|\overline{x}| \, |\overline{y}| + |\overline{y}|^2$

$\therefore \quad \overline{x} \cdot \overline{y} = |\overline{x}| \, |\overline{y}|$

$\therefore$ From the equality in Cauchy-Schwartz inequality, $\overline{x} = k\overline{y}$, $k > 0$.

$\therefore \quad \overline{x}$ and $\overline{y}$ are in the same direction.

Theorem 6.5 : Non-zero vectors $\overline{x}$ and $\overline{y}$ of R^3 are collinear if and only if $\overline{x} \times \overline{y} = \overline{0}$.

Proof : Since, $\overline{x}$ and $\overline{y}$ are collinear $\overline{x} = k\overline{y}$, $k \in R - \{0\}$, $\overline{x} \neq \overline{0}$, $\overline{y} \neq \overline{0}$

$\therefore \quad \overline{x} \times \overline{y} = (k\overline{y} \times \overline{y}) = k(\overline{y} \times \overline{y}) = k\overline{0} = \overline{0}$

Conversely, let $\overline{x} \times \overline{y} = \overline{0}$.

$\therefore \quad |\overline{x} \cdot \overline{y}| = |\overline{x}| \, |\overline{y}|$ **(Lagrange's identity)**

$\therefore$ Cauchy Schwarz inequality gives $\bar{x} = k\bar{y}$, $k \in R - \{0\}$ as $\bar{x} \neq \bar{0}$.

$\therefore$ $\bar{x}, \bar{y}$ are collinear.

Coplanar Vectors : Let $\bar{x}$, $\bar{y}$ and $\bar{z}$ be vectors of R^3. If we can find $\alpha, \beta, \gamma \in R$ with at least one of them non-zero, such that $\alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} = \bar{0}$, then $\bar{x}$, $\bar{y}$ and $\bar{z}$ are said to be coplanar vectors.

If $\bar{x}$, $\bar{y}$, $\bar{z}$ are not coplanar, they are called non-coplanar or linearly independent vectors. Thus if $\bar{x}$, $\bar{y}$ and $\bar{z}$ are non-coplanar vectors, then

$\alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} = \bar{0} \Rightarrow \alpha = 0, \beta = 0$ and $\gamma = 0$.

Theorem 6.6 : Distinct non-zero vectors $\bar{x}$, $\bar{y}$, $\bar{z}$ of R^3 are coplanar if and only if $[\bar{x} \;\; \bar{y} \;\; \bar{z}] = 0$.

Proof : Suppose $\bar{x}$, $\bar{y}$, $\bar{z}$ are coplanar.

$\therefore$ We can find α, β, γ with at least one non-zero in R such that $\alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} = \bar{0}$.

Let us assume that $\gamma \neq 0$

$\therefore$ $\bar{z} = \left(\frac{-\alpha}{\gamma}\right)\bar{x} + \left(\frac{-\beta}{\gamma}\right)\bar{y}$

$$\therefore [\bar{x} \;\; \bar{y} \;\; \bar{z}] = (\bar{x} \times \bar{y}) \cdot \bar{z} = (\bar{x} \times \bar{y}) \cdot \left[\left(\frac{-\alpha}{\gamma}\right)\bar{x} + \left(\frac{-\beta}{\gamma}\right)\bar{y}\right]$$

$$= (\bar{x} \times \bar{y}) \cdot \left(\frac{-\alpha}{\gamma}\right)\bar{x} + (\bar{x} \times \bar{y}) \cdot \left(\frac{-\beta}{\gamma}\right)\bar{y}$$

$$= \left(\frac{-\alpha}{\gamma}\right)((\bar{x} \times \bar{y}) \cdot \bar{x}) + \left(\frac{-\beta}{\gamma}\right)((\bar{x} \times \bar{y}) \cdot \bar{y})$$

$$= 0 + 0 = 0$$

$\therefore$ $[\bar{x} \;\; \bar{y} \;\; \bar{z}] = 0$

Conversely, suppose $[\bar{x} \;\; \bar{y} \;\; \bar{z}] = 0$.

$\therefore$ $\bar{x} \cdot (\bar{y} \times \bar{z}) = 0$

If $\bar{y} \times \bar{z} = \bar{0}$, then $\bar{y}$ and $\bar{z}$ are collinear.

$\therefore$ $\bar{y} = k\bar{z}$, $k \neq 0$

$\therefore$ $0\bar{x} + 1\bar{y} - k\bar{z} = \bar{0}$

Comparing it with $\alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} = \bar{0}$, $\alpha = 0$, $\beta = 1$ and $\gamma = -k \neq 0$

$\therefore$ $\bar{x}$, $\bar{y}$, $\bar{z}$ are coplanar.

