Ratio and Proportion

Concept of Ratios

There are situations, when we need to compare two quantities.

For example, Swaminathan and Rohan both are in the same class. Their respective marks in mathematics are 96 and 48.

Marks scored by Swaminathan and Rohan can be compared by two methods.

1. Subtraction method

In this method, we subtract one quantity from other to find that one is how much more than the other.

Now,

Marks scored by Swaminathan – Marks scored by Rohan = 96 – 48 = 48

So, it can be said that Swaminathan scored 48 marks more than Rohan in mathematics.

2. Division method

In this method, we divide one quantity by other to find that one is how many times the other.

Now,

$$\frac{\text{Marks scored by Swaminathan}}{\text{Marks scored by Rohan}} = \frac{96}{48} = \frac{2}{1}$$

So, it can be said that the marks scored by Swaminathan are twice the marks scored by Rohan.

When two quantities are compared using division method, the quotient obtained is called "ratio".

First term of a ratio is called "**antecedent**" and the second term is called "**consequent**".

For example, in the ratio $x : y$, x is antecedent and y is consequent.

Remember

- **Comparison is made between the quantities carrying the same units.**
- **Comparison cannot be made between the quantities which are not similar.**
- **Ratio does not have any unit.**

If *x* and *y* are two quantities in a particular ratio, one should not be confused between *x* : *y* and *y* : *x*.

The ratio *x* : *y* means $\frac{x}{y}$ and *y* : *x* means $\frac{y}{x}$.

Conversion of a Fractional Ratio into a Whole Number Ratio

Example: Convert $\frac{1}{5} : \frac{1}{3}$ into ratio in simple form

There are two methods of converting a fractional ratio into a whole number ratio. They are:

Method I: Dividing the first quantity by the second

Solution: We are given the ratio as $\frac{1}{5} : \frac{1}{3}$.
We simply divide the first quantity by the second.

$$\frac{\frac{1}{5}}{\frac{1}{3}} = \frac{1}{5} \times \frac{3}{1} = \frac{3}{5} = 3 : 5$$

Method II:

(i) Find the LCM of the denominators.
So, LCM of 5 and 3 will be 15

(ii) Multiply the terms of the given ratio with the LCM and simplify.

$$\frac{1}{5} \times 15 : \frac{1}{3} \times 15 = 3 : 5$$

Let us now look at an example to understand this concept better.

Example:

Identify the cases out of the following in which a comparison can be

made using ratios.

1. **The ratio between the price of a book and the price of a shirt**
2. **The ratio between the age of a person and the amount of money he has**
3. **The ratio of the length of a park to its breadth**

Solution:

1. The price of a book and the price of a shirt are of the same type. Therefore, in this case, comparison can be made using ratios.

2. The age and money are of different types. Therefore, in this case, we cannot compare the quantities.

3. The quantities length and breadth are of the same type. Therefore, in this case, comparison can be made using ratios.

Application of Ratios in Solving Problems

Ratios are used to compare quantities. They are widely applied in many day-to-day situations.

Consider the case where Seema and Sheetal wrote a test and scored 40 marks and 30 marks respectively.

Now, ratio of Seema's and Sheetal's scores $= \frac{\text{Seema's score}}{\text{Sheetal's score}} = \frac{40}{30} = \frac{4}{3}$

The ratio of two quantities is denoted by '**:**'.

Thus, the ratio of Seema's and Sheetal's scores $\left(\frac{4}{3}\right)$ can be written as 4:3.

Now, what information does this ratio give us?

Since the ratio of their scores is 4:3, it tells us that for every 4 marks that Seema scored, Sheetal scored 3 marks. Thus, even if we do not know their actual scores, but only the ratio of the scores, we can still tell who got more marks.

Let us take another case. Let us suppose Meenu has 60 marbles, out of which, 25 are red in colour, and the rest are black in colour. Now, we have to find the ratios of the numbers of red and black marbles out of the total number of marbles.

We know that the total number of marbles is 60.

Number of red marbles = 25

Thus, ratio of the number of red marbles to the total number of marbles

$$= \frac{\text{Number of red marbles}}{\text{Total number of marbles}} = \frac{25}{60} = \frac{5}{12} = 5:12$$

Now, we can find the number of black marbles by subtracting the number of red marbles from the total number of marbles.

Thus, number of black marbles = 60 – 25 = 35

Thus, ratio of the number of black marbles to the total number of marbles

$$= \frac{\text{Number of black marbles}}{\text{Total number of marbles}} = \frac{35}{60} = \frac{7}{12} = 7:12$$

We can, in fact, find the ratio of the number of black marbles to the total number of marbles in a different way. We know that the marbles are either red or black in colour.

Think of ratios as parts of a whole, just like fractions. Thus, the sum of the ratios of the numbers of red and black marbles to the total number of marbles would be 1.

Thus, ratio of the number of black marbles to the total number of marbles

= 1 – ratio of the number of red marbles to the total number of marbles

$$= 1 - \frac{5}{12} = \frac{12-5}{12} = \frac{7}{12} = 7:12$$

If the ratio between two quantities is *a* : *b*, then we cannot write the ratio as *b* : *a*.

∴ $a : b \neq b : a$

Let us now solve some more problems to understand this concept better.

