# * Linear Programing * (LP)

"work of George B. Dantzing father of linear programing

linear programing (LP) is used for optimization of our limited resources when there are a number of alternate solution posible for the problem. It is a ~~mathematical~~ mathematical technique and term linear used for variable and it simply means that the relationship between different variable can be represented in form of state line

Requirement of L.P.

1. Objective function:- it is a main function which we need to optimize and it should be clearly identifiable and measurable ~~and~~ in quantitative term like maximization of profil, sale, or minimization of cost.

2. Constraint or Condition:- These are the limited resources within which we need to optimize our objective function

3. All the variables in the objective function and constraint should be linear and non-negative.

# Assumptions/Rule/law's in L.P.

1. Law of certainity ⇒ The varies parameter like objective function coefficient, constraint, resources are ~~not~~ know exactly and their value do not change with time

2. Law of Proportionality →

   if $P =$ Rs $70/\text{unit}$

   by 6 unit

   Total profit $= 70 \times 6 =$ Rs. 420

Linear Programming

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3. Law of Addition/summation →

4. law of continuity or divisibility →

   In L.P. model decesion variable are continuous i.e. they are permitted to take any non negative value that satisfy all constraints.

## General statement of L.P.

objective function → $\max. Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

Constraint

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \le b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m \end{cases}$$

Non-Negative condition → $x_1, x_2, x_3 \cdots + x_n \ge 0$

Non-Negative Condition $x_1, x_2, x_3 - - x_n \geq 0$

where $a_{ij}$, $b_i$ & $c_j$ are constant & $x_j$ is variable

$i = 1, 2 - - m$

$j = 1, 2 - - - n$

$a_{ij}$ - Technological Coefficient or Substution

$b_i$ - Resource Value

$c_j$ - Profit Coefficient

$x_j$ - Decesion / choice Variable

Linear Programming (LP)

06/2016

Graphical Method:

Steps for Graphical method

1) Define Decision variable, objective function and Constraint

2) Draw a graph that include all the Constraint and identify the common fessiable region.

3) find the point within the fessiable region the optimize the objective function this point gives the final solution

Problem (workbook) Q.33
ESE 2007 (Page-83)

| Mc → / ↓Product | | P | Q | R | Profit/Unit |
|---|---|---|---|---|---|
| $x_1$ → | A | 10 | 6 | 5 | 60 |
| $x_2$ → | B | 7.5 | 9 | 13 | 70 |
| max. hr/week → | | 75 | 54 | 65 | |

- key decesion is to determine number of units produce of product A, B in a week
  let these are $x_1$ & $x_2$ repectivity
- Feasible alternative are All the value of $x_1, x_2 \geq 0$
- objective is to maximize weekly profit when the profit per unit give so the objective function

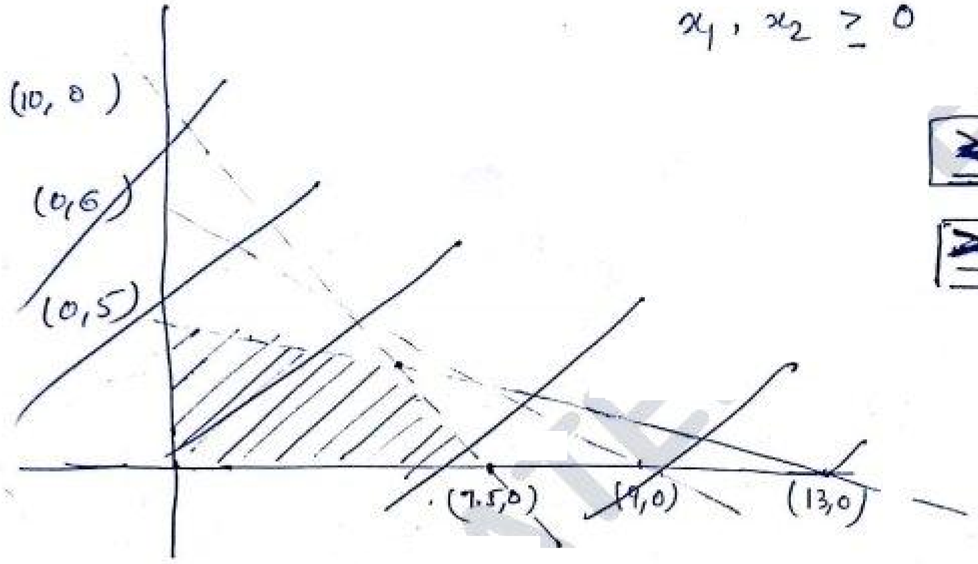
  Max. $Z = 60x_1 + 70x_2$ objective function
- Restriction is of the maximum machine time available for the three machine in a week so the constraints are

$$P \rightarrow 10x_1 + 7.5x_2 \leq 75$$

$$Q \rightarrow 6x_1 + 9x_2 \leq 54$$

$$R \rightarrow 5x_1 + 13x_2 \leq 65$$

all the constraints are ploted on graph to get feasiable ~~reagon~~ region




$x_1, x_2 \geq 0$

$\boxed{\leq}$ towards origion

$\boxed{\geq}$ Away from origion

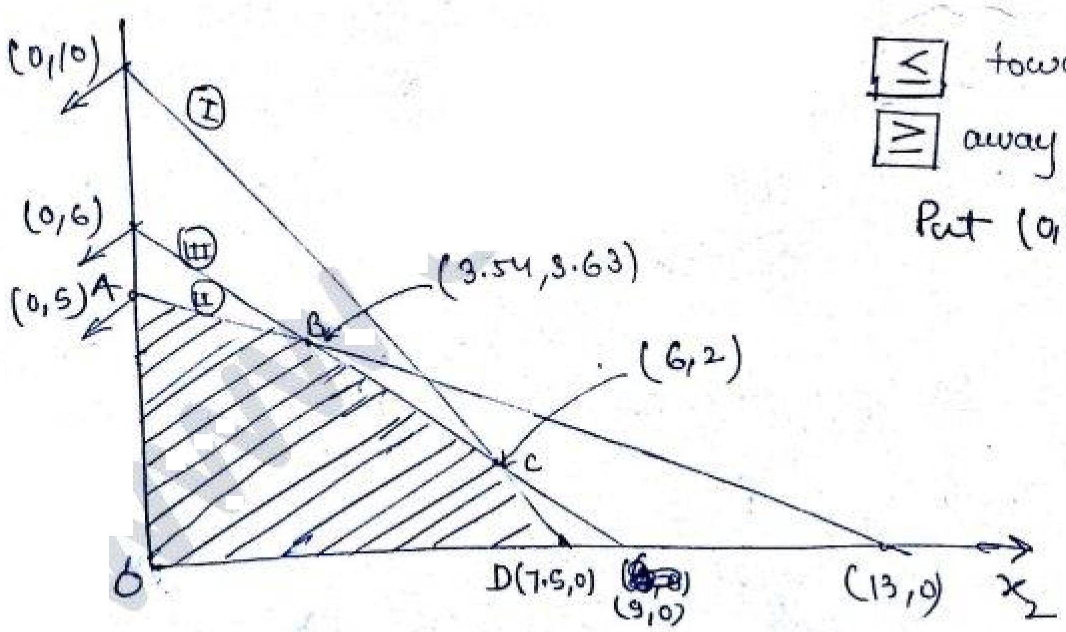
$P \rightarrow 10x_1 + 7.5x_2 \leq 75$

$R \rightarrow 10x_1 + 26x_2 \leq 130$

$x_2 =$

$10x_1 + 7.5x_2 = 75$, $6x_1 + 9x_2 = 54$, $5x_1 + 13x_2 = 65$

$\frac{x_1}{7.5} + \frac{x_2}{10} = 1$ $\quad$ $\frac{x_1}{9} + \frac{x_2}{6} = 1$ $\quad$ $\frac{x_1}{13} + \frac{x_2}{5} = 1$