Now suppose $\bar{y} \times \bar{z} \neq \bar{0}$.

$\therefore$ At least one of the numbers $y_1z_2 - y_2z_1$, $y_2z_3 - y_3z_2$ and $y_1z_3 - y_3z_1$ is non-zero.

Assume that $y_1z_2 - y_2z_1 \neq 0$

Now, we will prove $\bar{x} - \alpha\bar{y} - \beta\bar{z} = \bar{0}$ for some $\alpha, \beta \in R$ **(i)**

Consider the equations $\alpha y_1 + \beta z_1 - x_1 = 0$ **(ii)**

$\alpha y_2 + \beta z_2 - x_2 = 0$ **(iii)**

and $\alpha y_3 + \beta z_3 - x_3 = 0$ **(iv)**

Since $y_1z_2 - y_2z_1 \neq 0$, we can solve (ii) and (iii) to find α and β and these α and β satisfy (iv) as $[\bar{x}\ \bar{y}\ \bar{z}] = 0$.

$\therefore$ We can find $\alpha, \beta \in$ R such that $\alpha\bar{y} + \beta\bar{z} = \bar{x}$.

Here $1\bar{x} - \alpha\bar{y} - \beta\bar{z} = \bar{0}$

Also $1 \neq 0$.

$\therefore$ $\bar{x} - \alpha\bar{y} - \beta\bar{z} = \bar{0}$ with at least one coefficient $1 \neq 0$.

Thus, $\bar{x}, \bar{y}, \bar{z}$ are coplanar.

Example 11 : Prove that $(-1, 0, -1)$, $(0, -1, 1)$ and $(-1, 1, 0)$ are non-coplanar and that every $\bar{x} \in R^3$ can be written as $\bar{x} = \alpha(-1, 0, -1) + \beta(0, -1, 1) + \gamma(-1, 1, 0)$ for some real numbers α, β and γ.

Solution : $\begin{vmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = -1(-1) + 0 - 1(-1) = 2 \neq 0$

$\therefore$ Vectors $(-1, 0, -1)$, $(0, -1, 1)$ and $(-1, 1, 0)$ are non-coplanar.

Now, let $\bar{x} = \alpha(-1, 0, -1) + \beta(0, -1, 1) + \gamma(-1, 1, 0)$ for some $\alpha, \beta, \gamma \in$ R,

where $\bar{x} = (x_1, x_2, x_3)$.

$\therefore$ $(x_1, x_2, x_3) = (-\alpha - \gamma, -\beta + \gamma, -\alpha + \beta)$

$-\alpha - \gamma = x_1, \quad -\beta + \gamma = x_2, \quad -\alpha + \beta = x_3$

Solving them, we get

$$\alpha = -\frac{x_1 + x_2 + x_3}{2},\ \beta = \frac{x_3 - x_1 - x_2}{2},\ \gamma = \frac{x_2 + x_3 - x_1}{2}$$

$$\therefore \quad \bar{x} = -\frac{x_1 + x_2 + x_3}{2}(-1, 0, -1) + \frac{x_3 - x_1 - x_2}{2}(0, -1, 1) + \frac{x_2 + x_3 - x_1}{2}(-1, 1, 0)$$

Example 12 : Give one example of vectors $\bar{x}$ and $\bar{y}$ such that $|\bar{x} \cdot \bar{y}| < |\bar{x}||\bar{y}|$.

Solution : Let, $\bar{x} = (1, -1, 2)$ and $\bar{y} = (2, 1, -2)$ **(choose $\bar{x} \neq k\bar{y}$)**

$\bar{x} \cdot \bar{y} = 2 - 1 - 4 = -3$

$\therefore$ $|\bar{x} \cdot \bar{y}| = 3$ **(i)**

$|\bar{x}||\bar{y}| = \sqrt{6} \cdot \sqrt{9}$

$= 3\sqrt{6}$ **(ii)**

From results (i) and (ii), we have $|\bar{x} \cdot \bar{y}| < |\bar{x}||\bar{y}|$, since $3 < 3\sqrt{6}$.

Example 13 : When is $|\bar{x} + \bar{y}| = |\bar{x}| + |\bar{y}|$? Verify your answer by giving one example of $\bar{x}$ and $\bar{y}$.

Solution : If $\bar{x}$ and $\bar{y}$ are in the same direction, then $|\bar{x} + \bar{y}| = |\bar{x}| + |\bar{y}|$.

Let $\bar{x} = (1, -1, 1)$ and $\bar{y} = (2, -2, 2)$

Here, $\bar{x} = \frac{1}{2}\bar{y}$; $\frac{1}{2} > 0$, so $\bar{x}$ and $\bar{y}$ are in the same direction.

