7.4 Properties of a Parallelogram

Let us perform an activity.

Cut out a parallelogram from a sheet of paper and cut it along a diagonal (see Fig. 7.7). You obtain two triangles. What can you say about these triangles?

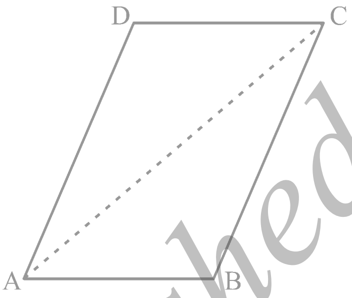


Fig. 7.7

Place one triangle over the other. Turn one around, if necessary. What do you observe?

Observe that the two triangles are congruent to each other.

Repeat this activity with some more parallelograms. Each time you will observe that each diagonal divides the parallelogram into two congruent triangles.

Let us now prove this result.

Theorem 7.1 : *A diagonal of a parallelogram divides it into two congruent triangles.*

Proof : Let ABCD be a parallelogram and AC be a diagonal (see Fig. 7.8). Observe that the diagonal AC divides parallelogram ABCD into two triangles, namely, Δ ABC and Δ CDA. We need to prove that these triangles are congruent.

In Δ ABC and Δ CDA, note that BC || AD and AC is a transversal.

So, ∠ BCA = ∠ DAC (Pair of alternate angles)

Also, AB || DC and AC is a transversal.

So, ∠ BAC = ∠ DCA (Pair of alternate angles)

and AC = CA (Common)

So, Δ ABC ≅ Δ CDA (ASA rule)

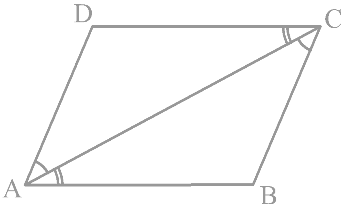


Fig. 7.8

or, diagonal AC divides parallelogram ABCD into two congruent triangles ABC and CDA.

Now, measure the opposite sides of parallelogram ABCD. What do you observe?

You will find that AB = DC and AD = BC.

This is another property of a parallelogram stated below:

Theorem 7.2 : *In a parallelogram, opposite sides are equal.*

You have already proved that a diagonal divides the parallelogram into two congruent triangles; so what can you say about the corresponding parts say, the corresponding sides? They are equal.

So, AB = DC and AD = BC

Now what is the converse of this result? You already know that whatever is given in a theorem, the same is to be proved in the converse and whatever is proved in the theorem it is given in the converse. Thus, Theorem 7.2 can be stated as given below :

If a quadrilateral is a parallelogram, then each pair of its opposite sides is equal. So its converse is :

Theorem 7.3 : *If each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.*

Can you reason out why?

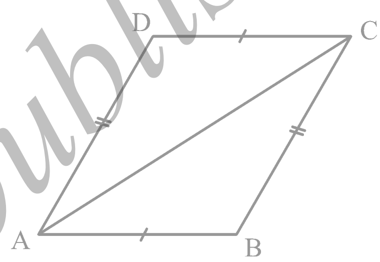


Fig. 7.9

Let sides AB and CD of the quadrilateral ABCD be equal and also AD = BC (see Fig. 8.9). Draw diagonal AC.

Clearly, Δ ABC $\cong$ Δ CDA (Why?)

So, $\angle$ BAC = $\angle$ DCA

and $\angle$ BCA = $\angle$ DAC (Why?)

Can you now say that ABCD is a parallelogram? Why?

You have just seen that in a parallelogram each pair of opposite sides is equal and conversely if each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram. Can we conclude the same result for the pairs of opposite angles?

Draw a parallelogram and measure its angles. What do you observe?

Each pair of opposite angles is equal.

Repeat this with some more parallelograms. We arrive at yet another result as given below.

Theorem 7.4 : *In a parallelogram, opposite angles are equal.*

Now, is the converse of this result also true? Yes. Using the angle sum property of a quadrilateral and the results of parallel lines intersected by a transversal, we can see that the converse is also true. So, we have the following theorem :

Theorem 7.5 : *If in a quadrilateral, each pair of opposite angles is equal, then it is a parallelogram.*

There is yet another property of a parallelogram. Let us study the same. Draw a parallelogram ABCD and draw both its diagonals intersecting at the point O (see Fig. 7.10).

Measure the lengths of OA, OB, OC and OD.

What do you observe? You will observe that

OA = OC and OB = OD.

or, O is the mid-point of both the diagonals.

Repeat this activity with some more parallelograms.

Each time you will find that O is the mid-point of both the diagonals.

So, we have the following theorem :

Theorem 7.6 : *The diagonals of a parallelogram bisect each other.*

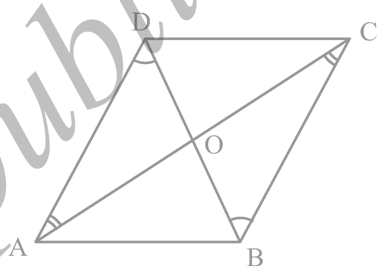


Fig. 7.10

Now, what would happen, if in a quadrilateral the diagonals bisect each other? Will it be a parallelogram? Indeed this is true.

This result is the converse of the result of Theorem 7.6. It is given below:

Theorem 7.7 : *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.*

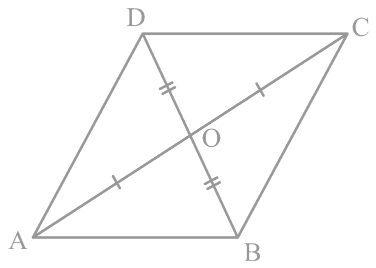


Fig. 7.11

You can reason out this result as follows:

Note that in Fig. 7.11, it is given that OA = OC and OB = OD.

So, Δ AOB $\cong$ Δ COD (Why?)

Therefore, $\angle$ ABO = $\angle$ CDO (Why?)

From this, we get AB || CD

Similarly, BC || AD

Therefore ABCD is a parallelogram.

Let us now take some examples.

Example 1 : Show that each angle of a rectangle is a right angle.

Solution : Let us recall what a rectangle is.

A rectangle is a parallelogram in which one angle is a right angle.

Fig. 7.12

Let ABCD be a rectangle in which ∠ A = 90°.

We have to show that ∠ B = ∠ C = ∠ D = 90°

We have, AD || BC and AB is a transversal (see Fig. 7.12).

So, ∠ A + ∠ B = 180° (Interior angles on the same side of the transversal)

But, ∠ A = 90°

So, ∠ B = 180° – ∠ A = 180° – 90° = 90°

Now, ∠ C = ∠ A and ∠ D = ∠ B

(Opposite angles of the parallellogram)

So, ∠ C = 90° and ∠ D = 90°.

Therefore, each of the angles of a rectangle is a right angle.

Example 2 : Show that the diagonals of a rhombus are perpendicular to each other.

Solution : Consider the rhombus ABCD (see Fig. 7.13).

You know that AB = BC = CD = DA (Why?)

Now, in Δ AOD and Δ COD,

OA = OC (Diagonals of a parallelogram bisect each other)

OD = OD (Common)

AD = CD

Therefore, Δ AOD ≅ Δ COD

(SSS congruence rule)

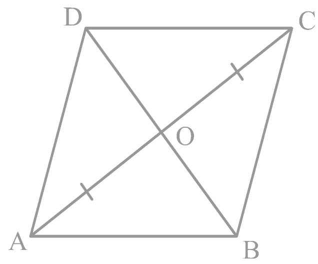


Fig. 7.13

This gives, ∠ AOD = ∠ COD (CPCT)

But, ∠ AOD + ∠ COD = 180° (Linear pair)

So, 2∠ AOD = 180°

or, ∠ AOD = 90°

So, the diagonals of a rhombus are perpendicular to each other.

Example 3 : ABC is an isosceles triangle in which AB = AC. AD bisects exterior angle PAC and CD || AB (see Fig. 7.14). Show that

(i) ∠ DAC = ∠ BCA and (ii) ABCD is a parallelogram.

Solution : (i) Δ ABC is isosceles in which AB = AC (Given)

So, ∠ ABC = ∠ ACB (Angles opposite to equal sides)

Also, ∠ PAC = ∠ ABC + ∠ ACB
(Exterior angle of a triangle)

or, ∠ PAC = 2∠ ACB (1)

Now, AD bisects ∠ PAC.

So, ∠ PAC = 2∠ DAC (2)

Therefore,

2∠ DAC = 2∠ ACB [From (1) and (2)]

or, ∠ DAC = ∠ ACB

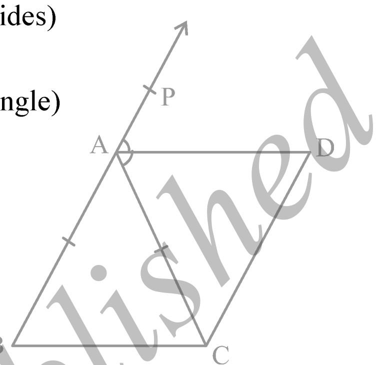


Fig. 7.14

(ii) Now, these equal angles form a pair of alternate angles when line segments BC and AD are intersected by a transversal AC.

So, BC || AD

Also, BA || CD (Given)

Now, both pairs of opposite sides of quadrilateral ABCD are parallel.

So, ABCD is a parallelogram.

Example 4 : Two parallel lines *l* and *m* are intersected by a transversal *p* (see Fig. 7.15). Show that the quadrilateral formed by the bisectors of interior angles is a rectangle.

Solution : It is given that PS || QR and transversal *p* intersects them at points A and C respectively.

The bisectors of ∠ PAC and ∠ ACQ intersect at B and bisectors of ∠ ACR and ∠ SAC intersect at D.

We are to show that quadrilateral ABCD is a rectangle.

Now, ∠ PAC = ∠ ACR

(Alternate angles as *l* || *m* and *p* is a transversal)

So, $\frac{1}{2}\angle \text{PAC} = \frac{1}{2}\angle \text{ACR}$

i.e., ∠ BAC = ∠ ACD

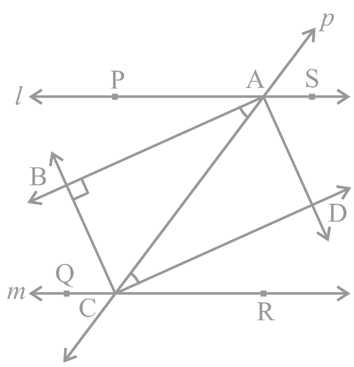


Fig. 7.15

These form a pair of alternate angles for lines AB and DC with AC as transversal and they are equal also.