$\boxed{\leq}$ toward origion

$\boxed{\geq}$ away from origion

Put (0,0) then

The shaded region OABCD is region is feasible solution and any point within this region can be our solution.

Optimality

Now we put the values of corner point of the feasible region in the objective function the point which optimizes the objective function give the final solution.

$Z(A) = 60 \times 0 + 70 \times 5 = 350$

$Z(B) = 60 \times 3.54 + 70 \times 3.63 = 466.5$

$Z(C) = 60 \times 6 + 70 \times 2 = 500$

$Z(D) = 60 \times 7.5 + 70 \times 0 = 450$

$Z(O) = 60 \times 0 + 70 \times 0 = 0$

So optimal point is B

i.e. 6 unit of A & 2 unit of B

* One of the vertex of feasible region give the final solution because objective function is the state line with a costant slope and as it moves away from origion it magnitude increases and optimum value will be at one of the corner extreme point

Binding & Non-Binding Constraint.

$x_1 = 6 \quad , \quad x_2 = 2$

$$\left\{\begin{array}{lll} 10x_1 + 7.5x_2 \le 75 & 75 = 75 & \rightarrow \text{Binding} \\ 6x_1 + 9x_2 \le 54 & 54 = 54 & \rightarrow \text{Binding} \\ 5x_1 + 13x_2 \le 65 & 56 < 65 & \rightarrow \text{Non-Binding} \end{array}\right.$$

~~Binding~~:- where we put the values of optimum Solution in the constraint and LHS = RHS the constraint is termed as binding otherwise non-binding. Final solution is always obtained from the binding constraint

