Relations and Functions

Short Answer Type Questions

1. **Let A = {0, 1, 2, 3} and define a relation R on A as follows:**
R = {(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)}.
Is R reflexive? symmetric? transitive?

Sol. R is reflexive and symmetric, but not transitive since for (1, 0) ∈ R and (0, 3) ∈ R whereas (1, 3) ∉ R.

2. **For the set A = {1, 2, 3}, define a relation R in the set A as follows:**
R = {(1, 1), (2, 2), (3, 3), (1, 3)}.
Write the ordered pairs to be added to R to make it the smallest equivalence relation.

Sol. (3, 1) is the single ordered pair which needs to be added to R to make it the smallest equivalence relation.

3. **Let R be the equivalence relation in the set Z of integers given by R = {(a, b) : 2 divides a– b}. Write the equivalence class [0].**

Sol. [0] = {0, ± 2, ± 4, ± 6, ...}

4. **Let the function** f **: R → R be defined by** $f(x) = 4x - 1, \forall x \in R$. **Then, show that** f **is one-one.**

Sol. For any two elements x_1, x_2 ∈ R such that $f(x_1) = f(x_2)$, we have

$4x_1 - 1 = 4x_2 - 1$

$\Rightarrow 4x_1 = 4x_2, \quad i.e., x_1 = x_2$

Hence f is one-one.

5. **If** f **= {(5, 2), (6, 3)},** g **= {(2, 5), (3, 6)}, write** $f \; o \; g$**.**

Sol. $f \; o \; g$ = {(2, 2), (3, 3)}

6. **Let** f **: R → R be the function defined by** $f(x) = 4x - 3 \forall x \in R$. **Then write** f^{-1}

Sol. Given that $f(x) = 4x - 3 = y(say)$, then

$4x = y + 3 \Rightarrow x = \frac{y+3}{4}$

Hence $f^{-1}(y) = \frac{y+3}{4} \Rightarrow f^{-1}(x) = \frac{x+3}{4}$

7. **Is the binary operation * defined on Z (set of integer) by** $m * n = m - n + mn$ $\forall m, n \in Z$ **commutative?**

Sol. No. Since for 1, $2 \in Z$, $1*2 = 1 - 2 + 1.2 = 1$ while $2*1 = 2 - 1 + 2.1 = 3$ so that $1*2 \neq 2*1$.

8. If f = {(5, 2), (6, 3)} and g = {(2, 5), (3, 6)}, write the range of f and g.

Sol. The range of f = {2, 3} and the range of g = {5, 6}.

9. If A = {1, 2, 3} and f, g are relations corresponding to the subset of $A \times A$ indicated against them, which of f, g is a function? Why?

f = {(1, 3), (2, 3), (3, 2)}

g = {(1, 2), (1, 3), (3, 1)}

Sol. f is a function since each element of A in the first place in the ordered pairs is related to only one element of A in the second place while g is not a function because 1 is related to more than one element of A, namely, 2 and 3.

10. If A = {a, b, c, d} and f = {a, b), (b, d), (c, a), (d, c)}, show that f is one-one from A onto A. Find f^{-1}

Sol. f is one-one since each element of A is assigned to distinct element of the set A. Also, f is onto since f (A) = A. Moreover, f^{-1} = {(b, a), (d, b), (a, c), (c, d)}.

11. In the set N of natural numbers, define the binary operation *by $m*n = g.c.d\ (m,n), m,n \in N$. Is the operation * commutative and associative?

Sol. The operation is clearly commutative since

m*n= g.c.d (m, n) = g.c.d(n, m) = n * m $\forall$ m, n $\in$ N.

It is also associative because for l, m, n $\in$ N, we have

l *(m*n) = g. c. d (l, g.c.d (m, n))

= g.c.d. (g. c. d(l, m), n)

= (l*m) *n.

<u>Long Answer Type Questions</u>

12. In the set of natural numbers N, define a relation R as follows:

$\forall$ n, m $\in$ N, nRm if on division by 5 each of the integers n and m leaves the remainder less than 5, i.e. one of the numbers 0, 1, 2, 3 and 4. Show that R is equivalence relation. Also, obtain the pairwise disjoint subsets determined by R.

Sol. R is reflexive since for each a $\in$ N, aRa. R is symmetric since if aRb, then bRa for a, b $\in$ N. Also, R is transitive since for a, b, c $\in$ N, if aRb and bRc, then aRc.

Hence R is an equivalence relation in N which will partition the set N into the pairwise disjoint subsets. The equivalent classes are as mentioned below:

A_0= {5, 10, 15, 20 ...}

A_1= {1, 6, 11, 16, 21 ...}

A_2= {2, 7, 12, 17, 22, ...}

A_3= {3, 8, 13, 18, 23, ...}
A_4= {4, 9, 14, 19, 24, ...}
It is evident that the above five sets are pairwise disjoint and

$$A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{i=0}^{4} A_i = N.$$

13. Show that the function f : R → R defined by $f(x) = \frac{x}{x^2+1}, \forall x \in R$, is neither one-one nor onto.

Sol. For $x_1, x_2 \in$ R, consider

$f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow x_1x_2^2 + x_1 = x_2x_1^2 + x_2$$

$$\Rightarrow x_1x_2(x_2 - x_1) = x_2 - x_1$$

$$\Rightarrow x_1 = x_2 \ \ or \ \ x_1x_2 = 0$$

We note that there are point, x_1 *and* x_2 with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$ for instance, if we take $x_1 = 2$ and $x_2 = \frac{1}{2}$, then we have $f(x_1) = \frac{2}{5}$ and $f(x_2) = \frac{2}{5}$ but $2 \neq \frac{1}{2}$. Hence f is not one-one. Also, f is not onto for if so then for $1 \in R \exists x \in R$ such that f (x) = 1 which gives $\frac{x}{x^2+1} = 1$. But there is no such x in the domain **R,** since the equation $x^2 - x + 1 = 0$ does not give any real value of x.

14. Let f , g : R → R be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x$ ∀ x ∈ R. Then, find $f \, o \, g$ and $g \, o f$.

Sol. Here $f(x) = |x| + x$ which can be redefined as

$$f(x) = \begin{cases} 2x \text{ if } x \geq 0 \\ 0 \text{ if } x \leq 0 \end{cases}$$

Similarly, the function g defined by $g(x) = |x| - x$ may be redefined as

$$g(x) = \begin{cases} 0 \text{ if } x \geq 0 \\ -2 \text{ if } x \leq 0 \end{cases}$$

Therefore, $g \, o f$ gets defined as:

For x ≥0, ($g \, o f$) (x) = g (f (x) = g (2x) = 0

and for x <0, ($g \, o f$) (x) = g (f (x) = g (0) = 0.

Consequently, we have ($g \, o f$) (x) = 0, ∀ x ∈ R.

Similarly, $f \, o \, g$ gets defined as:

For x ≥0, ($f\ o\ g$) (x) = f (g (x) = f (0) = 0,
and for x <0, ($f\ o\ g$) (x) = f (g (x)) = f (–2 x) = – 4x.

i.e. $(f\ o\ g)(x)=\begin{cases}0\ , x>0\\ -4x,\ \ x<0\end{cases}$

15. Let R be the set of real numbers and $f: R \to R$ be the function defined by f (x) = 4x+ 5. Show that f is invertible and find f^{-1} .