So, $AB \parallel DC$

Similarly, $BC \parallel AD$ (Considering $\angle$ ACB and $\angle$ CAD)

Therefore, quadrilateral ABCD is a parallelogram.

Also, $\angle PAC + \angle CAS = 180°$ (Linear pair)

So, $\frac{1}{2} \angle PAC + \frac{1}{2} \angle CAS = \frac{1}{2} \times 180° = 90°$

or, $\angle BAC + \angle CAD = 90°$

or, $\angle BAD = 90°$

So, ABCD is a parallelogram in which one angle is 90°.

Therefore, ABCD is a rectangle.

Example 5 : Show that the bisectors of angles of a parallelogram form a rectangle.

Solution : Let P, Q, R and S be the points of intersection of the bisectors of $\angle$ A and $\angle$ B, $\angle$ B and $\angle$ C, $\angle$ C and $\angle$ D, and $\angle$ D and $\angle$ A respectively of parallelogram ABCD (see Fig. 7.16).

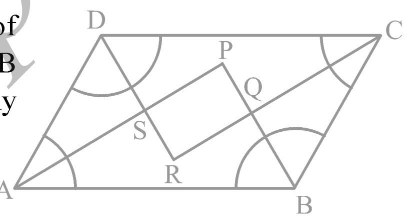


Fig. 7.16

In Δ ASD, what do you observe?

Since DS bisects $\angle$ D and AS bisects $\angle$ A, therefore,

$$\angle DAS + \angle ADS = \frac{1}{2} \angle A + \frac{1}{2} \angle D$$

$$= \frac{1}{2} (\angle A + \angle D)$$

$$= \frac{1}{2} \times 180°$$ ($\angle$ A and $\angle$ D are interior angles on the same side of the transversal)

$$= 90°$$

Also, $\angle DAS + \angle ADS + \angle DSA = 180°$ (Angle sum property of a triangle)

or, $90° + \angle DSA = 180°$

or, $\angle DSA = 90°$

So, $\angle PSR = 90°$ (Being vertically opposite to $\angle$ DSA)

Similarly, it can be shown that $\angle$ APB = 90° or $\angle$ SPQ = 90° (as it was shown for $\angle$ DSA). Similarly, $\angle$ PQR = 90° and $\angle$ SRQ = 90°.

So, PQRS is a quadrilateral in which all angles are right angles.

Can we conclude that it is a rectangle? Let us examine. We have shown that $\angle$ PSR = $\angle$ PQR = 90° and $\angle$ SPQ = $\angle$ SRQ = 90°. So both pairs of opposite angles are equal.

Therefore, PQRS is a parallelogram in which one angle (in fact all angles) is 90° and so, PQRS is a rectangle.

7.5 Another Condition for a Quadrilateral to be a Parallelogram

You have studied many properties of a parallelogram in this chapter and you have also verified that if in a quadrilateral any one of those properties is satisfied, then it becomes a parallelogram.

We now study yet another condition which is the least required condition for a quadrilateral to be a parallelogram.

It is stated in the form of a theorem as given below:

Theorem 7.8 : *A quadrilateral is a parallelogram if a pair of opposite sides is equal and parallel.*

Look at Fig 7.17 in which AB = CD and AB || CD. Let us draw a diagonal AC. You can show that Δ ABC ≅ Δ CDA by SAS congruence rule.

So, BC || AD (Why?)

Let us now take an example to apply this property of a parallelogram.

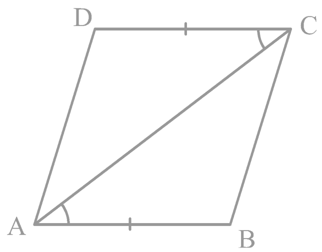


Fig. 7.17

Example 6 : ABCD is a parallelogram in which P and Q are mid-points of opposite sides AB and CD (see Fig. 7.18). If AQ intersects DP at S and BQ intersects CP at R, show that:

(i) APCQ is a parallelogram.

(ii) DPBQ is a parallelogram.

(iii) PSQR is a parallelogram.

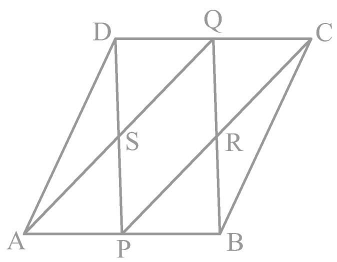


Fig. 7.18

Solution : (i) In quadrilateral APCQ,

$$AP \parallel QC \qquad \text{(Since } AB \parallel CD\text{)} \quad (1)$$

$$AP = \frac{1}{2}AB, \quad CQ = \frac{1}{2}CD \qquad \text{(Given)}$$

Also, $AB = CD$ (Why?)

So, $AP = QC$ (2)

Therefore, APCQ is a parallelogram [From (1) and (2) and Theorem 7.8]

(ii) Similarly, quadrilateral DPBQ is a parallelogram, because

$$DQ \parallel PB \text{ and } DQ = PB$$

(iii) In quadrilateral PSQR,

$SP \parallel QR$ (SP is a part of DP and QR is a part of QB)

Similarly, $SQ \parallel PR$

So, PSQR is a parallelogram.

EXERCISE 7.1

1. The angles of quadrilateral are in the ratio 3 : 5 : 9 : 13. Find all the angles of the quadrilateral.
2. If the diagonals of a parallelogram are equal, then show that it is a rectangle.
3. Show that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.
4. Show that the diagonals of a square are equal and bisect each other at right angles.
5. Show that if the diagonals of a quadrilateral are equal and bisect each other at right angles, then it is a square.
6. Diagonal AC of a parallelogram ABCD bisects $\angle$ A (see Fig. 7.19). Show that
   (i) it bisects $\angle$ C also,
   (ii) ABCD is a rhombus.

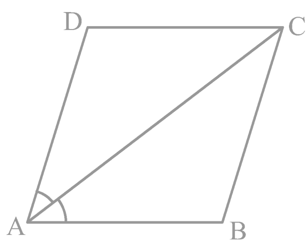


Fig. 7.19

7. ABCD is a rhombus. Show that diagonal AC bisects $\angle$ A as well as $\angle$ C and diagonal BD bisects $\angle$ B as well as $\angle$ D.
8. ABCD is a rectangle in which diagonal AC bisects $\angle$ A as well as $\angle$ C. Show that: (i) ABCD is a square (ii) diagonal BD bisects $\angle$ B as well as $\angle$ D.

9. In parallelogram ABCD, two points P and Q are taken on diagonal BD such that DP = BQ (see Fig. 7.20). Show that:

(i) Δ APD $\cong \Delta$ CQB

(ii) AP = CQ

(iii) Δ AQB $\cong \Delta$ CPD

(iv) AQ = CP

(v) APCQ is a parallelogram

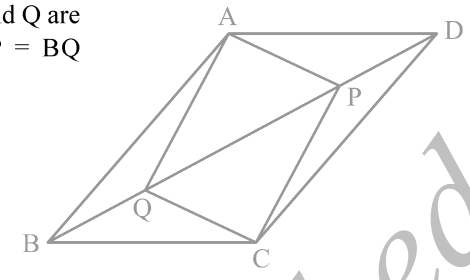


Fig. 7.20

10. ABCD is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD (see Fig. 7.21). Show that

(i) Δ APB $\cong \Delta$ CQD

(ii) AP = CQ

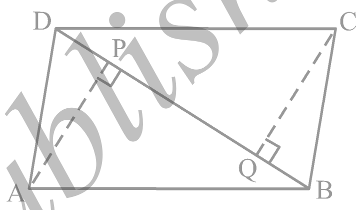


Fig. 7.21

11. In Δ ABC and Δ DEF, AB = DE, AB || DE, BC = EF and BC || EF. Vertices A, B and C are joined to vertices D, E and F respectively (see Fig. 7.22). Show that

(i) quadrilateral ABED is a parallelogram

(ii) quadrilateral BEFC is a parallelogram

(iii) AD || CF and AD = CF

(iv) quadrilateral ACFD is a parallelogram

(v) AC = DF

(vi) Δ ABC $\cong \Delta$ DEF.

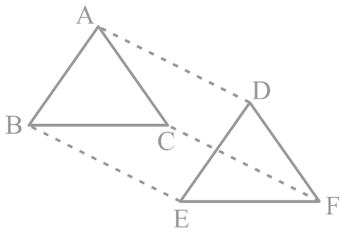


Fig. 7.22

12. ABCD is a trapezium in which AB || CD and AD = BC (see Fig. 7.23). Show that

(i) $\angle$ A = $\angle$ B

(ii) $\angle$ C = $\angle$ D

(iii) Δ ABC $\cong \Delta$ BAD

(iv) diagonal AC = diagonal BD

[*Hint* : Extend AB and draw a line through C parallel to DA intersecting AB produced at E.]

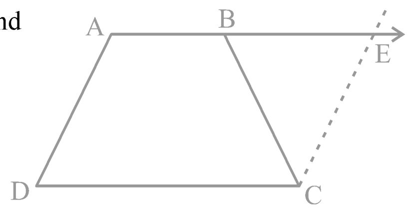


Fig. 7.23

7.6 The Mid-point Theorem

You have studied many properties of a triangle as well as a quadrilateral. Now let us study yet another result which is related to the mid-point of sides of a triangle. Perform the following activity.

Draw a triangle and mark the mid-points E and F of two sides of the triangle. Join the points E and F (see Fig. 7.24).

Measure EF and BC. Measure ∠ AEF and ∠ ABC.

What do you observe? You will find that :

$$EF = \frac{1}{2} BC \text{ and } \angle AEF = \angle ABC$$

so, EF || BC

Repeat this activity with some more triangles.