Problem 37: Solving the following LP probem for minimization $z = 6x_1 + 4x_2$

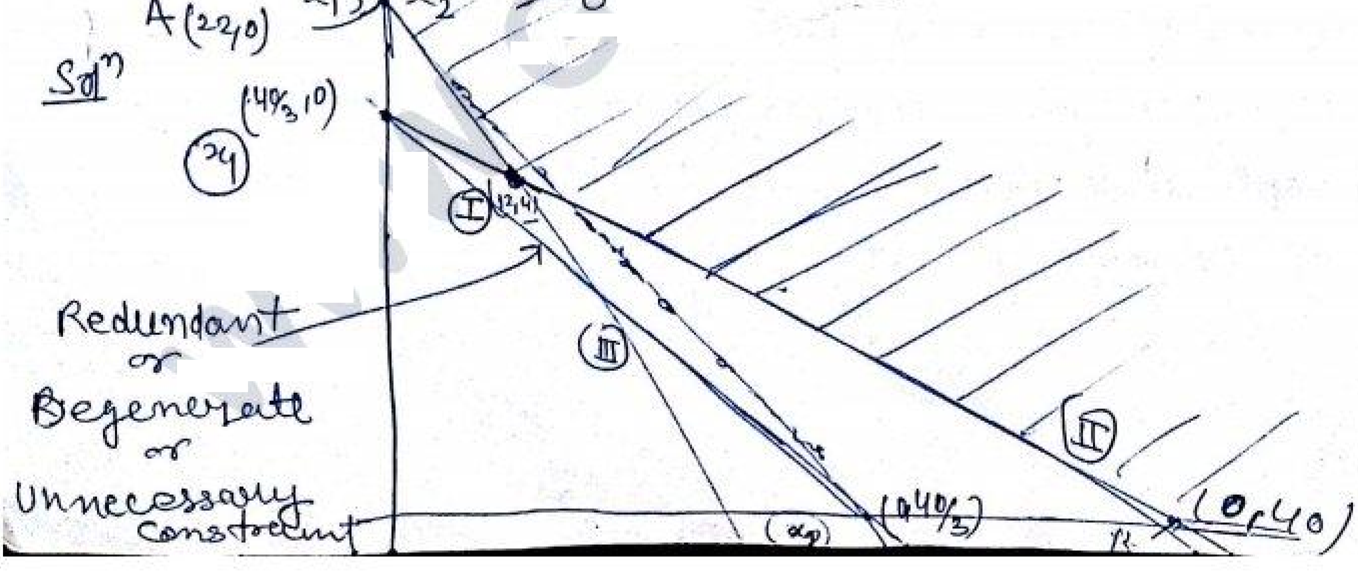
I $\quad 3x_1 + 3x_2 \ge 40$

II $\quad 3x_1 + x_2 \ge 40$

III $\quad 2x_1 + 5x_2 \ge 44$

$x_1, x_2 \ge 0$

Sol^n



$$3x_1 + x_2 = 40$$
$$2x_1 + 5x_2 = 44$$

$$2x_1 + 5(40 - 3x_1) = 4$$

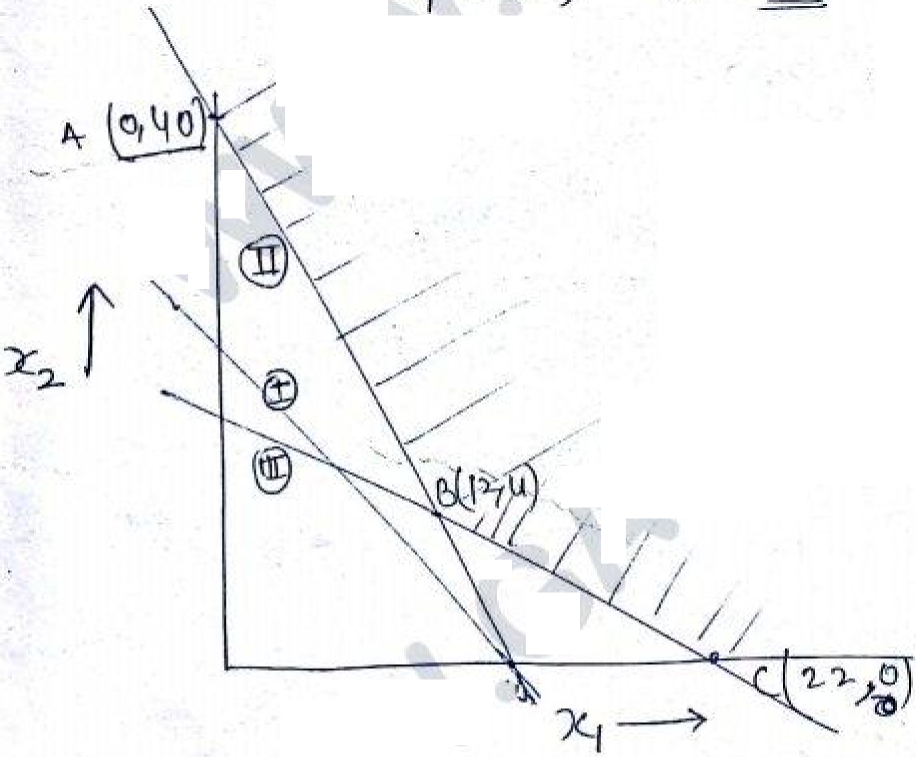
$$x_1 = 12$$
$$x_2 = 4$$

$$Z(\min) = 6 \times 12 + 4 \times 4$$

$$Z(\min) = 88$$

$$Z(A) = 160$$
$$Z(B) = 132$$

$$x_1 = 12, \quad x_2 = \underline{4}$$



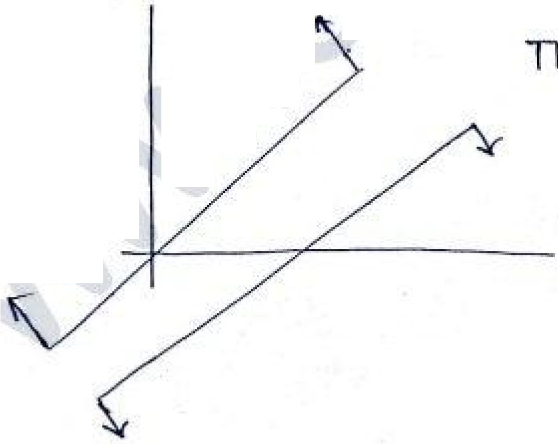

<u>Redundant</u>:- Constraint which does not become part of boundary making the feasible region is termed as redundant constrain. Inclusion or exclusion of such constraint doesnot have any effect on the final solution of problems

* Special cases.

1. Infinite or multioptimum solution:-

Infinite number of solution means we get the same optimum value of our objective function for different varying variable. We always get a unique solution when slope of objective functi is different from constraint.
Infinite no. of solution obtained when objective function slope become equal to one of binding constraint.

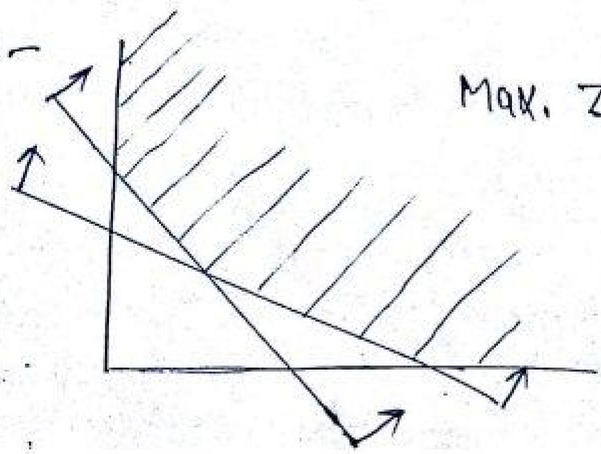
2. No Solution or Infeasibility:-



There is no common point.

In some condition contrains may be inconsistent in such a manner that it is not possible to find a feasible solution which satisfy all the constraint. There is no solution of such problem.

3. Unbounded Solution:-



Max. Z

In some condition the highest value of objective function goes upto infinite and it simply means that common feasible region is not bounded by the limit on the constraint it is termed as unbounded solution.

## Simplex method :-

it is a step by step procedure in which we proceed in a systematic manner from an intial feasible solution with an improve upon that Initial solution until certain in Number of steps we reach optimal solution This method also check the corner point of feasible region but in multidimension depending upon the variable

Standard form for simplex →

① All the resource value for the given constraint should be non-negative

e.g. $4x_1 - 3x_2 \leq -48$

$\Rightarrow -4x_1 + 3x_2 \geq 48$

② All the inequalities of the given constraint should be converted into equalities

e.g. $5x_1 - 2x_2 \leq 40$

$5x_1 - 2x_2 + S_1$

Slack variable

e.g. $3x_1 + x_2 \geq 75$

$3x + x_2 - S_2 = 75$

Surplus Variable

3. each of the decesion variable for the constraint and objective function should be non-negative and linear.

$$\boxed{x_j \geq 0}$$

$$\text{Max. } Z = c_1 x_1 + c_2 x_2 - - - + c_n x_n$$

Constrain

$$\left\{\begin{array}{l} a_{11} x_1 + a_{12} x_2 - - - + a_{1n} x_n \leq b_1 \\ a_{21} x_1 + a_{22} x_2 - - - + a_{2n} x_n \leq b_2 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 - - - + a_{mn} x_n \leq b_m \end{array}\right.$$

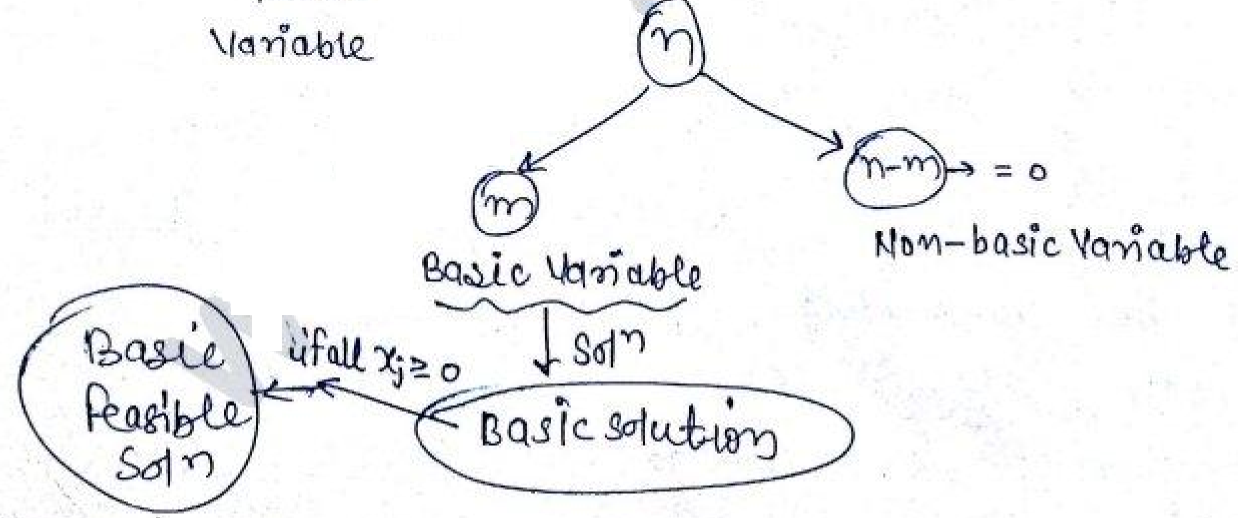
$$x_j \geq 0$$

4.