Sol. Here the function $f: R \to R$ is defined as $f(x) = 4x+5 = y$ (say). Then

$4x = y - 5$ or $x=\dfrac{y-5}{4}$

This leads to a function $g: R \to R$ defined as

$g(y)=\dfrac{y-5}{4}$

Therefore, $(g\ o f)\ (x) = g(f\ (x) = g(4x+5) =$

$=\dfrac{4x+5-5}{4}=x$

$or\ g\ o f = I_R$

Similarly, ($f\ o\ g$) (y) = f (g (y))

$= f\left(\dfrac{y-5}{4}\right)$

$= 4\left(\dfrac{y-5}{4}\right)+5=y$

or $f\ o\ g = I_R$.

Hence f is invertible and $f^{-1}=g$ which is given by $f^{-1}(x)=\dfrac{x-5}{4}$

16. Let * be a binary operation defined on Q. Find which of the following binary operations are associative.
(i) a *b= a– b for a, b $\in$ Q.
(ii) $a*b=\dfrac{ab}{4}\ for\ a,\ b\in Q.$
(iii) a *b= a– b+ ab for a, b $\in$ Q.
(iv) $a*b=ab^2$ **for a, b $\in$ Q.**

Sol. (i) * is not associative for if we take a= 1, b= 2 and c= 3, then
(a*b) *c= (1 *2) * 3 = (1 – 2) * 3 = – 1 – 3 = – 4 and
a *(b*c) = 1 *(2 *3) = 1 *(2 – 3) = 1 – (– 1) = 2.
Thus (a*b) *c ≠ a*(b*c) and hence * is not associative.
(ii) * is associative since Q is associative with respect to multiplication.
(iii) * is not associative for if we take a= 2, b= 3 and c = 4, then

(a*b) *c= (2 *3) * 4 = (2 – 3 + 6) *4 = 5*4 = 5 – 4 + 20 = 21, and
a*(b*c) = 2 *(3*4) = 2*(3 – 4 + 12) = 2 *11 = 2 – 11 + 22 = 13
Thus (a*b) * c ≠ a*(b*c) and hence * is not associative.
(iv) * is not associative for if we take a= 1, b= 2 and c= 3, then (a*b) * c=(1 * 2) * 3 = 4 * 3 = 4 × 9 = 36 and a*(b*c) = 1 * (2 * 3) = 1*18 =$1 \times 18^2 = 324$.
Thus (a * b) * c ≠ a * (b * c) and hence*is not associative.

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 17 to 25.

17. Let R be a relation on the set N of natural numbers defined by nRm if n divides m. Then R is
(A) Reflexive and symmetric
(B) Transitive and symmetric
(C) Equivalence
(D) Reflexive, transitive but not symmetric

Sol. The correct choice is (D).
Since n divides n, $\forall$ n $\in$ N, R is reflexive. R is not symmetric since for 3, 6 $\in$ N, 3 R 6 ≠6 R 3. R is transitive since for n, m, r whenever n/m and m/r ⇒ n/r, i.e., n divides m and m divides r, then n will divide r.

18. Let L denote the set of all straight lines in a plane. Let a relation R be defined by lRm if and only if l is perpendicular to m $\forall$ l, m $\in$ L. Then R is
(A) reflexive
(B) symmetric
(C) transitive
(D) none of these

Sol. The correct choice is (B).

19. Let N be the set of natural numbers and the function f : N → N be defined by f (n) = 2n+ 3 $\forall$ n $\in$ N. Then f is
(A) surjective
(B) injective
(C) bijective
(D) none of these

Sol. (B) is the correct option.

20. Set A has 3 elements and the set B has 4 elements. Then the number of injective mappings that can be defined from A to B is
(A) 144 (B) 12 (C) 24 (D) 64

Sol. The correct choice is (C). The total number of injective mappings from the set containing 3 elements into the set containing 4 elements is ${}^4P_3 = 4! = 24$.

21. **Let f: R → R be defined by f(x) = sin x and g: R → R be defined by $g(x) = x^2$, then $f \circ g$ is**

(A) $x^2 sin\, x$
(B) **(sin x)²**
(C) $sin\, x^2$
(D) $\frac{\sin x}{x^2}$

Sol. (C) is the correct choice.

22. **Let f : R → R be defined by $f(x) = 3x - 4$. Then f^{-1}(x) is given by**

(A) $\frac{x+4}{3}$
(B) $\frac{x}{3} - 4$
(C) $3x + 4$
(D) None of these

Sol. (A) is the correct choice.

23. **Let f : R → R be defined by f(x) = x² + 1. Then, pre-images of 17 and – 3, respectively, are**

(A) φ, $\{4, -4\}$
(B) $\{3, -3\}$, φ
(C) $\{4, -4\}$, φ
(D) {4, – 4, {2, – 2}

Sol. (C) is the correct choice since for $f^{-1}(17) = x \Rightarrow f(x) = 17$ or $x^2 + 1 = 17$ ⇒ x= ± 4 or f^{-1}(17) = {4, – 4} and for $f^{-1}(-3) = x \Rightarrow f(x) = -3 \Rightarrow x^2 + 1 = -3 \Rightarrow x^2 = -4$ and hence $f^{-1}(-3) = \varphi$.

24. **For real numbers x and y, define xRy if and only if $x - y + \sqrt{2}$ is an irrational number. Then the relation R is**

(A) reflexive
(B) symmetric
(C) transitive
(D) none of these

Sol. (A) is the correct choice.

Fill in the blanks in each of the Examples 25 to 30.

25. **Consider the set A = {1, 2, 3} and R be the smallest equivalence relation on A, then R = _______**

Sol. R = {(1, 1), (2, 2), (3, 3)}.

26. The domain of the function f : R→R defined by $f(x) = \sqrt{x^2 - 3x + 2}$ is ___.

Sol. Here $x^2 - 3x + 2 \geq 0$

$\Rightarrow (x-1)(x-2) \geq 0$

$\Rightarrow x \leq 1$ or $x \geq 2$

Hence the domain of $f = (-\infty, 1] \cup [2, \infty)$

27. Consider the set A containing n elements. Then, the total number of injective functions from A onto itself is _______.

Sol. n!

28. Let Z be the set of integers and R be the relation defined in Z such that aRb if a– b is divisible by 3. Then R partitions the set Z into _______ pairwise disjoint subsets.

Sol. Three.

29. Let R be the set of real numbers and *be the binary operation defined on R as a * b= a + b– ab $\forall$ a, b $\in$ R. Then, the identity element with respect to the binary operation * is ______.

Sol. 0 is the identity element with respect to the binary operation *.

State True or False for the statements in each of the Examples 30 to 34.

30. Consider the set A = {1, 2, 3} and the relation R = {(1, 2), (1, 3)}. R is a transitive relation.

Sol. True.

31. Let A be a finite set. Then, each injective function from A into itself is not surjective.

Sol. False.

32. For sets A, B and C, let f : A →B, g : B →C be functions such that $g o f$ is injective. Then both f and g are injective functions.

Sol. False.

33. For sets A, B and C, let f : A →B, g : B →C be functions such that $g o f$ is surjective. Then g is surjective.

Sol. True.

34. Let N be the set of natural numbers. Then, the binary operation * in N defined as $a * b = a + b, \forall a, b \in$ N has identity element.

Sol. False.