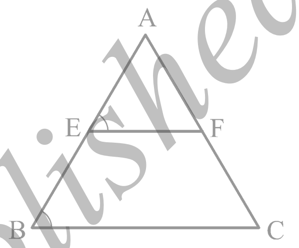


Fig. 7.24

So, you arrive at the following theorem:

Theorem 7.9 : *The line segment joining the mid-points of two sides of a triangle is parallel to the third side.*

You can prove this theorem using the following clue:

Observe Fig 7.25 in which E and F are mid-points of AB and AC respectively and CD || BA.

Δ AEF ≅ Δ CDF (ASA Rule)

So, EF = DF and BE = AE = DC (Why?)

Therefore, BCDE is a parallelogram. (Why?)

This gives EF || BC.

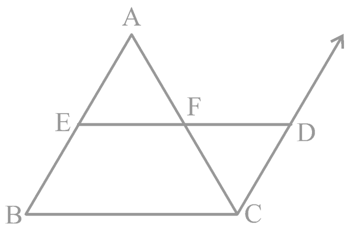


Fig. 7.25

In this case, also note that $EF = \frac{1}{2} ED = \frac{1}{2} BC$.

Can you state the converse of Theorem 7.9? Is the converse true?

You will see that converse of the above theorem is also true which is stated as below:

Theorem 7.10 : *The line drawn through the mid-point of one side of a triangle, parallel to another side bisects the third side.*

In Fig 7.26, observe that E is the mid-point of AB, line l is passsing through E and is parallel to BC and CM || BA.

Prove that AF = CF by using the congruence of Δ AEF and Δ CDF.

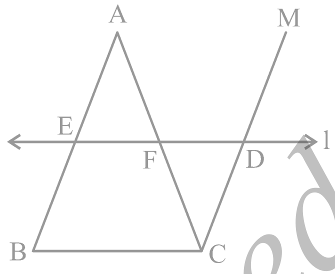


Fig. 7.26

Example 7 : In Δ ABC, D, E and F are respectively the mid-points of sides AB, BC and CA (see Fig. 7.27). Show that Δ ABC is divided into four congruent triangles by joining D, E and F.

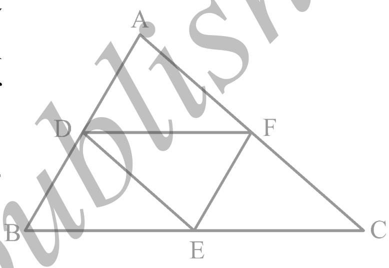


Fig. 7.27

Solution : As D and E are mid-points of sides AB and BC of the triangle ABC, by Theorem 7.9,

DE || AC

Similarly, DF || BC and EF || AB

Therefore ADEF, BDFE and DFCE are all parallelograms.

Now DE is a diagonal of the parallelogram BDFE,

therefore, Δ BDE ≅ Δ FED

Similarly Δ DAF ≅ Δ FED

and Δ EFC ≅ Δ FED

So, all the four triangles are congruent.

Example 8 : l, m and n are three parallel lines intersected by transversals p and q such that l, m and n cut off equal intercepts AB and BC on p (see Fig. 7.28). Show that l, m and n cut off equal intercepts DE and EF on q also.

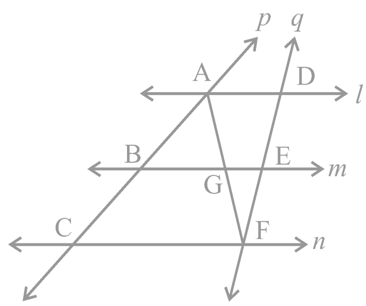


Fig. 7.28

Solution : We are given that AB = BC and have to prove that DE = EF.

Let us join A to F intersecting m at G..

The trapezium ACFD is divided into two triangles;

namely Δ ACF and Δ AFD.

In Δ ACF, it is given that B is the mid-point of AC (AB = BC)

and BG || CF (since *m* || *n*).

So, G is the mid-point of AF (by using Theorem 7.10)

Now, in Δ AFD, we can apply the same argument as G is the mid-point of AF, GE || AD and so by Theorem 7.10, E is the mid-point of DF,

i.e., DE = EF.

In other words, *l*, *m* and *n* cut off equal intercepts on *q* also.

EXERCISE 7.2

1. ABCD is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA (see Fig 7.29). AC is a diagonal. Show that :

(i) SR || AC and SR = $\frac{1}{2}$ AC

(ii) PQ = SR

(iii) PQRS is a parallelogram.

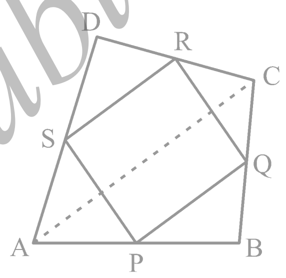


Fig. 7.29

2. ABCD is a rhombus and P, Q, R and S are ©wthe mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral PQRS is a rectangle.

3. ABCD is a rectangle and P, Q, R and S are mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral PQRS is a rhombus.

4. ABCD is a trapezium in which AB || DC, BD is a diagonal and E is the mid-point of AD. A line is drawn through E parallel to AB intersecting BC at F (see Fig. 7.30). Show that F is the mid-point of BC.

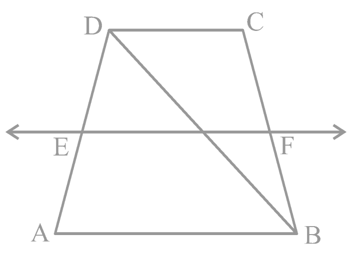


Fig. 7.30

5. In a parallelogram ABCD, E and F are the mid-points of sides AB and CD respectively (see Fig. 7.31). Show that the line segments AF and EC trisect the diagonal BD.

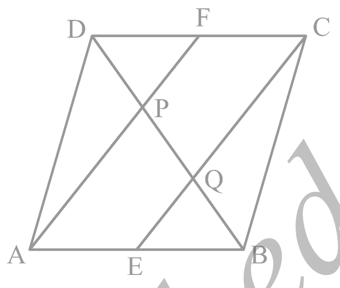


Fig. 7.31

6. Show that the line segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.

7. ABC is a triangle right angled at C. A line through the mid-point M of hypotenuse AB and parallel to BC intersects AC at D. Show that

   (i) D is the mid-point of AC (ii) $MD \perp AC$

   (iii) $CM = MA = \frac{1}{2}AB$

7.7 Summary

In this chapter, you have studied the following points :

1. Sum of the angles of a quadrilateral is 360°.
2. A diagonal of a parallelogram divides it into two congruent triangles.
3. In a parallelogram,

   (i) opposite sides are equal (ii) opposite angles are equal

   (iii) diagonals bisect each other
4. A quadrilateral is a parallelogram, if

   (i) opposite sides are equal or (ii) opposite angles are equal

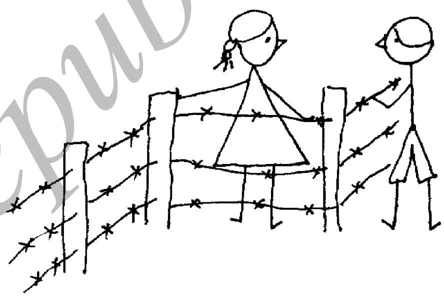
   or (iii) diagonals bisect each other

   or (iv)a pair of opposite sides is equal and parallel
5. Diagonals of a rectangle bisect each other and are equal and vice-versa.
6. Diagonals of a rhombus bisect each other at right angles and vice-versa.
7. Diagonals of a square bisect each other at right angles and are equal, and vice-versa.
8. The line-segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.
9. A line through the mid-point of a side of a triangle parallel to another side bisects the third side.
10. The quadrilateral formed by joining the mid-points of the sides of a quadrilateral, in order, is a parallelogram.

APPENDIX 1

PROOFS IN MATHEMATICS

A1.1 Introduction



Suppose your family owns a plot of land and there is no fencing around it. Your neighbour decides one day to fence off his land. After he has fenced his land, you discover that a part of your family's land has been enclosed by his fence. How will you prove to your neighbour that he has tried to encroach on your land? Your first step may be to seek the help of the village elders to sort out the difference in boundaries. But, suppose opinion is divided among the elders. Some feel you are right and others feel your neighbour is right. What can you do? Your only option is to find a way of establishing your claim for the boundaries of your land that is acceptable to all. For example, a government approved survey map of your village can be used, if necessary in a court of law, to prove (claim) that you are correct and your neighbour is wrong.

Let us look at another situation. Suppose your mother has paid the electricity bill of your house for the month of August, 2005. The bill for September, 2005, however, claims that the bill for August has not been paid. How will you disprove the claim made by the electricity department? You will have to produce a receipt proving that your August bill has been paid.

You have just seen some examples that show that in our daily life we are often called upon to prove that a certain statement or claim is true or false. However, we also accept many statements without bothering to prove them. But, in mathematics we only accept a statement as true or false (except for some axioms) if it has been proved to be so, according to the logic of mathematics.

In fact, proofs in mathematics have been in existence for thousands of years, and they are central to any branch of mathematics. The first known proof is believed to have been given by the Greek philosopher and mathematician Thales. While mathematics was central to many ancient civilisations like Mesopotamia, Egypt, China and India, there is no clear evidence that they used proofs the way we do today.

In this chapter, we will look at what a statement is, what kind of reasoning is involved in mathematics, and what a mathematical proof consists of.

A1.2 Mathematically Acceptable Statements

In this section, we shall try to explain the meaning of a mathematically acceptable statement. A 'statement' is a sentence which is not an order or an exclamatory sentence. And, of course, a statement is not a question! For example,

"What is the colour of your hair?" is not a statement, it is a question.

"Please go and bring me some water." is a request or an order, not a statement.

"What a marvellous sunset!" is an exclamatory remark, not a statement.

However, "The colour of your hair is black" is a statement.

In general, statements can be one of the following:

- *always true*
- *always false*
- *ambiguous*

The word 'ambiguous' needs some explanation. There are two situations which make a statement ambiguous. The first situation is when we cannot decide if the statement is always true or always false. For example, "Tomorrow is Thursday" is ambiguous, since enough of a context is not given to us to decide if the statement is true or false.

The second situation leading to ambiguity is when the statement is subjective, that is, it is true for some people and not true for others. For example, "Dogs are intelligent" is ambiguous because some people believe this is true and others do not.

Example 1 : State whether the following statements are always true, always false or ambiguous. Justify your answers.