$$\boxed{n > m}$$

$m \leftarrow$ No. of equality Constraint

$n \leftarrow$ No. of Variable



If there are m equality constrain- and n is the number of variable and $n > m$, then we need to put $(n-m)$ variable equal to zero known as non-basic variable and solve the remaining. m basic variable to give basic solution This step reducos the no. of alternate solution whos maximum limit is given by—

$$\boxed{{}^{n}C_{m} = \frac{n!}{m!\,(n-m)!}}$$

e.g. $x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

$n = 5$

$m = 3$ So total no. of max sol$^n$ = ${}^{5}C_{3} = 10$

$n - m = 2 \rightarrow 0$

Problem (workbook)

Q.35 Page. 83 Max. $Z = 40x_1 + 35x_2$

$\rightarrow$ max. $Z = 40x_1 + 35x_2 + 0.s_1 + 0.s_2$

Raw material $\rightarrow$ $2x_1 + 3x_2 \leq 60$

labour hour $\rightarrow$ $4x_1 + 3x_2 \leq 96$

need slack variable

$2x_1 + 3x_2 + s_1 = 60$

$n = 4$

$4x_1 + 3x_2 + s_2 = 96$

$m = 2$

$n - m = 2 = 0$

start from origion

$x_1 = 0$ ; $x_2 = 0$ (Non basic variable)

1st feasible solⁿ

$x_1 = 0, \; x_2 = 0$

$S_1 = 60, \; S_2 = 96$

So $Z = Rs\ 0$

| (Profit coeff.) $e_i$ | Basis | $x_1$ | $x_2$ | $S_1$ | $S_2$ | Current solⁿ $b_i$ | $\theta_i = b_i / a_{ij}$ Replacement ratio (Corrospording to key Column) |
|---|---|---|---|---|---|---|---|
| 0 | $S_1$ | 2 | 3 | 1 | 0 | 60 | 30 (60/2) |
| 0 | $S_2$ | (4)* | 3 | 0 | 1 | 96 | 24 (96/4) ← Key Raw; mini |
| Profit coeff. $C_j$ | | 40 | 35 | 0 | 0 | | |
| $Z_j = \sum e_j a_{ij}$ | | 0 | 0 | 0 | 0 | | |
| $\Delta_j = C_j - Z_j$ | | 40 (max; key Column) | 35 | 0 | 0 | | |

Calculate $\Delta_j$ value as the difference of $\Delta_j = C_j - Z_j$ raw and it is termed as Net evaluation Raw (NER) for Net opportunity raw. The value of $\Delta_j$ raw give the amount of increase or decrease in the objective function that would occur if 1 unit represented by column head is brought into current solution. A simplex table indicate the current solution to be optimum when all the values in $\Delta_j$ raw are

1) Negative or zero when L.P. for maximization

2) Positive or zero when L.P. for minimize

As the current problem for maximize we select the highest positive value and selected column called key column with the variable in column head as incoming variable.

Now we devide $b_i$ value from the corresponding element of key column ($a_{ij}$) to get replacement column. In this column we select the minimum positive value and selected raw is called key raw and the variable in raw as outgoing variable.

The element of intersection of key Raw and key column is termed as key element

Step

1. key element is converted to unity by multiplying or deviding key raw by a common multiplying factor. All the element in the key column are made zero expect element which will be unity this is done by adding subtracting the proper multples of key raw from other raw. In the new table outgoing variable replace by incoming variable.

| $c_i$ | Basis | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $b_i$ |
|---|---|---|---|---|---|---|
| 0 | $s_1$ | 2 | 3 | 1 | 0 | 60 |
| 40 | $x_1$ | 4 | 3 | 0 | 1/4 | 96 |

$R_2 \to R_2 \div 4$

$R_1 \to R_1 - 2R_2$

Replace $S_2 \to x_1$

| $c_i$ | basic | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $b_i'$ |
|---|---|---|---|---|---|---|
| 0 | $S_1$ | 0 | $(3/2)^*$ | 1 | $-1/2$ | 12 |
| 40 | $x_1$ | 1 | $3/4$ | 0 | $1/4$ | 24 |

key element

2nd feasible solution

$x_1 = 24$, $x_2 = 0$ $\quad Z = \text{Rs.}960$

$S_1 = 12$, $S_2 = 0$

|  | $x_1$ | $x_2$ |  |  | $\theta_i = b_i / a_{ij}$ |
|---|---|---|---|---|---|
| $c_j$ | 40 | 35 | 0 | 0 | $8 \left(\frac{12}{3/2}\right)$ key Row |
| $z_j = \sum c_i . a_{ij}$ | 40 | 30 | 0 | 10 | $32 \left(\frac{24}{3/4}\right)$ |
| $\Delta_j = c_j - z_j$ | 0 | 5 | 0 | $-10$ |  |

Still one +ve Value

key Column

Now make key element (3/2) unity & rest zero

Replace $S_1 \to x_2$ in new table

$R_1 \rightarrow R_1 \times 2/3 \quad R_2 \rightarrow R_2 - \frac{3}{4} \cdot R_1$

| $c_i$ | basis | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $b_i$ |
|---|---|---|---|---|---|---|
| 35 | $x_2$ | 0 | 1 | $2/3$ | $-1/3$ | 8 |
| 40 | $x_1$ | 1 | 0 | $-1/2$ | $1/2$ | 18 |
| | $c_j$ | 40 | 35 | 0 | 0 | |
| | $z_j = \sum c_i \cdot a_{ij}$ | 40 | 35 | $10/3$ | $25/3$ | |
| | $\Delta_j = z_j - c_j$ | 0 | 0 | $-10/3$ | $-25/3$ | |

shadow cost or price

3rd feasible soln

$\Rightarrow x_1 = 18, \; x_2 = 8$

$S_1 = 0, \; S_2 = 0$

$Z = \text{Rs } 1000$

So the value of $\Delta_j$ is Negative & zero

So current soln is optimum

$x_1 = 18, \; x_2 = 8$

$Z = \text{Rs } 1000$ → ~~this price is called~~

Big M method :-

$\geq$ or $=$ (only for these types of constraint

$$4x_1 - 3x_2 \geq 80$$

$$4x_1 - 3x_2 - S_1 = 80$$

$$x_1 = 0,\ x_2 = 0,\ S_1 = -80$$

$$4x_1 - 3x_2 - S_1 + A_1 = 80$$

Non basic

$x_1 = 0$
$x_2 = 0$
$S_1 = 0$

$A_1 = 80$

It is modified form of simplex method it is always require whenever the contraint are $\geq$ or $=$ type irrespective of whether the problem is maximization of for minimization In these condition we introduce an artificial variable in the current solution to get an initial working Metrix. These artificial variable must not apear in final solution and that is ensure by proving an extreme -ve value to their profit coefficient in the objective function

