INTERMEDIATE EXAMINATION - 2018

(ANNUAL)

MATHEMATICS

Time- $3\frac{1}{4}$ **Hours**

Full Marks: 100

Instruction for the candidates:

1) Candidates are required to give their answers in their own words as far as practicable.
2) Figures in the right hand margin indicate full marks.
3) 15Minutes of extra time has been allotted for the candidates to read the questions carefully.
4) This question paper is divided into two **section A** and **section B.**
5) In **section-A,** there are **50 objective type questions** which are compulsory, each carry **1 mark.** Darken the circle with blue/black ball pen against the correct option on **OMR** answer Sheet provided to you. **Do not use whitener/Liquid/Blade/Nall etc. on OMR Sheet; otherwise the result will be invalid.**
6) In **Section-B,** there **are 25 short answer** type question (each carrying **2 marks**),out of which **any 15** questions are to be answered. Apart this, there are **8 Long Answer** Type question (Each Carrying **5 Marks**), Out of which **any 4** questions to be answered.
7) Use of any electronic appliances is strictly prohibited.

Section-I : (Objective Type)

For the following Question Nos. 1 to 50 there is only one correct answer against each question. Mark the correct option on the answer sheet. [50 X 1=50]

1.let $\vec{a} = \vec{i} + \vec{j} + 2\vec{k}$ How many binary operation can be defined on this set ?

(a)8 (b)10

(c)-16 (d)20

Sol:

Correct option is C

2.let $\{1,2,3\}$ which of the following function $f : A \to A$ does not an inverse function ?

(A) $\{(1,1),(2,2),(3,3)\}$ (b) $\{(1,2),(2,1),(3,1)\}$

(c) $\{(1,3),(3,2),(2,1)\}$ (d) $\{(1,2),(2,3),(3,1)\}$

Sol:

Correct option is B

3.If $A=\{1,2,3\}$ $B=\{6,7,8\}$ and $f:A\to B$ is a function such that $f(x)=x+5$ then what type of a function is f ?

(a)into (b)one-one onto

(c)many-one onto (d)Constant function

SOL:

Correct option is B

4. What type of a relation is " Less then" in the set of real numbers ?

(a)Only symmetric (b) Only transitive

(c)Only reflexive (d) equivalence relation

SOL:

Correct option is B

5. $\cos^{-1}\left(\cos\frac{8\pi}{5}\right)=$

(a) $\frac{8\pi}{5}$ (b) $\frac{12\pi}{5}$

(c) $\frac{2\pi}{5}$ (d) $\frac{4\pi}{5}$

SOL:

Correct option is D

6. $\cos^{-1}(2x-1)=$

(a) $2\cos^{-1}x$ (b) $\cos^{-1}\sqrt{x}$

(c) $2\cos^{-1}\sqrt{x}$ (d) None of these

SOL:

Correct option is D

7. $2\cot^{-1}3+\cos^{-1}7=$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) π (d) $\frac{\pi}{6}$

SOL:

Correct option is B

8. $\tan^{-1}(1)+\cos^{-1}\left(\frac{-1}{2}\right)+\sin^{-1}\left(\frac{-1}{2}\right)=$

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$

(c) $-\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

SOL:

Correct option is B

9.If $\lambda\in R$ and $\Delta=\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ then $\lambda\Delta=$

(a) $\begin{vmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{vmatrix}$ (b) $\begin{vmatrix} \lambda a & b \\ c & d \end{vmatrix}$

(c) $\begin{vmatrix} \lambda a & b \\ \lambda c & d \end{vmatrix}$ (d) None of these

SOL:

Correct option is A

10. If a,b and c aren in A.P then $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} =$

a)3 (b) -3

(c)0 (d) 1

SOL:

Correct option is C

11. If 7 and 2 are two roots of the equation $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ then the third root is:

(a) -9 (b) 14

(c) $\frac{1}{2}$ (d) None of these

SOL:

Correct option is A

12. If $\omega \neq 1, \omega^3 = 1$ and $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$ then x =

(a)1 (b) ω

(c) ω^2 (d) 0

SOL:

Correct option is D

13. If $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ and $A + A' = I_2$ then $\alpha =$

(a) π (b) $\frac{\pi}{3}$

(c) $\frac{3\pi}{2}$ (d) $\frac{\pi}{6}$

SOL:

Correct option is B

14.If A be a square matrix. Then $A+A'$ will be………..

(a)Symmetric matrix (b) Skew symmetric matrix

(c) Null matrix (d) Unit matrix

SOL:

Correct option is A

15. If A is a matrix of order $A+A'$ such that $A+A'$ then $A+A'$ is equal to?

(a) I_3 (b)A

(c)3A (d) $I_3 - A$

SOL:

Correct option is A

16.Let A be a non-singular matrix of the order 2×2 then $|adjA| =$………..

(a) $2|A|$ (b) $|A|$

(c) $|A|^2$ (d) $|A|^3$

SOL:

Correct option is C

17. $\frac{d}{dx}\left[\log\left(\sec x+\tan x\right)\right]=$

(a) $\frac{1}{\sec x+\tan x}$ (b) $\sec x$

(c) $\tan x$ (d) $\sec x+\tan x$

SOL:

Correct option is A

18. If $x^2 y^3 = (x+y)^5$ then, $\frac{dy}{dx}$

(a) $\frac{x}{y}$ (b) $\frac{y}{x}$

(c) $-\frac{y}{x}$ (d) $-\frac{x}{y}$

SOL:

Correct option is B

19. $\frac{d}{dx}\left[\tan^{-1}\sqrt{1+x^2} - \cot^{-1}\left(-\sqrt{1+x^2}\right)\right] =$

(a) π (b) 1

(c)0 (d) $\frac{2x}{\sqrt{1+x^2}}$

SOL:

Correct option is C

20. $\frac{d(2^x)}{d(3^x)} =$

(a) $\left(\frac{2}{3}\right)^x$ (b) $\frac{2^{x-1}}{3^{x-1}}$

(c) $\left(\frac{2}{3}\right)^x \log_3 2$ (d) $\left(\frac{2}{3}\right)^x \log_2 3$

SOL:

Correct option is C

21. $f(x) = \sqrt{3}\sin x + \cos x$ is maximum then value of x=…………..