Example 1:

There are 35 boys and 30 girls in a class. Find the ratio of

1. **the number of boys to the total number of students in the class.**
2. **the total number of students to the number of girls in the class.**
3. **the number of boys to the number of girls in the class.**

Solution:

Number of boys in the class = 35

Number of girls in the class = 30

Thus, total number of students in the class = 35 + 30 = 65

(i) Ratio of number of boys to total number of students $= \frac{35}{65} = \frac{35 \div 5}{65 \div 5} = \frac{7}{13} = 7:13$

(ii) Ratio of total number of students to number of girls $= \frac{65}{30}$

$= \frac{65 \div 5}{30 \div 5} = \frac{13}{6} = 13:6$

(iii) Ratio of number of boys to number of girls $= \frac{35}{30}$

$= \frac{35 \div 5}{30 \div 5} = \frac{7}{6} = 7:6$

Example 2:

The length of a table is 1.5 m and its breadth is 75 cm. Find the ratio of the length of the table to its breadth.

Solution:

Length of the table = 1.5 m = (1.5 × 100) cm = 150 cm

Breadth of the table = 75 cm

Thus, ratio of the length of the table to its breadth $= \frac{\text{Length of table}}{\text{Breadth of table}}$

$$= \frac{150 \text{ cm}}{75 \text{ cm}} = \frac{2}{1} = 2:1$$

Example 3:

Naina has 15 chocolates. She wants to divide these chocolates between Prabha and Priyanka in the ratio 3:2. How many chocolates will each get?

Solution:

There are two parts. One is 3 and another is 2.

Therefore, there are a total of 3 + 2 = 5 parts.

This means that Naina has to divide 15 chocolates into 5 parts. Out of these 5 parts, Prabha will get 3 parts and Priyanka will get 2 parts.

Therefore, number of chocolates that Prabha got $= \frac{3}{5} \times 15 = 9$

Similarly, number of chocolates that Priyanka got $= \frac{2}{5} \times 15 = 6$

Example 4:

The ratio of Nayan's height to Tarun's height is 12:13. Who is taller?

Solution:

The ratio of their heights is 12:13. Now, 12 < 13.

Since the "13" part of the ratio corresponds to Tarun's height, Tarun is taller than Nayan.

Example 5:

Sanju and Rupam started a business. Sanju invested Rs.15000 and Rupam invested Rs.30000 in the business. If the profit of the business was Rs 60000, then divide the profit between them in the ratio of their investment.

Solution:

Amount invested by Sanju = Rs 15000

Amount invested by Rupam = Rs 30000

Thus, ratio of Sanju's investment to Rupam's investment $= \frac{\text{Rs } 15000}{\text{Rs } 30000}$

$= \frac{1}{2} = 1:2$

Therefore, there are a total of 1 + 2 = 3 parts, out of which, Sanju will get 1/3 part and Rupam will get 2/3 parts of the total profit.

Thus, Sanju's share of the profit $= \text{Rs } \left(\frac{1}{3} \times 60000\right) = \text{Rs } 20000$

Similarly, Rupam's share of the profit $= \text{Rs } \left(\frac{2}{3} \times 60000\right) = \text{Rs } 40000$

Example 6:

The ratio of milk and water in a 21 L solution is 5:2. Now, 3 L milk and 3 L water is added to the solution. What is the new ratio of milk and water in the solution?

Solution:

The two parts of the ratio are 5 and 2.

Therefore, sum of the parts = 5 + 2 = 7

This means that if there is a 7 L solution, then the amount of milk = 5 L and the amount of water = 2 L.

Out of 1 L solution, the amount of milk is 5/7 L and the amount of water is 2/7L.

Out of 21 L solution, the amount of milk is $\left(\frac{5}{7} \times 21\right) \text{L} = 15 \text{ L}$ and the amount of water is $\left(\frac{2}{7} \times 21\right) \text{L} = 6 \text{ L}$.

When 3 L of both milk and water is added to the solution, then the amount of milk = (15 + 3) L = 18 L and the amount of water = (6 + 3) L = 9 L.

Therefore, ratio of milk and water in new solution = $\frac{18\text{ L}}{9\text{ L}} = \frac{2}{1} = 2:1$

Example 7:

The population of two countries A and B are 800 lakhs and 1350 lakhs respectively. The respective areas of these countries are 4 lakh sq. km and 5 lakh sq. km. Which of these countries is less populated?

Solution:

The country will be less populated if its population per sq. km (population density) is less.

$$\text{Population density of country A} = \frac{\text{Population of country A}}{\text{Area of country A}}$$

$$= \frac{800\text{ lakh}}{4\text{ lakh sq. km}} = 200\text{ lakh/sq. km}$$

$$\text{Similarly, population density of country B} = \frac{\text{Population of country B}}{\text{Area of country B}}$$

$$= \frac{1350\text{ lakh}}{5\text{ lakh sq. km}} = 270\text{ lakh/sq. km}$$

From the above calculation, we can see that the population density of state A is less than that of state B. Therefore, state A is less populated.

Concept of Equivalent Ratios

Let a:b and c:d be two ratios. If a:b = c:d, i.e., if $\frac{a}{b} = \frac{c}{d}$, then a:b and c:d are called **equivalent ratios.**

For example, consider the case where the number of boys in a class is 30 and the number of girls in the class is 35. Now, the ratio of the number of boys to the number of girls in the class is $\frac{30}{35} = \frac{30 \div 5}{35 \div 5} = \frac{6}{7}$. Now, we say that the required ratio is 6:7. Are both the ratios same?