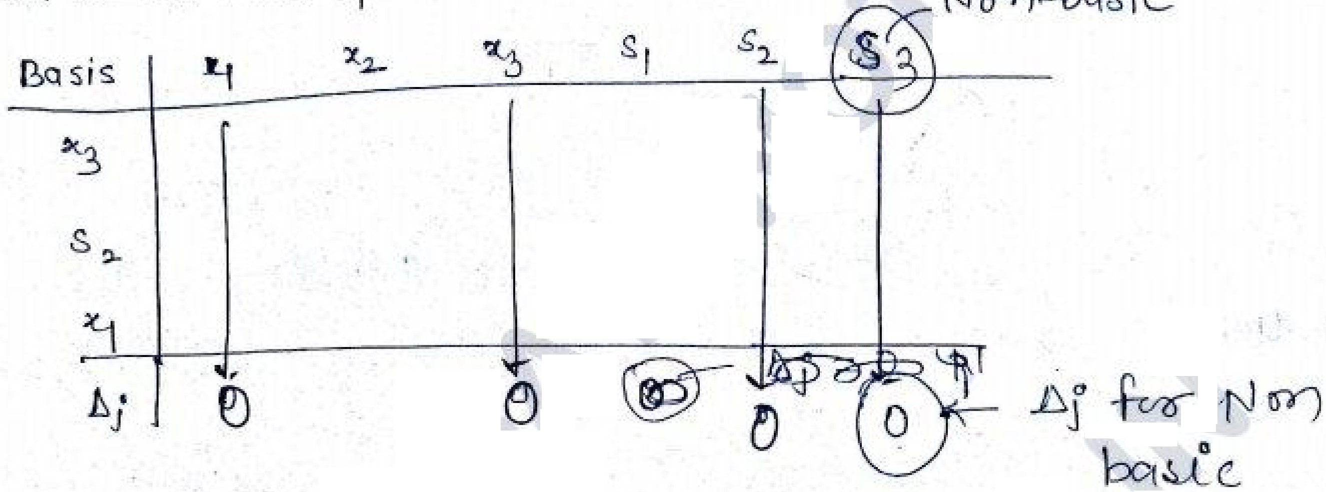
$$Max = -M \cdot A_1$$

$$Min = +M \cdot A_1$$

where M is the number higher than any finite Number

Special Cases :—

1. Infinite or Multi optimum Solution :-




when for a non basic variable in the optimum solution has zero value for $\Delta_j$ raw then the solution is not unique and it indicate that the problem has infinite no. of solutions.

2. Un-bounded Solution

$$\theta_j = \frac{b_i}{a_{ij}'} = -ve \text{ or } \infty \quad \text{(Unbounded Solution)}$$

If in a case all the value in the replacement ratio($\theta_j$) Column are either $-ve$ or infinite then the solution ~~then~~ terminate and it indicates that the problem has unbounded solution.

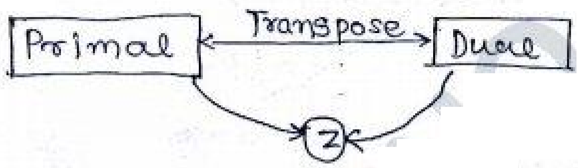
3. No solution of Infeasibility only in $\geq$ or $=$

when in the final solution artificial variable remain in the basis then their is no feasible solution to the problem.

4. Degenerate solution—

when one or more of the basic become equal to zero during calculation then the solution is called degenerate and the condition called degeneracy. In a degenerate solution the number of basic variable become less then equality constraint

## Duality




The initial given problem is termed as Primal and the problem obtained by transposing row and column but having same optimum value of the objective function is termed as dual

Primal

a) maximization $\leq$ type constraint

b) minimization $\geq$ type constraint

Dual of a Dual is primal

| Primal | | Dual |
|---|---|---|
| maximization | | Minimization |
| $m$ | $\longleftrightarrow$ | $n$ |
| $n$ | $\longleftrightarrow$ | $m$ |
| $b_i$ | $\longleftrightarrow$ | $C_j$ |
| $C_j$ | $\longleftrightarrow$ | $b_i$ |
| $\leq$ type Constraint | $\longleftrightarrow$ | Non negative Variable |
| $=$ type Constraint | $\longleftrightarrow$ | Unrestricted Sign Variable. |

Problem 38: Find the dual of following L.P. problem.

$$\min. \; z = 5x_1 - 9x_2 + 12x_3$$

$$3x_1 - x_2 + 7x_3 \geq 8$$

$$x_1 - 2x_3 \leq 9$$

$$5x_2 + 3x_3 \geq 10$$

$$2x_1 + 3x_2 \geq 4$$

$$4x_1 - 7x_2 + 4x_3 \leq 15$$

$$x_1, x_2, x_3 \geq 0$$

Sol$^n$

All Constraint $\geq$ type

| | |
|---|---|
| $y_1 \rightarrow 3x_1 - x_2 + 7x_3 \geq 8$ | $y_4 \rightarrow 2x_1 + 3x_2 \geq 4$ |
| $y_2 \rightarrow -x_1 + 2x_3 \geq -9$ | $y_5 \rightarrow -4x_1 + 7x_2 - 4x_3 \geq -15$ |
| $y_3 \rightarrow 5x_2 + 3x_3 \geq 10$ | |

| Primal | | Dual |
|---|---|---|
| $n = 3$ | | $n = 5$ |
| $m = 5$ | | $m = 3$ |

Dual

$$\text{Max. } z = 8y_1 - 9y_2 + 10y_3 + 4y_4 + 15y_5$$

Pick $x_1$ coefficient for 1st Constraint

Profit Value

$$3y_1 - y_2 + 2y_4 - 4y_5 \le 5$$

$x_2$ coeff.

$$-y_1 + 5y_3 + 3y_4 + 7y_5 \le -9$$

$x_3$ coeffi.

$$7y_1 + 2y_2 + 3y_3 - 4y_5 \le 12$$

$$y_1, y_2, y_3, y_4, y_5 \ge 0$$

# Transportation

Transportation

Destination $\rightarrow$

Factories or Source

| | $D_1$ | $D_2$ | | $D_n$ | Supply ↓ |
|---|---|---|---|---|---|
| $F_1$ | $x_{11}$ ($c_{11}$) | $x_{12}$ ($c_{12}$) | - - - - | $x_{1n}$ ($c_{1n}$) | $a_1$ |
| $F_2$ | $x_{21}$ ($c_{21}$) | $x_{22}$ ($c_{22}$) | - - - - | $x_{2n}$ ($c_{2n}$) | $a_2$ |
| ⋮ | ⋮ | ⋮ | - - - - | ⋮ | ⋮ |
| $F_m$ | $x_{m1}$ ($c_{m1}$) | $x_{m2}$ ($c_{m2}$) | - - - - | $x_{mn}$ ($c_{mn}$) | $a_m$ |
| Demand → | $b_1$ | $b_2$ | - - - - | $b_n$ | |

No. of variable = $m \times n$

No. of Constraint = $m + n - 1$

Condition

$$\boxed{\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j}$$

8(i)

Objective Function→ $\text{Min } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$

The aim of transportation problem is to meet the demand and supply requirement in most optimum and effective menner to minimize total transportation cost.