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

SOL:

Correct option is C

22. If $y = \log \cos x^2$ then $\frac{dy}{dx}$ at $x = \sqrt{x}$ has the value

(a)1 (b) $\frac{\pi}{4}$

(c)0 (d) $\sqrt{\pi}$

SOL:

Correct option is C

23.Equation of the tangent to the curve $x^2 + y^2 = a^2$ at (x_1, y_1) is

(a) $xx_1 - yy_1 = 0$ (b) $xx_1 + yy_1 = 0$

(c) $xx_1 - yy_1 = a^2$ (d) $xx_1 + yy_1 = a^2$

SOL:

Correct option is D

24. $\frac{d}{dx}\left[\lim_{x \to a} \frac{x^5 - a^5}{x - a}\right] =$

(a) $5a^4$ (b) $5x^4$

(c)1 (d) 0

SOL:

Correct option is D

25. $\int \sqrt{1 + \cos 2x}\, dx =$

(a) $\sqrt{2} \cos x + c$ (b) $\sqrt{2} \sin x + c$

(c) $-\cos x - \sin x + c$ (d) $\sqrt{2}\sin\frac{x}{2} + c$

SOL:

Correct option is B

26. $\int x^2 \cdot e^{x^3} dx =$

(a) $e^{x^3} + c$ (b) $\frac{1}{3}e^{x^3} + c$

(c) $e^{x^2} + c$ (d) $\frac{1}{3}e^{x^2} + c$

SOL:

Correct option is B

27. $\int \frac{xe^x}{(x+1)^2} dx =$

(a) $\frac{e^x}{(x+1)^2} + c$ (b) $\frac{-e^x}{x+1} + c$

(c) $\frac{e^x}{x+1} + c$ (d) $\frac{-e^x}{(x+1)^2} + c$

SOL:

Correct option is C

28 $\int \frac{dx}{a^2 + x^2} =$

(a) $\frac{1}{a}\tan^{-1}\frac{a}{x} + c$ (b) $\tan^{-1}\frac{x}{a} + c$

(c) $\frac{1}{a}\tan^{-1}\frac{x}{a} + c$ (d) $\frac{1}{a}\tan^{-1} x + c$

SOL:

Correct option is C

29. If $f(-x) = -f(x)$ then $\int_{-a}^{a} f(x)dx =$

(a) $2\int_{0}^{a} f(x)dx$ (b) 0

(c)1 (d) -1

SOL:

Correct option is B

30. $\int_{\alpha}^{\beta} \varphi(x)dx + \int_{\beta}^{\alpha} \varphi(x)dx =$

(a)1 (b) $2\int_{\alpha}^{\beta} \varphi(x)dx$

(c) $-2\int_{\beta}^{\alpha} \varphi(x)dx$ (d) 0

SOL:

Correct option is D

31.Area between the x-axis and the curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$:

(a)2 (b) -1

(c)1 (d) None of these

SOL:

Correct option is C

32. $\int_{0}^{1} xdx = ...$

(a)0 (b) 1

(c)2 (d) $\frac{1}{2}$

SOL:

Correct option is D

33. The differential equation $1+\left(\frac{dy}{dx}\right)^2=\left(\frac{d^2y}{dx^2}\right)^3$ is of order =……….and degree=………..

(a) Order = 2,degree=3

(b) Order = ,degree= 2

(c) Order = 2,degree= 2

(d) None of these

SOL:

Correct option is A

34. solution of the differential equation $ydx - xdy = xydx$ is

(a) $\frac{y^2}{2}-\frac{x^2}{2}=xy+c$

(b) $x = kye^x$

(c) $x = kye^y$

(d) None of these

SOL:

Correct option is B

35 Integrating factor (I.F) of differential equation $\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$ is

(a) $\log x$

(b) x

(c) $\frac{1}{x}$

(d) None of these

SOL:

Correct option is B

36.solution of $\frac{xdy - ydx}{x^2+y^2}=0$ is:

(a) $\frac{x^2}{2}+\tan^{-1}\frac{x}{y}=k$

(b) $\frac{x^2}{2}+\tan^{-1}\frac{y}{x}=k$

(c) $\frac{x^2}{2}-\tan^{-1}\frac{x}{y}=k$

(d) $\frac{x^2}{2}-\tan^{-1}\frac{y}{x}=k$

SOL:

Correct option is B

37. If $\vec{a} = \vec{i} + \vec{j} + 2\vec{k}$,then the corresponding unit vector $\hat{a}$ in the direction of $\vec{a}$ =

(a) $\frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{6}}$ (b) $\frac{\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{6}}$

(c) $\frac{\vec{i} + \vec{j} + 2\vec{k}}{6}$ (d) None of these

SOL:

Correct option is B

38.The direction cosines of the vector $3\vec{i} - 4\vec{j} + 12\vec{k}$ is

(a) $\frac{3}{13}, \frac{4}{13}, \frac{12}{13}$ (b) $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$

(c) $\frac{3}{\sqrt{13}}, \frac{4}{\sqrt{13}}, \frac{12}{\sqrt{13}}$ (d) $\frac{3}{\sqrt{13}}, \frac{-4}{\sqrt{13}}, \frac{12}{\sqrt{13}}$

SOL:

Correct option is B

39. $x\vec{i} - 3\vec{j} + 5\vec{k}, -x\vec{i} + x\vec{j} + 2\vec{k}$ are perpendicular to each other then value of x =

(a) $-2, 5$ (b) $2, 5$

(c) $-2, -5$ (d) $2, -5$

SOL:

Correct option is D

40 $\vec{i} \times (\vec{i} \times \vec{j}) + \vec{j} \times (\vec{j} \times \vec{k}) + \vec{k} (\vec{k} \times \vec{i})$

(a) $\vec{i} + \vec{j} + \vec{k}$ (b) 0

(c)1 (d) $-(\vec{i} + \vec{j} + \vec{k})$

SOL:

Correct option is C

41.The direction Cosines of y axis are:

(a) $(1,0,1)$ (b) $(0,1,0)$

(c) $\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$ (d) None of these

SOL:

Correct option is B

42. The equation of the xy-plane is:

(a) $x=0, y=0$ (b) $z=0$

(c) $x=0, y\neq 0$ (d) None of these

SOL:

Correct option is B

43. If two planes $-2x-4y+3z=5$ and $x+2y+\lambda z=12$ are perpendicular to each other, then $\lambda =$

(a)-2 (b) 2

(c)3 (d) None of these

SOL:

Correct option is B

44. The distance between (4,3,7) and (1,-1,-5) is:

(a)13 (b) 15

(c)12 (d) 5

SOL:

Correct option is A

45.If A' and B' are independent events then:

(a) $P(A'B')=P(A).P(B)$ (b) $P(A'B')=P(A')+P(B')$

(c) $P(A'B') = P(A').P(B')$ (d) $P(A'B') = P(A') - P(B')$

SOL:

Correct option is C

46.If events A and B are mutually exclusive then:

(a) $P\left(A \cap B\right) = P(A) \cdot P(B)$ (b) $P\left(A \cap B\right) = 0$

(c) $P\left(A \cap B\right) = 1$ (d) $P\left(A \cup B\right) = 0$

SOL:

Correct option is B

47.If $P\left(A\right) = \frac{3}{8}, P(B) = \frac{1}{2}$ and $P\left(A \cap B\right) = \frac{1}{4}$ then $P\left(\frac{A}{B}\right) =$

(a) 2 (b) $\frac{1}{2}$

(c) $\frac{2}{3}$ (d) $\frac{3}{2}$

SOL:

Correct option is C

48. If A and B are two events such that $P(A) \neq 0$ and $P\left(\frac{B}{A}\right) = 1$

(a) $B \subset A$ (b) $A \subset B$

(c) $B = \varphi$ (d) $A \cap B = \varphi$

SOL:

Correct option is B

49. $\int \frac{dx}{x + \sqrt{x}}$

(a) $\log x + \log\left(1 + \sqrt{x}\right) + C$ (b) $2\log\left(1 + \sqrt{x}\right) + C$

(c) $\log\left(1+\sqrt{x}\right)+C$ (d) $\log\left(\sqrt{x}\right)+C$

SOL:

Correct option is B

50. solution of $\lim_{n\to\infty}\left[\frac{e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+........+e^{\frac{n}{n}}}{n}\right] is$

(a) $1-e$ (b) $e-1$

(c) e (d) 1

SOL:

Correct option is C

SECTION-B (Non-Objective Type Question)

Question NO.1 to 22 are short answer type. Answer any 15 question. Each question carries 2 marks.

1.Examine whether the function $f:R\to R$ is one-one(injective) if $f(x)=x^3, x\in R$

Sol:

Given that function $f:R\to R$ is denote as $f(x)=x^3, x\in R$

Let $x_1, x_2 \in R$

$$\Rightarrow f(x_1)=f(x_2)$$
$$\Rightarrow x_1^3=x_2^3$$
$$\Rightarrow x_1=x_2$$

Thus the given function is one-one or injective.

2.Prove that $\tan\left[\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\frac{1-x^2}{1+x^2}\right]=\frac{2x}{1-x^2}$

Sol:

Given that,

$$2tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}$$
$$=\cos^{-1}\frac{1-x^2}{1+x^2}$$
$$=\tan^{-1}\frac{2x}{1+x^2}$$

Then,

$$L.H.S=\tan\left[\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\frac{1-x^2}{1+x^2}\right]$$
$$=\tan\left[\frac{1}{2}.2\tan^{-1}x+\frac{1}{2}.2\tan^{-1}x\right]$$
$$=\tan\left[\tan^{-1}x+\tan^{-1}x\right]$$
$$=\tan\left[2\tan^{-1}x\right]=\tan\left[\tan^{-1}\frac{2x}{1-x^2}\right]$$
$$=\frac{2x}{1-x^2}\left[\tan\left[\tan^{-1}x\right]\right]=x$$

Hence, LHS = RHS proved.

3.Prove that $\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{16}{65}$

Sol:

Let,

$$\alpha = \sin^{-1}\frac{3}{5}$$

$$\Rightarrow \sin\alpha = \frac{3}{5}$$

$$\therefore \cos\alpha = \sqrt{1-\sin^2\alpha}$$

$$\Rightarrow \cos\alpha = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} ----------(1)$$

and,

$$\beta = \cos^{-1}\frac{12}{13} \Rightarrow \cos\beta = \frac{12}{13}$$

$$\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13} ------(2)$$

Now, we know that: $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Putting the value of (1) and(2), we get

$$\sin(\alpha-\beta) = \frac{3}{5}\times\frac{12}{13} - \frac{4}{5}\times\frac{5}{13} = \frac{36}{65} - \frac{20}{65} = \frac{16}{65}$$

$$\Rightarrow \sin(\alpha-\beta) = \frac{16}{65}$$

$$\therefore \alpha+\beta = \sin^{-1}\frac{16}{65}$$

$$\Rightarrow \sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{16}{65}$$

4.Find the values of x and y when

$$\begin{vmatrix} 2 & 3 \\ y & x \end{vmatrix} \text{and} \begin{vmatrix} x & y \\ 2 & 1 \end{vmatrix} = \frac{7}{2}$$

Sol:

Given that,

$$\begin{vmatrix} 2 & 3 \\ y & x \end{vmatrix} \text{and} \begin{vmatrix} x & y \\ 2 & 1 \end{vmatrix} = \frac{7}{2}$$

Now,

$$\begin{vmatrix} 2 & 3 \\ y & x \end{vmatrix} = 4 \Rightarrow 2x - 3y = 4$$

And,

$$\begin{vmatrix} x & y \\ 2 & 1 \end{vmatrix} = \frac{7}{2}$$

$$\Rightarrow x - 2y = \frac{7}{2}$$

Multiplying equation (2) by 2 and subtract from equation (1), we get

$$2x - 3y - 2x + 4y = 4 - \frac{14}{2} = \frac{8-14}{2}$$

$$\Rightarrow y = \frac{-6}{2} = -3$$

Putting the value of y in (i), we get

$$2x - 3y = 4 \Rightarrow 2x = 4 + 3y = 4 - 9 = -5$$

$$\Rightarrow 2x = -5$$

$$\Rightarrow x = \frac{-5}{2}$$

Thus, $x = \frac{-5}{2}$ and $y = -3$

5.Prove that: $\begin{bmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{bmatrix} = x^2(x+a+b+c)$

Sol:

Let,

$$LHS = \begin{bmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{bmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\begin{bmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & c \\ x+a+b+c & b & x+c \end{bmatrix}$$

$$= (x+a+b+c)\begin{bmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{bmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ we get

$$= (x+a+b+c)\begin{bmatrix} 1-1 & b-x-b & c-c \\ 1-1 & x+b & c-x-c \\ 1 & b & x+c \end{bmatrix}$$

$$= (x+a+b+c)\begin{bmatrix} 0 & -x & 0 \\ 0 & x+b & -x \\ 1 & b & x+c \end{bmatrix}$$

$$= (x+a+b+c)\begin{bmatrix} -x & 0 \\ x & -x \end{bmatrix} = (x+a+b+c)(x^2-0)$$

$$= x^2(x+a+b+c)$$

6.If $2\begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then prove that:

$f(x+y) = f(x), f(y)$

Sol:

Given that,

$$f(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow f(y)=\begin{bmatrix}\cos y & -\sin y & 0\\ \sin y & \cos y & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\cos x\cos y-\sin x\sin y & -\cos x\sin y & 0\\ \sin x\cos y+\cos x\sin y & -\sin x\sin y & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}\cos(x+y) & -\sin(x+y) & 0\\ \sin(x+y) & \cos(x+y) & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= f(x+y)=RHS$$