Relations and Functions
<u>Objective Type Questions</u>

28. Let T be the set of all triangles in the Euclidean plane, and let a relation R on T be defined as aRb if a is congruent to $b, \forall a,\ b \in T$. Then R is

(A) reflexive but not transitive
(B) transitive but not symmetric
(C) equivalence
(D) none of these

Sol. (C) Consider that aRb, if a is congruent to b, $\forall a, b \in T$.

Then, $aRa \Rightarrow a \cong a$,

Which is true for all $a \in T$

So, R is reflexive, ……….(i)

Let $aRb \Rightarrow a \cong b$

$\Rightarrow b \cong a \Rightarrow b \cong a$

$\Rightarrow bRa$

So, R is symmetric. ………(ii)

Let aRb and bRc

$\Rightarrow a \cong b \ \ and \ \ b \cong c$

$\Rightarrow a \cong c \Rightarrow aRc$

So, R is transitive. ………..(iii)

Hence, R is equivalence relation.

29. Consider the non-empty set consisting of children in a family and a relation R defined as aRb if ais brother of b. Then R is

(A) symmetric but not transitive
(B) transitive but not symmetric
(C) neither symmetric nor transitive
(D) both symmetric and transitive

Sol. (b) Given,

$aRb \Rightarrow$ a is brother of b.

$\therefore aRa \Rightarrow a$ is brother of a, which is not true.

So, R is not reflexive.

$aRb \ \Rightarrow a$ is brother of b.

This does not mean b is also a brother of a and b can be a sister of a.

Hence, R is not symmetric

$aRb \ \Rightarrow a$ is brother of b

And $bRc \ \Rightarrow b$ is a brother of c.

So, a is brother of c.

Hence, R is transitive.

30. The maximum number of equivalence relations on the set A = {1, 2, 3} are

(A) 1
(B) 2
(C) 3
(D) 5

Sol. (d) Given that, A = {1,2,3}

Now, number of equivalence, relations as follows

$R_1 = \{(1,1),(2,2),(3,3)\}$

$R_2 = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

$R_3 = \{(1,1),(2,2),(3,3),(1,3),(3,1)\}$

$R_4 = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$

$R_5 = \{(1,2,3) \Leftrightarrow A \times A = A^2\}$

$\therefore$ Maximum number of equivalence relation on the set $A = \{1,2,3\} = 5$

31. If a relation R on the set {1, 2, 3} be defined by R = {(1, 2)}, then R is
(A) reflexive
(B) transitive
(C) symmetric
(D) none of these

Sol. (b) R on the set {1, 2, 3} be defined by $R = \{(1,2)\}$

It is clear that R is transitive.

32. Let us define a relation R in R as aRb if a ≥ b. Then R is
(A) an equivalence relation
(B) reflexive, transitive but not symmetric
(C) symmetric, transitive but
(D) neither transitive nor reflexive but symmetric.

Sol. (b) Given that, $aRb \text{ if } a \geq b$

$\Rightarrow aRa \Rightarrow a \geq a$ which is true

$let\ aRb, a \geq b,$ then b ≥ a which is not true R is not symmetric.

But $aRb\ and$ b R c

$\Rightarrow$ a ≥ b and b ≥ c

$\Rightarrow$ a ≥ c

Hence, R is transitive.

33. Let A = {1, 2, 3} and consider the relation R = {1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1,3)}.
Then R is
(A) reflexive but not symmetric
(B) reflexive but not transitive
(C) symmetric and transitive
(D) neither symmetric, nor transitive

Sol. (a) Given that A = {1, 2, 3}

And R = {1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1,3)}.

$\because\ (1,1),(2,2),(3,3)\in R$

Hence, R is reflexive.

$(1,2)\in R$ but $(2,1)\notin R$

Hence, R is not symmetric.

$(1,2)\in R$ and $(2,3)\in R$

$\Rightarrow (1,3)\in R$

Hence, R is transitive.

34. The identity element for the binary operation * defined on Q - {0} $as\ a*b=\frac{ab}{2},\forall a,\ b\ \in\ Q\sim\{0\}\ is$

(A) 1

(B) 0

(C) 2

(D) none of these

Sol. (c) Given that, $a*b=\frac{ab}{2},\forall a,b\in Q-\{0\}$

Let e be the identity element for *

$\therefore\ \text{a*e}=\frac{ae}{2}$

$\Rightarrow\ \frac{ae}{2}\Rightarrow e=2$

35. If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is

(A) 720

(B) 120

(C) 0

(D) none of these

Sol. (c) We know that, if A and B are two non-empty finite set containing m and n elements respectively, then the number of one-one and onto mapping from A to B is

$n!$ if m=n

$0, if\ m\neq n$

Given that, m=5 and n=6

$\therefore\ m\neq n$

Number of mapping=0

36. Let A = {1, 2, 3, …n} and B = {a, b}. Then the number of surjections from A into B is

(A) ${}^{n}P_{2}$

(B) $2^{n} - 2$

(C) $2^{n} - 1$

(D) None of these

Sol. (d) Given that A = {1, 2, 3, ...n} and B = {a, b}

We know that, if A and B are two non-empty finite sets containing m and n elements respectively, then the number of surjection from A into B is

${}^{n}C_{m} \times m!$ if n ≥ m

$0, if\ m < n$

Here, m=2

$\therefore$ Number of surjection from A into B is ${}^{m}C_{2} \times 2! = \frac{n!}{2!(n-2)!} \times 2!$

$= \frac{n(n-1)(n-2)!}{2 \times 1(n-2)} \times 2! = n^{2} - n$

37. Let f : R → R be defined by $f(x) = \frac{1}{x}, \forall \in R.$ Then f is

(A) one-one

(B) onto

(C) bijective

(D) f is not defined

Sol. Given that, $f(x) = \frac{1}{x}, \forall \in R.$

For x=0,

$f(x)$ is not defined.

Hence, $f(x)$ is a not define function.

38. Let f : R → R be defined by $f(x) = 3x^{2} - 5$ and g : R → R by $g(x) = \frac{x}{x^{2}+1}$.

Then gof is

(A) $\frac{3x^{2}-5}{9x^{4}-30x^{2}+26}$

(B) $\frac{3x^{2}-5}{9x^{4}-6x^{2}+26}$

(C) $\frac{3x^{2}}{x^{4}+2x^{2}-4}$

(D) $\frac{3x^{2}}{9x^{4}+30x^{2}-2}$

Sol. (a) Given that $f(x) = 3x^{2} - 5$ and $g(x) = \frac{x}{x^{2}+1}$.

$gof = g\{f(x)\} = g(3x^2-5)$

$= \dfrac{3x^2-5}{(3x^2-5)^2+1} = \dfrac{3x^2-5}{9x^4-30x^2+25+1}$

$= \dfrac{3x^2-5}{9x^4-30x^2+26}$

39. Which of the following functions from Z into Z are bijections?

(A) $f(x) = x^3$

(B) $f(x) = x+2$

(C) $f(x) = 2x+1$

(D) $f(x) = x^2 + 1$

Sol. (b) Here, $f(x) = x+2 \Rightarrow f(x_1) = f(x_2)$

$x_1 + 2 = x_2 + 2 \Rightarrow x_1 = x_2$

Let $y = x+2$

$x = y-2 \in Z, \forall y \in x$

Hence, $f(x)$ is one-one and onto.