Yes. The two ratios 30:35 and 6:7 are same and are known as equivalent ratios. Thus, we can use either of the two ratios. However, we always express a ratio in its lowest terms. That is why we expressed the ratio as 6:7, and not as 30:35.

A ratio is always expressed in its lowest terms.

A ratio written in its lowest terms is said to be in its simplest form.

Generally, if $\frac{a}{b}$ is in its lowest terms, then the ratio *a*:*b* is said to be in its simplest form. Now, how did we convert the ratio 30:35 to its equivalent ratio 6:7?

We did this by dividing both the numerator and the denominator of the ratio by the same number. In fact, we can also multiply the numerator and the denominator of a ratio by the same number to get its equivalent ratio. However, the same is not true for addition and subtraction operations.

If we multiply or divide the numerator and the denominator of a ratio by the same non-zero number, then we will get the equivalent ratios of that ratio.

Mathematically, if we have a ratio *a*:*b* and a non zero-number *k,* then

$$a:b = \frac{a}{b} = \frac{ak}{bk} = ak:bk \text{ and}$$
$$a:b = \frac{a}{b} = \frac{a \div k}{b \div k} = (a \div k):(b \div k)$$

Let us find some equivalent ratios of 20:25.

$$20:25 = \frac{20}{25} = \frac{20 \times 2}{25 \times 2} = \frac{40}{50} = 40:50$$
$$20:25 = \frac{20}{25} = \frac{20 \times 3}{25 \times 3} = \frac{60}{75} = 60:75$$
$$20:25 = \frac{20}{25} = \frac{20 \div 5}{25 \div 5} = \frac{4}{5} = 4:5$$

Thus, 40:50, 60:75, and 4:5 are equivalent ratios of 20:25. In this way, we can find infinite number of equivalent ratios of any ratio.

Suppose we divide a number in the ratio *a*:*b* and in the ratio *c*:*d*. Here, we will get two sets in each case. An important point to be noted here is that, if *a*:*b* and *c*:*d* are equivalent ratios, then both the sets obtained in the two cases will be the same. Let us try to understand this with the help of an example.

We know that 1:2 and 2:4 are equivalent ratios. Suppose we need to divide the number 24 in each of the two ratios.
First consider the ratio 1:2. In this case, we can divide the number 24 into two parts as:

$$\frac{1}{1+2}\times 24=\frac{1}{3}\times 24=8 \text{ and } \frac{2}{1+2}\times 24=\frac{2}{3}\times 24=16$$

Thus, the number 24 is divided in the ratio 1:2 as 8:16.
Let us now divide the number 24 in the ratio 2:4. Here, we can divide the number 24 as:

$$\frac{2}{2+4}\times 24=\frac{2}{6}\times 24=8 \text{ and } \frac{4}{2+4}\times 24=\frac{4}{6}\times 24=16$$

As we can see, we again divided the number 24 into the same sets, i.e., 8 and 16.
This is because the two ratios that we took were equivalent ratios.

Note: We can say that two ratios are equivalent, if the product of the numerator of the first ratio and the denominator of the other ratio is equal to the product of the denominator of first ratio and the numerator of the other ratio.
For example, to check the equivalence of the ratios, 14:49 and 6:21, we have to check whether $\frac{14}{49}$ and $\frac{6}{21}$ are equivalent or not.

Then, 14 × 21 = 294 = 6 × 49

Therefore, $\frac{14}{49}$ and $\frac{6}{21}$ are equivalent fractions.
Hence, 14:49 and 6:21 are equivalent ratios.

Finding missing values and numbers in given ratio equations

If two or more equivalent ratios are given, we can find the missing numbers in the ratios. to understand this concept better.

Compare quantities with the help of ratios

While comparing the fractions, if the two fractions are equal, then the given ratios are an example of **equivalent ratios**.

Let us solve some examples based on the above discussed concepts.

Example 1:
Find four equivalent ratios of 88:102.

Solution:

$$88:102 = \frac{88}{102} = \frac{88\times 2}{102\times 2} = \frac{176}{204} = 176:204$$

$$88:102 = \frac{88}{102} = \frac{88\times 3}{102\times 3} = \frac{264}{306} = 264:306$$

$$88:102 = \frac{88}{102} = \frac{88\times 4}{102\times 4} = \frac{352}{408} = 352:408$$

$$88:102 = \frac{88}{102} = \frac{88\div 2}{102\div 2} = \frac{44}{51} = 44:51$$

Thus, four equivalent ratios of 88:102 are 176:204, 264:306, 352:408, and 44:51.

Example 2:

Find the missing numerator and denominator in the ratio equation $\frac{52}{39} = \frac{?}{3} = \frac{260}{?}$.

Solution:

In $\frac{52}{39} = \frac{?}{3}$, division of 39 by 13 gives 3.

$$\therefore \frac{52}{39} = \frac{52\div 13}{39\div 13} = \frac{4}{3} = \frac{?}{3}$$

Thus, the missing numerator is 4.

Similarly, multiplication of 52 by 5 gives 260.

$$\therefore \frac{52}{39} = \frac{52\times 5}{39\times 5} = \frac{260}{195} = \frac{260}{?}$$

Thus, the missing denominator is 195.