7.Find the value of x, such that: $\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}\begin{bmatrix}x\\ 4\\ 1\end{bmatrix}(x-5-1)=0$

Sol:

Given that,

$$\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}_{3\times 3}\begin{bmatrix}x\\ 4\\ 1\end{bmatrix}_{3\times 1}[x-5-1]_{1\times 3}=0$$

Arranged the given matrices as law of multiplication of matrix. We get

$$[x-5-1]_{1\times 3}\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}_{3\times 3}\begin{bmatrix}x\\ 4\\ 1\end{bmatrix}_{3\times 1}=0$$

$$\Rightarrow\left[x\times 1-5\times 0-1\times 2 \quad x\times 0-5\times 2-1\times 0 \quad x\times 2-5\times 1-1\times 3\right]\begin{bmatrix}x\\ 4\\ 1\end{bmatrix}$$

$$\Rightarrow\left[x-2-10 \quad 2x-8\right]\begin{bmatrix}x\\ 4\\ 1\end{bmatrix}=0$$

$$\Rightarrow\left[x(x-2)-10(4)+(2x-8)\times 1\right]_{1\times 1}=\left[0\right]_{1\times 1}$$

$$\Rightarrow\left[x^2-48\right]_{[x]}=[0]_{[x]}$$

$$\Rightarrow x^2-48=0$$

$$\Rightarrow x^2=48$$

$$\Rightarrow x^2=16\times 3$$

$$x=4\pm\sqrt{3}$$

8.Find $\frac{dy}{dx}$ if $x=y\log(xy)$

Sol:

Given that,

$y=y\log(xy)$

Differentiate w.r.to x in both side,

$$\frac{dx}{dx} = y\frac{d}{dy}\left[\log(xy)\right] + \log(xy)\frac{dy}{dx}$$

$$\Rightarrow 1 = y\frac{d[\log(xy)]}{d(xy)} \times \frac{d(xy)}{dx} + \log(xy)\frac{dy}{dx}$$

$$\Rightarrow 1 = y \times \frac{1}{xy}\left(x\frac{dy}{dx} + y \times 1\right) + \log(xy)\frac{dy}{dx}$$

$$\Rightarrow 1 = \frac{dy}{dx} + \frac{y}{x} + \frac{x}{y}\frac{dy}{dx}[\,from(1)]$$

$$\Rightarrow 1 = \frac{y}{x}\frac{dy}{dx}\left(1 + \frac{x}{y}\right)$$

$$\Rightarrow \frac{x-y}{x} = \frac{dy}{dx}\left(\frac{x+y}{y}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(x-y)}{x(x+y)}$$

9.if $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ then find $\frac{dy}{dx}$

Sol:

Given that,

$$y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$$

$$y = \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right]$$

$$y = \tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]$$

$$\left[\because \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \, and \, 1+\cos A = 2\cos^2\frac{A}{2}\right]$$

$$y = \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]$$

$$y = \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]$$

$$y = \frac{\pi}{4}-\frac{x}{2}$$

$[\because \tan^{-1}(\tan x) = x]$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\frac{\pi}{4}-\frac{x}{2}\right] = \frac{-1}{2}$$

10.If $y = \sqrt{x+\sqrt{x+\sqrt{x+............to\ \infty}}}$ then find $\frac{dy}{dx}$

Sol:

Given that,

$$y = \sqrt{x+\sqrt{x+\sqrt{x+............to\ \infty}}}$$

Let,

$$\sqrt{x+\sqrt{x+..........}} = y$$

Then,

$y = \sqrt{x+y}$

$\Rightarrow y^2 = x + y$

$\Rightarrow 2y\dfrac{dy}{dx} = 1 + \dfrac{dy}{dx}$

$\Rightarrow (2y-1)\dfrac{dy}{dx} = 1$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{2y-1}$

11.Integrate $\int \sec^n \theta . \tan\theta d\theta$

sol:

We know that

$\int \tan^m x \sec^n x$

Here $m=1$i.e power or degree of $\sec x$ is odd positive integer

Then let $\sec x \Rightarrow dz = \sec x \tan x dx$

Now,

$\int \sec^n \theta \tan\theta d\theta$

$= \int \sec^{n-1}\theta \sec\theta \tan\theta d\theta$

$= \int z^{n-1} dz = \dfrac{z^{n-1+1}}{n+1+1} + c$

$= \dfrac{z^n}{n} + c$

$= \dfrac{\sec^n \theta}{n} + c$

$\therefore \int \sec^n \theta \tan\theta d\theta = \dfrac{\sec^n \theta}{n} + c$

12.prove that $\int_0^{2\pi} |\cos x|\, dx = 4$

Given that,

$L.H.S = \int_0^{2\pi} |\cos x| dx$

Here $|\cos x|$ line between $0 \le x \le \frac{\pi}{2}, \frac{\pi}{2} \le x \frac{3\pi}{2}$ and $\frac{3\pi}{2} \le x \le 2\pi$

i.e, $0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$

$$\therefore \int_0^{2\pi} |\cos x| dx = \int_0^{\frac{\pi}{2}} |\cos x|\, dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |\cos x|\, dx + \int_{\frac{3\pi}{2}}^{2\pi} |\cos x|\, dx$$

$$\int_0^{\frac{\pi}{2}} \cos x dx = \int_0^{\frac{\pi}{2}} |\cos x|\, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x dx + \int_{\frac{3\pi}{2}}^{2\pi} |\cos x|\, dx$$

$$= [\sin]_0^{\frac{\pi}{2}} - [\sin]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + [\sin]_{\frac{3\pi}{2}}^{2\pi}$$

$$= \left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right] + \left[\sin 2\pi - \sin\frac{3\pi}{2}\right]$$

$$= [1-0] - [-1-0] + [1+1]$$

$$= 1+1+2 = 4$$

13.Evaluate $\therefore \lim_{x\to\infty} \sum_{r=1}^{n} \frac{n+r}{n^2+r^2}$

Sol:

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n+r}{n^2+r^2} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n\left(1+\frac{r}{n}\right)}{n^2\left(1+\frac{r^2}{n^2}\right)}$$

Now, putting $\frac{r}{n} = x \Rightarrow \frac{r^2}{n^2} = x^2$ and $\frac{1}{n} = dx$