40. Let f : R → R be the functions defined by $f(x) = x^3 + 5$. Then $f^{-1}(x)$ is

(A) $(x+5)^{\frac{1}{3}}$

(B) $(x-5)^{\frac{1}{3}}$

(C) $(5-x)^{\frac{1}{3}}$

(D) 5 – x

Sol. (b) Given that, $f(x) = x^3 + 5$

Let $y = x^3 + 5 \Rightarrow x^3 = y-5$

$x = (y-5)^{\frac{1}{3}} \Rightarrow f(x)^{-1} = (x-5)^{\frac{1}{3}}$

41. Let f : A → B and g : B →C be the bijective functions. Then $(gof)^{-1}$ is

(A) $f^{-1}\, og^{-1}$

(B) fog

(C) $g^{-1}of^{-1}$

(D) gof

Sol. (a) Given that, $f : A \to B$ *and* $g : B \to C$ be the bijective functions.

$(gof)^{-1} = f^{-1}og^{-1}$

42. **Let** $f: R-\left\{\frac{3}{5}\right\} \to R$ **be defined by** $f(x) = \frac{3x+2}{5x-3}$ **, Then**

(A) $f^{-1}(x) = f(x)$

(B) $f^{-1}(x) = -f(x)$

(C) $(fof)x = -x$

(D) $f^{-1}(x) = \frac{1}{19} f(x)$

Sol. (a) Given that, $f(x) = \frac{3x+2}{5x-3}$

Let $y = \frac{3x+2}{5x-3}$

$3x+2 = 5xy-3y \Rightarrow x(3-5y) = -3y-2$

$x = \frac{3y+2}{5y-3} \Rightarrow f^{-1}(x) = \frac{3x+2}{5x-3}$

$f^{-1}(x) = f(x)$

43. **Let** f**: [0, 1] → [0, 1] be defined by** $f(x) = \begin{cases} x, & if\ x\ is\ rational \\ 1-x, & if\ x\ is\ irrational \end{cases}$ **then** $(fof)x$ **is**

(A) constant

(B) 1 + x

(C) x

(D) none of these

Sol. (c) Given that $f: [0,\ 1] \to [0,\ 1]$ be defined by

$f(x) = \begin{cases} x, & if\ x\ is\ rational \\ 1-x, & if\ x\ is\ irrational \end{cases}$

$\therefore\ (fof)x = f(f(x)) = x$

44. **Let** $f: [2,\ \infty) \to R$ **be the function defined by** $f(x) = x^2 - 4x + 5,$ **then the range of** f **is**

(A) R

(B) $[1,\ \infty)$

(C) $[4,\ \infty)$

(D) $[5,\ \infty)$

Sol. (b) Given that, $f(x) = x^2 - 4x + 5,$

Let $y = x^2 - 4x + 5$

$\Rightarrow y = x^2 - 4x + 4 + 1 = (x-2)^2 + 1$

$\Rightarrow \ (x-2)^2 = y - 1 \Rightarrow x - 2 = \sqrt{y-1}$

$\Rightarrow$ x=2+$\sqrt{y-1}$

$\therefore$ y-1 ≥ 0, y ≥ 1

Range=$[1, \infty)$

45. **Let f: N →R be the function defined by $f(x) = \frac{2x-1}{2}$ and $g: Q \to R$ be another function defined by $g(x) = x + 2$. Then, $(gof)\frac{3}{2}$ is**

(A) 1

(B) 1

(C) $\frac{7}{2}$

(D) none of these

Sol. (D) Given that $f(x) = \frac{2x-1}{2}$ and $g(x) = x + 2$

$$(gof)^{\frac{3}{2}} = g\left[f\left(\frac{3}{2}\right)\right] = g\left(\frac{2\times\frac{3}{2}-1}{2}\right)$$

$$= g(1) = 1 + 2 = 3$$

46. **Let f : R → R be defined by $f(x) = \begin{cases} 2x : x > 3 \\ x^2 : 1 < x \leq 3 \\ 3x : x \leq 1 \end{cases}$ Then $f(-1) + f(2) + f(4)$ is**

(A) 9 **(B) 14** **(C) 5** **(D) none of these**

Sol. (a) Given that $f(x) = \begin{cases} 2x : x > 3 \\ x^2 : 1 < x \leq 3 \\ 3x : x \leq 1 \end{cases}$

$$f(-1) + f(2) + f(4) = 3(-1) + (2)^2 + 2\times 4$$

$$= -3 + 4 + 8 = 9$$

47. **Let f : R → R be given by $f(x) = tan\ x$. Then $f^{-1}(1)$ is**

(A) $\frac{\pi}{4}$

(B) $\left\{n\pi+\frac{\pi}{4}: n\in Z\right\}$

(C) does not exist

(D) none of these

Sol. (a) Given that, $f\left(x\right)=tan\ x$

Let $y=\tan x\Rightarrow x=\tan^{-1} y$

$\Rightarrow\ \mathrm{f}^{-1}\left(x\right)=\tan^{-1} x\Rightarrow f^{-1}\left(1\right)=\tan^{-1} 1$

$\Rightarrow\ \tan^{-1}\tan\frac{\pi}{4}=\frac{\pi}{4}\ \left[\because \tan\frac{\pi}{4}=1\right]$

Fill in the blanks in each of the Exercises 48 to 52.

48. Let the relation R be defined in N by aRb if 2a+ 3b= 30. Then R =………

Sol. Given that, 2a+ 3b= 30

3b=30-2a

$b=\frac{30-2a}{3}$

For a=3, b=8

a=6, b=6

a=9, b=4

a= 12, b=2

$R=\left\{\left(3,8\right),\left(6,6\right),\left(9,4\right),\left(12,2\right)\right\}$

49. Let the relation R be defined on the set $A=\left\{1,2,3,4,5\right\}$ **by** $R=\{\left(a,\ b\right):\ \left|a^2-b^2\right|<8\}$**. Then R is given by ______.**

Sol. Given $A=\left\{1,2,3,4,5\right\}$

$R=\{\left(a,\ b\right):\ \left|a^2-b^2\right|<8\}$

$R=\left\{\left(1,1\right),\left(1,2\right),\left(2,1\right),\left(2,2\right),\left(2,3\right),\left(3,2\right),\left(3,3\right),\left(4,3\right),\left(3,4\right),\left(4,4\right),\left(5,5\right)\right\}$

50. Let f **= {(1, 2), (3, 5), (4, 1) and** g **= {(2, 3), (5, 1), (1, 3)}. Then** $gof=$…………… **and** $fog=$……………

Sol. Given that, f = {(1, 2), (3, 5), (4, 1) and g = {(2, 3), (5, 1), (1, 3)}

$gof(1)=g\left\{f\left(1\right)\right\}=g\left(2\right)=3$

$gof(3)=g\left\{f\left(3\right)\right\}=g\left(5\right)=3$

$gof(4)=g\left\{f\left(4\right)\right\}=g\left(1\right)=3$

$gof=\left\{\left(1,3\right),\left(3,1\right),\left(4,3\right)\right\}$

Now $fog\left(2\right)=f\left\{g\left(2\right)\right\}=f\left(3\right)=5$

$fog(5) = f\{g(5)\} = f(1) = 2$

$fog(1) = f\{g(1)\} = f(3) = 5$

$fog = \{(2,5),(5,2),(1,5)\}$

51. **Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \dfrac{x}{\sqrt{1+x^2}}$, then $(fofof)(x) = ...$**