(i) There are 8 days in a week.

(ii) It is raining here.

(iii) The sun sets in the west.

(iv) Gauri is a kind girl.

(v) The product of two odd integers is even.

(vi) The product of two even natural numbers is even.

Solution :

(i) This statement is always false, since there are 7 days in a week.

(ii) This statement is ambiguous, since it is not clear where 'here' is.

(iii) This statement is always true. The sun sets in the west no matter where we live.

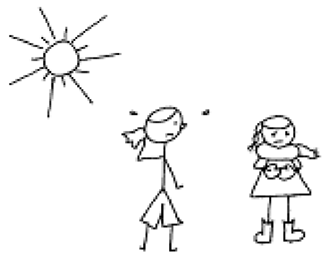
(iv) This statement is ambiguous, since it is subjective–Gauri may be kind to some and not to others.

(v) This statement is always false. The product of two odd integers is always odd.

(vi) This statement is always true. However, to justify that it is true we need to do some work. It will be proved in Section A1.4.

As mentioned before, in our daily life, we are not so careful about the validity of statements. For example, suppose your friend tells you that in July it rains everyday in Manantavadi, Kerala. In all probability, you will believe her, even though it may not have rained for a day or two in July. Unless you are a lawyer, you will not argue with her!

As another example, consider statements we often make to each other like "it is very hot today". We easily accept such statements because we know the context even though these statements are ambiguous. 'It is very hot today' can mean different things to different people because what is very hot for a person from Kumaon may not be hot for a person from Chennai.



But a mathematical statement cannot be ambiguous. *In mathematics, a statement is only acceptable or valid, if it is either true or false.* We say that a statement is true, if it is always true otherwise it is called a false statement.

For example, $5 + 2 = 7$ is always true, so '$5 + 2 = 7$' is a true statement and $5 + 3 = 7$ is a false statement.

Example 2 : State whether the following statements are true or false:

(i) The sum of the interior angles of a triangle is 180°.

(ii) Every odd number greater than 1 is prime.

(iii) For any real number x, $4x + x = 5x$.

(iv) For every real number x, $2x > x$.

(v) For every real number x, $x^2 \geq x$.

(vi) If a quadrilateral has all its sides equal, then it is a square.

Solution :

(i) This statement is true. You have already proved this in Chapter 6.

(ii) This statement is false; for example, 9 is not a prime number.

(iii) This statement is true.

(iv) This statement is false; for example, $2 \times (-1) = -2$, and -2 is not greater than -1.

(v) This statement is false; for example, $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, and $\frac{1}{4}$ is not greater than $\frac{1}{2}$.

(vi) This statement is false, since a rhombus has equal sides but need not be a square.

You might have noticed that to establish that a statement is not true according to mathematics, all we need to do is to find one case or example where it breaks down. So in (ii), since 9 is not a prime, it is an example that shows that the statement "Every odd number greater than 1 is prime" is not true. Such an example, that counters a statement, is called a *counter-example.* We shall discuss counter-examples in greater detail in Section A1.5.

You might have also noticed that while Statements (iv), (v) and (vi) are false, they can be restated with some conditions in order to make them true.

Example 3 : Restate the following statements with appropriate conditions, so that they become true statements.

(i) For every real number x, $2x > x$.

(ii) For every real number x, $x^2 \geq x$.

(iii) If you divide a number by itself, you will always get 1.

(iv) The angle subtended by a chord of a circle at a point on the circle is 90°.

(v) If a quadrilateral has all its sides equal, then it is a square.

Solution :

(i) If $x > 0$, then $2x > x$.

(ii) If $x \leq 0$ or $x \geq 1$, then $x^2 \geq x$.

(iii) If you divide a number except zero by itself, you will always get 1.

(iv) The angle subtended by a diameter of a circle at a point on the circle is 90°.

(v) If a quadrilateral has all its sides and interior angles equal, then it is a square.

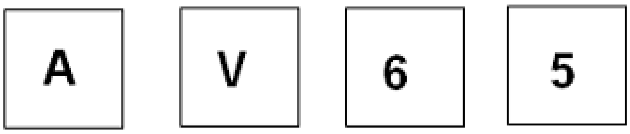
EXERCISE A1.1

1. State whether the following statements are always true, always false or ambiguous. Justify your answers.
   (i) There are 13 months in a year.
   (ii) Diwali falls on a Friday.
   (iii) The temperature in Magadi is 26^0 C.
   (iv) The earth has one moon.
   (v) Dogs can fly.
   (vi) February has only 28 days.
2. State whether the following statements are true or false. Give reasons for your answers.
   (i) The sum of the interior angles of a quadrilateral is 350°.
   (ii) For any real number x, $x^2 \geq 0$.
   (iii) A rhombus is a parallelogram.
   (iv) The sum of two even numbers is even.
   (v) The sum of two odd numbers is odd.
3. Restate the following statements with appropriate conditions, so that they become true statements.
   (i) All prime numbers are odd.
   (ii) Two times a real number is always even.
   (iii) For any x, $3x + 1 > 4$.
   (iv) For any x, $x^3 \geq 0$.
   (v) In every triangle, a median is also an angle bisector.

A1.3 Deductive Reasoning

The main logical tool used in establishing the truth of an **unambiguous** statement is *deductive reasoning.* To understand what deductive reasoning is all about, let us begin with a puzzle for you to solve.

You are given four cards. Each card has a number printed on one side and a letter on the other side.



Suppose you are told that these cards follow the rule:

"If a card has an even number on one side, then it has a vowel on the other side."

What is the **smallest number** of cards you need to turn over to check if the rule is true?

Of course, you have the option of turning over all the cards and checking. But can you manage with turning over a fewer number of cards?

Notice that the statement mentions that a card with an even number on one side has a vowel on the other. It does not state that a card with a vowel on one side must have an even number on the other side. That may or may not be so. The rule also does not state that a card with an odd number on one side must have a consonant on the other side. It may or may not.

So, do we need to turn over 'A'? No! Whether there is an even number or an odd number on the other side, the rule still holds.

What about '5'? Again we do not need to turn it over, because whether there is a vowel or a consonant on the other side, the rule still holds.

But you do need to turn over V and 6. If V has an even number on the other side, then the rule has been broken. Similarly, if 6 has a consonant on the other side, then the rule has been broken.

The kind of reasoning we have used to solve this puzzle is called **deductive reasoning**. It is called 'deductive' because we arrive at (i.e., deduce or infer) a result or a statement from a previously established statement using logic. For example, in the puzzle above, by a series of logical arguments we deduced that we need to turn over only V and 6.

Deductive reasoning also helps us to conclude that a particular statement is true, because it is a special case of a more general statement that is known to be true. For example, once we prove that the product of two odd numbers is always odd, we can immediately conclude (without computation) that 70001×134563 is odd simply because 70001 and 134563 are odd.

Deductive reasoning has been a part of human thinking for centuries, and is used all the time in our daily life. For example, suppose the statements "The flower Solaris blooms, only if the maximum temperature is above 28° C on the previous day" and "Solaris bloomed in Imaginary Valley on 15th September, 2005" are true. Then using deductive reasoning, we can conclude that the maximum temperature in Imaginary Valley on 14th September, 2005 was more than 28° C.

Unfortunately we do not always use correct reasoning in our daily life! We often come to many conclusions based on faulty reasoning. For example, if your friend does not smile at you one day, then you may conclude that she is angry with you. While it may be true that "if she is angry with me, she will not smile at me", it may also be true that "if she has a bad headache, she will not smile at me". Why don't you examine some conclusions that you have arrived at in your day-to-day existence, and see if they are based on valid or faulty reasoning?

EXERCISE A1.2

1. Use deductive reasoning to answer the following:

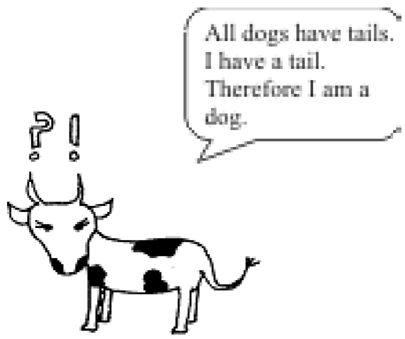
(i) Humans are mammals. All mammals are vertebrates. Based on these two statements, what can you conclude about humans?

(ii) Anthony is a barber. Dinesh had his hair cut. Can you conclude that Antony cut Dinesh's hair?

(iii) Martians have red tongues. Gulag is a Martian. Based on these two statements, what can you conclude about Gulag?

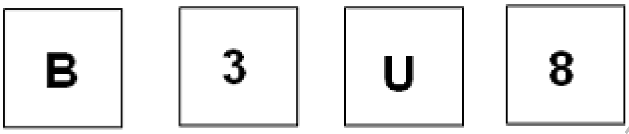
(iv) If it rains for more than four hours on a particular day, the gutters will have to be cleaned the next day. It has rained for 6 hours today. What can we conclude about the condition of the gutters tomorrow?

(v) What is the fallacy in the cow's reasoning in the cartoon below?



2. Once again you are given four cards. Each card has a number printed on one side and a letter on the other side. Which are the only two cards you need to turn over to check whether the following rule holds?

"If a card has a consonant on one side, then it has an odd number on the other side."



A1.4 Theorems, Conjectures and Axioms

So far we have discussed statements and how to check their validity. In this section, you will study how to distinguish between the three different kinds of statements mathematics is built up from, namely, a theorem, a conjecture and an axiom.

You have already come across many theorems before. So, what is a theorem? A mathematical statement whose truth has been established (proved) is called a *theorem*. For example, the following statements are theorems, as you will see in Section A1.5.

Theorem A1.1 : *The sum of the interior angles of a triangle is* 180°.

Theorem A1.2 : *The product of two even natural numbers is even.*

Theorem A1.3 : *The product of any three consecutive even natural numbers is divisible by* 16.

A *conjecture* is a statement which we believe is true, based on our mathematical understanding and experience, that is, our mathematical intuition. The conjecture may turn out to be true or false. If we can prove it, then it becomes a theorem. Mathematicians often come up with conjectures by looking for patterns and making intelligent mathematical guesses. Let us look at some patterns and see what kind of intelligent guesses we can make.