Also taking limits between 0 to 1, we get

$$\lim = \int_0^1 \frac{1+x}{1+x^2}\,dx = \int_0^1 \frac{dx}{1+x^2} + \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$\Rightarrow \lim = \left[\tan^{-1} x\right]_0^1 + \frac{1}{2}\left[\log\left(1+x^2\right)\right]_0^1$$

$$\Rightarrow \lim = \left[\tan^{-1}(1) - \tan^{-1}(0)\right] + \frac{1}{2}\left[\log 2 - \log 1\right]$$

$$\Rightarrow \lim = \left[\tan^{-1}\left(\tan\frac{\pi}{4}\right) - 0\right] + \frac{1}{2}\left[\log 2 - 0\right]$$

$$\Rightarrow \lim = \frac{\pi}{4} + \frac{1}{2}\log 2$$

$$\therefore \lim_{x\to\infty}\sum_{r=1}^{n} \frac{n+r}{n^2+r^2} = \frac{\pi}{4} + \frac{1}{2}\log 2$$

14.solve: $\left(x^2 - y^2\right)\frac{dy}{dx} = 2xy$

Given that,

$$\left(x^2 - y^2\right)\frac{dy}{dx} = 2xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

Putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Putting the value of y and $\frac{dy}{dx}$ in(i)we get

$$v + x\frac{dv}{dx} = \frac{2x.vx}{x^2 - v^2x^2} = \frac{2v.x^2}{(1-v^2)x^2}$$

$$= \frac{2v}{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v}{1-v^2} - v = \frac{2v - v + v^3}{1-v^2} = \frac{v+v^3}{1-v^2} = \frac{v(1+v^2)}{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v(1+v^2)}{1-v^2} \Rightarrow \frac{1-v^2}{v(1+v^2)}dv = \frac{dx}{x}$$

$$\Rightarrow \frac{dx}{x} = \frac{1+v^2-2v^2}{v(1+v^2)}dv = \frac{1+v^2}{v(1+v^2)}dv - \frac{2v^2dv}{v(1+v^2)}$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{v} - \frac{2vdv}{1+v^2}$$

$$\Rightarrow \int\frac{dx}{x} = \int\frac{dv}{v} - \int\frac{2v}{1+v^2}dv$$

$$\Rightarrow \log x = \log y - \log(1+v^2) + \log k$$

$$\Rightarrow \log vk - \log(1+v^2)$$

$$\Rightarrow x = \frac{\frac{ky}{x}}{\frac{x^2+y^2}{x^2}} = \frac{kxy}{x^2+y^2}$$

$$\Rightarrow x^2 + y^2 = ky$$

15.Evaluate $x\cos x\frac{dy}{dx} + y(x\sin x + \cos x) = 1$

Given that,

$$x\cos x\frac{dy}{dx} + y(x\sin x + \cos x) = 1$$

$$\Rightarrow \frac{dy}{dx} + \frac{y(x\sin x + \cos x)}{x\cos x} = \frac{1}{x\cos x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{x\sin x}{x\cos x} + \frac{\cos x}{x\cos x}\right)y = \frac{1}{x\cos x}$$

$$\Rightarrow \frac{dy}{dx}\left(\tan x + \frac{1}{x}\right)y = \frac{1}{x\cos x}$$

It is linear differential equation in the from $\frac{dy}{dx} + Py = Q$

where $P = \tan x + \frac{1}{x}$ and $Q = \frac{\sec x}{x}$

$\therefore (I.F) = e^{\int (\tan x + \frac{1}{x})dx} = e^{\int \tan x dx + \int \frac{dx}{x}} = e^{\log \sec x + \log x}$
$= e^{\log(x \sec x)} = x \sec x$

Now, multiplying (1) by I.F and integrating, we get

$$y \times I.F = \int Q \times I.F + C \Rightarrow yx \sec = \int \frac{\sec x}{x} \times x \sec x = \tan x + c$$

16. Prove by vector method, that the angle inscribed in a semi-circle is right angle.

Sol:

Let O be the centre of semicircle with BOA as its diameter.let P be any point. The $\angle BPA$ is and angle in a semicircle. Take O as the origin. Let the position vectors of A and P are $\vec{a}$ and $\vec{r}$ respectively. Then the position vector of B is $-\vec{a}$

Now,

$\overrightarrow{AP} = (\text{Position vector of P}) - (\text{Position vector of A})$
$\Rightarrow AP = \overrightarrow{OP} - \overrightarrow{OA} = (\vec{r} - \vec{a})$
$\overrightarrow{BP} = (\text{Position vector of P}) - (\text{Position vector of B})$
$\Rightarrow BP = (OP - OB) = [\vec{r} - (-\vec{a})] = (\vec{r} - \vec{a})$
$\therefore \overrightarrow{AP}.\overrightarrow{BP} = (\vec{r} - \vec{a})(\vec{r} + \vec{a})$
$\overrightarrow{AP}.\overrightarrow{BP} = \vec{r}.\vec{r} + \vec{r}.\vec{a} - \vec{a}.\vec{r} - \vec{a}.\vec{a}$
$\overrightarrow{AP}.\overrightarrow{BP} = |\vec{r}|^2 - |\vec{a}|^2 \left[\because \vec{a}.\vec{r} = \vec{r}.\vec{a}\right]$
$\overrightarrow{AP}.\overrightarrow{BP} = 0 \qquad \left[\because |\vec{r}| = |\vec{a}|\right]$

$\therefore BP \perp AP \Rightarrow \angle BPA = \frac{\pi}{2}$

Thus the angle inscribed in a semicircle is a right angle.

17.If $\vec{a}, \vec{b}, \vec{c}$ are three vectors and $\vec{a} + \vec{b} + \vec{c} = 0$ prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

sol:

Given that,

$\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \vec{b} = -\vec{a} - \vec{c}$

Multiplying both sides by cross product $\vec{a}$, we get

$$\vec{b} = \vec{a} - \vec{c} \Rightarrow \vec{a} \times \vec{b} = \vec{a} \times (-\vec{a} - \vec{c})$$
$$\Rightarrow \vec{a} \times \vec{b} = -\vec{a} \times \vec{a} + \vec{a}(-\vec{c}) \Rightarrow \vec{a} \times \vec{b} = -\vec{a} \times \vec{a} - \vec{a} \times \vec{c}$$
$$\Rightarrow \vec{a} \times \vec{b} = \vec{a} \times \vec{a} + \vec{c} \times \vec{a} \Rightarrow \vec{a} \times \vec{b} = 0 + \vec{c} \times \vec{a}$$
$$\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a} -------------(i)$$
$$[\because \vec{a} \times \vec{a} = 0, -\vec{a} \times \vec{c} = \vec{c} \times \vec{a}]$$

Similarly $\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \vec{c} = -\vec{a} - \vec{b}$