Sol. Given that, $f(x) = \dfrac{x}{\sqrt{1+x^2}}$

$$(fofof)(x) = f\left[f\{f(x)\}\right]$$

$$= f\left[f\left(\frac{x}{\sqrt{1+x^2}}\right)\right] = f\left(\frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{1+x^2}}}\right)$$

$$= f\left[\frac{x\sqrt{1+x^2}}{\sqrt{1+x^2}\left(\sqrt{2x^2}+1\right)}\right] = f\left(\frac{x}{\sqrt{1+2x^2}}\right)$$

$$= \frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{1+x^2}}} = \frac{x\sqrt{1+2x^2}}{\sqrt{1+2x^2}\sqrt{1+3x^2}}$$

$$= \frac{x}{\sqrt{1+3x^2}} = \frac{x}{\sqrt{3x^2+1}}$$

52. **If $f(x) = \left[4-(x-7)^3\right]$, then $f^{-1}(x) = ...$**

Sol. Given that, $f(x) = \left[4-(x-7)^3\right]$,

Let $y = \left[4-(x-7)^3\right]$

$(x-7)^3 = 4-y$

$\Rightarrow (x-7) = (4-y)^{1/3}$

$\Rightarrow x = 7+(4-y)^{1/3}$

$f^{-1}(x) = 7+(4-x)^{1/3}$

True/ False

53. **Let $\mathbf{R} = \{(3,\ 1),\ (1,\ 3),\ (3,\ 3)\}$ be a relation defined on the set A = {1, 2, 3}. Then R is symmetric, transitive but not reflexive.**

Sol. **False**

Given that, $\mathbf{R}=\{(\mathbf{3},\ \mathbf{1}),\ (\mathbf{1},\ \mathbf{3}),\ (\mathbf{3},\ \mathbf{3})\}$ be defined on the set $A=\{1,2,3\}$

$(1,1)\notin R$

So, R is not reflexive. $(3,1)\notin R,(1,3)\in R$

Hence, R is symmetric.

Since, $(3,1)\in R,(1,3)\in R$

But $(1,1)\notin R$

Hence, R is not transitive.

54. **Let $f:R\to R$ be the function defined by. Then $f(x)=sin(3x+2)\ \forall x\in R.$ Then, f is invertible.**

Sol. **False**

Given $f(x)=sin(3x+2)\ \forall x\in R.$ is not one-one function for all $x\in R.$

So, f is not invertible.

55. **Every relation which is symmetric and transitive is also reflexive.**

Sol. **False**

Let R be a relation defined by

$R=\{(1,2),(2,1),(1,1),(2,2)\}$ on the set $A=\{1,2,3\}$.

It is clear that $(3,3)\in R$. So, it is not reflexive.

56. **An integer m is said to be related to another integer n, if m is a integral multiple of n. This relation in Z is reflexive, symmetric and transitive.**

Sol. **False**

The given relation is reflexive and transitive but not symmetric.

57. **Let $\mathbf{A}=\{\mathbf{0},\ \mathbf{1}\}$ and N be the set of natural numbers. Then, the mapping $\mathrm{f}:N\to A$ defined by $f(2n-1)=0$, $f(2n)=1$, $\forall n\in N$, is onto.**

Sol. **True**

Given, A={0, 1}

$f(2n-1)=0, f(2n)=1,\forall n\in N$

So, the mapping $f:N\to A$ is onto.

58. **The relation R on the set A={1,2,3} defined as $\mathrm{R}=\{(1,1),(1,2),(2,1),(3,3)\}$ is reflexive, symmetric and transitive.**

Sol. **False**

Given that, $\mathrm{R}=\{(1,1),(1,2),(2,1),(3,3)\}$

$(2,2)\notin R$

So, R is not reflexive.

59. The composition of functions is commutative.

Sol. False

Let $f(x)=x^2$

And $g(x)=x+1$

$fog(x)=f\{g(x)\}=f(x+1)$

$=(x+1)^2=x^2+2x+1$

$gof(x)=g\{f(x)\}=g(x^2)=x^2+1$

$\therefore$ $\mathrm{fog}(x)\neq gof(x)$

60. The composition of functions is associative.

Sol. **True**

Let $f(x)=x, g(x)=x+1$

And $h(x)=2x-1$

Then, $fo\{goh(x)\}=f\left[g\{h(x)\}\right]$

$= f\{g(2x-1)\}$

$= f(2x-1)+1$

$= f(2x)=2x$

$\therefore$ $(fog)oh(x)=(fog)\{h(x)\}$

$=$ $(fog)(2x-1)$

$=$ $f\{g(2x-1)\}$

$=$ $f(2x-1+1)$

$= f(2x)=2x$

61. Every function is invertible.

Sol. **False**

Only bijective functions are invertible.

62. A binary operation on a set has always the identity element.

Sol. **False**

'+' is a binary operation on the set N but it has no identity element.

Relations and Functions

<u>Short Answer Type Questions</u>

1. **Let A = {a, b, c} and the relation R be defined on A as follows:**
R = {(a, a), (b, c), (a, b)}.
Then, write minimum number of ordered pairs to be added in R to make R reflexive and transitive.

Sol. Given relation, R = {(a, a), (b, c), (a, b)}.
To make R is reflexive we must add (b, b) and (c, c) to R. Also, to make R is transitive we must add (a, c) to R.
So, minimum number of ordered pair is to be added are (b, b), (c, c), (a, c).

2. **Let D be the domain of the real valued function f defined by $f(x) = \sqrt{25-x^2}$. Then, write D.**

Sol. Given function is $f(x) = \sqrt{25-x^2}$.
For real valued of $f(x)$ $\quad 25-x^2 \geq 0$
$x^2 \leq 25$
$-5 \leq x \leq +5$
$D = [-5, 5]$

3. **Let f, g: R → R be defined by $f(x) = 2x+1$ and $g(x) = x^2-2$, $g(x) = x^2-2$, respectively. Then, find $g o f$.**

Sol. Given that, $f(x) = 2x+1$ and $g(x) = x^2-2$, ∀ x ∈ R
$gof = g\{f(x)\}$
$= g(2x+1) = (2x+1)^2 - 2$
$= 4x^2 + 4x + 1 - 2$
$= 4x^2 + 4x - 1$

4. **Let f : R → R be the function defined by $f(x) = 2x-3, \forall x \in R$. write f^{-1}.**

Sol. Given that $f(x) = 2x-3, \forall x \in R$.
Now, let $y = 2x-3$
$2x = y+3$
$x = \frac{y+3}{2}$
$f^{-1}(x) = \frac{x+3}{2}$

5. **If** $A = \{a, b, c, d\}$ **and the function** $f = \{(a,b), (b,d), (c,a), (d,c)\}$**, write** f^{-1}**.**

Sol. Given that $A = \{a, b, c, d\}$

And $f = \{(a,b), (b,d), (c,a), (d,c)\}$

$f^{-1} = \{(b,a), (d,b), (a,c), (c,d)\}$

6. **If** $f: \mathbf{R} \to \mathbf{R}$ **is defined by** $f(x) = x^2 - 3x + 2$, **write** $f(f(x))$

Sol. Given that $f(x) = x^2 - 3x + 2$,

$f(f(x)) = f(x^2 - 3x + 2)$,

$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$

$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$

$= x^4 + 10x^2 - 6x^3 - 3x$

$f(f(x)) = x^4 - 6x^3 + 10x^2 - 3x$

7. **Is** $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ **a function? If** g **is described by** $g(x) = \alpha x + \beta$**, then what value should be assigned to** α **and** β**.**

Sol. Given that $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$

Here, each element of domain has unique image. So, g is a function.