Example 4 : Take any three consecutive even numbers and add them, say,

$2 + 4 + 6 = 12$, $4 + 6 + 8 = 18$, $6 + 8 + 10 = 24$, $8 + 10 + 12 = 30$, $20 + 22 + 24 = 66$.

Is there any pattern you can guess in these sums? What can you conjecture about them?

Solution : One conjecture could be :

(i) the sum of three consecutive even numbers is even.

Another could be :

(ii) the sum of three consecutive even numbers is divisible by 6.

Example 5 : Consider the following pattern of numbers called the Pascal's Triangle:

Line		Sum of numbers
1	1	1
2	1 1	2
3	1 2 1	4
4	1 3 3 1	8
5	1 4 6 4 1	16
6	1 5 10 10 5 1	32
7	: :	:
8	: :	:

What can you conjecture about the sum of the numbers in Lines 7 and 8? What about the sum of the numbers in Line 21? Do you see a pattern? Make a guess about a formula for the sum of the numbers in line n.

Solution : Sum of the numbers in Line 7 = $2 \times 32 = 64 = 2^6$

Sum of the numbers in Line 8 = $2 \times 64 = 128 = 2^7$

Sum of the numbers in Line 21 = 2^{20}

Sum of the numbers in Line $n = 2^{n-1}$

Example 6 : Consider the so-called triangular numbers T_n:

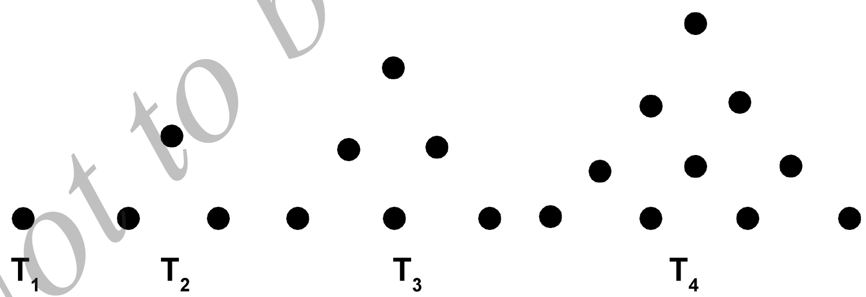


Fig. A1.1

The dots here are arranged in such a way that they form a triangle. Here $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, and so on. Can you guess what T_5 is? What about T_6? What about T_n?

Make a conjecture about T_n.

It might help if you redraw them in the following way.

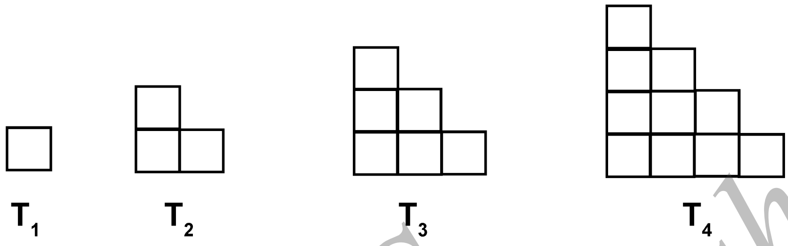


Fig. A1.2

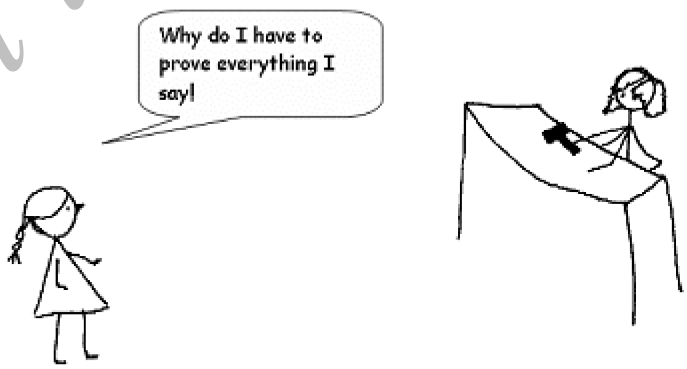
Solution : $T_5 = 1 + 2 + 3 + 4 + 5 = 15 = \frac{5 \times 6}{2}$

$$T_6 = 1 + 2 + 3 + 4 + 5 + 6 = 21 = \frac{6 \times 7}{2}$$

$$T_n = \frac{n \times (n+1)}{2}$$

A favourite example of a conjecture that has been open (that is, it has not been proved to be true or false) is the Goldbach conjecture named after the mathematician Christian Goldbach (1690 – 1764). This conjecture states that "*every even integer greater than* 4 *can be expressed as the sum of two odd primes*." Perhaps you will prove that this result is either true or false, and will become famous!

You might have wondered – do we need to prove everything we encounter in mathematics, and if not, why not?



The fact is that every area in mathematics is based on some statements which are assumed to be true and are not proved. These are 'self-evident truths' which we take to be true without proof. These statements are called *axioms*. In Chapter 5, you would have studied the axioms and postulates of Euclid. (We do not distinguish between axioms and postulates these days.)

For example, the first postulate of Euclid states:

A straight line may be drawn from any point to any other point.

And the third postulate states:

A circle may be drawn with any centre and any radius.

These statements appear to be perfectly true and Euclid assumed them to be true. Why? This is because we cannot prove everything and we need to start somewhere. We need some statements which we accept as true and then we can build up our knowledge using the rules of logic based on these axioms.

You might then wonder why we don't just accept all statements to be true when they appear self-evident. There are many reasons for this. Very often our intuition can be wrong, pictures or patterns can deceive and the only way to be sure that something is true is to prove it. For example, many of us believe that if a number is multiplied by another, the result will be larger than both the numbers. But we know that this is not always true: for example, $5 \times 0.2 = 1$, which is less than 5.

Also, look at the Fig. A1.3. Which line segment is longer, AB or CD?

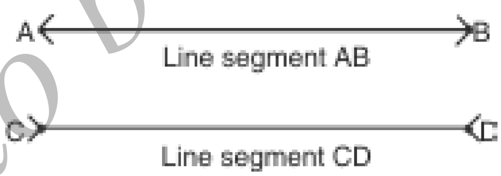


Fig. A1.3

It turns out that both are of exactly the same length, even though AB appears shorter!

You might wonder then, about the validity of axioms. Axioms have been chosen based on our intuition and what appears to be self-evident. Therefore, we expect them to be true. However, it is possible that later on we discover that a particular axiom is not true. What is a safeguard against this possibility? We take the following steps:

(i) Keep the axioms to the bare minimum. For instance, based on only axioms and five postulates of Euclid, we can derive hundreds of theorems.

(ii) Make sure the axioms are consistent.

We say a collection of axioms is *inconsistent,* if we can use one axiom to show that another axiom is not true. For example, consider the following two statements. We will show that they are inconsistent.

Statement1: No whole number is equal to its successor.

Statement 2: A whole number divided by zero is a whole number.

(Remember, **division by zero is not defined**. But just for the moment, we assume that it is possible, and see what happens.)

From Statement 2, we get $\frac{1}{0} = a$, where a is some whole number. This implies that, $1 = 0$. But this disproves Statement 1, which states that no whole number is equal to its successor.

(iii) A false axiom will, sooner or later, result in a contradiction. We say that *there is a contradiction, when we find a statement such that, both the statement and its negation are true.* For example, consider Statement 1 and Statement 2 above once again.

From Statement 1, we can derive the result that $2 \neq 1$.

Now look at $x^2 - x^2$. We will factorise it in two different ways as follows:

(i) $x^2 - x^2 = x(x - x)$ and

(ii) $x^2 - x^2 = (x + x)(x - x)$

So, $x(x - x) = (x + x)(x - x)$.

From Statement 2, we can cancel $(x - x)$ from both sides.

We get $x = 2x$, which in turn implies $2 = 1$.

So we have both the statement $2 \neq 1$ and its negation, $2 = 1$, true. This is a contradiction. The contradiction arose because of the false axiom, that a whole number divided by zero is a whole number.

So, the statements we choose as axioms require a lot of thought and insight. We must make sure they do not lead to inconsistencies or logical contradictions. Moreover, the choice of axioms themselves, sometimes leads us to new discoveries. From Chapter 5, you are familiar with Euclid's fifth postulate and the discoveries of non-Euclidean geometries. You saw that mathematicians believed that the fifth postulate need not be a postulate and is actually a theorem that can be proved using just the first four postulates. Amazingly these attempts led to the discovery of non-Euclidean geometries.

We end the section by recalling the differences between an axiom, a theorem and a conjecture. An **axiom** is a mathematical statement which is assumed to be true without proof; a **conjecture** is a mathematical statement whose truth or falsity is yet to be established; and a **theorem** is a mathematical statement whose truth has been logically established.

EXERCISE A1.3

1. Take any three consecutive even numbers and find their product; for example, $2 \times 4 \times 6 = 48$, $4 \times 6 \times 8 = 192$, and so on. Make three conjectures about these products.
2. Go back to Pascal's triangle.

   Line 1 : 1 = 11^0

   Line 2 : 1 1 = 11^1

   Line 3 : 1 2 1 = 11^2

   Make a conjecture about Line 4 and Line 5. Does your conjecture hold? Does your conjecture hold for Line 6 too?
3. Let us look at the triangular numbers (see Fig.A1.2) again. Add two consecutive triangular numbers. For example, $T_1 + T_2 = 4$, $T_2 + T_3 = 9$, $T_3 + T_4 = 16$.

   What about $T_4 + T_5$? Make a conjecture about $T_{n-1} + T_n$.
4. Look at the following pattern:

   $1^2 = 1$

   $11^2 = 121$

   $111^2 = 12321$

   $1111^2 = 1234321$

   $11111^2 = 123454321$

   Make a conjecture about each of the following:

   $111111^2 =$

   $1111111^2 =$

   Check if your conjecture is true.
5. List five axioms (postulates) used in this book.