Multiplying both sides by $\vec{b}$ cross product, we get

$$\vec{c} = -\vec{a} - \vec{b} \Rightarrow \vec{b} \times \vec{c} = \vec{b} \times (-\vec{a} - \vec{b})$$
$$\Rightarrow \vec{b} \times \vec{c} = -\vec{b} \times \vec{a} + \vec{b} \times (-\vec{b}) \Rightarrow \vec{b} \times \vec{c} = \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$$
$$\Rightarrow \vec{b} \times \vec{c} = \vec{a} \times \vec{b} - 0 \Rightarrow \vec{b} \times \vec{c} = \vec{a} \times \vec{b} ---------(ii)$$
$$[\because -\vec{b} \times \vec{a} = \vec{a} \times \vec{b}, \vec{b} \times \vec{b} = 0]$$

From equation (1)and (2),we get

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

18.Find the value of P, if:

$$\left(2\vec{i} + 6\vec{j} + 27\vec{k}\right) \times \left(\vec{i} + 3\vec{j} + p\vec{k}\right) = 0$$

sol:

Given that,

$$\left(2\vec{i} + 6\vec{j} + 27\vec{k}\right) \times \left(\vec{i} + 3\vec{j} + p\vec{k}\right) = 0$$

let $\vec{a} = 2\vec{i} + 6\vec{j} + 27\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j} + p\vec{k}$

then,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 6 & 27\\ 1 & 3 & p\end{vmatrix}$$

$$\Rightarrow \vec{a}\times\vec{b}=(6p-81)\vec{i}-(2p-27)\vec{j}+(6-6)\vec{k}=0$$

$$6p-81=0,-(2p-27)=0$$

$$6p-81=0\Rightarrow 6p=81$$

$$p=\frac{81}{6}$$

$$p=\frac{27}{2}$$

and,

$$-2p+27=0$$

$$\Rightarrow 2p=27$$

$$p=\frac{27}{2}$$

Thus $\vec{a}\times\vec{b}=0$ when $p=\frac{27}{2}$.

19. Prove by direction numbers, that the point $(1,-1,3)(2,-4,5)$ and $(5,-13,11)$ are in a straight line.

Sol:

Let the three given points are $A(1,-1,3), B(2,-4,5)$ and $C(5,-13,11)$ The direction ratios of the line AB joining the points A and B:

$$2-1,-4-(-1),5-3=2-1,-4+1,5-3=1,-3,2---(i)$$

Similarly, the direction rations of the line BC joining the points B and C

$$5-2,-13-(-4),11-5=3,-9,6-----(ii)$$

It is clear from (1) and (2) that direction ratios of AB and BC are proportional i.e AB is parallel to BC. But the point B is common to both AB and BC. Therefore A,B and C are collinear points.

20. Find the distance of the point $(4,-5,6)$ from the plane $\vec{r}\left(4\vec{i}-4\vec{j}+7\vec{k}\right)=-6$

Sol:

Here $\vec{a}=4\vec{i}-5\vec{j}+6\vec{k}, \vec{N}=4\vec{i}-4\vec{j}+7\vec{k}$ and $d=-6$

We know that the point $\vec{a}$ plane $\vec{N}$ and distance d. i.e, the plane in the form of $\vec{r}.\vec{N} = d$ where $\vec{N}$ is normal to the plane. Then the perpendicular distance

$$\frac{\left|\vec{a}.\vec{N} - d\right|}{\left|\vec{N}\right|} = \frac{\left|\left(4\vec{i} - 5\vec{j} + 6\vec{k}\right).\left(4\vec{i} - 4\vec{j} + 7\vec{k}\right) + 6\right|}{\left|4\vec{i} - 4\vec{j} + 7\vec{k}\right|}$$
$$= \frac{|16 + 20 + 42 + 6|}{\sqrt{4^2 + (-4)^2 + (7)^2}} = \frac{|84|}{\sqrt{81}}$$
$$= \frac{84}{9}$$
$$= \frac{2\sqrt{21}}{9} units.$$

21If A and B are independent events then prove that:

$$P\left(A \cup B\right) = 1 - P(A')P.(B')$$

Sol:

Since A and B are two independent events, therefore

$$P\left(A \cup B\right) = P(A) + P(B) - P(A \cap B)$$
$$\Rightarrow P\left(A \cup B\right) = P(A) + P(B) - P(A).P(B)$$
$$P\left(A \cup B\right) = P(A) + P(B)[1 - P(A)]$$
$$P\left(A \cup B\right) = P(A) + P(B).P(A')$$
$$P\left(A \cup B\right) = 1 - P(A') + P(B)P(A')$$
$$P\left(A \cup B\right) = 1 - P(A')[1 - P(B)]$$
$$P\left(A \cup B\right) = 1 - P(A')P(B').$$

22.Odds are 8:5 against a man, who is 55 years old, living till he is 75 and 4:3 against his wife who is now 48, living till she is 68. Find the probability that the Couple will be alive 20 years hence.

Sol:

Let A = the event of husband will be alive 20 years and B = the event of wife will be alive 20 years. Clearly A and B are independent events,

Let $P(A) = a$ and $P(A') = b$ then odds against of $A = \frac{P(A')}{P(A)} = \frac{b}{a}$

By question $b:a = 8:5 \Rightarrow b = \frac{8}{5}a$

Again, since $P(A) + P(A') = 1$ therefore $a + b = 1 \Rightarrow b = 1 - a$

Putting this value of b in (i), we get

$$b = \frac{8}{5}a \Rightarrow 1 - a = \frac{8}{5}a$$
$$\Rightarrow \frac{8}{5}a + a = 1$$
$$\Rightarrow \frac{8a + 5a}{5} = 1$$
$$\Rightarrow 13a = 5$$
$$a = \frac{5}{13}$$

$\Rightarrow P(A) = \frac{5}{13}$, Similarly $P(B) = \frac{3}{7}$

Then, the probability of the couple will be alive 20 years

$$= P\left(A \cap B\right) = P(A).P(B) = \frac{5}{13} \times \frac{3}{7} = \frac{15}{91}$$

Long Answer Type Question

Question No. 23 to 33 are long Answer Type Question. Each question carries 5 marks. Each question has an alternative as ''or''. You have to the answer each question or its alternative.