Example 3:

Arrange the following ratios in ascending order of magnitude.
1:3, 4:5, 3:12, 3:8

Solution:
The given ratios can be written as fractions as follows:

$\frac{1}{3}, \frac{4}{5}, \frac{3}{12}$, and $\frac{3}{8}$

LCM of 3, 5, 12, and 8 = 120

$$\frac{1}{3} = \frac{1\times 40}{3\times 40} = \frac{40}{120}$$
$$\frac{4}{5} = \frac{4\times 24}{5\times 24} = \frac{96}{120}$$
$$\frac{3}{12} = \frac{3\times 10}{12\times 10} = \frac{30}{120}$$
$$\frac{3}{8} = \frac{3\times 15}{8\times 15} = \frac{45}{120}$$
$$\frac{30}{120} < \frac{40}{120} < \frac{45}{120} < \frac{96}{120}$$

Or, $\frac{3}{12} < \frac{1}{3} < \frac{3}{8} < \frac{4}{5}$

Thus, the given ratios in the ascending order are:

3:12 < 1:3 < 3:8 < 4:5

Example 4:

If $\frac{p}{q} = \frac{r}{s}$ then show that $\frac{7p-5r}{7q-5s} = \frac{9p+4r}{9q+4s}$.

Solution:

Let $\frac{p}{q} = \frac{r}{s} = k$

∴ $p = qk$ and $r = sk$

We have to show that

$$\frac{7p-5r}{7q-5s} = \frac{9p+4r}{9q+4s}$$

L.H.S. = $\frac{7p-5r}{7q-5s} = \frac{7qk-5sk}{7q-5s} = \frac{k(7q-5s)}{7q-5s} = k$

R.H.S. = $\frac{9p+4r}{9q+4s} = \frac{9qk+4sk}{9q+4s} = \frac{k(9q+4s)}{9q+4s} = k$

So, L.H.S. = R.H.S.

$\therefore \frac{7p-5r}{7q-5s} = \frac{9p+4r}{9q+4s}$

Concept of Compounded Ratios

We know what are ratios and how to find ratios in given situations.

Using this ratio, we can also find many other types of ratios.

For example, $2:3 \text{ and } 1:5$ can be multiplied together as $2 \times 1 : 3 \times 5 = 2:15$

Here, $2:15$ is the compounded ratio.

Thus, when two or more ratios are multiplied together, the ratio obtained is known as **compounded ratio**.

That is, if $a:b$ and $c:d$ are two ratios, then their compounded ratio is $ac:bd$.

The compounded ratio of three ratios $a:b$, $c:d$ and $e:f$ is the ratio $ace:bdf$.

Duplicate, sub-duplicate, triplicate, and sub-triplicate are also compounded ratios.

When a ratio is compounded with itself, it is called **duplicate ratio** of the given ratio.

For example, 16:25 is the duplicate ratio of 4:5.

The **sub-duplicate ratio** of $a:b$ is $\sqrt{a}:\sqrt{b}$.

Similarly, we can write triplicate ratio.

A ratio multiplied with itself three times is called **triplicate ratio**.

For example, The triplicate of a ratio $a:b$ is $a^3:b^3$.

The **sub-triplicate ratio** of $a:b$ is $\sqrt[3]{a}:\sqrt[3]{b}$.

The **reciprocal ratio** of $a:b$ is $b:a$.

Let us now solve some examples to understand the concept of compounded ratios better.

Example 1:

Find the compounded ratio of:

(i) 22:7, 5:3, and 9:11

(ii) 2*a* + 3*b*: 2*a* − 3*b* and 2*a* − 3*b*: 7*a* − 4*b*

Solution:

(i) Compounded ratio $= \frac{22}{7} \times \frac{5}{3} \times \frac{9}{11}$

= 30/7

(ii) Compounded ratio $= \frac{(2a+3b)}{(2a-3b)} \times \frac{(2a-3b)}{(7a-4b)}$

$= \frac{2a+3b}{7a-4b}$

Example 2:

Determine the following:

(i) Duplicate of 8:25

(ii) Triplicate of 2:7

(iii) Sub-duplicate of 8:121

Solution:

1. Duplicate ratio of 8:45

$= \frac{8}{25} \times \frac{8}{25}$

$= \frac{64}{625} = 64:625$

(ii) Triplicate ratio of 2:7

$$= \frac{2}{7} \times \frac{2}{7} \times \frac{2}{7}$$

$$= \frac{8}{343} = 8:343$$

(iii) Sub-duplicate ratio of 8:121

$$= \frac{\sqrt{8}}{\sqrt{121}}$$
$$= \frac{2\sqrt{2}}{11}$$
$$= 2\sqrt{2}:11$$

Concept of Proportion

Ravi has 15 pens and Sumit has 10 pens. What is the ratio of the number of pens with Ravi to the number of pens with Sumit?

Ratio = $15:10 = \frac{3}{2}$ = 3:2

This ratio is equivalent to a ratio 6:4 i.e., the ratio 3:2 is same as the ratio 6:4.

The numbers 3, 2, 6, and 4 are said to be in **proportion**.

> “**If two ratios are equal, then the numbers or values in the ratios are said to be in proportion”.**

In general, if *a*, *b*, *c*, and *d* are any four numbers and $\frac{a}{b} = \frac{c}{d}$, then *a*, *b*, *c*, and *d* are said to be in proportion.

Proportion is denoted by the symbol ‘**=**’ or ‘**::**’ and is placed between two ratios.