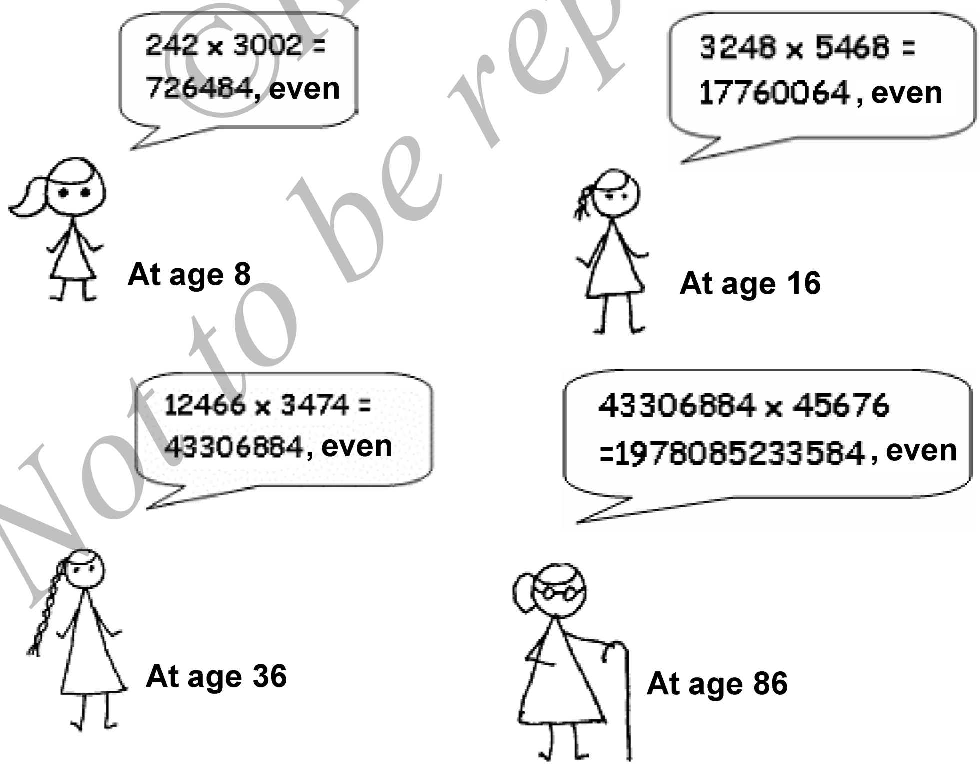
A1.5 What is a Mathematical Proof?

Let us now look at various aspects of proofs. We start with understanding the difference between verification and proof. Before you studied proofs in mathematics, you were mainly asked to verify statements.

For example, you might have been asked to verify with examples that "the product of two even numbers is even". So you might have picked up two random even numbers, say 24 and 2006, and checked that 24 × 2006 = 48144 is even. You might have done so for many more examples.

Also, you might have been asked as an activity to draw several triangles in the class and compute the sum of their interior angles. Apart from errors due to measurement, you would have found that the interior angles of a triangle add up to 180°.

What is the flaw in this method? There are several problems with the process of verification. While it may help you to make a statement you believe is true, you cannot be *sure* that it is true in *all* cases. For example, the multiplication of several pairs of even numbers may lead us to guess that the product of two even numbers is even. However, it does not ensure that the product of all pairs of even numbers is even. You cannot physically check the products of all possible pairs of even numbers. If you did, then like the girl in the cartoon, you will be calculating the products of even numbers for the rest of your life. Similarly, there may be some triangles which you have not yet drawn whose interior angles do not add up to 180°. We cannot measure the interior angles of all possible triangles.



Moreover, verification can often be misleading. For example, we might be tempted to conclude from Pascal's triangle (Q.2 of Exercise A1.3), based on earlier verifications, that $11^5 = 15101051$. But in fact $11^5 = 161051$.

So, you need another approach that does not depend upon verification for some cases only. There is another approach, namely 'proving a statement'. A process which can establish the truth of a mathematical statement based purely on logical arguments is called a *mathematical proof*.

In Example 2 of Section A1.2, you saw that to establish that a mathematical statement is false, it is enough to produce a single counter-example. So while it is not enough to establish the validity of a mathematical statement by checking or verifying it for thousands of cases, it is enough to produce one counter-example to *disprove* a statement (i.e., to show that something is false). Th g.

To show that a mathematical statement is false, it is enough to find a single counter-example.

So, 7 + 5 = 12 is a counter-example to the statement that the sum of two odd numbers is odd.

Let us now look at the list of basic ingredients in a proof:

(i) To prove a theorem, we should have a rough idea as to how to proceed.

(ii) The information already given to us in a theorem (i.e., the hypothesis) has to be clearly understood and used.

For example, in Theorem A1.2, which states that the product of two even numbers is even, we are given two even natural numbers. So, we should use their properties. In the Factor Theorem (in Chapter 2), you are given a polynomial $p(x)$ and are told that $p(a) = 0$. You have to use this to show that $(x - a)$ is a factor of $p(x)$. Similarly, for the converse of the Factor Theorem, you are given that $(x - a)$ is a factor of $p(x)$, and you have to use this hypothesis to prove that $p(a) = 0$.

You can also use constructions during the process of proving a theorem. For example, to prove that the sum of the angles of a triangle is 180°, we draw a line parallel to one of the sides through the vertex opposite to the side, and use properties of parallel lines.

(iii) A proof is made up of a successive sequence of mathematical statements. Each statement in a proof is logically deduced from a previous statement in the proof, or from a theorem proved earlier, or an axiom, or our hypothesis.

(iv) The conclusion of a sequence of mathematically true statements laid out in a logically correct order should be what we wanted to prove, that is, what the theorem claims.

To understand these ingredients, we will analyse Theorem A1.1 and its proof. You have already studied this theorem in Chapter 6. But first, a few comments on proofs in geometry. We often resort to diagrams to help us prove theorems, and this is very important. However, each statement in the proof has to be established **using only logic**. Very often, we hear students make statements like "Those two angles are equal because in the drawing they look equal" or "that angle must be 90^0, because the two lines look as if they are perpendicular to each other". Beware of being deceived by what you see (remember Fig A1.3)! .

So now let us go to Theorem A1.1.

Theorem A1.1 : *The sum of the interior angles of a triangle is* 180°.

Proof : Consider a triangle ABC (see Fig. A1.4).

We have to prove that $\angle$ ABC + $\angle$ BCA + $\angle$ CAB = 180° (1)

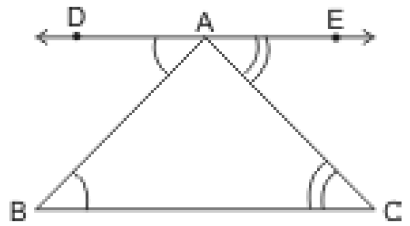


Fig A 1.4

Construct a line DE parallel to BC passing through A. (2)

DE is parallel to BC and AB is a transversal.

So, ∠ DAB and ∠ ABC are alternate angles. Therefore, by Theorem 6.2, Chapter 6, they are equal, i.e. ∠ DAB = ∠ ABC (3)

Similarly, ∠ CAE = ∠ ACB (4)

Therefore, ∠ ABC + ∠ BAC+ ∠ ACB = ∠ DAB +∠ BAC + ∠ CAE (5)

But ∠ DAB +∠ BAC + ∠ CAE = 180°, since they form a straight angle. (6)

Hence, ∠ ABC + ∠ BAC+ ∠ ACB = 180°. ■ (7)

Now, we comment on each step of the proof.

Step 1 : Our theorem is concerned with a property of triangles, so we begin with a triangle.

Step 2 : This is the key idea – the intuitive leap or understanding of how to proceed so as to be able to prove the theorem. Very often geometric proofs require a construction.

Steps 3 and 4 : Here we conclude that ∠ DAE = ∠ ABC and ∠ CAE = ∠ ACB, by using the fact that DE is parallel to BC (our construction), and the previously proved Theorem 6.2, which states that if two parallel lines are intersected by a transversal, then the alternate angles are equal.

Step 5 : Here we use Euclid's axiom (see Chapter 5) which states that: "If equals are added to equals, the wholes are equal" to deduce

∠ ABC + ∠ BAC+ ∠ ACB = ∠ DAB +∠ BAC + ∠ CAE.

That is, the sum of the interior angles of the triangle are equal to the sum of the angles on a straight line.

Step 6 : Here we use the Linear pair axiom of Chapter 6, which states that the angles on a straight line add up to 180°, to show that ∠ DAB +∠ BAC + ∠ CAE = 180°.

Step 7 : We use Euclid's axiom which states that "things which are equal to the same thing are equal to each other" to conclude that ∠ ABC + ∠ BAC + ∠ ACB = ∠ DAB +∠ BAC + ∠ CAE = 180°. Notice that Step 7 is the claim made in the theorem we set out to prove.

We now prove Theorems A1.2 and A1.3 without analysing them.

Theorem A1.2 : *The product of two even natural numbers is even.*

Proof : Let x and y be any two even natural numbers.

We want to prove that xy is even.

Since x and y are even, they are divisible by 2 and can be expressed in the form

$x = 2m$, for some natural number m and $y = 2n$, for some natural number n.

Then $xy = 4\,mn$. Since $4\,mn$ is divisible by 2, so is xy.

Therefore, xy is even. ■

Theorem A1.3 : *The product of any three consecutive even natural numbers is divisible by* 16.

Proof : Any three consecutive even numbers will be of the form $2n$, $2n + 2$ and $2n + 4$, for some natural number n. We need to prove that their product $2n(2n + 2)(2n + 4)$ is divisible by 16.

Now, $2n(2n + 2)(2n + 4) = 2n \times 2(n + 1) \times 2(n + 2)$

$= 2 \times 2 \times 2n(n + 1)(n + 2) = 8n(n + 1)(n + 2)$.

Now we have two cases. Either n is even or odd. Let us examine each case.

Suppose n is even : Then we can write $n = 2m$, for some natural number m.

And, then $2n(2n + 2)(2n + 4) = 8n(n + 1)(n + 2) = 16m(2m + 1)(2m + 2)$.

Therefore, $2n(2n + 2)(2n + 4)$ is divisible by 16.

Next, suppose n is odd. Then $n + 1$ is even and we can write $n + 1 = 2r$, for some natural number r.

We then have :

$$\begin{aligned} 2n(2n + 2)(2n + 4) &= 8n(n + 1)(n + 2) \\ &= 8(2r - 1) \times 2r \times (2r + 1) \\ &= 16r(2r - 1)(2r + 1) \end{aligned}$$

Therefore, $2n(2n + 2)(2n + 4)$ is divisible by 16.