23.Find the maximum and minimum values $x^3 - 2x^2 + x + 6$

Sol:

Let,

$$y = x^3 - 2x^2 + x - 6$$
$$\Rightarrow \frac{dy}{dx} = 3x^2 - 4x + 1 = 0$$
$$\frac{d^2y}{dx^2} = 6x - 4$$

Now, for maximum and minimum value of y, $\frac{dy}{dx}=0$

$$3x^2-4x+1=0$$
$$\Rightarrow 3x^2-3x-x+1=0$$
$$\Rightarrow 3x(x-1)-1(x-1)=0$$
$$\Rightarrow (3x-1)(x-1)=0$$
$$x=1,\frac{1}{3}$$

Again $\frac{d^2y}{dx^2}$ at $x=1$

$\frac{d^2y}{dx^2}=6x-4=6\times1-4=2>0$and $\frac{d^2y}{dx^2}\,at\ x=\frac{1}{3}$

$$\frac{d^2y}{dx^2}=6x-4=6\times\frac{1}{3}-4=2-4=-2<0$$

It is clear that maximum and minimum value of y at $x=\frac{1}{3}$ and $x=1$ respectively

$\therefore$ putting $x=\frac{1}{3}$ in $y=x^3-2x^2+x+6$,we get

$$\Rightarrow y=\left(\frac{1}{3}\right)^3-2\times\left(\frac{1}{3}\right)^2+\frac{1}{3}+6$$
$$=\frac{1}{27}-\frac{2}{9}+\frac{1}{3}+6$$
$$=\frac{1-6+9+162}{27}$$
$$=\frac{172-6}{27}=\frac{166}{27}=6.15$$

Again at $x=1, y=1^3-2.1^2+1+6=1-2+1+6=8-2=6$

Thus maximum and minimum value of y is $=\frac{166}{27}=6.15$ and 6 respectively.

OR,

If $x^m y^n = (x+y)^{m+n}$ then prove that $\frac{dy}{dx} = \frac{y}{x}$

Sol:

Given that,

$x^m y^n = (x+y)^{m+n}$

Taking logarithm on the both sides, we get

$$\log\left(x^m . y^n\right) = \log(x+y)^{m+n}$$
$$\Rightarrow m\log x + n\log y = (m+n)\log(x+y)$$

Differentiating both sides, with respect to x, we get

$$\frac{m}{n} + \frac{n}{y}\frac{dy}{dx} = \frac{(m+n)}{(x+y)}\left[1 + \frac{dy}{dx}\right]$$
$$\Rightarrow \frac{m}{n} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}\frac{dy}{dx}$$
$$\Rightarrow \left(\frac{n}{y} - \frac{m+n}{x+y}\right)\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$
$$\Rightarrow \left[\frac{n(x+y) - y(m+n)}{y(x+y)}\right]\frac{dy}{dx} = \frac{x(m+n) - m(x+y)}{x(x+y)}$$
$$\Rightarrow \left(\frac{nx + ny - my - ny}{y(x+y)}\right)\frac{dy}{dx} = \frac{mx + nx - mx - my}{x(x+y)}$$
$$\Rightarrow \left(\frac{nx - my}{y(x+y)}\right)\frac{dy}{dx} = \frac{nx - my}{x(x+y)}$$
$$\therefore \frac{dy}{dx} = \frac{nx - my}{x(x+y)} \times \frac{y(x+y)}{nx - my}$$
$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

Hence, LHS = RHS proved.

24.Prove that: $\int_0^{\frac{\pi}{2}} \log(\tan\theta + \cot\theta)d\theta = \pi\log 2$

Sol:

$$LHS = I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \log \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \log \frac{d\theta}{\sin\theta\cos\theta} = -\int_0^{\frac{\pi}{2}} \log(\sin\theta\cos\theta) d\theta$$

$$= \left[\int_0^{\frac{\pi}{2}} \log\sin\theta d\theta + \int_0^{\frac{\pi}{2}} \log\cos\theta d\theta\right] ----(A)$$

$$I = -\left[I_1 - I_2\right] --------(B)$$

Now,

$$I_1 = \int_0^{\frac{\pi}{2}} \log\sin\theta d\theta ----------(1)$$

$$\Rightarrow I_1 = \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - \theta\right) d\theta = I_1 = \int_0^{\frac{\pi}{2}} \log\cos\theta d\theta -----(2)$$

Adding (1) and (2), we get

$$2I_1 = \int_0^{\frac{\pi}{2}} \log\left(\sin\theta\cos\theta\right) d\theta = \int_0^{\frac{\pi}{2}} \log\left(\frac{2\sin\theta\cos\theta}{2}\right) d\theta$$

$$2I_1 = \int_0^{\frac{\pi}{2}} \log\frac{\sin 2\theta}{2} d\theta - \log\int_0^{\frac{\pi}{2}} \log\sin 2\theta d\theta - \int_0^{\frac{\pi}{2}} \log 2 d\theta$$

$$2I_1 = \int_0^{\frac{\pi}{2}} \log\sin 2\theta d\theta - \log 2 \int_0^{\frac{\pi}{2}} d\theta$$

$$2I_1 = I_3 - \log 2\left[\theta\right]_0^{\frac{\pi}{2}} = I_3 - \frac{\pi}{2}\log 2$$

$$2I_1 = I_3 - \frac{\pi}{2}\log 2 --------(3)$$

for I_3 let $2\theta = z \Rightarrow 2d\theta$

$= dz \Rightarrow d\theta = \frac{dz}{2}$ also when θ= 0 Then z = 0

and when θ = $\frac{\pi}{2}$ Then z = π

$$\therefore I_3 = \log \int_0^{\frac{\pi}{2}} \log \sin \theta d\theta - \int_0^{\pi} \log \sin z \frac{dz}{2}$$

$$\Rightarrow I_3 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log \sin z dz = \frac{1}{2}\int_0^{2-\frac{\pi}{2}} \log \sin z dz$$

$$\Rightarrow I_3 = \frac{2}{z}\int_0^{\frac{\pi}{2}} \log \sin z dz = \int_0^{\frac{\pi}{2}} \log \sin \theta d\theta$$

$$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \ if \ f(2a-x) = f(x)$$

$$and \int_0^{a} f(x)dx = \int_0^{a} f(y)dy$$

$\therefore$ from (3), we get

$$2I = I_1 - \frac{\pi}{2}\log 2$$

$$\Rightarrow I_1 = -\frac{\pi}{2}\log 2$$

Similarly,

$$I_2 = -\frac{\pi}{2}\log 2$$

Lastly from (A), we get

$$I_1 = \left[-\frac{\pi}{2}\log 2 - \frac{\pi}{2}\log 2\right] = \frac{\pi}{2}\log 2 + \frac{\pi}{2}\log 2$$