<u>Definations/Term</u>

① <u>Feasible Solution</u> :- A set of non-negative individual allocation with satisfy all the given constraint is termed as feasible solution.

② <u>Basic Feasible Solution</u> :- In $(m \times n)$ transportation problem if the total number of allocation become exactly equal to $(m+n-1)$ then the solution is called <u>basic feasible</u>.

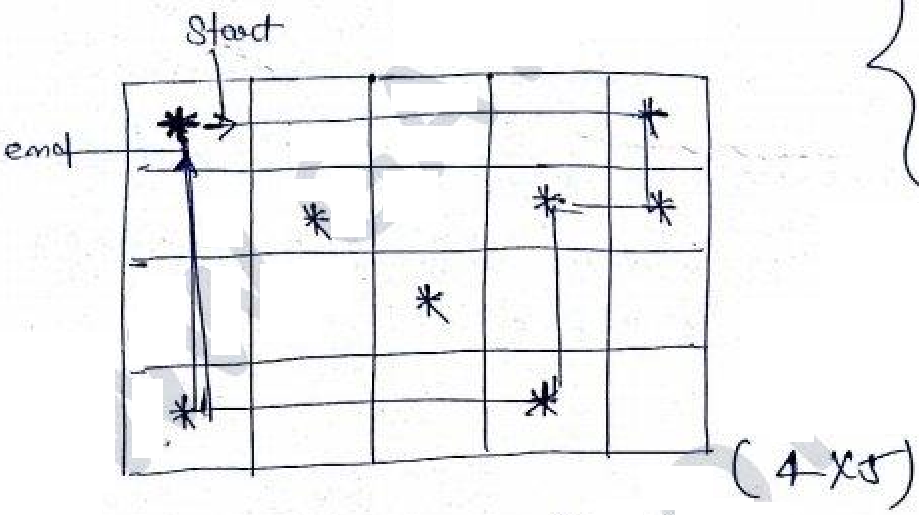
less than $(m+n-1)$ can <u>also come</u>

③ Non-degenerate Basic Feasible Solution :—

In a m×n transportation problem solution is called Non-degenerate when following two conditions are satisfy

i) Total no. of allocation become exactly equal to (m+n-1)

ii) these (m+n-1) allocations must be at independend possition.




it less than (m+n-1) Degenracy

(looping) Dependend

By Independend position we mean that it is always imposible to form a closed loop by joining these allocation, by the series of horizontal and vertical line from one allocated cell to another

Note:- Optimality Can only be preform when intial solution is ~~not~~ non-degenerate.

## Balanced and Unbalanced Transportation Problem:

If the total supply from all the factories equal to total demand from all the destination problem is called balanced otherwise unbalanced.
If the given problem is unbalanced, balance it by adding dummy source or destination.

e.g.

| | | | | | |
|---|---|---|---|---|---|
| dummy | 0 | 0 | 0 | 0 | 70 |
| | | | | | |
| | | | | | $\Sigma a_i = 280$ |
| | | | | | |

$\Sigma b_j = 350$

Problem 39 ~~Product~~ A Company transport product from three factories to four destination as given in balance. find optimum allocation to minimize transportation cost.

| | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| $P_1$ | 20 | 30 | 50 | 17 | 7 |
| $P_2$ | 70 | 35 | 40 | 60 | 10 |
| $P_3$ | 40 | 12 | 60 | 25 | 18 |
| | 5 | 8 | 7 | 15 | 35 / 35 |

balanced

Solⁿ

initial Solution

a) North-West Corner Rule :-

| | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| $P_1$ | 5 - start | 2 | | | 7/2/0 |
| $P_2$ | | 6 | 4 | | 10/4/0 |
| $P_3$ | | | 3 | 15 end | 18/15/0 |
| | 5/0 | 8/6/0 | 7/3/0 | 15/0 | |

$Z = 5\times20 + 30\times2 + 35\times6 + 40\times4 + 60\times3 + 25\times15$

$Z =$ Rs 1085

b) Raw - Minima :-

| | 1 | 2 | 3 | 4 Min. | |
|---|---|---|---|---|---|
| $P_1$ | 20 | 30 | 50 | 17 (7) | 7/0 |
| $P_2$ | 70 | 35 (8) | 40 (2) | 60 | 10/2/0 |
| $P_3$ | 40 (5) | 12 | 60 (5) | 25 (8) | 18/10/5/0 |
| | 5/0 | 8/0 | 7/5/0 | 15/8/0 | |

$Z = 7\times17 + 8\times35 + 2\times40 + \cancel{8}\,5\times40 + 5\times60 + 8\times25$

$Z =$ Rs 1179

(c) Column minima :-

| | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| $P_1$ | 20 (5) | 30 | 50 | 17 (2) | 7 |
| $P_2$ | 70 | 35 | 40 (7) | 60 (3) | 10/3 |
| $P_3$ | 40 | 12 (8) | 60 | 25 (10) | 18/10 |
| | 5/0 | 8/0 | 7/0 | 15/13/3/0 | |

$Z = 5 \times 20 + 8 \times 12 + 7 \times 40 + 2 \times 17 + 3 \times 60 + 10 \times 25$

$Z =$ Rs 940

(d) least Cost Method / Method of matrix minima.
सबसे कम वाला पहले

| | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| $P_1$ | 20 | 30 | 50 | 17 (7) | 7/0 |
| $P_2$ | 70 (3) | 35 | 40 (7) | 60 | 10/3/0 |
| $P_3$ | 40 (2) | 12 (8) | 60 | 25 (8) | 18/10/2/0 |
| | 5/3/0 | 8/0 | 7/0 | 15/8/0 | |

if same cost value then where max. possible allocation prefer that

$Z = 7 \times 17 + 3 \times 70 + 7 \times 40 + 2 \times 40 + 8 \times 12 + 8 \times 25$

$Z =$ Rs 985

imp.

(e) Vogel's Approximation Method (VAM) or Unit cost penalty — Most (Prefered Method)

| | 1 | 2 | 3 | 4 | | | |
|---|---|---|---|---|---|---|---|
| $P_1$ | 20 (5) | 30 | 50 | 17 (2) | 7/2/0 | 3, 13, 33 | 33 |
| $P_2$ | 70 | 35 | 40 (7) | 60 (3) | 10/3/0 | 5, 5, 20 | 20 |
| $P_3$ | 40 | 12 (8) | 60 | 28 (10) | 18/10/0 | 13, 13, 35 | |
| | 5/0 | 8/0 | 7/0 | 15/5/3/0 | | | |
| | 20 (I) | 18 | 10 | 8 | | | |
| | | 18 (II) | 10 | 8 | | | |
| | | | 10 | 8 → 10 | 43 | | |

Difference between smallest & 2nd smallest element in every raw and column

No select max. diff after full fill requirement delet that raw/column. Allocate at mini cost

Again take difference

if diff. equal give preference at min cost.

if cost same then prefer where max. allocation possible

$$Z = 5\times20 + 17\times2 + 40\times7 + 60\times3 + 12\times8 + 28\times10$$

$$Z = \text{Rs. } 940$$

# Vogel's Approximation Method

In this method we write the difference between smallest and second smallest element in each raw and Column. ~~betwo~~ below the respective raw and Column

Then we select the highest individual difference and the max. possibile allocation is done in the minimum cost cell of the selected raw or column. The raw or column whose requirment become zero is strike off so it can not be considered again Continue in the similar manner until all the allocation are done.

optimality

2 conditions

(i) Allocation $= m+n-1 = 4+3-1 = 6$

(2) should be independend

As the total no. of allocation is exactly equal to $(m+n-1) = 6$ and at independent position so optimality test can be perform.