So, in both cases we have shown that the product of any three consecutive even numbers is divisible by 16. ■

We conclude this chapter with a few remarks on the difference between how mathematicians discover results and how formal rigorous proofs are written down. As mentioned above, each proof has a key intuitive idea (sometimes more than one). Intuition is central to a mathematician's way of thinking and discovering results. Very often the proof of a theorem comes to a mathematician all jumbled up. A mathematician will often experiment with several routes of thought, and logic, and examples, before she/he can hit upon the correct solution or proof. It is only after the creative phase subsides that all the arguments are gathered together to form a proper proof.

It is worth mentioning here that the great Indian mathematician Srinivasa Ramanujan used very high levels of intuition to arrive at many of his statements, which he claimed were true. Many of these have turned out be true and are well known theorems. However, even to this day mathematicians all over the world are struggling to prove (or disprove) some of his claims (conjectures).

Srinivasa Ramanujan (1887–1920)
Fig. A1.5

EXERCISE A1.4

1. Find counter-examples to disprove the following statements:
   (i) If the corresponding angles in two triangles are equal, then the triangles are congruent.
   (ii) A quadrilateral with all sides equal is a square.
   (iii) A quadrilateral with all angles equal is a square.
   (iv) For integers a and b, $\sqrt{a^2 + b^2} = a + b$
   (v) $2n^2 + 11$ is a prime for all whole numbers n.
   (vi) $n^2 - n + 41$ is a prime for all positive integers n.
2. Take your favourite proof and analyse it step-by-step along the lines discussed in Section A1.5 (what is given, what has been proved, what theorems and axioms have been used, and so on).
3. Prove that the sum of two odd numbers is even.
4. Prove that the product of two odd numbers is odd.
5. Prove that the sum of three consecutive even numbers is divisible by 6.
6. Prove that infinitely many points lie on the line whose equation is $y = 2x$.
   (*Hint* : Consider the point $(n, 2n)$ for any integer n.)
7. You must have had a friend who must have told you to think of a number and do various things to it, and then without knowing your original number, telling you what number you ended up with. Here are two examples. Examine why they work.
   (i) Choose a number. Double it. Add nine. Add your original number. Divide by three. Add four. Subtract your original number. Your result is seven.
   (ii) Write down any three-digit number (for example, 425). Make a six-digit number by repeating these digits in the same order (425425). Your new number is divisible by 7, 11 and 13.

A1.6 Summary

In this Appendix, you have studied the following points:

1. In mathematics, a statement is only acceptable if it is either always true or always false.
2. To show that a mathematical statement is false, it is enough to find a single counter-example.
3. Axioms are statements which are assumed to be true without proof.
4. A conjecture is a statement we believe is true based on our mathematical intuition, but which we are yet to prove.
5. A mathematical statement whose truth has been established (or proved) is called a theorem.
6. The main logical tool in proving mathematical statements is deductive reasoning.
7. A proof is made up of a successive sequence of mathematical statements. Each statement in a proof is logically deduced from a previously known statement, or from a theorem proved earlier, or an axiom, or the hypothesis.

ANSWERS/HINTS

EXERCISE 1.1

1. Yes. $0 = \frac{0}{1} = \frac{0}{2} = \frac{0}{3}$ etc., denominator q can also be taken as negative integer.

2. There can be infinitely many rationals betwen numbers 3 and 4, one way is to take them

$3 = \frac{21}{6+1}, 4 = \frac{28}{6+1}\cdot$ Then the six numbers are $\frac{22}{7}, \frac{23}{7}, \frac{24}{7}, \frac{25}{7}, \frac{26}{7}, \frac{27}{7}$.

3. $\frac{3}{5} = \frac{30}{50}, \frac{4}{5} = \frac{40}{50}\cdot$ Therefore, five rationals are : $\frac{31}{50}, \frac{32}{50}, \frac{33}{50}, \frac{34}{50}, \frac{35}{50}$.

4. (i) True, since the collection of whole numbers contains all the natural numbers.

(ii) False, for example -2 is not a whole number.

(iii) False, for example $\frac{1}{2}$ is a rational number but not a whole number.

EXERCISE 1.2

1. (i) True, since collection of real numbers is made up of rational and irrational numbers.

(ii) False, 'm' not only be a natural number, it may be a possitive rational.

(iii) False, for example 2 is real but not irrational.

2. No. For example, $\sqrt{4} = 2$ is a rational number.

3. Repeat the procedure as in Fig. 1.8 several times. First obtain $\sqrt{4}$ and then $\sqrt{5}$.

EXERCISE 1.3

1. (i) 0.36, terminating. (ii) $0.\overline{09}$, non-terminating repeating.
(iii) 4.125, terminating. (iv) $0.\overline{230769}$, non-terminating repeating.
(v) $0.\overline{18}$ non-terminating repeating. (vi) 0.8225 terminating.

2. $\frac{2}{7} = 2 \times \frac{1}{7} = 0.\overline{285714}$, $\frac{3}{7} = 3 \times \frac{1}{7} = 0.\overline{428571}$,

$\frac{4}{7} = 4 \times \frac{1}{7} = 0.\overline{571428}$, $\frac{5}{7} = 5 \times \frac{1}{7} = 0.\overline{714285}$,

$\frac{6}{7} = 6 \times \frac{1}{7} = 0.\overline{857142}$

3. (i) $\frac{2}{3}$ [Let $x = 0.666\ldots$ So $10x = 6.666\ldots$ or, $10x = 6 + x$ or, $x = \frac{6}{9} = \frac{2}{3}$]

(ii) $\frac{43}{90}$ (iii) $\frac{1}{999}$

4. 1 [Let $x = 0.9999\ldots$ So $10\,x = 9.999\ldots$ or, $10\,x = 9 + x$ or, $x = 1$]
5. $0.\overline{0588235294117647}$
6. The prime factorisation of q has only powers of 2 or powers of 5 or both.
7. 0.01001000100001…, 0.202002000200002…, 0.003000300003…
8. 0.75075007500075000075…, 0.767076700767000767…, 0.808008000800008…
9. (i) and (v) irrational; (ii), (iii) and (iv) rational.

EXERCISE 1.4

1. Proceed as in Section 1.4 for 2.665.
2. Proceed as in Example 11.

EXERCISE 1.5

1. (i) Irrational (ii) Rational
(iii) Rational (iv) Irrational
(v) Irrational

2. (i) $6 + 3\sqrt{2} + 2\sqrt{3} + \sqrt{6}$ (ii) 6

(iii) $7 + 2\sqrt{10}$ (iv) 3

3. There is no contradiction. Remember that when you measure a length with a scale or any other device, you only get an approximate rational value. So, you may not realise that either *c* or *d* is irrational.

4. Refer Fig. 1.17.

5. (i) $\frac{\sqrt{7}}{7}$ (ii) $\sqrt{7} + \sqrt{6}$

(iii) $\frac{\sqrt{5} - \sqrt{2}}{3}$ (iv) $\frac{\sqrt{7} + 2}{3}$

EXERCISE 1.6

1. (i) 8 (ii) 2 (iii) 5

2. (i) 27 (ii) 4 (iii) 8

(iv) $\frac{1}{5}\left[(125)^{-\frac{1}{3}} = \left(5^3\right)^{-\frac{1}{3}} = 5^{-1}\right]$

3. (i) $2^{\frac{13}{15}}$ (ii) 3^{-21} (iii) $11^{\frac{1}{4}}$ (iv) $56^{\frac{1}{2}}$

EXERCISE 2.1

1. (i) False. This can be seen visually by the student.

(ii) False. This contradicts Axiom 2.1.

(iii) True. (Postulate 2)

(iv) True. If you superimpose the region bounded by one circle on the other, then they coincide. So, their centres and boundaries coincide. Therefore, their radii will coincide.

(v) True. The first axiom of Euclid.

3. There are several undefined terms which the student should list. They are consistent, because they deal with two different situations — (i) says that given two points A and

B, there is a point C lying on the line in between them; (ii) says that given A and B, you can take C not lying on the line through A and B.

These 'postulates' do not follow from Euclid's postulates. However, they follow from Axiom 2.1.

4. $$AC = BC$$

So, $$AC + AC = BC + AC \quad \text{(Equals are added to equals)}$$

i.e., $$2AC = AB \quad \text{(BC + AC coincides with AB)}$$

Therefore, $$AC = \frac{1}{2}AB$$

5. Make a temporary assumption that different points C and D are two mid-points of AB. Now, you show that points C and D are not two different points.

6. $$AC = BD \quad \text{(Given)} \quad (1)$$

$$AC = AB + BC \quad \text{(Point B lies between A and C)} \quad (2)$$

$$BD = BC + CD \quad \text{(Point C lies between B and D)} \quad (3)$$

Substituting (2) and (3) in (1), you get

$$AB + BC = BC + CD$$

So, $$AB = CD \quad \text{(Subtracting equals from equals)}$$

7. Since this is true for any thing in any part of the world, this is a universal truth.

EXERCISE 2.2

1. Any formulation the student gives should be discussed in the class for its validity.

2. If a straight line l falls on two straight lines m and n such that sum of the interior angles on one side of l is two right angles, then by Euclid's fifth postulate the line will not meet on this side of l. Next, you know that the sum of the interior angles on the other side of line l will also be two right angles. Therefore, they will not meet on the other side also. So, the lines m and n never meet and are, therefore, parallel.

EXERCISE 3.1

1. 30°, 250° **2.** 126° **4.** Sum of all the angles at a point = 360°

5. ∠ QOS = ∠ SOR + ∠ ROQ and ∠ POS = ∠ POR – ∠ SOR. **6.** 122°, 302°

EXERCISE 3.2

1. 130°, 130° **2.** 126° **3.** 126°, 36°, 54° **4.** 60° **5.** 50°, 77°

6. Angle of incidence = Angle of reflection. At point B, draw BE ⊥ PQ and at point C, draw CF ⊥ RS.

EXERCISE 3.3

1. 65° **2.** 32°, 121° **3.** 92° **4.** 60° **5.** 37°, 53°

6. Sum of the angles of Δ PQR = Sum of the angles of Δ QTR and ∠ PRS = ∠ QPR + ∠ PQR.

EXERCISE 4.1

1. (i) and (ii) are polynomials in one variable, (v) is a polynomial in three variables, (iii), (iv) are not polynomials, because in each of these exponent of the variable is not a whole number.