For example, if 2, 4, 5, and 10 are in proportion, then we can denote this by writing

2:4 **=** 5:10

Or, 2:4 **::** 5:10

Now, consider one more example. Suppose there are 6 males and 2 females in a car. And there are 27 males and 9 females in a bus. Now, ratio of the number of males to the number of females in the car $= \frac{6}{2} = \frac{3}{1}$

Ratio of the number of males to the number of females in the bus $= \frac{27}{9} = \frac{3}{1}$

Therefore, we can write, $\frac{6}{2} = \frac{27}{9}$

Thus, 6, 2, 27, and 9 are in proportion.

Here, the numbers 6, 2, 27, and 9 are called **respective terms.**

The four numbers or values involved in a statement of proportion when taken in order are called respective terms. If the numbers are not taken in order, then the numbers are not called respective terms.

The first and the fourth terms of the respective terms are called extreme terms or extremes. The second and the third terms are called middle terms or means.

For example, in the above example of proportion $\frac{6}{2} = \frac{27}{9}$, the numbers 6, 2, 27, and 9 are respective terms; however, 6, 27, 2, and 9 or 6, 2, 9, and 27 are not respective terms.

The numbers 6 and 9 are called extreme terms, while 2 and 27 are called middle terms.

Remember

- **If *p*, *q*, *r*, and *s* are four numbers and *p*:*q* ≠ *r*:*s*, then *p*, *q*, *r*, and *s* are not in proportion.**

  For example, 2:4 ≠ 3:8, therefore, 2, 4, 3, and 8 are not in proportion.

- **If *x*:*y* :: *a*:*b*, then it is read as *x* is to *y* as *a* is to *b*.**

  Let us look at some more examples to understand this concept better.

Example 1:

Determine which of the following numbers are in proportion when taken in the given order.

1. **16, 32, 25, 50**
2. – **22, 55, –35, 48**
3. **72, 24, 12, 36**
4. – **84, – 60, – 56, – 40**

Solution:

1. 16, 32, 25, 50

$$16:32=\frac{16}{32}=\frac{1}{2}=1:2$$
$$25:50=\frac{25}{50}=\frac{1}{2}=1:2$$

Therefore, $16:32=25:50$

Hence, 16, 32, 25, and 50 are in proportion.

2. – 22, 55, –35, 48

$$-22:55=\frac{-22}{55}=-\frac{2}{5}=-2:5$$
$$-35:48=\frac{-35}{48}=-35:48$$

Therefore, $22:55\neq 35:48$

Hence, 22, 55, 35, and 48 are not in proportion.

3. 72, 24, 12, 36

$$72:24=\frac{72}{24}=\frac{3}{1}=3:1$$
$$12:36=\frac{12}{36}=\frac{1}{3}=1:3$$

Therefore, $72:24\neq 12:36$

Hence, 72, 24, 12, and 36 are not in proportion.

4. – 84, – 60, – 56, – 40

$$-84:-60=\frac{-84}{-60}=\frac{7}{5}=7:5$$
$$-56:-40=\frac{-56}{-40}=\frac{7}{5}=7:5$$

Therefore, $-84:-60=-56:-40$

Hence, – 84, – 60, – 56, and – 40 are in proportion.

Example 2:

Find the extreme terms and the middle terms of the following proportions.

1. *a*:*b* = *c*:*d*
2. 7:9 :: 28:36
3. $\frac{18}{30}=\frac{36}{60}$

Solution:

1. *a*:*b* = *c*:*d*

   Extreme terms = *a*, *d*

   Middle terms = *b*, *c*

2. 7:9 :: 28:36

   Extreme terms = 7, 36

   Middle terms = 9, 28

3. $\frac{18}{30}=\frac{36}{60}$

   Extreme terms = 18, 60

   Middle terms = 30, 36

Example 3:

The numbers 42, 54, *x*, and 108 are in proportion. Find the value of *x*.

Solution:

42, 54, x, and 108 are in proportion.

Therefore, $\frac{42}{54} = \frac{x}{108}$

Multiplication of 54 by 2 gives 108.

To find the value of x, we have to multiply 42 by 2.

$\therefore x = 42 \times 2 = 84$

Example 4:

Are the ratios 6 kg to 4800 g and Rs 75 to Rs 60 in proportion?

Solution:

6 kg to 4800 g $= \frac{6\,\text{kg}}{4800\,\text{g}} = \frac{6000\,\text{g}}{4800\,\text{g}} = \frac{6000}{4800} = \frac{5}{4} = 5:4$

Rs 75 to Rs 60 $= \frac{75}{60} = \frac{5}{4} = 5:4$

Therefore, the ratios 6 kg to 4800 g and Rs 75 to Rs 60 are in proportion.

Example 5:

Rita was asked that if a bus travelled 225 km in 5 hours, then what distance will it travel in 15 hours. She replied that the bus will travel 675 km in 15 hours. Was she correct?

Solution:

Rita will be correct, if the ratios 5 hours to 15 hours and 225 km to 675 km are in proportion.

5 hours to 15 hours $= \frac{5}{15} = \frac{1}{3} = 1:3$

225 km to 675 km $= \frac{225}{675} = \frac{1}{3} = 1:3$

Therefore, 5 hours to 15 hours and 225 km to 675 km are in proportion.

Thus, Rita's answer was **correct**.

Example 6:

If the cost of 6 umbrellas is Rs 144, then what is the cost of 8 umbrellas?

Solution:

Let the cost of 8 umbrellas be Rs x.