## Stepping Stone Method-

| | | | |
|---|---|---|---|
| 20 (5) | 30 | 50 | 17 (2) |
| 70 | +3 35 | 40 (7) | −3 60 (3) |
| 40 | −3 12 (8) | 60 | +3 25 (10) |

Choose empty shell

60, 40,

| | | | | | |
|---|---|---|---|---|---|
| +60 | +40 | +30 | +35 | +70 | +50 |
| −25 | −20 | −17 | −60 | −20 | −17 |
| +60 | +17 | +25 | +25 | +17 | +60 |
| −40 | −25 | −12 | −12 | −60 | −40 |
| +55 | +12 | +26 | −12 | 7 | 53 |

In this method we allocated 1 unit in a unallocated empty shell and compute the effect on cost of matrix. it is an hit and trial method in which chances of making error are more and much prefforced.

## Modifield Distribution (MODI-method) or U-V method

Steps:-

① Develop Cost matrix for allocated shells only

② Computing $U_i$ & $V_j$ Value by taking $V_1 = 0$

| | $V_1$ | $V_2$ | $V_3$ | $V_4$ |
|---|---|---|---|---|
| $u_1$ | 20 | | | 17 |
| $u_2$ | | | 40 | 60 |
| $u_3$ | | 12 | | 25 |

$\boxed{V_1 = 0}$ $u_1 + V_1 = 20$ ; $u_1 + V_4 = 17$

$u_2 + V_3 = 40$ ; $u_2 + V_4 = 60$

$u_3 + V_2 = 12$ ; $u_3 + V_4 = 25$

| | 0 | -16 | -23 | -3 |
|---|---|---|---|---|
| 20 | 20 | | | 12 |
| 63 | | | 40 | 60 |
| 28 | | 12 | | 25 |

③ Develop $u_i + V_j$ matrix for unallocated shell by entering the summation of $u_i + V_j$ value for unallocated shell

| | 0 | -16 | -23 | -3 |
|---|---|---|---|---|
| 20 | – | 4 | -3 | – |
| 63 | 63 | 47 | – | – |
| 28 | 28 | – | 5 | – |

$u_i + V_j$ matrix for unallocated cells

(4) subtract the cell value of $u_i + v_j$ matrix for unallocated shell from the origional cost matrix to get cell evaluation matrix.

| | | | |
|---|---|---|---|
| | 26 | 53 | |
| 7 | -12 | | |
| 12 | | 55 | |

cell evaluation matrix

(5) If any of the cell value in the cell evaluation matrix is negative then the current solution is not optimum

(6) In cell evaluation matrix identify the cell with the most negative value, mark it and it is termed as identified cell

(7) Trace of a path in matrix such that it start from identified cell and corner of path should already have allocation

(8) Make identify cell positive and each other cell at a cornor of path alternatively -ve, +ve and so on.

9) make a new allocation in the identified cell by entering the smallest allocation on the path that has been assign a negative sign. The basic cell whose allocation become zero lead the solution.

Identifield cell

| | | | |
|---|---|---|---|
| | 26 | 53 | |
| 7 | −12 +3 | | −3 |
| 12 | −3 | 55 | +3 |

New allocated cell

| | | | |
|---|---|---|---|
| 20 (5) | 30 | 50 | 17 (2) |
| 70 | 35 (3) | 40 (7) | 60 |
| 40 | 12 (5) | 60 | 25 (13) |

Now $Z = RS\ 904$

Again $u_i + v_j$ matrix

| $u_i \downarrow$ \ $v_j \rightarrow$ | 0 | −16 | −11 | −3 |
|---|---|---|---|---|
| 20 | ~~20~~ | 4 | 9 | ~~17~~ |
| 51 | 51 | ~~35~~ | ~~40~~ | 48 |
| 28 | 28 | ~~12~~ | 17 | ~~25~~ |

Cell evaluation matrix

| | | | |
|---|---|---|---|
| | 26 | 41 | |
| 19 | | | 12 |
| 12 | | 43 | |

All cell value are positive so this our optimum solution

Special Cases of transportation.

1) Degeneracy: when the number of allocation become less then $(m+n-1)$ then optimality test can not be performed and such a solution called degenerate and the condition is know as degeneracy.

2) Maximization Problem:-

max. $\xrightarrow{100-\text{Matrix}}$ min.

| | | | |
|---|---|---|---|
| 40 | 70 | 20 | 50 |
| 90 | 40 | 100 | 70 |
| 10 | 80 | 50 | 90 |

| | | | |
|---|---|---|---|
| 60 | 30 | 80 | 50 |
| 10 | 60 | 0 | 30 |
| 90 | 20 | 50 | 10 |

maximization problems are solved by converting it into minimization this is done by subtracting from the highest element, all the element of matrix.

Problem 40: Unit transportation cost in Rs are give in the cost matrix below determine the initial feasible solution using Vogel's Apporximation and find the optimum distribution possible for Company.

| | D | E | F | G | |
|---|---|---|---|---|---|
| A | 44 | 50 | 40 | 39 | 180 |
| B | 42 | 51 | 54 | 53 | 170 |
| C | 41 | 40 | 42 | 45 | 200 |
| | 90 | 100 | 120 | 180 | 490 / 550 |

Sol^n first we check the problem is balance or not since problem is unbalanced a dummy destination

| | D | E | F | G | H | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| A | 44 | 50 | 40 | 39 180 | 0 | 180/0 | 39 | 1 | 1 | 1 |
| B | 42 90 | 51 | 54 20 | 53 | 0 60 | 170/110/20 | 42 ← | 9 | 11 ← | 1 |
| C | 41 | 40 100 | 42 100 | 45 | 0 | 200/100/0 | 40 | 1 | 1 | 3 |
| | 90/0 | 100/0 | 120/20/0 | 180/0 | 60/0 | | | | | |
| | 1 | 10 | 2 | 6 | 0 | | | | | |
| | 1 | 10 ↑ | 2 | 6 | | | | | | |
| | 1 | | 2 | 6 | | | | | | |
| | | | 2 | 6 ↑ | | | | | | |

Z = Rs 20080

As the number of allocation is 6 which is less the $(m+n-1) = (5+3-1) = 7$ So the current solution is degenerate