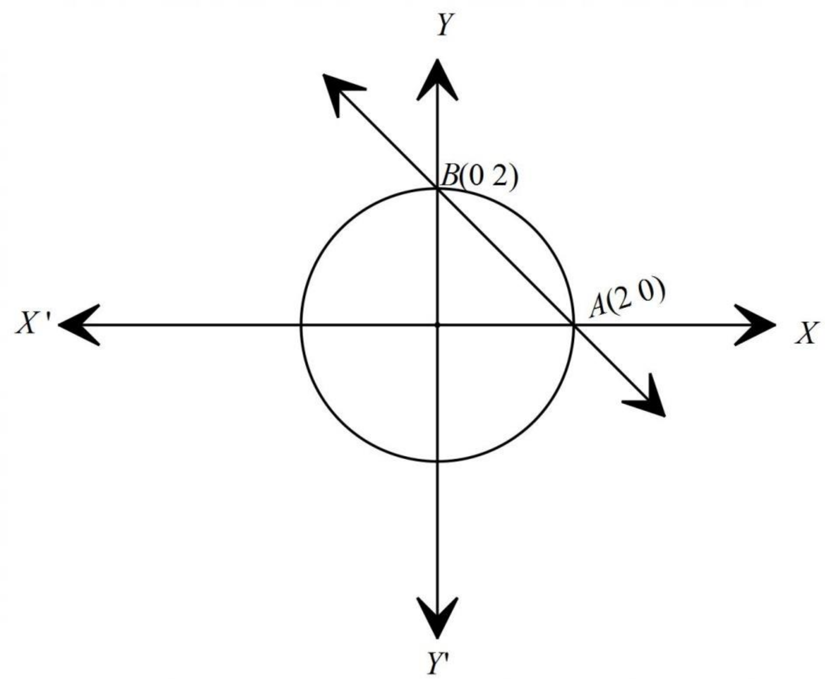
$$= \frac{2\pi}{2}\log 2 = \pi \log 2 = RHS$$

OR,

Find the area of the Smaller portion of the circle $x^2 + y^2 = 4$ cut of by the line $x + y = 2$.

Sol:

Let $x^2 + y^2 = 4$ represent a circle whose centre (0,0) and radius 2 units. AB : $x + y = 2$ be the line which passes through circle at the points(2,0)and(0,2)



$\therefore$ Area of region ACB = Area of quadrant OAB - Area of $\triangle OAB$-----(1)

Now $x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2 \Rightarrow y = \sqrt{4 - x^2}$

$\therefore$ Area of quadrant OAB

$$OAB = \int_0^2 \sqrt{4 - x^2}\, dx$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{a} \right]_0^2$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]$$

$$= 0 + 2\sin^{-1}\frac{2}{2} - 0 = 2\sin^{-1}\frac{\pi}{2}$$

$$= 2\sin^{-1}\left(\sin\frac{\pi}{2} \right) = 2 \times \frac{\pi}{2} = \pi sq.units$$

Area of $\triangle OAB$= Area of the region bounded by AB.

$\therefore AB : x + y = 2 \Rightarrow y = 2 - x \; and \; x = 0, y = 0$

$\therefore$ Area of $\triangle OAB$

$$\int_0^2 (2-x)dx = \left[2x - \frac{x^2}{2}\right]_0^2 = \left[4 - \frac{4}{2}\right] - 0 = 2$$

Area of region ACB = Area of quadrant OAB – Area of $\triangle OAB$

$\therefore$ Area of region ACB = $(\pi - 2)$ sq. units.

25.Prove by Vector method, that in any $\triangle ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Sol:

let ABC be a triangle, whose sides are $\overrightarrow{BC} = \vec{a}, \overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$.

Then $\vec{a} + \vec{b} + \vec{c} = \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$

$$\vec{a} + \vec{b} + \vec{c} = 0$$
$$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times 0$$
$$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$$
$$\vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0 \qquad [\because \vec{a} \times \vec{a} = 0]$$
$$\vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$$

Now,
$$\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \qquad [\because -\vec{a} \times \vec{c} = \vec{c} \times \vec{a}] - - - - - (1)$$

Similarly,

$$\vec{a} + \vec{b} + \vec{c} = 0$$
$$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times 0$$
$$\vec{b} \times \vec{a} + \vec{b} \times \vec{b} + \vec{b} \times \vec{c} = \vec{b} \times \vec{c} = -\vec{b} \times \vec{a} = \vec{a} \times \vec{b}$$
$$\vec{b} \times \vec{c} = \vec{a} \times \vec{b} - - - - - - - - - - - (2)$$
$$[\vec{b} \times \vec{b} = 0, -\vec{b} \times \vec{a} = \vec{a} \times \vec{b}]$$

From (1) and (2), we get $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

$\Rightarrow \left|\vec{a} \times \vec{b}\right| = |b \times c| = |c \times a|$

$\Rightarrow ab\sin(\pi - C) = bc\sin(\pi - A) = ca\sin(\pi - B)$

$\Rightarrow ab\sin C = bc\sin A = ca\sin B [\because \sin(\pi - \theta) = \sin\theta]$

$\Rightarrow \frac{ab\sin C}{abc} = \frac{bc\sin A}{abc} = \frac{ca\sin B}{abc}$

$\Rightarrow \frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$

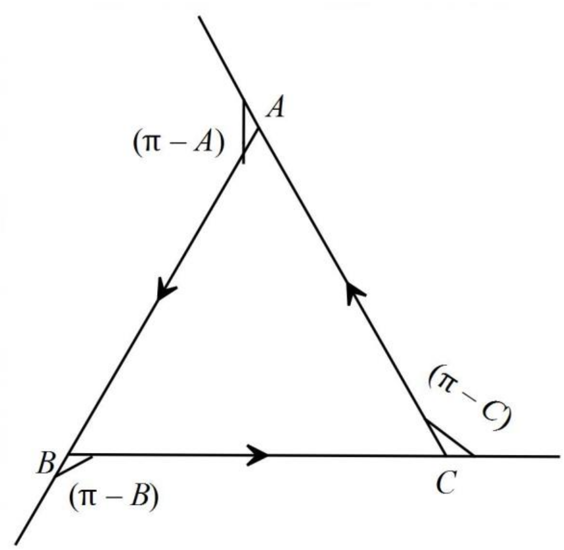
$\Rightarrow \frac{c}{\sin C} = \frac{a}{\sin A} = \frac{b}{\sin B}$

$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

OR,

Show that the line joining the point $(4,7,8),(2,3,4)$ is parallel to the line joining the points $(2,4,10),(-2,-4,2)$

Sol:



Let the given point are $A(4,7,8), B(2,3,4)$ and $C(2,4,10), D(-2,-4,2)$ .Then direction ratios of the line AB:

$2-4, 3-7, 4-8 = -2, -4, -4 \Rightarrow a_1 = -2, b_1 = -4, c_1 = -4$

And the direction ratios of the line CD:

$$-2-2, -4-4, 2-10 = -4, -8, -8$$
$$\Rightarrow a_2 = -4, b_2 = -8, c_2 = -8$$

Now, two lines AB and CD are parallel if its direction ratios are in the from $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$$\frac{-2}{-4} = \frac{-4}{-8} = \frac{-4}{-8} \Rightarrow \frac{1}{2} = \frac{1}{2} = \frac{1}{2}$$

Thus $AB \,||\, CD$