Now, the ratios of 8 umbrellas to 6 umbrellas and Rs x to Rs 144 must be in proportion,

i.e., 8 umbrellas to 6 umbrellas = Rs x to Rs 144

$\Rightarrow \frac{8}{6} = \frac{x}{144}$

Multiplication of 6 with 24 gives 144, therefore, we have to multiply 8 with 24 to find the value of x. On multiplying 8 with 24, we obtain

$8 \times 24 = 192$

$\therefore x = 192$

Thus, the cost of 8 umbrellas is Rs 192.

Example 7:

On a sunny day, Preetam observed that a 1 m long stick standing vertically on the ground makes a shadow of length 30 cm. What will be the height of a tree if the length of its shadow is 3 m 60 cm at that time?

Solution:

Let x be the height of the tree.

We know 1 m = 100 cm

$\therefore$ 3 m 60 cm = $(3 \times 100 + 60)$ cm = 360 cm

Ratio of height of stick and length of its shadow = 100 cm : 30 cm

Similarly, ratio of height of tree and length of its shadow = *x* : 360 cm

At the same time, the ratio of the height of the stick and the length of its shadow and the ratio of the height of the tree and the length of its shadow are equal. Thus, the two ratios are in proportion.

∴ 100 cm: 30 cm = *x* : 360 cm

$$\frac{100 \text{ cm}}{30 \text{ cm}} = \frac{x}{360 \text{ cm}}$$
$$x = \left(\frac{100 \times 360}{30}\right) \text{ cm}$$
$$x = 1200 \text{ cm}$$
$$x = 12 \text{ m}$$

Therefore, the height of the tree is 12 m.

Continued and Mean Proportion

We know that the numbers in the ratio are said to be in **proportion** if the two ratios are equal.

For example, 2, 5, 10 and 25 are said to be in proportion as 2: 5 = 10: 25

Three quantities of the same kind *a*, *b*, and *c* are said to be in continued proportion, if

$$\frac{a}{b} = \frac{b}{c}$$

i.e., the ratio of *a* to *b* is equal to *b* to *c*.

We can also generalise this for more than three ratios.

Let *a*, *b*, *c*, *d*… are some (non-zero) quantities of the same kind. *a*, *b*, *c*, *d* … are said to be in **continued proportion,** if

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \dots$$

For example, 2, 6 and 18 are in continued proportion as $\frac{2}{6} = \frac{6}{18}$.

If *a*, *b*, and *c* are in continued proportion, then *a*, *b*, and *c* are called first proportional, mean proportional, and third proportional respectively.

If we know any two proportional, then we can find the third one.

For example: What is the mean proportional to 6 and 96?

Let the mean proportional be *x*.

Since 6, *x*, and 96 are in continued proportion.

$$\frac{6}{x}=\frac{x}{96}$$
$$6\times 96=x\times x$$
$$x^2=6\times 96$$
$$x=\sqrt{6\times 96}$$
$$x=24$$

Thus, 24 is the mean proportional to 6 and 96.

Now, let us solve some more examples to understand the concept better.

Let us understand the concept better with the help of an example.

Example 1:

What are the values of *a* and *b* for which *a*, 5, 35, and *b* are in continued proportion?

Solution:

a, 5, 35, and *b* are in continued proportion.

$$\therefore \frac{a}{5}=\frac{5}{35}=\frac{35}{b}$$

$\frac{a}{5}=\frac{5}{35}$ (Taking first two terms)

$$a=\frac{5\times 5}{35}=\frac{5}{7}$$

$\frac{5}{35} = \frac{35}{b}$ (Taking last two terms)

$$5 \times b = 35 \times 35$$
$$b = \frac{35 \times 35}{5}$$
$$b = 245$$

Thus, the values of *a* and *b* are 5/7 and 245 respectively.

Example 2:

What number must be added to the numbers 27, 111, and 363 so that they are in continued proportion?

Solution:

Let the required number be *x*.

Then, 27 + *x*, 111 + *x*, and 363 + *x* are in continued proportion.

$$\therefore \frac{27+x}{111+x} = \frac{111+x}{363+x}$$
$$(27+x)(363+x) = (111+x)(111+x)$$
$$x^2 + 390x + 9801 = x^2 + 222x + 12321$$
$$390x + 9801 = 222x + 12321$$
$$390x - 222x = 12321 - 9801$$
$$168x = 2520$$
$$x = 15$$

Thus, the required number is 15.

Example 3:

If *a*, *b*, *c*, and *d* are in continued proportion, then prove that:

$$(a+c)^3 : (b+d)^3 :: a : d$$

Solution:

Since *a*, *b*, *c*, and *d* are in continued proportion,

$\therefore \frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k$ (say)

⇒ $a = bk$, $b = ck$, $c = dk$

Then,

$$(a+c)^3 : (b+d)^3$$
$$= \frac{(a+c)^3}{(b+d)^3}$$
$$= \frac{(bk+dk)^3}{(b+d)^3}$$
$$= \frac{k^3 (b+d)^3}{(b+d)^3}$$
$$= k^3 \qquad ...(1)$$

And,

$$\frac{a}{d} = \frac{bk}{d}$$
$$= \frac{(ck)k}{d}$$
$$= \frac{(dk)k^2}{d}$$
$$= k^3 \qquad ...(2)$$

From equations (1) and (2), we obtain

$$(a+c)^3 : (b+d)^3 :: a : d$$

Example 4:

Six numbers are in continued proportion. If the first and the fourth terms of proportion are 4 and 108 then find the numbers.

Solution:

Let the required numbers be a, ak, ak^2, ak^3, ak^4 and ak^5.