Now given that,

$g(x) = \alpha x + \beta$

$g(1) = \alpha + \beta$

$\alpha + \beta = 1 \ldots.(i)$

$g(2) = 2\alpha + \beta$

$2\alpha + \beta = 3 \ldots.(ii)$

From Eqs. (i) and (ii)

$\Rightarrow 2(1 - \beta) + \beta = 3$

$2 - 2\beta + \beta = 3$

$\Rightarrow 2 - \beta = 3$

$\beta = -1$

$If\ \beta = -1, then\ \alpha = 2$

$\alpha = 2, \beta = -1$

8. **Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective.**

(i) $\{(x, y) : x\ is\ a\ person,\ y\ is\ the\ mother\ of\ x\}$**.**

(ii) $\{(a,\ b):\ a \text{ is a person},\ b \text{ is an ancestor of } a\}$.

Sol. (i) Given set of ordered pair is $\{(x, y): x \text{ is a person},\ y \text{ is the mother of } x\}$

It represents a function. Here, the image of distinct elements of x under f are not distinct, so it is not injective but it is a surjective.

(ii) Set of ordered pairs = $\{(a,\ b):\ a \text{ is a person},\ b \text{ is an ancestor of } a\}$.

Here, each element of domain does not have a unique image. So, it does not represent function.

9. If the mappings f and g are given by $f=\{(1,\ 2),(3,\ 5),(4,\ 1)\}$ and $g=\{(2,\ 3),(5,\ 1),(1,\ 3)\}$, write $f\,o\,g$.

Sol. Given that, $f=\{(1,\ 2),(3,\ 5),(4,\ 1)\}$

And $g=\{(2,\ 3),(5,\ 1),(1,\ 3)\}$,

$Now, fog(2)=f\{g(2)\}=f(3)=5$

$fog(5)=f\{g(5)\}=f(1)=2$

$fog(1)=f\{g(1)\}=f(3)=5$

$fog=\{(2,5),(5,2),(1,5)\}$

10. Let C be the set of complex numbers. Prove that the mapping f : C → R given by $f(z)=|z|,\ \forall z\in C$, is neither one-one nor onto.

Sol. The mapping $f:C\to R$

Given, $f(z)\ =\ |z|,\ \forall z\ \in\ C$

$f(1)\ =\ |1|=1$

$f(-1)\ =\ |-1|=1$

$f(1)=f(-1)$

$But\, 1\neq -1$

So, $f(z)$ is not one-one. Also, $f(z)$ is not onto as there is no pre-image for any negative element of R under the mapping $f(z)$.

11. Let the function $f:R\to R$ be defined by $f(x)=\cos x,\ \forall\ x\in R$. Show that f is neither one-one nor onto.

Sol. Given function, $f(x)=\cos x,\ \forall\ x\in R$

Now, $f\left(\frac{\pi}{2}\right)=\cos\frac{\pi}{2}=0$

$\Rightarrow\ f\left(\frac{-\pi}{2}\right)=\cos\frac{\pi}{2}=0$

$\Rightarrow\ f\left(\frac{\pi}{2}\right)=f\left(\frac{-\pi}{2}\right)$

But $\frac{\pi}{2}\neq\frac{-\pi}{2}=0$

So, f (x) is not one-one

Now, $f(x)=\cos x,\ \forall\ x\in R$ is not onto as there is no pre-image for any real number. Which does not belong to the intervals [-1, 1], the range of cos x.

12. **Let X = {1, 2, 3}and Y = {4, 5}. Find whether the following subsets of $X\times Y$ are functions from X to Y or not.**
(i) f = {(1, 4), (1, 5), (2, 4), (3, 5)}
(ii) g = {(1, 4), (2, 4), (3, 4)}
(iii) h = {(1,4), (2, 5), (3, 5)}
(iv) k = {(1,4), (2, 5)}.

Sol. Given that
X = {1, 2, 3} and Y = {4, 5}.
$X\times Y=\{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$
(i) f = {(1, 4), (1, 5), (2, 4), (3, 5)}
f is not a function because f has not unique image.
(ii) g = {(1, 4), (2, 4), (3, 4)}
Since, g is a function as each element of the domain has unique image.
(iii) h = {(1,4), (2, 5), (3, 5)}
It is clear that h is a function.
(iv) k = {(1,4), (2, 5)}.
k is not a function as 3 has not any image under the mapping.

13. **If functions $f: A\rightarrow B\ and\ g: B\rightarrow A$ satisfy $gof=I_A$, then show that f is one-one and g is onto.**

Sol. Given that, $f: A\rightarrow B\ and\ g: B\rightarrow A$ satisfy $gof=I_A$,

$\because gof=I_A$

$\Rightarrow gof\{f(x_1)\}=gof\{f(x_2)\}$

$\Rightarrow g(x_1)=g(x_2)$ $[\because gof=I_A]$

$\therefore x_1=x_2$

Hence, f is one-one and g is onto.

14. **Let f : R → R be the function defined by $f\ (x)\ =\frac{1}{2-cosx},\ \forall x\in R$. Then, find the range of f.**

Sol. Given Function, $f\ (x)\ =\frac{1}{2-cosx},\ \forall x\in R$

Let $y = \dfrac{1}{2-\cos x}$

$\Rightarrow 2y - y\cos x = 1$

$\Rightarrow y\cos x = 2y - 1$

$\Rightarrow \cos x = \dfrac{2y-1}{y} = 2 - \dfrac{1}{y} \Rightarrow \cos x = 2 - \dfrac{1}{y}$

$\Rightarrow -1 \le \cos x \le 1 \Rightarrow -1 \le 2 - \dfrac{1}{y} \le 1$

$\Rightarrow -3 \le \dfrac{1}{y} \le 1 \Rightarrow 1 \le \dfrac{1}{y} \le 3$

$\Rightarrow 3 \le \dfrac{1}{y} \le 1$

So, y range is $\left[\dfrac{1}{3}, 1\right]$

15. Let n be a fixed positive integer. Define a relation R in Z as follows $\forall\, a,\ b \in Z,$ aRb if and only if a– b is divisible by n. Show that R is an equivalence relation.

Sol. Given that, $\forall\, a,\ b \in Z,$ aRb if and only a-b is divisible by n

Now

I. Reflexive

$aRa \Rightarrow (a-a)$ is divisible by n, which is true for any integer a as '0' is divisible by n. Hence, R is reflexive.

II. Synnetric

aRb

$\Rightarrow$ a-b is divisible by n.

$\Rightarrow$ -b+a is divisible by n.

$\Rightarrow$ -$(b-a)$ *is* divisible by n.

$\Rightarrow$ $(b-a)$ *is divisible by n.*

$\Rightarrow$ bRa

Hence, R is symmetric.

III. Transitive

Let aRb and bRa

$\Rightarrow$ (a-b) is divisible by n and (b-c) is divisible by n

$\Rightarrow$ (a-b) +(b-c) is divisible by n

$\Rightarrow$ (a-c) is divisible by n

$\Rightarrow$ aRc

Hence, R is transitive.