Now, $\bar{x} + \bar{y} = (3, -3, 3)$

$\therefore$ $|\bar{x} + \bar{y}| = 3\,|(1, -1, 1)| = 3\sqrt{3}$

$\therefore$ $|\bar{x} + \bar{y}| = 3\sqrt{3}$ **(i)**

$|\bar{x}| = \sqrt{3}$, $|\bar{y}| = 2\sqrt{3}$

$\therefore$ $|\bar{x}| + |\bar{y}| = \sqrt{3} + 2\sqrt{3} = 3\sqrt{3}$

Hence, $|\bar{x} + \bar{y}| = |\bar{x}| + |\bar{y}|$.

6.9 Angle Between Two Non-zero Vectors

If two non-zero vectors in R^3 are given, then the measure of the angle between their corresponding bound vectors is defined as the measure of the angle between the given vectors.

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be the corresponding bound vectors of $\bar{a}$ and $\bar{b}$ respectively. The measure of the angle between $\bar{a}$ and $\bar{b}$ is the measure of the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

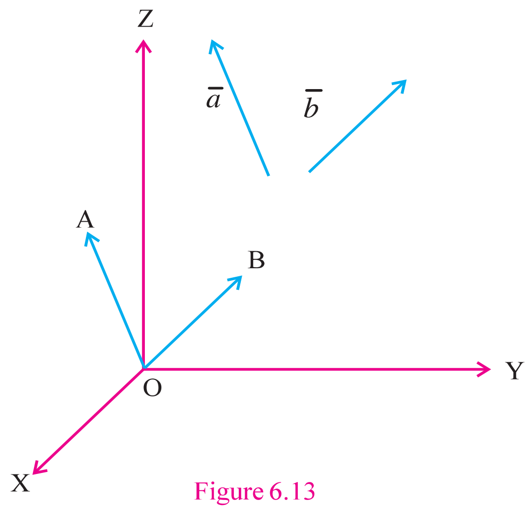


Figure 6.13

Let $\bar{x}$ and $\bar{y}$ be two non-zero vectors.

(1) **If $\bar{x} = k\bar{y}$, $k > 0$, then $\bar{x}$ and $\bar{y}$ have the same directions and so the measure of the angle between them is defined to be 0.**

(2) **If $\bar{x} = k\bar{y}$, $k < 0$, then $\bar{x}$ and $\bar{y}$ have opposite directions and so the measure of the angle between them is defined to be π.**

(3) Now, suppose that $\bar{x}$ and $\bar{y}$ have different directions. So by Cauchy-Schwartz inequality, $|\bar{x} \cdot \bar{y}| < |\bar{x}||\bar{y}|$.

$\therefore \quad -|\bar{x}||\bar{y}| < \bar{x} \cdot \bar{y} < |\bar{x}||\bar{y}|$ **($|x| < a \Leftrightarrow -a < x < a$)**

$\therefore \quad -1 < \dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} < 1$

$\therefore$ **There is a unique $\alpha \in (0, \pi)$ such that,**

$$cos^{-1} \frac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = \alpha$$

The number α is defined to be the measure of the angle between $\bar{x}$ and $\bar{y}$. It is denoted by $\alpha = (\bar{x}, \hat{\ } \bar{y})$.

Thus $(\bar{x}, \hat{\ } \bar{y}) = cos^{-1} \dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|}$ if $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$.

Also, if $|\bar{x} \cdot \bar{y}| = |\bar{x}||\bar{y}|$, then $\bar{x} \cdot \bar{y} = |\bar{x}||\bar{y}|$ or $\bar{x} \cdot \bar{y} = -|\bar{x}||\bar{y}|$. The directions of $\bar{x}$ and $\bar{y}$ are same or opposite respectively. Hence respective measure of the angle between $\bar{x}$ and $\bar{y}$ is 0 or π.

Let us justify.

If $\bar{x}$ and $\bar{y}$ have same direction, then $\bar{x} = k\bar{y}$, $k > 0$.

Now $\dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = \dfrac{(k\bar{y}) \cdot \bar{y}}{|k\bar{y}||\bar{y}|} = \dfrac{k(\bar{y} \cdot \bar{y})}{|k||\bar{y}||\bar{y}|} = \dfrac{k|\bar{y}|^2}{k|\bar{y}|^2} = 1$ **($k > 0$)**

$\therefore \quad cos^{-1} \dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = cos^{-1} 1 = 0$

If $\bar{x}$ and $\bar{y}$ have opposite directions, then $\bar{x} = k\bar{y}$, $k < 0$.