We have $a = 4$ and $ak^3 = 108$

$\therefore \frac{ak^3}{a} = \frac{108}{4}$

$\Rightarrow k^3 = 27$

$\Rightarrow k = 3$

Therefore, the remaining numbers can be obtained as follows:

$ak = 4 \times 3 = 12$

$ak^2 = 4 \times 3^2 = 4 \times 9 = 36$

$ak^4 = 4 \times 3^4 = 4 \times 81 = 324$

$ak^5 = 4 \times 3^5 = 4 \times 243 = 972$

Thus, the required numbers are 4, 12, 36, 108, 324 and 972.

Example 5:

If x, y, z are in continued proportion then prove that $\frac{z}{y-z} = \frac{y+z}{x-z}$.

Solution:

It is given that x, y, z are in continued proportion.

$\therefore \frac{x}{y} = \frac{y}{z}$

Let $\frac{x}{y} = \frac{y}{z} = k$

$\therefore x = yk$ and $y = zk$

$\Rightarrow x = zk^2$

We have to prove that

$$\frac{z}{y-z} = \frac{y+z}{x-z}$$

$$\text{L.H.S.} = \frac{z}{y-z} = \frac{z}{zk-z} = \frac{z}{z(k-1)} = \frac{1}{k-1}$$

$$\text{R.H.S.} = \frac{y+z}{x-z} = \frac{zk+z}{zk^2-z} = \frac{z(k+1)}{z\left(k^2-1\right)} = \frac{(k+1)}{(k-1)(k+1)} = \frac{1}{k-1}$$

So, L.H.S. = R.H.S.

$$\therefore \frac{z}{y-z} = \frac{y+z}{x-z}$$

Example 6:
If the cost of 6 wallets is Rs 1500, then find the cost of 20 wallets.

Solution:

Method I:

Let the cost of 20 wallets be Rs *x*.
Ratio of wallets = 6 : 20

Ratio of cost = 1500 : *x*

6 : 20 :: 1500 : *x*

$$\frac{6}{20} = \frac{1500}{x}$$
$$6x = 1500 \times 20$$
$$x = \frac{1500 \times 20}{6}$$
$$x = 5000$$

Thus, the cost of 20 wallets = Rs 5000.

Method II:
This is a case of direct proportion, so it will be indicated by downward arrow.

Number of wallets	Cost
$\downarrow$ 6	$\uparrow$ 1500
20	x

The product of the numbers at the arrow heads is equal to the product of the numbers at the arrow tails.

$$6x = 1500 \times 20$$
$$x = \frac{1500 \times 20}{6}$$
$$x = 5000$$

Method III (Unitary method):
Cost of 6 wallets = Rs 1500

∴ The cost of 1 wallet = $\text{Rs } \frac{1500}{6} = \text{Rs } 250$

∴ Cost of 20 wallets = Rs (250 × 20) = Rs 5000

Properties of Ratio and Proportion

Let us suppose that two ratios *a*:*b* and *c*:*d* are equal or in proportion.

$$\frac{a}{b} = \frac{c}{d}$$

Then what can we conclude from this?

Yes, we can conclude that:

$$b : a :: d : c$$
$$(a+b) : b :: (c+d) : d$$
$$(a-b) : b :: (c-d) : d$$

And many more results

These are the properties related to ratio and proportion. Now, let us learn these properties one by one in detail.

(1) If *a*:*b*::*c*:*d*, then *b*:*a*::*d*:*c*

This property is called **invertendo.**

(2) If *a*:*b*::*c*:*d*, then *a*:*c*::*b*:*d*

This property is called **alternendo.**

(3) If *a*:*b*::*c*:*d*, then (*a* + *b*): *b*:: (*c* + *d*):*d*

This property is called **componendo.**

(4) If $a:b::c:d$, then $(a - b): b:: (c - d):d$

This property is called **dividendo.**

(5) If $a:b::c:d$, then $(a + b): (a - b) :: (c + d) : (c - d)$

This property is called **componendo and dividendo.**

(6) If $a:b::c:d$, then $a: (a - b)::c: (c - d)$

This property is called **convertendo.**

Theorem on equal ratios:

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$ and $p, q, r\ldots$ are non zero numbers such that $pb+qd+rf\ldots\neq 0$ then

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=\frac{pa+qc+re+\ldots}{pb+qd+rf+\ldots}$$

Proof:

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=k$

Then $a=bk$, $c=dk$ and $e=fk$

$$\therefore \frac{pa+qc+re+\ldots}{pb+qd+rf+\ldots}=\frac{pbk+qdk+rfk+\ldots}{pb+qd+rf+\ldots}$$

$$\Rightarrow \frac{pa+qc+re+\ldots}{pb+qd+rf+\ldots}=\frac{k(pb+qd+rf+\ldots)}{pb+qd+rf+\ldots}$$

$$\Rightarrow \frac{pa+qc+re+\ldots}{pb+qd+rf+\ldots}=k$$

$$\therefore \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=\frac{pa+qc+re+\ldots}{pb+qd+rf+\ldots}=k$$

In particular, we have following formula which is commonly used.

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{a+c+e}{b+d+f}$$

Now let us solve some examples based on these properties.