2. (i) 1 (ii) –1 (iii) $\frac{\pi}{2}$ (iv) 0

3. $3x^{35} - 4$; $\sqrt{2}\,y^{100}$ (You can write some more polynomials with different coefficients.)

4. (i) 3 (ii) 2 (iii) 1 (iv) 0

5. (i) quadratic (ii) cubic (iii) quadratic (iv) linear (v) linear (vi) quadratic (vii) cubic

EXERCISE 4.2

1. (i) 3 (ii) –6 (iii) –3

2. (i) 1, 1, 3 (ii) 2, 4, 4 (iii) 0, 1, 8 (iv) –1, 0, 3

3. (i) Yes (ii) No (iii) Yes (iv) Yes (v) Yes (vi) Yes

(vii) $-\frac{1}{\sqrt{3}}$ is a zero, but $\frac{2}{\sqrt{3}}$ is not a zero of the polynomial (viii) No

4. (i) –5 (ii) 5 (iii) $\frac{-5}{2}$ (iv) $\frac{2}{3}$

(v) 0 (vi) 0 (vii) $-\frac{d}{c}$

EXERCISE 4.3

1. (i) 0 (ii) $\frac{27}{8}$

(iii) 1 (iv) $-\pi^3 + 3\pi^2 - 3\pi + 1$ (v) $-\frac{27}{8}$

2. $5a$ **3.** No, since remainder is not zero.

EXERCISE 4.4

1. $(x + 1)$ is a factor of (i), but not the factor of (ii), (iii) and (iv).

2. (i) Yes (ii) No (iii) Yes

3. (i) –2 (ii) $-\left(2 + \sqrt{2}\right)$ (iii) $\sqrt{2} - 1$ (iv) $\frac{3}{2}$

4. (i) $(3x - 1)(4x - 1)$ (ii) $(x + 3)(2x + 1)$ (iii) $(2x + 3)(3x - 2)$ (iv) $(x + 1)(3x - 4)$

5. (i) $(x - 2)(x - 1)(x + 1)$ (ii) $(x + 1)(x + 1)(x - 5)$

(iii) $(x + 1)(x + 2)(x + 10)$ (iv) $(y - 1)(y + 1)(2y + 1)$

EXERCISE 4.5

1. (i) $x^2 + 14x + 40$ (ii) $x^2 - 2x - 80$ (iii) $9x^2 - 3x - 20$

(iv) $y^4 - \frac{9}{4}$ (v) $9 - 4x^2$

2. (i) 11021 (ii) 9120 (iii) 9984

3. (i) $(3x + y)(3x + y)$ (ii) $(2y - 1)(2y - 1)$ (iii) $\left(x + \frac{y}{10}\right)\left(x - \frac{y}{10}\right)$

4. (i) $x^2 + 4y^2 + 16z^2 + 4xy + 16yz + 8xz$

(ii) $4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz$

(iii) $4x^2 + 9y^2 + 4z^2 - 12xy + 12yz - 8xz$

(iv) $9a^2 + 49b^2 + c^2 - 42ab + 14bc - 6ac$

(v) $4x^2 + 25y^2 + 9z^2 - 20xy - 30yz + 12xz$

(vi) $\frac{a^2}{16} + \frac{b^2}{4} + 1 - \frac{ab}{4} - b + \frac{a}{2}$

5. (i) $(2x + 3y - 4z)(2x + 3y - 4z)$ (ii) $\left(-\sqrt{2}\,x + y + 2\sqrt{2}\,z\right)\left(-\sqrt{2}\,x + y + 2\sqrt{2}\,z\right)$

6. (i) $8x^3 + 12x^2 + 6x + 1$ (ii) $8a^3 - 27b^3 - 36a^2b + 54ab^2$

(iii) $\frac{27}{8}x^3 + \frac{27}{4}x^2 + \frac{9}{2}x + 1$ (iv) $x^3 - \frac{8}{27}y^3 - 2x^2y + \frac{4xy^2}{3}$

7. (i) 970299 (ii) 1061208 (iii) 994011992

8. (i) $(2a + b)(2a + b)(2a + b)$ (ii) $(2a - b)(2a - b)(2a - b)$

(iii) $(3 - 5a)(3 - 5a)(3 - 5a)$ (iv) $(4a - 3b)(4a - 3b)(4a - 3b)$

(v) $\left(3p - \frac{1}{6}\right)\left(3p - \frac{1}{6}\right)\left(3p - \frac{1}{6}\right)$

10. (i) $(3y + 5z)(9y^2 + 25z^2 - 15yz)$ (ii) $(4m - 7n)(16m^2 + 49n^2 + 28mn)$

11. $(3x + y + z)(9x^2 + y^2 + z^2 - 3xy - yz - 3xz)$

12. Simiplify RHS.

13. Put $x + y + z = 0$ in Identity VIII.

14. (i) -1260. Let $a = -12$, $b = 7$, $c = 5$. Here $a + b + c = 0$. Use the result given in Q13.

(ii) 16380

15. (i) One possible answer is : Length = $5a - 3$, Breadth = $5a - 4$

(ii) One possible answer is : Length = $7y - 3$, Breadth = $5y + 4$

16. (i) One possible answer is : 3, x and $x - 4$.

(ii) One possible answer is : $4k$, $3y + 5$ and $y - 1$.

EXERCISE 5.1

1. They are equal. **6.** $\angle$ BAC = $\angle$ DAE

EXERCISE 5.2

6. $\angle$ BCD = $\angle$ BCA + $\angle$ DCA = $\angle$ B + $\angle$ D   **7.** each is of 45°

EXERCISE 5.3

3. **(ii)** From (i), $\angle$ ABM = $\angle$ PQN

EXERCISE 5.4

4. Join BD and show $\angle$ B > $\angle$ D. Join AC and show $\angle$ A > $\angle$ C.

5. $\angle$ Q + $\angle$ QPS > $\angle$ R + $\angle$ RPS, etc.

EXERCISE 7.1

1. 36°, 60°, 108° and 156°.

6. (i) From Δ DAC and Δ BCA, show $\angle$ DAC = $\angle$ BCA and $\angle$ ACD = $\angle$ CAB, etc.

(ii) Show $\angle$ BAC = $\angle$ BCA, using Theorem 7.4.

EXERCISE 7.2

2. Show PQRS is a parallelogram. Also show PQ || AC and PS || BD. So, $\angle$ P = 90°.

5. AECF is a parallelogram. So, AF || CE, etc.

EXERCISE A1.1

1. (i) False. There are 12 months in a year.

(ii) Ambiguous. In a given year, Diwali may or may not fall on a Friday.

(iii) Ambiguous. At some time in the year, the temperature in Magadi, may be 26° C.

(iv) Always true.

(v) False. Dogs cannot fly.

(vi) Ambiguous. In a leap year, February has 29 days.

2. (i) False. The sum of the interior angles of a quadrilateral is 360°.

(ii) True   (iii) True   (iv) True

(v) False, for example, 7 + 5 = 12, which is not an odd number.

3. (i) All prime numbers greater than 2 are odd. (ii) Two times a natural number is always even. (iii) For any $x > 1$, $3x + 1 > 4$. (iv) For any $x \geq 0$, $x^3 \geq 0$.

(v) In an equilateral triangle, a median is also an angle bisector.

EXERCISE A1.2

1. (i) Humans are vertebrates. (ii) No. Dinesh could have got his hair cut by anybody else. (iii) Gulag has a red tongue. (iv) We conclude that the gutters will have to be cleaned tomorrow. (v) All animals having tails need not be dogs. For example, animals such as buffaloes, monkeys, cats, etc. have tails but are not dogs.

2. You need to turn over B and 8. If B has an even number on the other side, then the rule has been broken. Similarly, if 8 has a consonant on the other side, then the rule has been broken.

EXERCISE A1.3

1. Three possible conjectures are:

(i) The product of any three consecutive even numbers is even. (ii) The product of any three consecutive even numbers is divisible by 4. (iii) The product of any three consecutive even numbers is divisible by 6.

2. Line 4: 1 3 3 1 = 11^3; Line 5: 1 4 6 4 1 = 11^4; the conjecture holds for Line 4 and Line 5; No, because $11^5 \neq 15101051$.

3. $T_4 + T_5 = 25 = 5^2$; $T_{n-1} + T_n = n^2$.

4. $111111^2 = 12345654321$; $1111111^2 = 1234567654321$

5. Student's own answer. For example, Euclid's postulates.

EXERCISE A1.4

1. (i) You can give any two triangles with the same angles but of different sides.

(ii) A rhombus has equal sides but may not be a square.

(iii) A rectangle has equal angles but may not be a square.

(iv) For $a = 3$ and $b = 4$, the statement is not true.

(v) For $n = 11$, $2n^2 + 11 = 253$ which is not a prime.

(vi) For $n = 41$, $n^2 - n + 41$ is not a prime.

2. Student's own answer.

3. Let x and y be two odd numbers. Then $x = 2m + 1$ for some natural number m and $y = 2n + 1$ for some natural number n.

$x + y = 2\,(m + n + 1)$. Therefore, $x + y$ is divisible by 2 and is even.

4. See Q.3. $xy = (2m + 1)(2n + 1) = 2\,(2mn + m + n) + 1$.

Therefore, $x\,y$ is not divisible by 2, and so it is odd.

5. Let $2n$, $2n + 2$ and $2n + 4$ be three consecutive even numbers. Then their sum is $6(n + 1)$, which is divisible by 6.

7. (i) Let your original number be n. Then we are doing the following operations:

$$n \to 2n \to 2\,n + 9 \to 2\,n + 9 + n = 3n + 9 \to \frac{3n + 9}{3} = n + 3 \to n + 3 + 4 = n + 7 \to n + 7 - n = 7.$$

(ii) Note that $7 \times 11 \times 13 = 1001$. Take any three digit number say, *abc*. Then *abc* × 1001 = *abcabc*. Therefore, the six digit number *abcabc* is divisible by 7, 11 and 13.

****