So, R is an equivalence relation.

Relations and Functions
Long Answer Type Questions

16. **If $A = \{1, 2, 3, 4\}$, define relations on A which have properties of being:**

(a) reflexive, transitive but not symmetric
(b) symmetric but neither reflexive nor transitive
(c) reflexive, symmetric and transitive.

Sol. Given that $A = \{1, 2, 3, 4\}$,

(i) Let $R_1 = \{(1,1),(1,2),(2,3),(2,2),(1,3),(3,3.)\}$

R_1 is reflexive, since, (1,1) (2,2) (3,3) lie is R_1

Now, $(1,2) \in R_1$ $(2,3) \in R_1 \Rightarrow (1,3) \in R_1$

Hence, R_1 is also transitive but $(1,2), \in R_1 \Rightarrow (2,1) \notin R_1$

So, it is not symmetric.

(ii) Let $R_2 = \{(1,2),(2,1)\}$

Now, $(1,2) \in R_2, (2,1) \in R_2$

So, it is symmetric.

(iii) Let $R_3 = \{(1,2),(2,1),(1,1),(2,2),(3,3),(1,3),(3,1),(2,3)\}$

Hence, R_3 is reflexive, symmetric and transitive.

17. **Let R be relation defined on the set of natural number N as follows: $R = \{(x, y): x \in N, y \in N, 2x + y = 41\}$. Find the domain and range of the relation R. Also verify whether R is reflexive, symmetric and transitive.**

Sol. Given that, $R = \{(x, y): x \in N, y \in N, 2x + y = 41\}$.

Domain = {1,2,3……….20}

Range = {1,3,5,7…………………39}

$R = \{(1,39),(2,37),(3,35)……(19,3),(20,1)\}$

R is not reflexive as (2, 2)$\notin R$

$2 \times 2 + 2 \neq 41$

So, R is not symmetric

As $(1,39) \in R\ but\ (39,1) \in R$

So, R is not transitive

As $(11,19) \in R, (19,3) \in R$

But $(11,3) \notin R$

Hence, R is neither reflexive, nor symmetric and nor transitive.

18. **Given, A = {2, 3, 4}, B = {2, 5, 6, 7}. Construct an example of each of the following:**
(a) an injective mapping from A to B

(b) a mapping from A to B which is not injective
(c) a mapping from B to A.

Sol. Given that A = {2, 3, 4}, B = {2, 5, 6, 7}

(i) Let $f : A \to B$ donote a mapping

$f = \{(x, y) : y = x+3\}$

i.e. $f = \{(2,5),(3,-6),(4,7)\}$ Which is an injective mapping.

(ii) Let $g : A \to B$ donote a mapping such that $g = \{(2,2),(3,5),(4,5)\}$ which is not an injective mapping.

(iii) Let $h : B \to A$ denote a mapping such that $h = \{(2,2),(5,3),(6,4),(7,4)\}$ which is a mapping from B to A.

19. Give an example of a map
(i) which is one-one but not onto
(ii) which is not one-one but onto
(iii) which is neither one-one nor onto.

Sol. (i) Let $f : N \to N,$ be a mapping defined by $f(x) = 2x$

Which is one-one

For $f(x) = f(x_2)$

$\Rightarrow \ 2x_1 = 2x_2$

$x_1 = x_2$

Further f is not onto, as for $1 \in N,$ there does not exist any x in N such that $f(x) = 2x+1.$

(ii) Let $f : N \to N,$ given by $f(1) = f(2) = 1 \ and \ f(x) = x-1$ for every $x > 2$ is onto but not one-one. f is not one-one as $f(1) = f(2) = 1.$ But f is onto.

(iii) The mapping $f : R \to R$ defined as $f(x) = x^2$, is neither one-one not onto.

20. Let $A = R - \{3\}, B = R - \{1\}.$ If $f :\ A \to B$ be defined by $f(x) = \dfrac{x-2}{x-3} \ \forall x \in A.$. Then, show that f is bijective.

Sol. Given that, $A = R - \{3\}, B = R - \{1\}.$

$f :\ A \to B$ is defined by $f(x) = \dfrac{x-2}{x-3} \ \forall x \in A.$

For injectivity

Let $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1 - 2}{x_1 - 3} = \dfrac{x_2 - 2}{x_2 - 3}$

$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$

$\Rightarrow x_1 x_2 - 3x_1 - 2x_2 + 6 = x_1 x_2 - 3x_2 - 2x_1 + 6$

$\Rightarrow -3x_1 - 2x_2 = -3x_2 - 2x_1$

$\Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$

So, f (x) is an injective function

For surjectivity

$Let\ y = \dfrac{x-2}{x-3} \Rightarrow x-2 = xy-3y$

$\Rightarrow \ \mathrm{x}(1-y) = 2-3y \Rightarrow x = \dfrac{2-3y}{1-y}$

$\Rightarrow \ \mathrm{x} = \dfrac{3y-2}{y-1} \in A, \forall y \in B$ [codomain]

So, $f(x)$ is surjective function.

Hence, $f(x)$ is a bijective function.

21. Let A = [–1, 1]. Then, discuss whether the following functions defined on A are one-one, onto or bijective:

(i) $f(x) = \dfrac{x}{2}$

(ii) $g(x) = |x|$

(iii) $h(x) = x|x|$

(iv) $k(x) = x^2$

Sol. Given that $A = [-1, 1]$

(i) $f(x) = \dfrac{x}{2}$

Let $f(x_1) = f(x_2)$

$\Rightarrow \ \dfrac{x_1}{2} = \dfrac{x_2}{2} \Rightarrow x_1 = x_2$

So, $f(x)$ is one-one.

Now, let $y = \dfrac{x}{2}$

$\Rightarrow$ x=2y $\notin$ A, $\forall$y $\in$ A

As for $y = 1 \in A$, $x = 2 \notin A$

So, $f(x)$ is not onto.

Also, $f(x)$ is not bijective as it is not onto.

(ii) $g(x) = |x|$

Let $g(x_1) = g(x_2)$

$\Rightarrow |x_1| = |x_2| \Rightarrow x_1 = \pm x_2$

So, $g(x)$ is not one-one.

Now, $y|x| \Rightarrow x = \pm y \notin A, \forall y \in A$

So, $g(x)$ is not onto, also, $g(x)$ is not bijective.

(iii) $h(x) = x|x|$

$\Rightarrow$ $x_1|x_1| = x_2|x_2| \Rightarrow x_1 = x_2$

So, $h(x)$ is one-one

Now, let $y = x|x|$

$\Rightarrow$ $y = x^2 \in A, \forall x \in A$

So, $h(x)$ is onto also, $h(x)$ is a bijective.

(iv) $k(x) = x^2$

Let $k(x_1) = k(x_2)$

$\Rightarrow$ $x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$

Thus, $k(x)$ is not one-one.

Now, let $y = x^2$

$\Rightarrow x\sqrt{y} \notin A, \forall\ y \in A$

As for $y = -1, x = \sqrt{-1} \notin A$

Hence, $k(x)$ is neither one-one nor onto.

22. **Each of the following defines a relation on N:**

(i) x is greater than $y,\ x,\ y \in N$

(ii) $x + y = 10,\ x,\ y \in N$

(iii) xy is square of an integer $x,\ y \in N$

(iv) $x + 4y = 10\ x,\ y \in N$.