Example 1:

If *a*:*b*::*c*:*d* then show that:

2a* + 9*b*: 2*c* + 9*d* :: 2*a* − 9*b*:2*c* − 9*d

Solution:

$$a : b :: c : d$$
$$\Rightarrow \frac{a}{b} = \frac{c}{d}$$

$$\Rightarrow \frac{2}{9} \times \frac{a}{b} = \frac{2}{9} \times \frac{c}{d}$$ (Multiplying both sides by 2/9)

$$\Rightarrow \frac{2a}{9b} = \frac{2c}{9d}$$

$$\Rightarrow \frac{2a+9b}{2a-9b} = \frac{2c+9d}{2c-9d}$$ (By componendo and dividendo)

$$\Rightarrow (2a+9b):(2a-9b)::(2c+9d):(2c-9d)$$

$$\Rightarrow (2a+9b):(2c+9d)::(2a-9b):(2c-9d)$$ (By alternendo)

Example 2:

If $(a+b):\sqrt{ab}::8:3$, then find *a*: *b*.

Solution:

$$(a+b):\sqrt{ab}::8:3$$
$$\Rightarrow \frac{a+b}{\sqrt{ab}} = \frac{8}{3}$$
$$\Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{4}{3}$$

$$\Rightarrow \frac{(a+b)+2\sqrt{ab}}{(a+b)-2\sqrt{ab}} = \frac{4+3}{4-3}$$ (By componendo and dividendo)

$$\Rightarrow \frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{\left(\sqrt{a}-\sqrt{b}\right)^2} = \frac{\sqrt{7}}{1}$$

Taking square roots on both sides, we obtain

$$\Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{7}}{1}$$

$$\Rightarrow \frac{\left(\sqrt{a}+\sqrt{b}\right)+\left(\sqrt{a}-\sqrt{b}\right)}{\left(\sqrt{a}+\sqrt{b}\right)-\left(\sqrt{a}-\sqrt{b}\right)} = \frac{\sqrt{7}+1}{\sqrt{7}-1}$$

$$\Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{\sqrt{7}+1}{\sqrt{7}-1}$$

Taking squares on both sides, we obtain

$$\Rightarrow \frac{a}{b} = \frac{\left(\sqrt{7}+1\right)^2}{\left(\sqrt{7}-1\right)^2}$$

$$\Rightarrow \frac{a}{b} = \frac{7+1+2\sqrt{7}}{7+1-2\sqrt{7}}$$

$$\Rightarrow \frac{a}{b} = \frac{2\left(4+\sqrt{7}\right)}{2\left(4-\sqrt{7}\right)}$$

$$\Rightarrow \frac{a}{b} = \frac{4+\sqrt{7}}{4-\sqrt{7}}$$

Or, $a:b = \left(4+\sqrt{7}\right):\left(4-\sqrt{7}\right)$

Example 3:

If $x = \frac{\sqrt{a+5}+\sqrt{a-5}}{\sqrt{a+5}-\sqrt{a-5}}$ **, then prove that $5x^2 - 2ax + 5 = 0$**

Solution:

$$x = \frac{\sqrt{a+5}+\sqrt{a-5}}{\sqrt{a+5}-\sqrt{a-5}}$$

Applying componendo and dividendo,

$$\frac{x+1}{x-1}=\frac{\left(\sqrt{a+5}+\sqrt{a-5}\right)+\left(\sqrt{a+5}-\sqrt{a-5}\right)}{\left(\sqrt{a+5}+\sqrt{a-5}\right)+\left(\sqrt{a+5}-\sqrt{a-5}\right)}$$

$$\Rightarrow \frac{x+1}{x-1}=\frac{2\sqrt{a+5}}{2\sqrt{a-5}}$$

$$\Rightarrow \frac{x+1}{x-1}=\frac{\sqrt{a+5}}{\sqrt{a-5}}$$

Squaring both sides, we obtain

$$\Rightarrow \frac{(x+1)^2}{(x-1)^2}=\frac{a+5}{a-5}$$

$$\Rightarrow \frac{x^2+1+2x}{x^2+1-2x}=\frac{a+5}{a-5}$$

Again applying componendo and dividendo, we obtain

$$\Rightarrow \frac{x^2+1+2x+x^2+1-2x}{x^2+1+2x-x^2-1+2x}=\frac{a+5+a-5}{a+5-a+5}$$

$$\Rightarrow \frac{2\left(x^2+1\right)}{4x}=\frac{2a}{10}$$

By cross multiplication, we obtain

$$5\left(x^2+1\right)=2ax$$

$$5x^2+5-2ax=0$$

$$5x^2-2ax+5=0$$

Example 4:

If $\frac{b+c}{a+2b+2c}=\frac{c+a}{b+2c+2a}=\frac{a+b}{c+2a+2b}$ **then show that each ratio is equal to** $\frac{2}{5}$**.**

Solution:

We have $\frac{b+c}{a+2b+2c}=\frac{c+a}{b+2c+2a}=\frac{a+b}{c+2a+2b}$

On applying theorem on equal ratios, we obtain

$$\frac{b+c}{a+2b+2c}=\frac{c+a}{b+2c+2a}=\frac{a+b}{c+2a+2b}=\frac{(b+c)+(c+a)+(a+b)}{(a+2b+2c)+(b+2c+2a)+(c+2a+2b)}$$

$$\Rightarrow \frac{b+c}{a+2b+2c}=\frac{c+a}{b+2c+2a}=\frac{a+b}{c+2a+2b}=\frac{2(a+b+c)}{5(a+b+c)}$$

$$\Rightarrow \frac{b+c}{a+2b+2c}=\frac{c+a}{b+2c+2a}=\frac{a+b}{c+2a+2b}=\frac{2}{5}$$

Hence proved.