→ Now allocating infinitly small but positive Value 'ε' at vacent minimum cost cell such that allocation remain at independent possition in the final solution we put $\epsilon = 0$

$u_i$ & $v_j$ matrix

| | 0 | 10 | 12 | -3 | -42 |
|---|---|---|---|---|---|
| 42 | | | $\epsilon$ | 39 | 0 ($\epsilon$) -6 |
| 42 | 42 (90) | [illegible] (20) | 0 54 (20) | | 0 60 |
| 30 | | (80) 40 (100) | (20) 42 (100) | | |

$$\begin{array}{r} +51 \\ -54 \\ +42 \\ -40 \\ \hline -1 \end{array}$$

$Z = 20080 - 20$

$Z = Rs\ 20060$

Assignment :-

1. square matrix $(n = m)$     Objective function

2. $x_{ij} = 0$ or $1$     $\left\{ \text{Min } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \cdot x_{ij} \right.$

at $a_i = 1$

& $b_j = 1$

$\square$ – Allocation $(x_{ij} = 1)$

$\times$ – Non Allocation $(x_{ij} = 0)$

Assignment problem are a special case of transportation where a matrix must be a square matrix and ~~...~~ every raw every column only one allocation is possible.

Problem 41 four technician ~~require~~ requir to perform a job
ESE 2009 whose time are given in hrs in table given below assign the job to technician to get total work time mi...

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $T_1$ | 20 | 36 | 31 | 27 |
| $T_2$ | 24 | 34 | 45 | 22 |
| $T_3$ | 22 | 45 | 38 | 18 |
| $T_4$ | 37 | 40 | 35 | 28 |

8(ii)

# Hungarian Method (Flood's technique)

1. Develope opportinuity Cost matrix

   a) Subtract the smallest element in each raw from every element of the corrosponding raw

| | | | |
|---|---|---|---|
| 0 | 16 | 11 | 7 |
| 2 | 12 | 23 | 0 |
| 4 | 27 | 20 | 0 |
| 9 | 12 | 7 | 0 |

   b) Subtract the smallest element in each Column from every element of the corrosponding Column.

| | | | |
|---|---|---|---|
| 0 | 4 | 4 | 7 |
| 2 | 0 | 16 | 0 |
| 4 | 15 | 13 | 0 |
| 9 | 0 | 0 | 0 |

Apportinuity Cost matrix
{Atleast one zero in every column & raw

2. Make allocation in the opportinity cost matrix

First Allocated in single 0 raw/ column.

| | | | |
|---|---|---|---|
| $\boxed{0}$ | 4 | 4 | 7 |
| 2 | $\boxed{0}$ | 16 | $\cancel{0}$ |
| 4 | 15 | 13 | $\boxed{0}$ |
| 9 | $\cancel{0}$ | $\boxed{0}$ | $\cancel{0}$ |

if No. of allocation = Size of matrix
4 = 4
So this out solution

$Z = 20 + 34 + 35 + 18$

$Z = 107$ hr

If the number of allocation is exactly equal to the size of matrix then the current solution optimum otherwise perform optimality

Problem 42 - Solve the following assignment problem for minimization of cost.

| | | | |
|---|---|---|---|
| 20 | 30 | 40 | 50 |
| 40 | 50 | 60 | 70 |
| 70 | 80 | 90 | 80 |
| 30 | 50 | 80 | 40 |

Sol^n

Raw wise substraction

| | | | |
|---|---|---|---|
| 0 | 10 | 20 | 30 |
| 0 | 10 | 20 | 30 |
| 0 | 10 | 20 | 10 |
| 0 | 20 | 50 | 10 |

→

Column wise subtraction

| | | | |
|---|---|---|---|
| ~~0~~ | $\boxed{0}$ | ~~0~~ | 20 |
| ~~0~~ | ~~0~~ | $\boxed{0}$ | 20 |
| ~~0~~ | ~~0~~ | ~~0~~ | $\boxed{0}$ |
| $\boxed{0}$ | 10 | 30 | ~~0~~ |

So $Z = 30 + 60 + 80 + 30 = Rs. 200$

search single zero if not find then search double and assign after the again search single zero and repeat

Problem 43: Solve the following Assignment for cost minimization

| | | | | | |
|---|---|---|---|---|---|
| 9 | 22 | 58 | 11 | 19 | 27 |
| 43 | 78 | 72 | 50 | 63 | 48 |
| 41 | 28 | 91 | 37 | 45 | 33 |
| 74 | 42 | 27 | 49 | 39 | 32 |
| 36 | 11 | 57 | 22 | 25 | 18 |
| 3 | 56 | 53 | 31 | 17 | 28 |

Soln After raw and column wise substraction opportinuty cost matrixe

$$
\begin{array}{cccccc|l}
\cancel{0} & 13 & 49 & \boxed{0} & \cancel{0} & 13 & \\
\cancel{0} & 35 & 29 & 5 & 10 & \boxed{0} & \checkmark (3a) \\
13 & \cancel{0} & 63 & 7 & 7 & \cancel{0} & \checkmark (1) \\
47 & 15 & \boxed{0} & 20 & 2 & \cancel{0} & \\
25 & \boxed{0} & 46 & 9 & 4 & 2 & \checkmark (3b) \\
\boxed{0} & 53 & 50 & 26 & 4 & 20 & \checkmark (5) \\
\hline
\checkmark (4) & (2a) \checkmark & & & & (2b) &
\end{array}
$$

matrix size = 6    No of Allocation = 5

As the size of matrix is 6 and the no. of allocation is 5, so the current solution is not optimum.

Now we proced to find the minimum Number of line require to cover all zero at least once and the step involved are

1. mark all raw for which allocation has not been made. (i.e. 3rd Raw)

2. mark all column which have unassigned zero in the marked raw. (2nd & 6th Column.)

3. Mark all raw which have assignment in the marked Column (2nd & 5th raw)

4. Continue step 2 and 3 until chain of mark end.

5. Draw the minimum number of lines through unmark raw and through marked Column to Cover all zero atleast once.

6. Select the smallest element that do not have line through them subtract it from all the element that do not have line through them, add it to every element at the intersection of two lines and leave the remaining element of matrix unchanged make allocation in New opportunity Cost matrix

| | | | | | |
|---|---|---|---|---|---|
| 4 | 17 | 49 | $\boxed{0}$ | $\cancel{0}$ | 17 |
| $\boxed{0}$ | 35 | 25 | 1 | 6 | $\cancel{0}$ |
| 13 | $\cancel{0}$ | 59 | 3 | 3 | $\boxed{0}$ |
| 51 | 19 | $\boxed{0}$ | 20 | 2 | 4 |
| 25 | $\boxed{0}$ | 42 | 9 | $\cancel{0}$ | 2 |
| $\cancel{0}$ | 53 | 46 | 22 | $\boxed{0}$ | 20 |

$Z = 43 + 11 + 27 + 11 + 33 + 17 = \underline{\underline{142}}$