Determine which of the above relations are reflexive, symmetric and transitive.

Sol. (i) x is greater than $y, x, y \in N$

$(x, x) \in R$

For xRx $x > x$ *is not true for any* $x \in N$.

Therefore, R is not reflexive.

Let $(x, y) \in R \Rightarrow xRy$

x>y

but y>x is not true for any $x, y \in N$

Thus, R is not symmetric.

Thus, R is not symmetric.

Let xRy and yRz

x>y and $y > z \Rightarrow x > z$

$\Rightarrow$ xRz

So, R is transitive.

(ii) $x+y=10, x, y \in N$

$R=\{(x,y); x+y=10x, y \in N\}$

$R=\{(1,9),(2,8),(3,7),(4,6),(5,5),(6,4),(7,3)(8,2),(9,1)\}$ $(1,1)\notin R$

So, R is not reflexive.

$(x,y)\in R \Rightarrow (y,x)\in R$

Therefore, R is symmetric.

$(1,9)\in R,(9,1)\in R \Rightarrow (1,1)\notin R$

Hence, R is not transitive.

(iii) Given xy, is square of an integer $x,\ y \in N$

$\Rightarrow \quad R=\{(x,y): xy$ is a square of an integer x,y $\in$ N$\}$

$(x,x)\in R, \forall x \in N$

As x^2 is square of an integer for any $x \in N$

Hence, R is reflexive.

If $(x,y)\in R \Rightarrow (y,x)\in R$

Therefore, R is symmetric.

If $(x,y)\in R(y,z)\in R$

So, xy is square of an integer and yz is square of an integer.

Let xy=m^2 and yz=n^2 for some $m, n \in Z$

$x=\frac{m^2}{Y}$ and z=$\frac{x^2}{Y}$

$xz=\frac{m^2n^2}{y^2}$, Which is square of an integer.

So, R is transitive.

(iv) $x+4y=10, x, y \in N$

$R=\{(x,y): x+4y=10, x, y \in N\}$

$R\{(2,2),(6,1)\}$

$(1,1),(3,3)\ldots\ldots\notin R$

Thus, R is not reflexive.

$(6,1)\in R \ \ but\ (1,6)\notin R$

Hence, R is not symmetric.

$(x,y)\in R \quad \Rightarrow x+4y=10 \ \ but\ \ (y,z)\in R$

$y+4z=10 \Rightarrow (x,z)\in R$

So, R is transitive.

23. **Let $A = \{1, 2, 3, \ldots 9\}$ and R be the relation in $A \times A$ defined by $(a, b)\ R\ (c, d)$ if $a + d = b + c$ for $(a, b), (c, d)$ in $A \times A$. Prove that R is an equivalence relation and also obtain the equivalent class [(2, 5)].**

Sol. Given that $A = \{1, 2, 3, \ldots 9\}$ $(a, b)\ R\ (c, d)$ $a + d = b + c$ for $(a,b) \in A \times A$ and

$(c,d) \in A \times A$.

Let $(a,b) R (a,b)$

$\Rightarrow a + b = b + a, \forall a, b \in A$

Which is true for any $a, b \in A$

Hence, R is reflexive.

Let (a, b) R (c, d)

a+d = b+c

$c + b = d + a \Rightarrow (c,d) R (a,b)$

So, R is symmetric.

Let $(a,b) R (c,d)\, and\, (c,d) R (e,f)$

$a + d = b + c$ and $c + f = d + e$

$a + d = b + c$ and $d + e = c + f$ $(a+d) - (d+e) = (b+c) - (c+f)$

$(a - e) = b - f$

$a + f = b + e$

$(a,b) R (e,f)$

Now, equivalence class containing $\left[(2,5)\ is \{(1,4),(2,5),(3,6),(4,7),(5,8),(6,9)\}\right]$

24. **Using the definition, prove that the function $f : A \rightarrow B$ is invertible if and only if f is both one-one and onto.**

Sol. A Function $f : \mathrm{x} \rightarrow \mathrm{y}$ is defined to be invertible, if there exist a function $g = y \rightarrow X$ such that $gof = I_x$ and $fog = I_y$. The function is called the inverse of f and is denoted by f^{-1}

A function $f = x \rightarrow y$ is invertible if f is bijective function.

25. **Functions $f, g : R \rightarrow R$ are defined, respectively, by $f\ (x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, find**

(i) fog

(ii) gof

(iii) fof

(iv) gog

Sol. Given that $f\ (x) = x^2 + 3x + 1$, $g(x) = 2x - 3$,

(i) $fog = f\{g(x)\} = f(2x-3)$

$= (2x-3)^2 + 3(2x-3) + 1$

$= 4x^2 + 9 - 12x + 6x - 9x + 1 = 4x^2 - 6x + 1$

(ii) $gof = g\{f(x)\} = g(x^2 + 3x + 1)$

$= 2(x^2 + 3x + 1) - 3$

$= 2x^2 + 6x + 2 - 3$

$= 2x^2 + 6x - 1$

(iii) $fof = f\{f(x)\} = f(x^2 + 3x + 1)$

$= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$

$= x^4 + 9x^2 + 1 + 6x^3 + 6x + 2x^2 + 3x^2 + 9x + 3 + 1$

$= x^4 + 6x^3 + 14x^2 + 15x + 5$

(iv) $gog = g\{g(x)\} = g(2x-3)$

$= 2(2x-3) - 3$

$= 4x - 6 - 3 = 4x - 9$

26. Let * be the binary operation defined on Q. Find which of the following binary operations are commutative

(i) $a*b = a - b, \forall a, b \in Q$

(ii) $a*b = a^2 + b^2, \forall a, b \in Q$

(iii) $a*b = a + ab, \forall a, b \in Q$

(iv) $a*b = (a-b)^2, \forall a, b \in Q$

Sol. Given that * be the binary operation defined on Q.

(i) $a*b = a - b, \forall a, b \in Q$ and b * a = b - a

So, $a*b \neq b*a [\because b - a \neq a - b]$

Hence, * is not commutative.

(ii) $a*b = a^2 + b^2$

$b*a = b^2 + a^2$

So, * is commutative. [since, '+' is on rational us commutative]

(iii) $a*b = a + ab$

$b*a = b + ab$

Clearly, $a + ab \neq b + ab$

So, * is not commutative.

(iv) $a*b = (a-b)^2, \forall a,b \in Q$

$b*a = (b-a)^2$

$\because (a-b)^2 = (b-a)^2$

Hence, * is commutative.

27. **If * be binary operation defined on R by $a*b=1+ab, \forall a,b\in R$. Then, the operation * is**

(i) commutative but not associative

(ii) associative but not commutative

(iii) neither commutative nor associative

(iv) both commutative and associative

Sol. (i) Given that $a*b=1+ab, \forall a,b\in R$

$a*b=ab+1=b*a$

So, * is a commutative binary Operation.

Also $a*(b*c)=a*(1+bc)=1+a(1+bc)$

$a*(b*c)=1+a+abc..........(1)$

$(a*b)*c=(1+ab)*c$

$=1+(1+ab)c=1+c+abc.........(ii)$

From Eqs. (i) and (ii)

$a*(b*c)\neq(a*b)*c$

So, * is not associative

Hence, * is commutative but not associative.