Now $\dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = \dfrac{(k\bar{y}) \cdot \bar{y}}{|k\bar{y}||\bar{y}|} = \dfrac{k(\bar{y} \cdot \bar{y})}{|k||\bar{y}||\bar{y}|} = \dfrac{k|\bar{y}|^2}{-k|\bar{y}|^2} = -1$ **($k < 0$)**

$\therefore \quad cos^{-1} \dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = cos^{-1}(-1) = \pi$

Thus, for all non-zero vectors $\overline{x}$ and $\overline{y}$, there exists $\alpha \in [0, \pi]$ such that,

$$\alpha = (\widehat{\overline{x}, \overline{y}}) = \cos^{-1} \frac{\overline{x} \cdot \overline{y}}{|\overline{x}||\overline{y}|}$$

Geometrical Interpretation : Our definition of the measure of the angle between two vectors is quite consistent with our understanding of the measure of the angle in geometry.

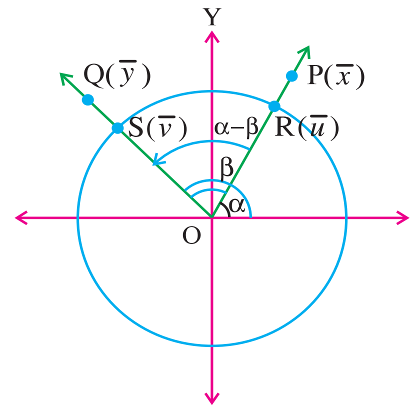


Figure 6.14

Suppose, position vectors of P and Q are $\overline{x}$ and $\overline{y}$ respectively, where $\overline{x} \neq \overline{0}$, $\overline{y} \neq \overline{0}$.

Let $\frac{\overline{x}}{|\overline{x}|} = \overline{u}$ and $\frac{\overline{y}}{|\overline{y}|} = \overline{v}$ be unit vectors in the direction of $\overline{x}$ and $\overline{y}$ respectively.

$(\widehat{\overline{x}, \overline{y}}) = (\widehat{\overline{u}, \overline{v}})$

Suppose $\overline{u}$ and $\overline{v}$ are the position vectors of R and S respectively. R and S are the points on the unit circle, so for some α and β with $0 \leq \alpha, \beta < 2\pi$, we would have $\overline{u} = (\cos\alpha, \sin\alpha)$ and $\overline{v} = (\cos\beta, \sin\beta)$.

Now if the radian measure of the angle formed by the rays $\overrightarrow{OR}$ and $\overrightarrow{OS}$ is θ, then it is clear that, $\theta = \alpha - \beta$ or $\beta - \alpha$.

$$\text{Now, } \cos(\widehat{\overline{x}, \overline{y}}) = \frac{\overline{x} \cdot \overline{y}}{|\overline{x}||\overline{y}|} = \overline{u} \cdot \overline{v}$$

$$= (\cos\alpha, \sin\alpha) \cdot (\cos\beta, \sin\beta)$$

$$= \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

$$= \cos(\alpha - \beta) \text{ or } \cos(\beta - \alpha)$$

$$= \cos\theta \qquad (0 < \theta < \pi, \ 0 < (\widehat{\overline{x}, \overline{y}}) < \pi)$$

$$\therefore \quad \theta = (\widehat{\overline{x}, \overline{y}}) = \cos^{-1} \frac{\overline{x} \cdot \overline{y}}{|\overline{x}||\overline{y}|}$$

Thus, the measure of angle θ formed by $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, as we understand from geometry is same as $(\widehat{\overline{x}, \overline{y}})$.

Orthogonal Vectors : If $\overline{x} \neq \overline{0}$, $\overline{y} \neq \overline{0}$ and $(\widehat{\overline{x}, \overline{y}}) = \frac{\pi}{2}$, then $\overline{x}$ and $\overline{y}$ are said to be orthogonal or perpendicular to each other. Perpendicularity of $\overline{x}$ and $\overline{y}$ denoted by $\overline{x} \perp \overline{y}$. We say $\overline{x}$ is perpendicular to $\overline{y}$.

Necessary and sufficient condition for two non-zero vectors to be perpendicular to each other :

Let $\overline{x}$ and $\overline{y}$ be two non-zero vectors.

$$\overline{x} \perp \overline{y} \Leftrightarrow (\widehat{\overline{x}, \overline{y}}) = \frac{\pi}{2}$$

$$\Leftrightarrow \cos(\widehat{\overline{x}, \overline{y}}) = \cos\frac{\pi}{2}$$

$$\Leftrightarrow \frac{\overline{x} \cdot \overline{y}}{|\overline{x}||\overline{y}|} = 0$$

$$\Leftrightarrow \overline{x} \cdot \overline{y} = 0$$

Thus $\overline{x}$ and $\overline{y}$ are orthogonal if and only if $\overline{x} \cdot \overline{y} = 0$.